A cluster category of type A_∞

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arXiv:0902.4125

Workshop on Derived Categories in Algebra, Topology and Geometry

Leicester, 17.09.2009

Outline

1. Motivation:
   Categorification of cluster algebras

2. Definition of the category D of type A_∞

3. Combinatorics on the AR-quiver

4. Cluster tilting subcategories of D and
   triangulations of the ∞-gon

5. Mutations and cluster structure

1. Categorification of cluster algebras

S. Fomin and A. Zelevinsky introduced cluster algebras around 2000.

Cluster algebra:

– commutative $\mathbb{Q}$-algebra $\subseteq \mathbb{Q}(x_1, \ldots, x_n)$

– generated by cluster variables

– cluster variables occur in clusters

– starting from an initial seed (containing generators), say $x_1, \ldots, x_n$, perform mutations to create new seeds, whence new generators

– iterate this mutation process

Cluster algebras from quivers
(symmetric, without coefficients)

Q quiver, without loops and oriented 2-cycles

Seed has the form (R, u) where R quiver, $u = \{u_1, \ldots, u_n\}$ algebraic independent generators of $\mathbb{Q}(x_1, \ldots, x_n)$

Mutation at k, giving new seed (R', u'):

– quiver mutation at vertex k of R

– $u' = u \setminus \{u_k\} \cup \{u'_k\}$ where

$$u_k u'_k = \prod_{i \to k} u_i + \prod_{k \to j} u_j$$

$\longrightarrow$ Cluster algebra $\mathcal{A}_Q$

Categorification of cluster algebras
(acyclic case)

Q quiver without oriented cycles, n vertices

cluster algebra $\mathcal{A}_Q$, $\quad K = \overline{K}$ field

Buan-Marsh-Reineke-Reiten-Todorov (2006):
cluster category $\mathcal{C}_Q = D^b(KQ)/(\tau^{-1} \circ \Sigma)$
where τ AR translation, Σ suspension

$\mathcal{C}_Q$ triangulated (Keller '05)

Object T in $\mathcal{C}_Q$ is rigid if $\mathrm{Ext}^1(T, T) = 0$.

$T_1, \ldots, T_n$ (pairwise non-isom.) form a cluster tilting set (c.t.s.) if $\mathrm{Ext}^1(T_i, T_j) = 0$ for all i, j.

Thm (BMRRT, Caldero-Keller) Bijections

rigid indec^s in $\mathcal{C}_Q \longleftrightarrow$ cluster variables of $\mathcal{A}_Q$

cluster tilting sets in $\mathcal{C}_Q \longleftrightarrow$ clusters of $\mathcal{A}_Q$

mutation of c.t.s. $\longleftrightarrow$ seed mutation in $\mathcal{A}_Q$

2. Definition of the category D

General setup

X topological space, k a field

Singular cochain complex $C^*(X, k)$

- $\mathbb{Z}$-graded k-algebra
- cup product satisfying Leibniz rule
  $\delta(r \cup s) = \delta(r) \cup s + (-1)^{|r|} r \cup \delta(s)$

$C^*(X, k)$ a Differential Graded algebra over k

R a Differential Graded (DG) algebra

$D(R)$ derived category of DG R-modules

$D^c(R)$ subcategory of compact DG R-modules

Specializing to spheres

2-sphere $X = S^2$

$C^*(S^2, k)$ singular cochain DG algebra

$C^*(S^2, k)$ has cohomology in degrees 0 and 2

$C^*(S^2, k)$ quasi-isomorphic to the DG algebra S obtained by placing k in cohomological degrees 0 and 2.

$D^c(C^*(S^2, k))$ equivalent to $D^c(S)$

'Koszul duality': $D^c(S)$ equivalent to the finite derived category of the DG algebra $R = k[T]$ with zero differential where T is placed in homological degree 1, i.e.

$$R : \ldots \to T^4 \xrightarrow{0} T^3 \xrightarrow{0} T^2 \xrightarrow{0} T \xrightarrow{0} k \to 0 \ldots.$$

Definition: D $:= D^f(k[T])$.

The finite derived category is formed by the DG R-modules having finite dimensional homology.

Properties of the category D (Jørgensen)

- D has finite-dimensional Hom spaces over k, split idempotents, so is a Krull-Schmidt category

- D is a 2-Calabi-Yau category, i.e. $S = \Sigma^2$ is a Serre functor (S^2 is a simply connected Poincaré duality space)

- D has Auslander-Reiten triangles, and Auslander-Reiten translation $\tau = S\Sigma^{-1} = \Sigma$

- D has AR-quiver $\mathbb{Z}A_\infty$.

Indecomposable objects of D

For each integer $r \geq 0$ there is a DG R-module

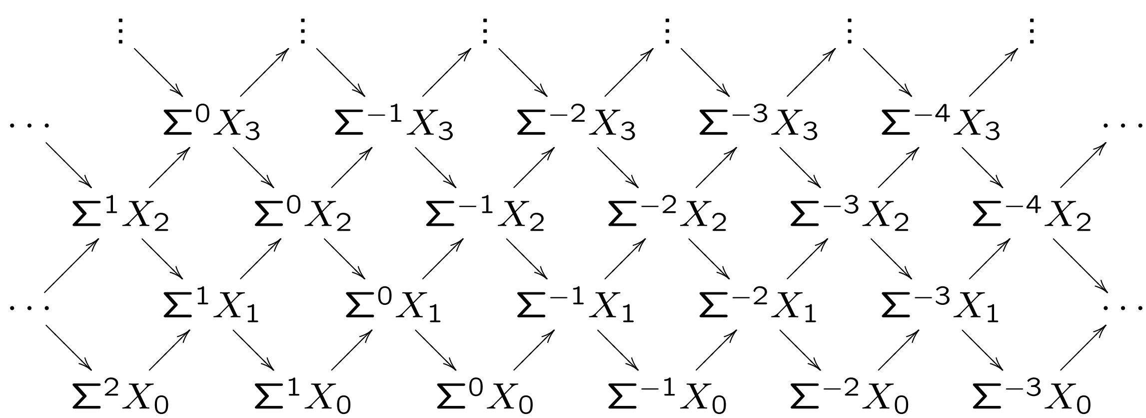
$$X_r := R/(T^{r+1})$$

concentrated in homological degrees 0 to r (and with zero differential).

