Pythagoras’ Theorem!

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Area of Different Shapes

One of the problems many students struggle with is the idea of how the area of shapes change as the lengths of their sides change.

What appears to be a simple problem can confuse students.

Let us consider a simple example!
What is the Area of this Shape?

What happens though if we wanted the area in cm\(^2\) ?

The area of a rectangle is length times width:

area = 5 \times 2 = 10 \text{ m}^2
What is the Area in cm$^2$?

1,000 cm$^2$

0.001 cm$^2$

0.1 cm$^2$

100,000 cm$^2$
What Happens to the Area if we Change all of the Length Scales?

• This is more challenging than it may sound.
• What happens to a simple square if we double, triple or quadruple the length scales?
• Let us investigate what happens to the area of a square, rectangle and triangle when we change the length scales.
Draw the square, and then move the slider to change the length of the square. What do you notice about the area?
Area of a Square - Explanation!

If we double the length scales, we get four times the number of squares, so the area is four times greater. If we triple the length scales, we get nine times the number of squares!
Area of a Rectangle!

Length of Rectangle: [ ]
Height: [ ]
Area: [ ]

Draw the rectangle, and then move the slider to change the length scales of the rectangle. What do you notice about the area?
Area of a Rectangle - Explanation!

If we double the length scales, we get four times the number of rectangles, so the area is four times greater. If we triple the length scales, we get nine times the number of rectangles!
Area of a Triangle!

Draw the triangle, and then move the slider to change the length scales of the triangle. What do you notice about the area?
If we double the length scales, we get four times the number of triangles, so the area is four times greater. If we triple the length scales, we get nine times the number of triangles!
Area of a Shape

• Hopefully it should be clear how the area of a shape scales when we change the length scales.

• Now let us try and explain why!
The Area of a Shape

The area of any shape is given by the following:

\[ \text{Area} = \text{Constant} \times \text{Length}^2 \]

- Where we have defined the length (or more simply \( L \)) as the longest side of the shape.
- And the constant is different for every shape we can think of!
- Let us find the result for a few different shapes.
- Move the slider to change the length scales!
Area of a Square
Constant for the Square

If the length of the square is:  \( L \text{ units} \)

The quantity \( L^2 \) has the value:  \( \text{units}^2 \)

The area of the square is:  \( \text{units}^2 \)

Therefore the constant has the value =  \( \text{units}^2 \)
As the constant is 1 we find the area of the square is simply

\[ \text{Area} = \text{Length}^2 \]

And it should be clear now, that if we double the length, then the area quadruples!
Area of a Rectangle
Constant for the Rectangle

If the length of the rectangle is: \[ \square \] units

The height of the rectangle is: \[ \square \] units

The quantity \( L^2 \) has the value: \[ \square \] units\(^2\)

The area of the rectangle is: \[ \square \] units\(^2\)

Therefore the constant has the value = \[ \square \]

Clear

Reveal
We should see that the area of a rectangle is given by:

\[ \text{Area} = C \times \text{Length}^2 \]

where \( C \) is the ratio of the height to the length!
Constant for a General Rectangle

If the length of the rectangle is: $\underline{\quad}$ units

And if the height of the rectangle is: $\underline{\quad}$ units

The quantity $L^2$ has the value: $\underline{\quad}$ units$^2$

The area of the rectangle is: $\underline{\quad}$ units$^2$

Therefore the constant has the value = $\underline{\quad}$

Clear
Area of a Triangle
Constant for a Triangle

If the length of the triangle is:                      units

The height of the triangle is:                        units

The quantity $L^2$ has the value:                     units$^2$

The area of the triangle is:                          units$^2$

Therefore the constant has the value =
We should see that the area of a triangle is given by:

\[ \text{Area} = C \times \text{Length}^2 \]

where \( C \) is half the ratio of the height to the length!
Constant for a General Triangle

If the length of the triangle is: \( L \) units

And if the height of the triangle is: \( H \) units

The quantity \( L^2 \) has the value: \( L^2 \) units\(^2\)

The area of the triangle is: \( \frac{1}{2} \times \text{base} \times \text{height} \) units\(^2\)

Therefore the constant has the value = \( \) units
Can You Find the Constant?

