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Gerasimov's theorem and Calabi-Yau algebras

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Aim: present a family of quadratic ($N = 2$) Koszul algebras and determine which of these algebras are Calabi-Yau.

PART I - Koszul algebras

V is a k-vector space, $T(V)$ is the tensor algebra, naturally graded (elements of V have degree 1),

R is a subspace of $V^{\otimes 2}$, $A = A(V,R) = T(V)/I(R)$ is a graded algebra, called a *quadratic algebra*.

The Koszul complex $K(A)$ is defined as follows

$$\cdots \longrightarrow K_i \xrightarrow{\delta_i} K_{i-1} \longrightarrow \cdots \longrightarrow K_2 \xrightarrow{\delta_2} K_1 \xrightarrow{\delta_1} K_0$$

in which $K_i = A \otimes W_i$,

$$W_i = \bigcap_{j+2+j'=m} V^{\otimes j} \otimes R \otimes V^{\otimes j'} \subseteq V^{\otimes i},$$

and δ_i is A-linear extending $W_i \hookrightarrow V \otimes W_{i-1}$.

$K(A)$ is a complex in the category A-grMod.

Definition (Priddy, 1970). The quadratic algebra A is said to be *Koszul* if the homology of $K(A)$ is 0 in any degree $i > 0$.

Basic example (Koszul, 1950's): polynomial algebras.

If A is Koszul, then

(i) $K(A) \xrightarrow{\epsilon} k$ is a *minimal* resolution in A-grMod, where $A \xrightarrow{\epsilon} k$ is the natural projection,

(ii) gl. $\dim A$ is the length of $K(A)$,

(iii) if $\dim(V) < \infty$, then $H_A(t) = c_A(t)^{-1}$ where

$$c_A(t) = \sum_{i \geq 0} (-1)^i \dim(W_i)\, t^i,$$

(iv) if moreover gl. $\dim A < \infty$, GK. $\dim A$ is finite if and only if the complex roots of the polynomial $c_A(t)$ have module 1, and in this case, GK. $\dim A$ is equal to the multiplicity of the root 1.

Let $A = A(V, R)$ be a quadratic algebra.

1. gl. $\dim A = 2 \implies A$ is Koszul,

2. Backelin's theorem:

A is Koszul $\iff A$ is distributive, i.e., for any i the sublattice of the lattice

$$\mathcal{L}(V^{\otimes i}) = \{subspaces\ of\ V^{\otimes i}\} \text{ ordered by } \subseteq,$$

generated by the $V^{\otimes j} \otimes R \otimes V^{\otimes j'}$, $j + 2 + j' = i$ is *distributive*: $E \cap (F + G) = (E \cap F) + (E \cap G)$ for any E, F, G in the sublattice,

3. Gerasimov's theorem (a special case):

$$\dim R = 1 \implies A \text{ is distributive,}$$

4. Consequence of 2. and 3.:

$$\dim R = 1 \implies A \text{ is Koszul.}$$

The paper contains a purely algebraic proof of 3., using 1., the hard implication in 2., and Bergman's confluence.

Actually, Gerasimov's theorem holds in a more general context.

Gerasimov's theorem (1993). Let $A = A(V,R)$ be an N-homogeneous algebra ($N \geq 2$), meaning that R is a subspace of $V^{\otimes N}$. Then

$$\dim R = 1 \implies A \text{ is distributive.}$$

Corollary (RB). If A is N-homogeneous and if $\dim R = 1$, then A is N-Koszul if and only if we have for $m = 2, \ldots, N-1$ the inclusion

$$(R \otimes V^{\otimes m}) \cap (V^{\otimes m} \otimes R) \subseteq V^{\otimes (m-1)} \otimes R \otimes V.$$

See the paper for new examples of N-Koszul algebras deriving from this corollary.