Indecomposable objects of D are

$$\Sigma^j X_r \quad \text{where} \;\; j, r \in \mathbb{Z},\; r \geq 0.$$

AR-quiver $\mathbb{Z}A_\infty$ has the form



Example

$$\begin{array}{rccccccccc}
X_1: & 0 & \to & T & \to & k & \to & 0 & \to & 0 \\
 & & & \downarrow & & \downarrow & & & & \\
\Sigma^{-1} X_2: & 0 & \to & T^2 & \to & T & \to & k & \to & 0
\end{array}$$

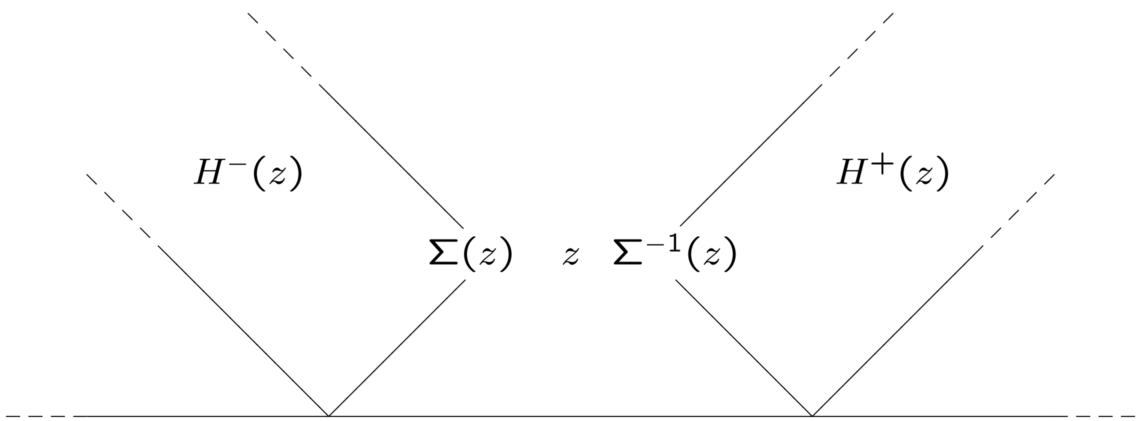
4. Combinatorics on the AR-quiver

Crucial proposition (Hom spaces)

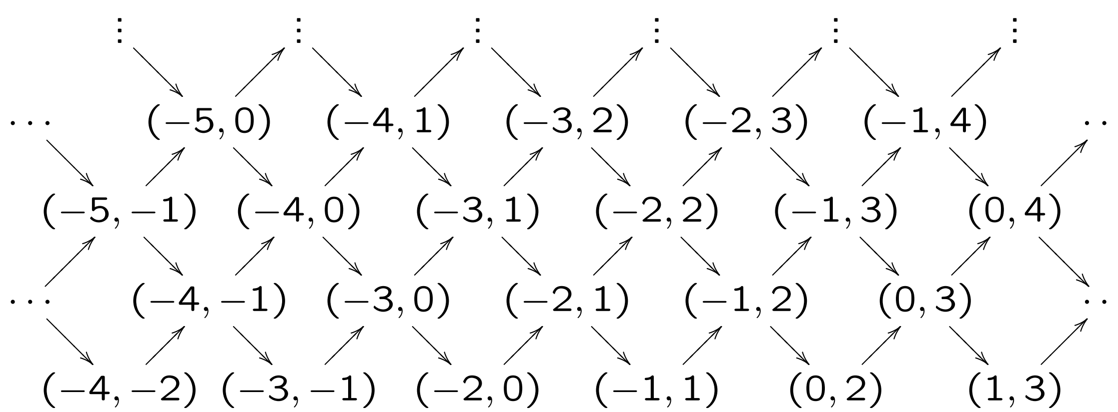
Let $x, y \in \mathsf{D}$ be indecomposable objects. Then

$$\mathrm{Hom}_{\mathsf{D}}(x,y) = \begin{cases} k & \text{for } y \in H(\Sigma(x)) \\ 0 & \text{otherwise} \end{cases}$$

where $H(z) = H^{+}(z) \cup H^{-}(z)$ for $z \in \mathsf{D}$.

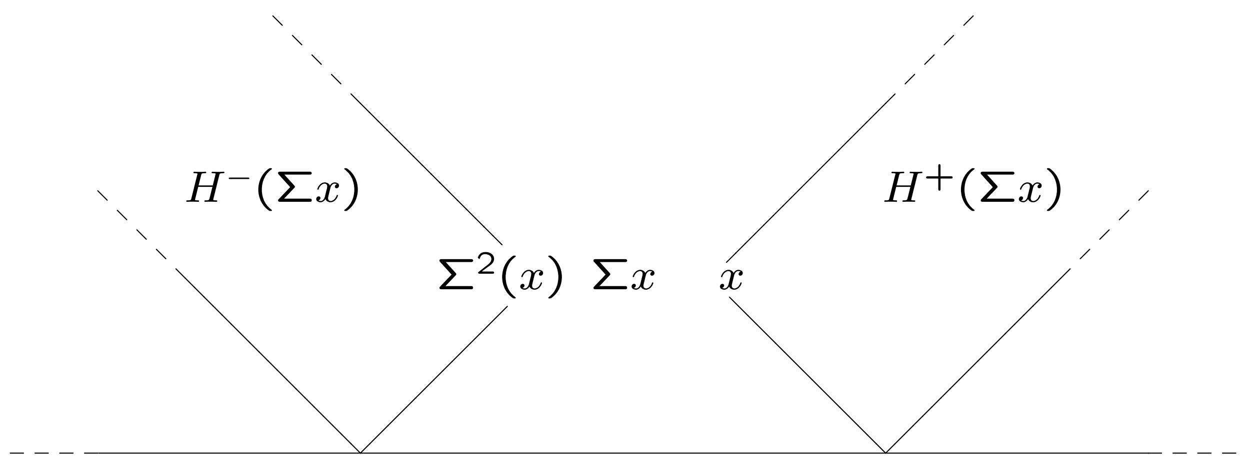


Coordinate system on AR quiver (Iyama)

