**GOAL:** To explain and compute an example of the following result of Kai Behrend:

Let $G$ be an algebraic group over $\mathbb{F}_p$, then the classifying stack $BG$ satisfies \( \#BG(\mathbb{F}_p) = \frac{1}{\#G(\mathbb{F}_p)} = p^{-\dim G} \text{tr}(\Phi_p \mid H^*_{et}(BG, \mathbb{Q}_l)). \)

Moreover, there is a complex analytic zeta function given by \( \zeta_{BG}(s) = \exp \left( \sum_{r=1}^{\infty} \#BG \left( \mathbb{F}_p^r \right) \frac{s^r}{r} \right). \)

### Groups

A group is a mathematical object that encodes symmetries. It has a set of elements and a way to combine them using a group operation to get another element of the group. The most common example is the integers $\mathbb{Z}$, along with addition $+$. 

### Modular Arithmetic

We use modular arithmetic every day. For example, we know that 13:00 is the same as 01:00 on a 24 hour clock. Mathematically we say that 13 is congruent 1 modulo 12. Of great interest are numbers modulo $p$, where $p$ is a prime number. The field $\mathbb{F}_p$ has the set of elements $0, 1, \ldots, p-1$ with addition and multiplication modulo $p$. An algebraic group over $\mathbb{F}_p$ is a group whose elements take values somehow in $\mathbb{F}_p$.

### Classifying Stacks

Given a group, we can look at special objects which interact with the group in a nice way, called principal bundles. We can ask when two such bundles are the same up to symmetry. The collection of all principal bundles up to isomorphism give rise to an abstract object called the classifying stack of the group. We denote the classifying stack as $BG$. Classifying stacks are useful as they contain all the information about the group and they have nice properties such as being a topological space. Classifying stacks can also interact with other algebraic stacks, objects of study in algebraic geometry.

### Order of Groups

If a group has a finite amount of elements, we can count them. We write this number as $#G$. The first statement in the theorem tells us how we can compute the number of elements in the classifying stack of $G$ in two different ways, each giving the same answer. We count the elements of a stack in an unusual way, which means that the answer is a fraction! We see that the first way of computing $#BG$ just uses the number $#G$ so is quite easy to compute.

### Cohomology

The second way to compute $#BG$ uses the theory of cohomology. The term $H^*_{et}(BG, \mathbb{Q}_l)$ is known as the $\ell$-adic cohomology of the classifying stack, and can be extremely difficult to compute. In short, cohomology is a way of assigning an algebraic cohomology to a topological space. The function $\Phi_p$, appearing in the theorem is an operator that we can apply to this cohomology. In particular, we are interested in the way the invariant changes if we multiply the elements of our group by a fixed number. This usually takes the form of an infinite sum of terms.

### Zeta Functions

Finally, the theorem tells us how to assemble a zeta function. A zeta function is a way of bundling the information about $#BG$ for a family of $\mathbb{F}_p$. These functions take values in complex numbers, and by evaluating it at special values we can infer information about our group.

### Example

We will look at the simplest example, $G = \mathbb{F}_p^\times$, which is just $\mathbb{F}_p$ with zero removed. This can be thought of as a finite version of a circle. We can see using the above that $#G = p-1$, therefore $#BG = (p-1)^{-1}$. However, we know that we can also compute $#BG$ using the $\ell$-adic cohomology. Eventually, after some deep mathematics that cannot be contained in a poster, we can compute using the $\ell$-adic cohomology that $#BG = p^{-1} + p^{-2} + p^{-3} + \cdots$. By the theorem we have therefore proved that:

$$ \frac{1}{p-1} = \sum_{d \mid (p-1)} \frac{1}{p^d} $$

which is a result which can be proved using 1st year analysis! Finally we can compute the zeta function to be, after some manipulation:

$$ \zeta_{BG}(s) = \prod_{d \mid (p-1)} (1 - p^{-d}z)^{-1} $$

(A plot of the absolute value of this function in the unit circle makes up the background of this poster!)

### Importance of Research

Zeta functions are of interest in many areas of mathematics where you may want to package up a collection of information. The most famous zeta function, the Riemann zeta function, is concerned with the properties of prime numbers which are highly mysterious. The Riemann Hypothesis (which asks for a proof of a certain property of the Riemann zeta function) is one of the Clay Institute Millennium Problems, and carries a prize of £1m. Although the zeta functions here are of a different nature, they are still closely linked with the Riemann zeta function. They are also interesting in their own right, classifying stacks see use in areas such as representation theory, string theory, and gauge theory.

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