The proof of the corollary also uses:

Proposition (RB, 2001). If A is N-homogeneous, A is N-Koszul if and only if for any $j \geq 1$, one has the distributivity of the triples (E, F, G) for $n \geq (j+1)N$ and of the triples (E', F', G') for $n \geq (j+1)N + 1$, where

$$E = V^{(n-jN)} \otimes W_{jN}$$

$$F = I(R)_{n-jN} \otimes V^{(jN)}$$

$$G = V^{(n-(j+1)N+1)} \otimes I(R)_{2N-2} \otimes V^{((j-1)N+1)}$$

$$E' = V^{(n-jN-1)} \otimes W_{jN+1}$$

$$F' = I(R)_{n-jN-1} \otimes V^{(jN+1)},$$

$$G' = V^{(n-(j+1)N)} \otimes R \otimes V^{(jN)}$$

and the following *extra condition* (non-distributivity condition, void if $N = 2$)

$$(V^{(N-1)} \otimes R) \cap (R \otimes V^{(N-1)} + \cdots + V^{(N-2)} \otimes R \otimes V)$$
$$\subseteq V^{(N-2)} \otimes R \otimes V.$$

The extra condition reduces to the inclusions in the corollary when $\dim R = 1$.

PART II - Algebras with a single quadratic relation

$\dim V = n < \infty$,

$\dim R = 1$: R is generated by $f \neq 0$, $f \in V \otimes V$.

According to 4. of Part 1, $A = A(V, R)$ is Koszul.

Fix a basis $x^t = (x_1, \ldots, x_n)$ of V, and write

$$f = \sum_{1 \leq i,j \leq n} f_{ij}\, x_i \otimes x_j,$$

$$M = M(f) = (f_{ij})_{1 \leq i,j \leq n},$$

$A^{p \times q}$= A-bimodule of the $p \times q$ matrices with entries in A.

Theorem 1. A is AS-Gorenstein if and only if M is invertible.

Definition (Artin-Schelter, 1987). Let A be a connected graded algebra of finite global dimension $n \geq 1$. We say that A is AS-Gorenstein if $\mathit{Ext}^i_A(k, A) = 0$ for $i \neq n$, and $\mathit{Ext}^n_A(k, A) \cong k$ as right A-modules.

Sketch of proof of Theorem 1:

Necessarily, gl. $\dim A = 2$ (otherwise, gl. $\dim A = \infty$ and M is not invertible).

$K(A)$ has the following matrix form in A-grMod

$$0 \longrightarrow A \xrightarrow{\cdot(x^t M)} A^{1\times n} \xrightarrow{\cdot x} A \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(-, A)$, we get in grMod-A

$$0 \longrightarrow A \xrightarrow{x\cdot} A^{n\times 1} \xrightarrow{(x^t M)\cdot} A \longrightarrow 0.$$

Consider the following commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{x\cdot} & A^{n\times 1} & \xrightarrow{(x^t M)\cdot} & A & \xrightarrow{\epsilon} & k & \longrightarrow & 0 \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle M\cdot} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\
0 & \longrightarrow & A & \xrightarrow{(Mx)\cdot} & A^{n\times 1} & \xrightarrow{x^t\cdot} & A & \xrightarrow{\epsilon} & k & \longrightarrow & 0
\end{array}$$

in which the second row is exact (since A is Koszul).

If M is invertible, the first row is exact, thus A is AS-Gorenstein.

Conversely, if A is AS-Gorenstein, the first row is a projective resolution in grMod-A, which is *minimal* because the arrows $x\cdot$ and $(x^t\,M)\cdot$ are homogeneous of degree 1 (so they vanish after applying $-\otimes_A k$).

Since a morphism between two minimal resolutions is an isomorphism, we conclude that M is invertible. $\square$

Remarks. 1. gl. $\dim A = 2$ or ∞, and gl. $\dim A = \infty$ iff M is symmetric of rank 1.

2. If gl. $\dim A = 2$, then

$$H_A(t) = (1 - nt + t^2)^{-1}$$

and GK. $\dim A = 2$ if $n = 2$, ∞ otherwise. According to the Stephenson-Zhang Theorem, A is not Noetherian if GK. $\dim A = \infty$.

3. If gl. $\dim A = \infty$, then

$$H_A(t) = (1 - nt + t^2 - t^3 + \cdots)^{-1}$$

and GK. $\dim A = 0$ if $n = 1$, ∞ otherwise.

Theorem 2. A is Calabi-Yau if and only if M is invertible and antisymmetric.

Roughly speaking, for graded algebras,

Calabi-Yau = "AS-Gorenstein for bimodules".