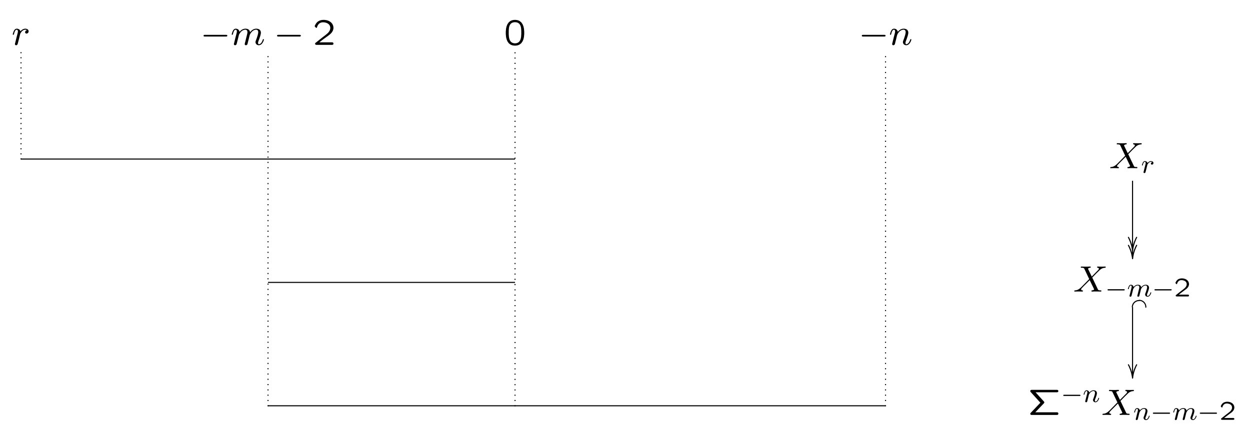


On the proof

(i) Forward morphisms $x \to H^{+}(\Sigma(x))$:



W.l.o.g. $x = X_r$ with coordinates $(-r-2, 0)$. Take $y = (m, n)$ in $H^{+}(\Sigma x)$, hence we have $-r-2 \leq m \leq -2$ and $0 \leq n$. The non-zero parts of the DG modules x and y overlap as follows, giving a canonical map



(2) Backward morphisms $H^{-}(\Sigma(x)) \leftarrow x$ can not be seen in the AR quiver, they are in the infinite radical of D.

However, their existence can be deduced by Serre duality: we have

$$\mathrm{Hom}_{\mathsf{D}}(a, x) \cong \mathrm{Hom}_{\mathsf{D}}(x, \Sigma^2 a).$$

The region of a's with non-zero forward morphisms $a \to x$ is by (1) precisely $H^{-}(\Sigma^{-1}x)$. Applying Serre duality gives non-zero morphisms from x to the region

$$\Sigma^2(H^{-}(\Sigma^{-1}x)) = H^{-}(\Sigma^2\Sigma^{-1}x) = H^{-}(\Sigma x).$$

Corollary Let x, y in D be indecomposable. The following are equivalent:
(i) $\mathrm{Hom}_{\mathsf{D}}(x, y) \neq 0$
(ii) $y \in H(\Sigma x)$
(iii) $x \in H(\Sigma^{-1}y)$.

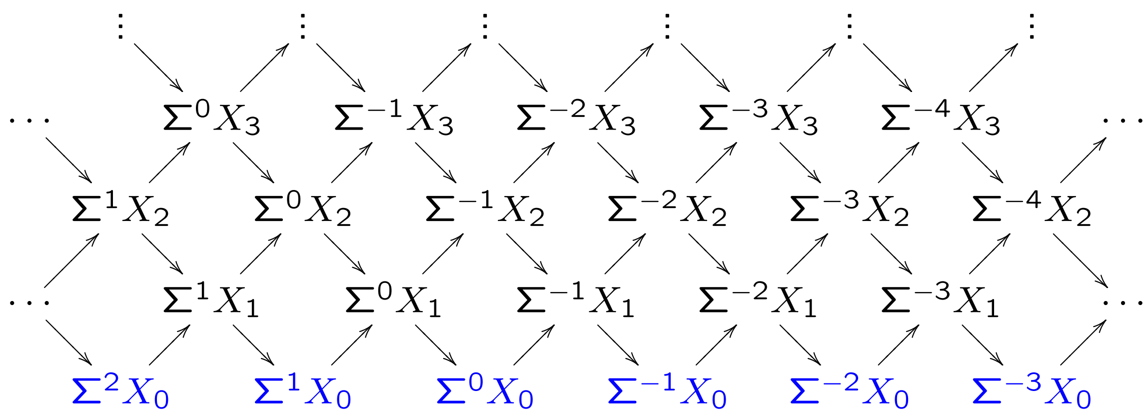
Application: spherical objects

T triangulated category (k-linear, algebraic etc.)

Object x in T is d-spherical if its Ext-algebra behaves like the homology of the d-sphere, i.e.

$$\mathrm{Ext}^i_{\mathsf{T}}(x,x) := \mathrm{Hom}_{\mathsf{T}}(x, \Sigma^i x) = \begin{cases} k & if\ i = 0, d \\ 0 & otherwise \end{cases}$$

In our 2-Calabi-Yau category D we have 2-spherical objects, namely $\{\Sigma^i X_0 | i \in \mathbb{Z}\}$, the bottom line of the AR-quiver.



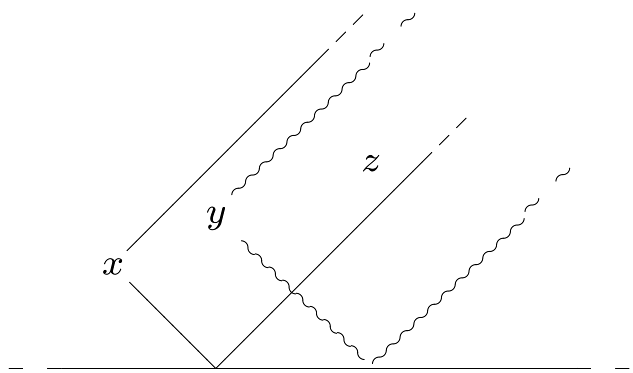
In particular, D is generated (as triangulated category) by a single 2-spherical object.

Keller-Yang-Zhou: there is a unique triangulated category generated by one d-spherical object.

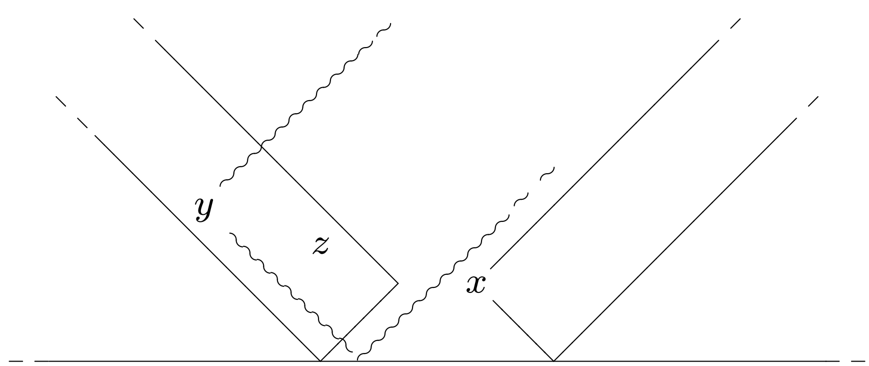
Proposition
(Composition/factorization of morphisms)
Let $x, y, z \in \mathsf{D}$ be indecomposable objects.

