Abstract: A regular operad on Dyck paths

For $m \geq 1$, the $m$-Dyck paths are a particular family of lattice paths counted by Fuss-Catalan numbers, which are connected with the (bivariate) diagonal coinvariant spaces of the symmetric group, also called Garsia-Haiman spaces. The cardinal of the set of 1-Dyck paths of size $n$ is the Catalan number $c_n$, and there exist a natural bijection between this type of lattice paths and the set of planar rooted binary trees with $n + 1$ leaves, the classical Tamari order is defined on both sets. Motivated by the combinatorics of the Garsia-Haiman spaces and by an enumerative formula of Chapoton counting intervals in the Tamari lattice, F. Bergeron introduced in [1] and [2] the $m$-Tamari lattice, where the case $m = 1$ is the usual Tamari lattice.

In a joint work with D. López N. and L.-P. Préville-Ratelle, we introduced a non-symmetric multiplicative operad Dyck$^m$ such that the space of $n$-ary operations of the theory is precisely the vector space $\mathbb{K}[\text{Dyck}^m_n]$, spanned by all the $m$-Dyck paths of size $n$, for any $m \geq 1$. When $m = 1$, we recover the operad of dendriform algebras, introduced by J.-L. Loday in [3].

Given an $m$-Dyck path of size $n$, there is a unique way to color its down steps with elements of the set $\{1, \ldots, n\}$ in such a way that F. Bergeron’s covering relation consists in increasing the level of a down step without changing its color. This condition characterizes the order and is the key ingredient of our construction. The operad Dyck$^m$ is spanned by $m + 1$ binary operations $\ast_0, \ldots, \ast_m$, which are given by intervals of F. Bergeron’s $m$-Tamari lattice.

For any composition $\underline{r}$ of $m + 1$ of length $s + 1$, there exists an operad morphism from Dyck$^s$ to Dyck$^m$, compatible under composition. The corresponding functor from the category of Dyck$^m$ algebras to the category of Dyck$^s$ algebras preserves free objects.

References


IMAFI, Universidad de Talca, Campus Norte, Camino Lircay s/n, Talca. Chile
E-mail address: mariaronco@inst-mat.utalca.cl