Definition (V. Ginzburg, 2007). Let A be a k-algebra, assumed to be *homologically smooth*, that is, having a finite projective A-bimodule resolution by bimodules of finite type. We say that A is a *Calabi-Yau algebra* of dimension $n \geq 1$ (or n-CY algebra) if $\mathit{Ext}^i_{A^e}(A, A^e) = 0$ for $i \neq n$, and $\mathit{Ext}^n_{A^e}(A, A^e) \cong A$ as right A^e-modules, where $A^e = A \otimes A^{op}$ and A^{op} is the opposite algebra of A.

For graded algebras, CY $\Longrightarrow$ AS-Gorenstein. So we assume that gl. $\dim A = 2$ for our algebras A. Then the bimodule Koszul resolution is defined as

$$0 \to A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \to 0$$

which has the following matrix form (without μ and zero maps)

$$A \otimes A \xrightarrow{\overset{r}{\cdot} x^t M + \overset{\ell}{\cdot} x^t M^t} (A \otimes A)^{1\times n} \xrightarrow{\overset{r}{\cdot} x - \overset{\ell}{\cdot} x} A \otimes A \ .$$

Notation for the action $\overset{\ell}{\cdot}$ or $\overset{r}{\cdot}$ is the following:

for a, b, c in A, set $(a \otimes b) \overset{\ell}{\cdot} c = a \otimes cb$ and $(a \otimes b) \overset{r}{\cdot} c = ac \otimes b$,

for $C = (c_{jk})$, $p \times q$ matrix with entries in A, $(A \otimes A)^{m\times p} \xrightarrow{\cdot C} (A \otimes A)^{m\times q}$ is defined by

$$(u_{ij}) \cdot C = (\sum_j u_{ij} \cdot c_{jk})$$

where $\cdot$ denotes $\overset{\ell}{\cdot}$ or $\overset{r}{\cdot}$ respectively.

The dual bimodule Koszul complex has the matrix form (without zero maps)

$$A \otimes A \xrightarrow{\overset{r}{\cdot} x^t - \overset{\ell}{\cdot} x^t} (A \otimes A)^{1\times n} \xrightarrow{\overset{r}{\cdot} Mx + \overset{\ell}{\cdot} M^t x} A \otimes A$$

computing Hochschild cohomology $\mathsf{HH}^i(A, A \otimes A)$ for $i = 0, 1, 2$ (the other spaces are zero).

It is easy to check that

$\mu \circ (\overset{r}{\cdot}\, Mx + \overset{\ell}{\cdot}\, M^t x) = 0 \Longleftrightarrow M$ is antisymmetric,

so M is antisymmetric if A is 2-CY.

Assume that M is antisymmetric. Then the commutative diagram

$$\begin{array}{ccccc}
A \otimes A & \xrightarrow{\overset{r}{\cdot} x^t - \overset{\ell}{\cdot} x^t} & (A \otimes A)^{1 \times n} & \xrightarrow{\overset{r}{\cdot} Mx - \overset{\ell}{\cdot} Mx} & A \otimes A \\
\downarrow {\scriptstyle id} & & \downarrow {\scriptstyle \cdot M} & & \downarrow {\scriptstyle id} \\
A \otimes A & \xrightarrow{\overset{r}{\cdot} x^t M - \overset{\ell}{\cdot} x^t M} & (A \otimes A)^{1 \times n} & \xrightarrow{\overset{r}{\cdot} x - \overset{\ell}{\cdot} x} & A \otimes A
\end{array}$$

shows that A is 2-CY iff M is invertible, by the *same* arguments as those used in the proof of Theorem 1. $\square$

Viewing f as the bilinear form $f = \sum_{1 \leq i,j \leq n} f_{ij}\, x_i x_j$, A is an associative algebra attached to geometry. Theorem 2 states that

A is Calabi-Yau iff f is a symplectic form.

Question. Why symplectic geometry ?

PART III - Poincaré-Birkhoff-Witt (PBW) deformations

We keep the notation and assumptions of Part II: f is a bilinear form and A is the Koszul quadratic algebra attached to f. We assume that gl. $\dim A = 2$ (the ∞ case is easy).

Aim: find all the PBW deformations of A and determine which of these deformations are CY.

Proposition. Let $A = A(V, R)$ be an N-Koszul graded algebra, $N \geq 2$. Let $\varphi : R \to F^{N-1}$ be k-linear, and $U = T(V)/I(P)$ where $P = (\mathrm{Id} - \varphi)(R)$. If the global dimension of A is 2, then the filtered algebra U is N-Koszul, that is, is a PBW deformation of A.