(i) Suppose $y, z \in H^{+}(\Sigma x)$, $z \in H^{+}(\Sigma y)$.

Then the composition of non-zero morphisms $x \to y$ and $y \to z$ is non-zero; each morphism $x \to z$ factors through any non-zero $y \to z$.

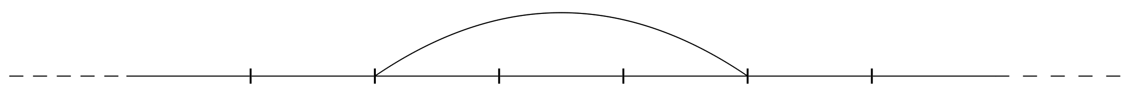


(ii) Suppose $y, z \in H^{-}(\Sigma x)$, $z \in H^{+}(\Sigma y)$. Let $f : y \to z$ be non-zero. Then each morphism $x \to z$ factors through f.



5. Cluster tilting subcategories and triangulations of the ∞-gon

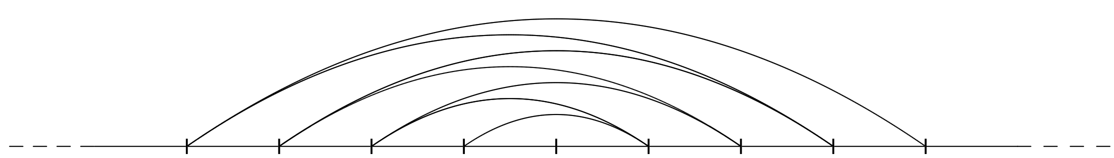
∞-gon: integers on real line

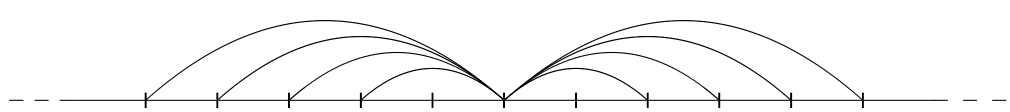


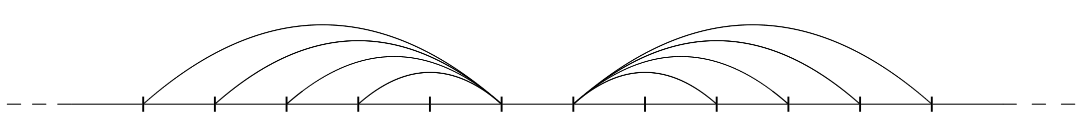
arc: pair (m, n) of integers with $m \leq n - 2$

Triangulation of the ∞-gon: maximal set of non-crossing arcs

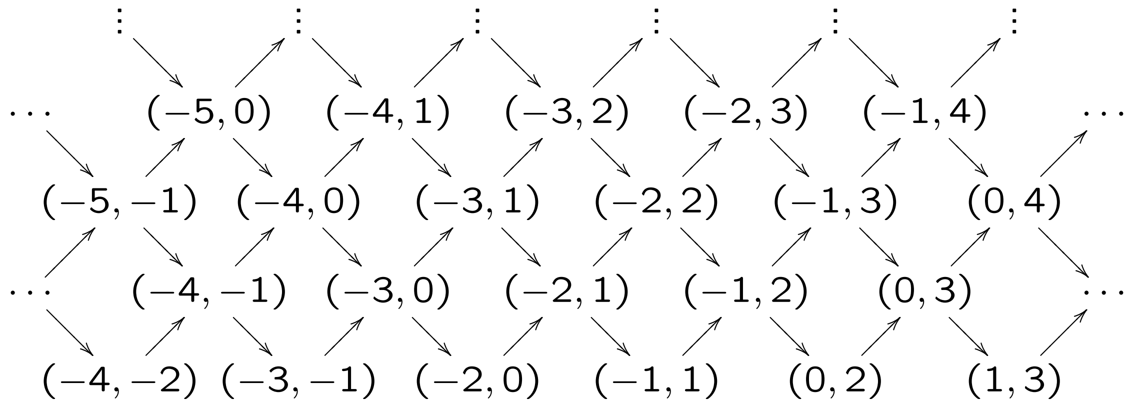
Examples:





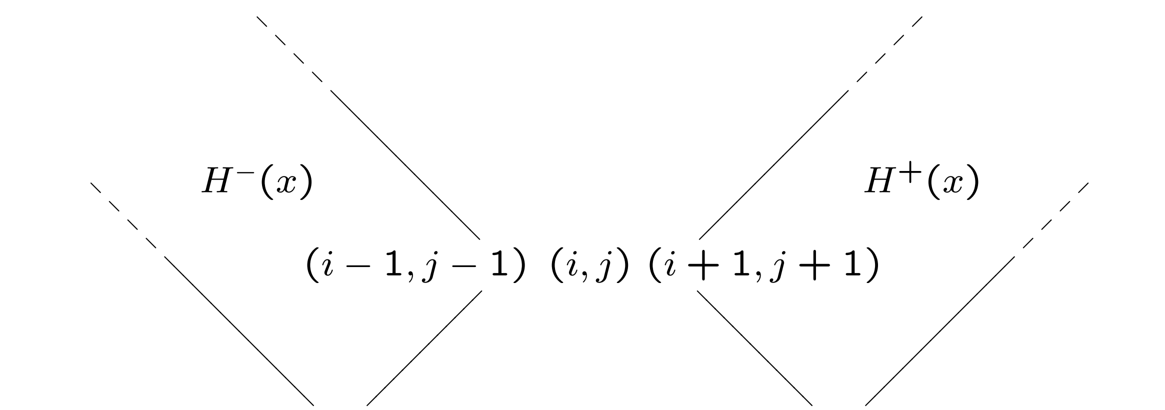


Recall the coordinate system



vertices of AR quiver $\longleftrightarrow$ arcs of ∞-gon

Crucial: $H(x) \longleftrightarrow$ arcs crossing $x = (i, j)$



Let $\mathcal{A} \subset \mathsf{D}$ be a subcategory. Set

$$\mathcal{A}^{\perp} := \{d \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{D}}(a, d) = 0 \text{ for all } a \in \mathcal{A}\},$$