In the next slide you will be given a series of shapes and some possible values for the constants.

Can you match up the constant with the shape?
Match the shape to the constant!

<table>
<thead>
<tr>
<th>Constant</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.375</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td></td>
</tr>
</tbody>
</table>
Proof For Pythagoras’ Theorem!

• We would first like to begin by giving a proof of Pythagoras’ Theorem.
• You will begin with a right angled triangle which you will draw three other right angled triangles on.
Proof of Pythagoras’ Theorem
Using Right Angled Triangles

STEP 1
We want to first draw similar right angled triangles to each side of our initial triangle.
Proof of Pythagoras’ Theorem Using Right Angled Triangles

**STEP 1**
We want to first draw similar right angled triangles to each side of our initial triangle.

**STEP 2**
What happens now if we fold the two smaller triangles over on to the initial triangle?
Proof of Pythagoras’ Theorem
Using Right Angled Triangles

STEP 1
We want to first draw similar right angled triangles to each side of our initial triangle.

STEP 2
What happens now if we fold the two smaller triangles over on to the initial triangle?

STEP 3
We should notice that when we fold over the two smaller triangles they cover the area of the whole triangle.
Proof of Pythagoras’ Theorem

• It should be clear that the area of the blue triangle is the same as the original black triangle.
• We can therefore see that the sum of the two smaller areas is equal to the area of the bigger triangle, i.e.

   \[ \text{Area Red} + \text{Area Green} = \text{Area Blue} \]

• This is the proof complete! Let us see why!
In the next few slides, you will be given a right angled triangle.

You will also be given a variety of different shapes, including a square, rectangle and triangle. You are required to fit a shape to each side of your right angled triangle.

It is important that you draw, place, rotate if needed each shape before you move to the next one.

What do you notice about the area of the three shapes?
How to Perform the Proof

• You will be asked to draw each shape by first clicking on the button. The shape appears nears the bottom left hand side of the screen.

• You can then use the sliders to change the size of the shape, the position on the screen (Horizontal and Vertical sliders) and the orientation of the shape. Remember to click on the slider button!

• Have a play and see if you can fit a shape to the edge of each side of the triangle.
Using Squares
Prove Pythagoras Using Squares

What do you notice about the area of the three squares? Can you explain why?

Area: 4

Area: 3

Area: 5
Using Rectangles
Prove Pythagoras Using Rectangles

What do you notice about the area of the three rectangles? Can you explain why?
Using Triangles
Prove Pythagoras Using Triangles

What do you notice about the area of the three triangles? Can you explain why?
The Proof

• What we have seen is that the sum of the areas of the two smaller shapes adds up to the total area of the bigger shape, so that

\[ \text{Area1} + \text{Area2} = \text{Area3} \]

• So we have

\[ C \times \text{Length1}^2 + C \times \text{Length2}^2 = C \times \text{Length3}^2 \]

\[ \text{Length1}^2 + \text{Length2}^2 = \text{Length3}^2 \]
We Could Actually Use Any Shape!

Draw Semi Circles

Radius: [ ]
Area: [ ]

As you can see it also works when we consider semi-circles!

Radius: [ ]
Area: [ ]
The Proof

• We should hopefully recognise this as Pythagoras’ Theorem!

\[ a^2 + b^2 = c^2 \]
An Alternative View!

\[ a^2 + b^2 = c^2 \]
An Alternative View - An Explanation!

\[ a^2 + b^2 = c^2 \]
Conclusion

• The area of any shape regardless of its orientation is given by

\[ \text{Area} = C \times \text{Length}^2 \]

• We can use this result to prove Pythagoras’ Theorem. It does not matter which shape we use in the proof as the constant will cancel each time!
Further Information

• Over the past few years here in the mathematics department we have been creating some interactive materials to help teach A-Level mathematics.
• You can find some examples of what we have been doing at the following pages:

Older materials
http://www2.le.ac.uk/departments/mathematics/undergraduate-courses/further-learning-resources

Updated materials
https://docs.google.com/open?id=0Byz1qbOsxdYnX0dIcXdXcXhhbEE

Or our new website
www.kineticmaths.com