Here $F^i = k \oplus V \oplus \cdots \oplus V^{\otimes i}$, $i \geq 0$, denotes the filtration of $T(V)$.

Proof: immediate by using the generalised PBW theorem (Fløystad-Vatne, V. Ginzburg-RB) and the fact that $W_{N+1} = 0$.

For our algebra A, U as in the Proposition is defined by a single relation $f = v + \lambda$, where $v \in V$ and $\lambda \in k$. Set $v = \sum_{1 \leq i \leq n} \lambda_i x_i$ and let $\bar{v}$ be the $n \times 1$ matrix with entries $\lambda_1, \ldots, \lambda_n$. Then

$$U \otimes U \xrightarrow{\overset{r}{\cdot} x^t M + \overset{\ell}{\cdot} x^t M^t - \cdot \bar{v}^t} (U \otimes U)^{1 \times n} \xrightarrow{\overset{r}{\cdot} x - \overset{\ell}{\cdot} x} U \otimes U$$

augmented by the multiplication of U and zero maps, is a projective resolution of U in the category U-filtMod-U, called the *bimodule Koszul resolution* of U.

Proof: it is a complex whose associated graded complex is the bimodule Koszul resolution of A.

The *dual bimodule Koszul complex* of U is the following

$$U \otimes U \xrightarrow{\overset{r}{\cdot} x^t - \overset{\ell}{\cdot} x^t} (U \otimes U)^{1 \times n} \xrightarrow{\overset{r}{\cdot} Mx + \overset{\ell}{\cdot} M^t x - \cdot \bar{v}} U \otimes U$$

Then

$$\mu \circ (\overset{r}{\cdot}\, Mx + \overset{\ell}{\cdot}\, M^t x - \cdot \bar{v}) = 0$$

iff M is antisymmetric and $v = 0$.

So M is antisymmetric and $v = 0$ if U is CY.

Assume that M is antisymmetric and $v = 0$. The commutative diagram

$$\begin{array}{ccccc}
U \otimes U & \xrightarrow{\overset{r}{\cdot}x^t - \overset{\ell}{\cdot}x^t} & (U \otimes U)^{1\times n} & \xrightarrow{\overset{r}{\cdot}Mx - \overset{\ell}{\cdot}Mx} & U \otimes U \\
\Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \cdot M} & & \Big\downarrow{\scriptstyle id} \\
U \otimes U & \xrightarrow{\overset{r}{\cdot}x^t M - \overset{\ell}{\cdot}x^t M} & (U \otimes U)^{1\times n} & \xrightarrow{\overset{r}{\cdot}x - \overset{\ell}{\cdot}x} & U \otimes U
\end{array}$$

shows that U is 2-CY iff M is invertible. We have obtained:

Theorem 3. Let $n \geq 2$ be even, and let U be the associative k-algebra defined by generators $x_1, \ldots, x_n$ subject to the single relation

$$\sum_{1 \leq i \leq n/2} [x_i, x_{i+n/2}] = v + \lambda,$$

where the bracket stands for the commutator, v is a linear combination of the x_i's, and $\lambda \in k$. Then the filtered algebra U is Koszul and we have $\mathrm{HH}^i(U, U \otimes U) = 0$ whenever $i \neq 2$. Furthermore, U is Calabi-Yau if and only if $v = 0$.

Remark. The implication "A is CY $\Longrightarrow U$ is CY" is false in general. It is true in the context of Theorem 3 when only a constant is added to the relation f of A. This phenomenon occurs in the following general context.

Theorem 4. Let $A = A(V, R)$ be an N-Koszul graded algebra, $N \geq 2$, with V finite-dimensional. Let $\varphi : R \to k$ be k-linear. Assume that $U = T(V)/I(P)$ is a PBW deformation of A, where $P = (\mathrm{Id} - \varphi)(R)$. If A is d-Calabi-Yau for a certain $d \geq 2$, then U is d-Calabi-Yau. For example, any Sridharan enveloping algebra of an n-dimensional abelian Lie algebra is n-Calabi-Yau; in particular the Weyl algebra A_n is $2n$-Calabi-Yau.