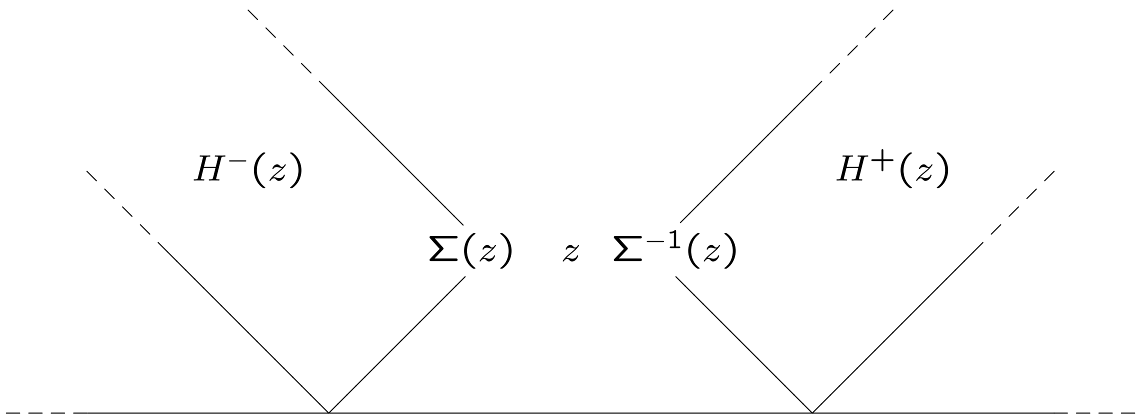
$$^{\perp}\mathcal{A} := \{d \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{D}}(d, a) = 0 \text{ for all } a \in \mathcal{A}\}.$$

$\mathcal{A}$ is called *weak cluster tilting* if $\mathcal{A} = (\Sigma^{-1}\mathcal{A})^{\perp}$ and $\mathcal{A} = {}^{\perp}(\Sigma\mathcal{A})$.

$\mathcal{A}$ is called *cluster tilting* if it is weak cluster tilting and functorially finite.

Theorem 1 (H-Jørgensen) *There is a bijection between weak cluster tilting subcategories of* D *and triangulations of the* ∞*-gon.*

Proof Let $\mathcal{A} \subseteq \mathsf{D}$ be a subcategory (closed under direct sums and direct summands).



$\mathcal{A}$ weak cluster tilting if and only if
$\mathcal{A} = (\Sigma^{-1}\mathcal{A})^{\perp}$ and $\mathcal{A} = {}^{\perp}(\Sigma\mathcal{A})$

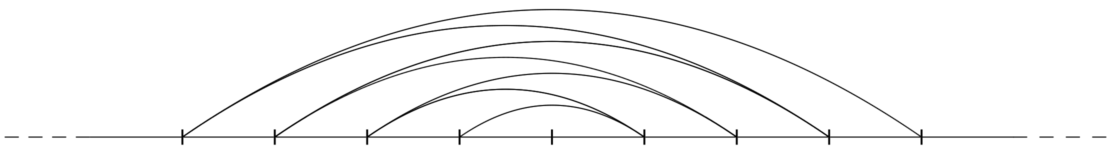
Let $z \in \mathcal{A}$. Then $\mathcal{A} = (\Sigma^{-1}\mathcal{A})^{\perp}$ implies that $\mathrm{Hom}_{\mathsf{D}}(\Sigma^{-1}z, x) = 0$ for all $x \in \mathcal{A}$.

$\rightsquigarrow$ forbidden regions $H(z) = H^{+}(z) \cup H^{-}(z)$

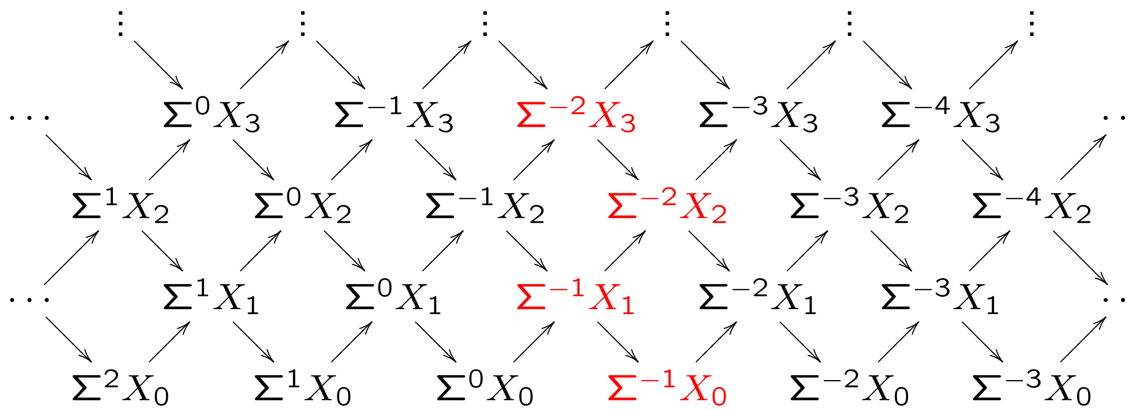
forbidden regions $\longleftrightarrow$ crossing arcs

weak cluster tilting subcategories
$\longleftrightarrow$ maximal sets of non-crossing arcs
$\longleftrightarrow$ triangulations of the ∞-gon $\square$

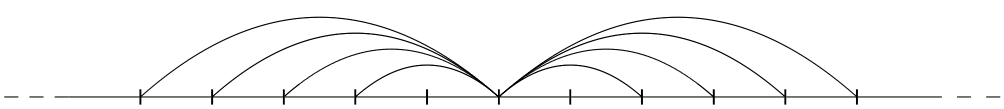
Example The 'leapfrog' triangulation



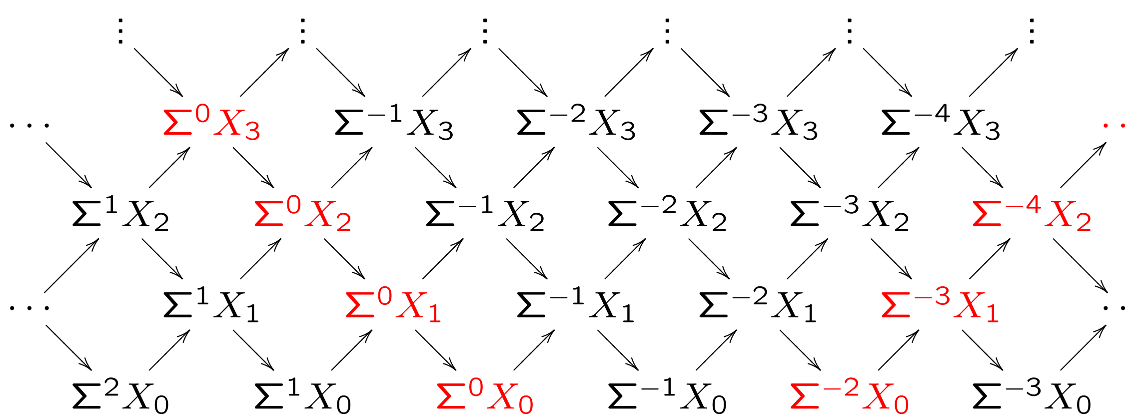
corresponds to the zig-zag

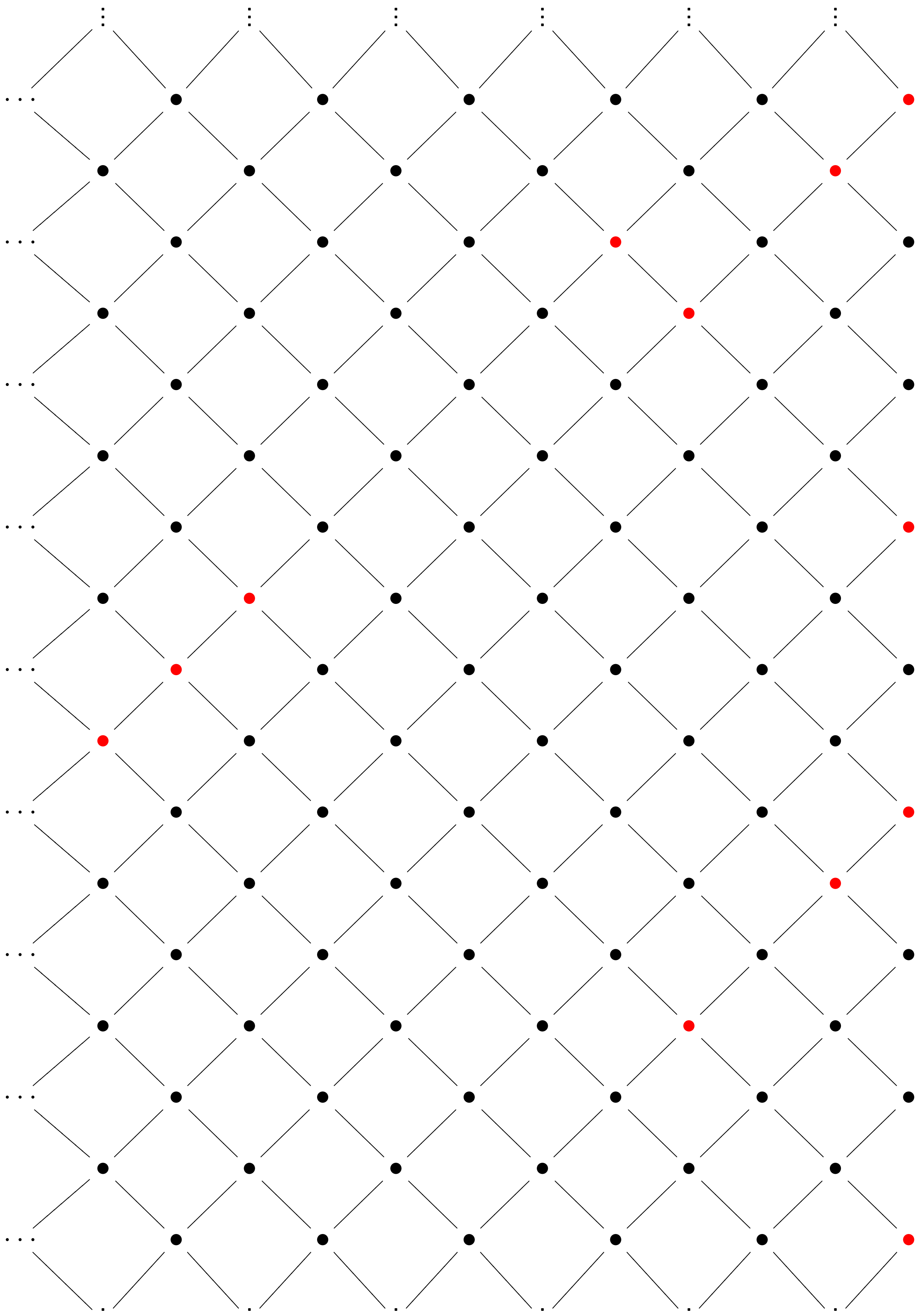


The 'fountain'



corresponds to





König-Zhu: For a weak cluster tilting subcategory $\mathcal{A} \subseteq$ D to be cluster tilting (i.e. functorially finite) it suffices to show that it is precovering (or preenveloping).

Precovering: for each object x in D there is a morphism $a \to x$ with a in $\mathcal{A}$ through which any morphism $a' \to x$ with a' in $\mathcal{A}$ factors.

Question Which weak cluster tilting subcategories are cluster tilting subcategories?

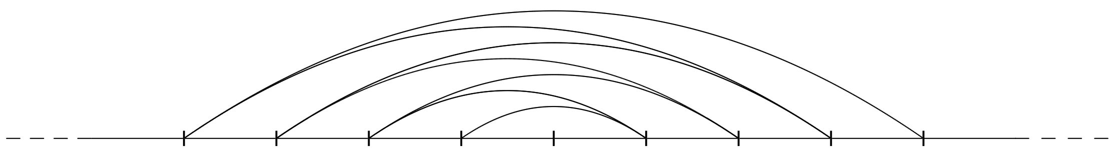
The answer can be given in terms of the combinatorics of the corresponding maximal set of non-intersecting arcs.

Let $\mathcal{A}$ be a set of arcs.

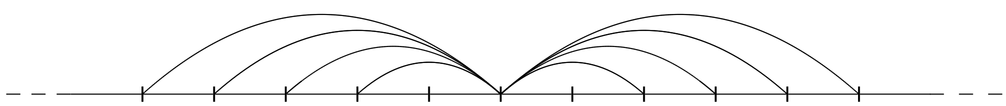
$\mathcal{A}$ is locally finite if for each integer n there are only finitely many arcs attached to n.

An integer n is called a fountain of $\mathcal{A}$ if there are infinitely many arcs of the form (m, n) and infinitely many arcs of the form (n, p).

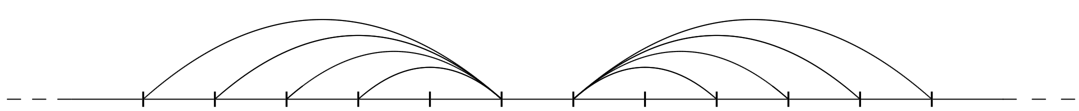
Examples



Locally finite, no fountain



Not locally finite, has a fountain

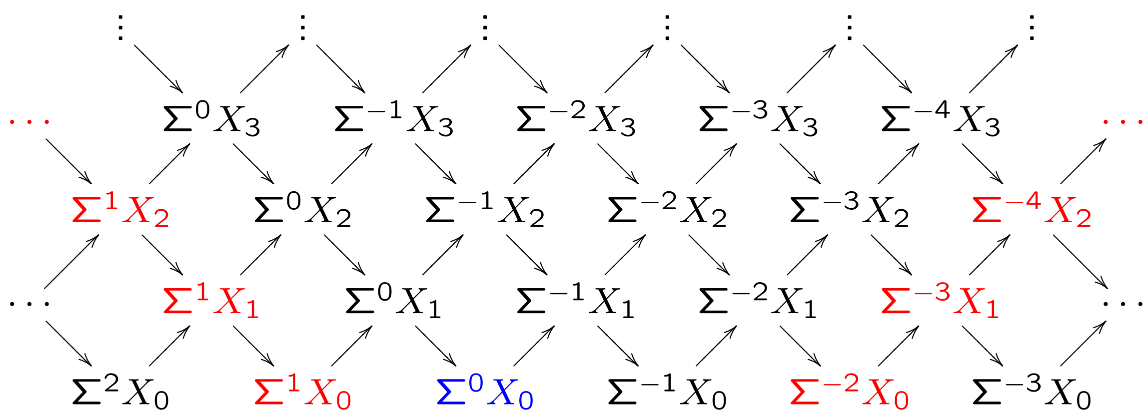


Not locally finite, no fountain

Theorem 2 (H-Jørgensen) *A weak cluster tilting subcategory of* D *is a cluster tilting subcategory if and only if the corresponding set of arcs is locally finite or has a fountain.*

Example

Weak cluster tilting, but not cluster tilting



No precover!

Proof is combinatorial but quite involved...

6. Mutation and cluster structure

Let $\mathcal{A}$ be a cluster tilting subcategory of D. Indecomposable objects $\operatorname{ind}\mathcal{A}$ are clusters

Theorem 3 (H-Jørgensen) *The clusters in* D *form a cluster structure (in the sense of B-I-R-S). In particular, for any cluster* $A = \operatorname{ind}\mathcal{A}$

(i) a *an indecomposable object in* A*, then there is a unique other indecomposable element* a^* *of* D *such that* $A^* := A\backslash\{a\}\cup\{a^*\}$ *is again a cluster (and* $\operatorname{add}(A^*)$ *a cluster tilting subcategory)*

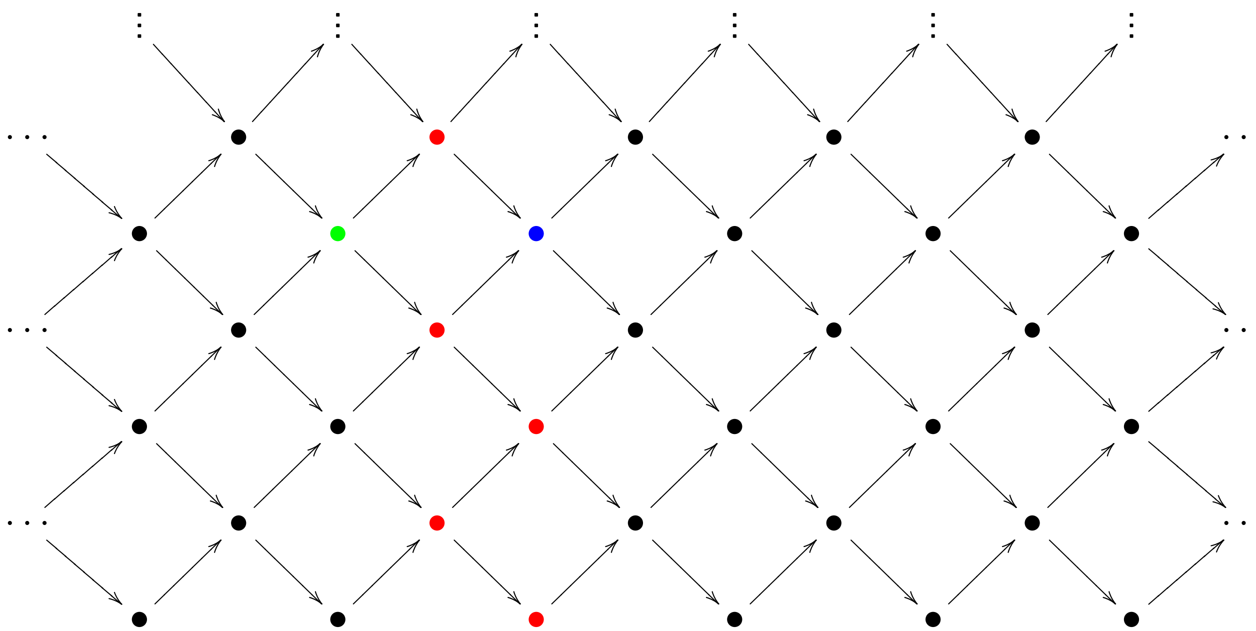
(ii) AR-quiver of $\operatorname{add} A$ *has no loops or 2-cycles*

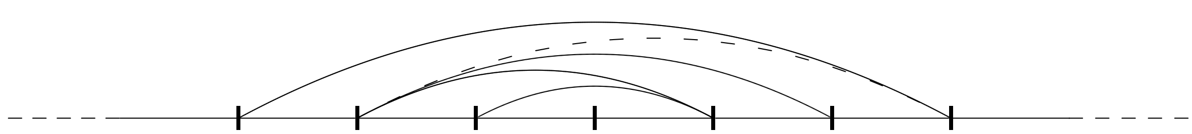
(iii) passing from the AR-quiver of $\operatorname{add} A$ *to the AR-quiver of* $\operatorname{add} A^*$ *given by Fomin-Zelevinsky quiver mutation.*

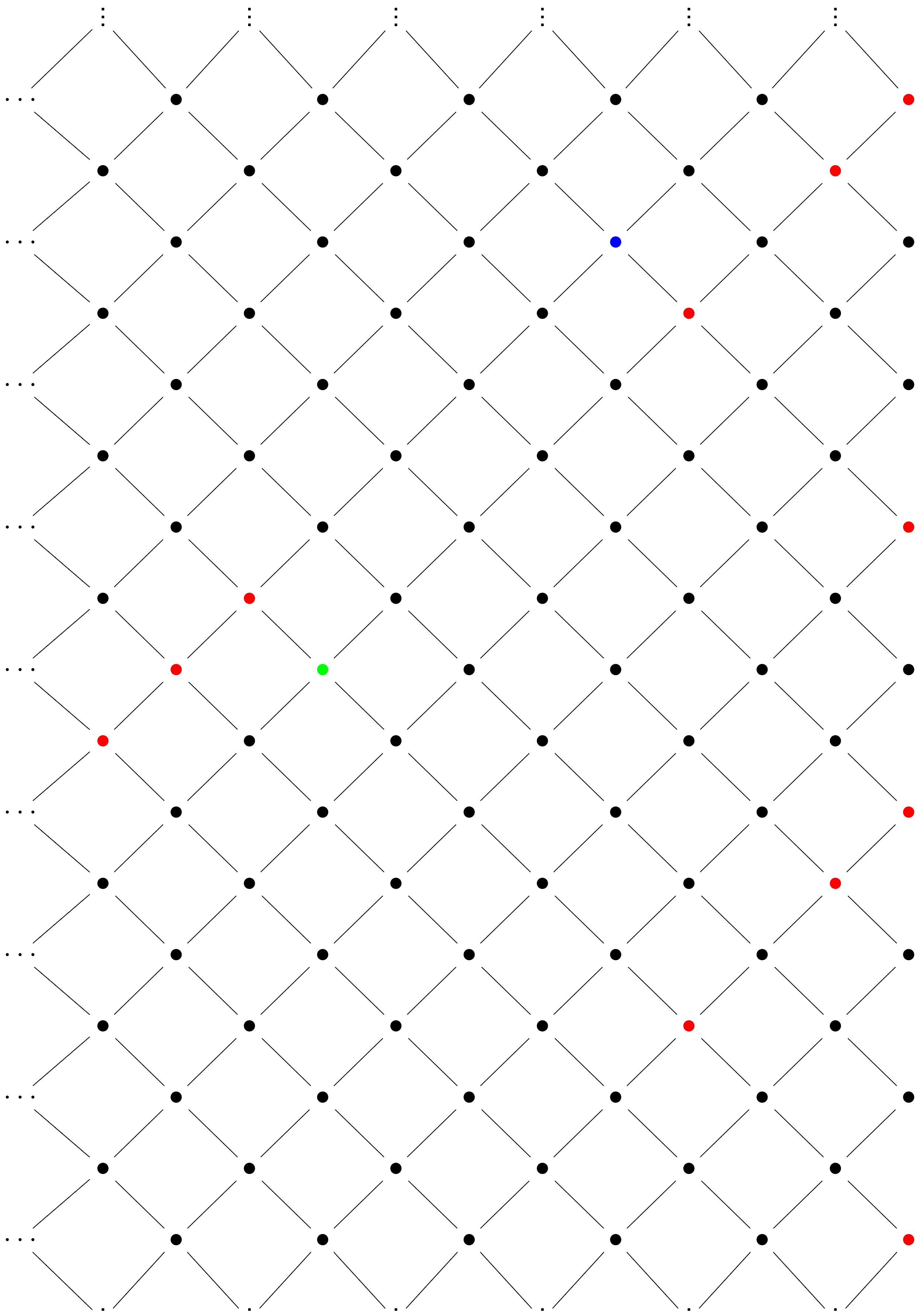
Exchange in clusters given by flips of arcs in the corresponding triangulation of the ∞*-gon.*

Remark In contrast to the cluster categories usually studied in the context of Fomin-Zelevinsky's cluster algebras, the 2-Calabi-Yau category D has clusters with infinitely many indecomposables.

Example







Proof of Theorem 3: Since D is 2-Calabi-Yau and contains cluster tilting subcategories, it suffices by [Buan-Iyama-Reiten-Scott] to show that for each cluster A, the AR quiver of the cluster tilting subcategory $\mathcal{A} = \operatorname{add} A$ has no loops or 2-cycles.

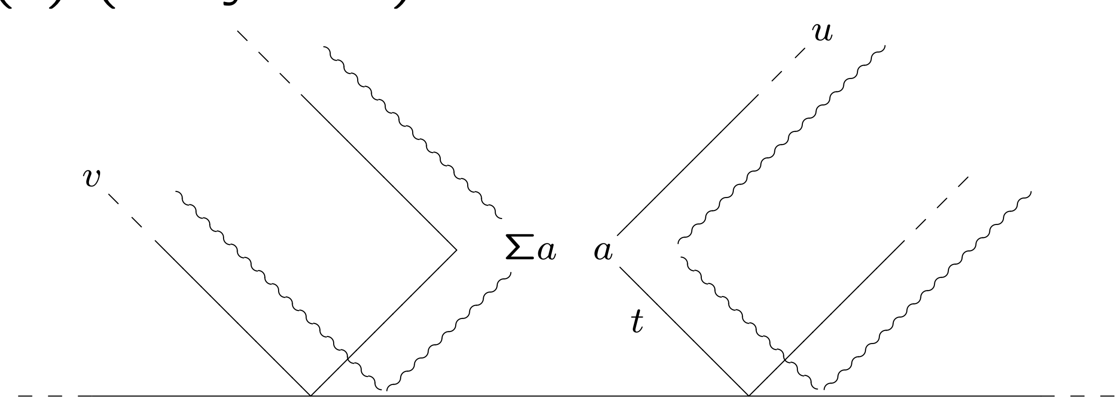
No loops: for a in $\mathcal{A}$ we have

$$\mathrm{Hom}_{\mathcal{A}}(a, a) = \mathrm{Hom}_{\mathsf{D}}(a, a) = k,$$

so each non-zero morphism is an isomorphism and thus not irreducible.

No 2-cycles: for indecomposables a, b in $\mathcal{A}$ we show that $\mathrm{Hom}_{\mathcal{A}}(a, b) \neq 0$ implies $\mathrm{Hom}_{\mathcal{A}}(b, a) = 0$.

Consider the regions $H(\Sigma a)$ (straight lines) and $H(a)$ (wavy lines).



Since $\text{Hom}_{\mathcal{A}}(a, b) \neq 0$ the object b is in the region $H(\Sigma a)$. On the other hand, a, b are both in the cluster tilting subcategory, thus b is outside the region $H(a)$.

So b lies on the line segment t or on one of the half lines u and v.

Then $\text{Hom}_{\mathcal{A}}(b, a) = 0$ follows by direct inspection. $\square$

Open questions

(1) Do triangulations of the ∞-gon occur in other (combinatorics) contexts?

(2) What are the connected components of the cluster structure on D?

(3) Are there higher cluster categories of type A_∞?

(4) D is a cluster category of type A_∞; is there a reasonable definition of a cluster algebra of type A_∞?

(5) Analogues for other infinite Dynkin quivers, e.g. D_∞?

etc.