Hilbert transform approach for pricing Bermudan options in Lévy models

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Spectral and cubature methods in finance and econometrics
University of Leicester, 6/19/2009
Lévy models

- Better fit to empirical data, explain volatility smiles
- Finite activity jump diffusion models
  *Merton 76 (normal jump diffusion), Kou 02 (double exponential jump diffusion)*
- Infinite activity pure jump Lévy models
  *Madan et al 90, 91, 98 (variance gamma), Barndorff-Nielsen 98 (normal inverse Gaussian), Carr, Geman, Madan & Yor 02 (CGMY), Eberlein, Keller & Prause 98 (Generalized hyperbolic)*
- Books on Lévy processes
  *Boyarchenko & Levendorskii 02, Cont & Tankov 04, Schoutens 03, Bertoin 96, Sato 99, Applebaum 04, Kyprianou 06*
Lévy-Khintchine theorem

- Lévy process: independent and stationary increments
- **Analytical tractability**: characteristic functions available via Lévy-Khintchine Theorem

\[ \phi_t(\xi) = \mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)} \]

where \( \psi \) is the characteristic exponent

\[ \psi(\xi) = \frac{1}{2} \sigma^2 \xi^2 - i\mu \xi + \int_{\mathbb{R}} (1 - e^{i\xi x} + i\xi x 1_{\{|x|\leq 1\}}) \Pi(dx) \]
Fourier transform method

**Fourier transform** method for European options: *Carr & Madan 99*

\[ c(k) = \frac{1}{2\pi} e^{-\alpha k} \int_{\mathbb{R}} e^{-i\xi k} \Phi(\xi) d\xi \]

where

\[ \Phi(\xi) = \frac{e^{-rT} \phi_T(\xi - i(\alpha + 1))}{(i\xi + \alpha)(i\xi + \alpha + 1)} \]

- Extended in *Lee 04*: more general payoffs, exponentially decaying errors
- Bermudan options: vanilla, discrete barrier/lookback options, **Hilbert transform** method
Hilbert transform

- **Hilbert transform** of \( f \in L^p(\mathbb{R}), 1 \leq p < \infty \)

\[
\mathcal{H}f(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy
\]

- For any \( f \in L^1(\mathbb{R}) \) with \( \hat{f} \in L^1(\mathbb{R}) \)

\[
\mathcal{F}(\text{sgn} \cdot f)(\xi) = i\mathcal{H}\hat{f}(\xi)
\]

\[
\mathcal{F}(1_{(l,\infty)} \cdot f)(\xi) = \frac{1}{2} \hat{f}(\xi) + \frac{i}{2} e^{i\xi l} \mathcal{H}(e^{-inl} \hat{f}(\eta))(\xi)
\]
Option pricing

- **European vanilla** options:

\[
\mathbb{E}[(K - S_T)^+] = K \mathbb{E}[\mathbf{1}_{S_T < K}] - \mathbb{E}[S_T \mathbf{1}_{S_T < K}]
\]

\[
P(X \leq x) = \int_{\mathbb{R}} \mathbf{1}_{(-\infty, x)}(y)p(y)dy
\]

- **European style discrete barrier** options: *Feng & Linetsky 08*

\[
f^j(x) = \mathbf{1}_{(l, u)} \cdot \mathbb{E}_{t, x}[f^{j+1}(X_{t+1})]
\]

- **European style discrete lookbacks**: *Feng & Linetsky 09*

\[
M_j - X_j = \max(M_{j-1}, X_j) - X_j = \max(0, M_{j-1} - X_{j-1} - (X_j - X_{j-1}))
\]
Discrete Hilbert transform

- **Discrete Hilbert transform** with step size $h > 0$

\[
\mathcal{H}_h f(x) = \sum_{m=-\infty}^{\infty} f(mh) \frac{1 - \cos[\pi(x - mh)/h]}{\pi(x - mh)/h}, \quad x \in \mathbb{R}
\]

- For $f$ analytic in a horizontal strip $\{z \in \mathbb{C} : |\Im(z)| < d\}$

\[
\|\mathcal{H} f - \mathcal{H}_h f\|_{L^\infty(\mathbb{R})} \leq \frac{Ce^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})}
\]

- Related to **Whittaker cardinal series (sinc) expansion**
Sinc expansion of analytic functions

- **Whittaker cardinal series** (sinc expansion)

\[
c(f, h)(x) = \sum_{-\infty}^{\infty} f(kh) \frac{\sin(\pi(x - kh)/h)}{\pi(x - kh)/h}
\]

- For entire functions of exponential type $\pi/h$, sinc expansion is exact: $c(f, h) = f$
- For functions analytic in a strip $\{z \in \mathbb{C} : |\Im(z)| < d\}$, Stenger 93

\[
\|f - c(f, h)\|_{L^\infty} \leq \frac{Ce^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})}
\]
Trapezoidal rule

• Take Hilbert transform on \( c(f, h) \), obtain discrete Hilbert transform

• Trapezoidal rule very accurate for \( f \) analytic in a strip \( \{ z \in \mathbb{C} : |\Im(z)| < d \} \)

\[
\left| \int_{\mathbb{R}} f(x) dx - \sum_{m=-\infty}^{\infty} f(kh)h \right| \leq \frac{Ce^{-2\pi d/h}}{1 - e^{-2\pi d/h}}
\]

• We use trapezoidal rule to compute Fourier inverse integral
Bermudan style vanilla options

- **Bermudan put**: payoff $G(S) = (K - S)^+$, discrete monitoring
  \[ T = \{ t_0, t_1, \cdots, t_N \} = \{ 0, \Delta, 2\Delta, \cdots, N\Delta = T \} \]

- **Exponential Lévy model**: $X_t$ a Lévy process in $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, start from $\ln(S_0/K)$, equivalent martingale measure $\mathbb{P}$ is given
  \[ S_t = Ke^{X_t} \]

- **Optimal stopping**
  \[ V^0(S_0) = \sup_{\tau} \mathbb{E}_0[e^{-r\tau} G(S_{\tau})] \]
Variable change $x = \ln(S/K)$,

$$g(x) = G(Ke^x) = K(1 - e^x)^+, \quad f^0(x) = V^0(Ke^x)$$

Backward induction

$$f^N(x) = g(x)$$

$$f^j(x) = \max \left( g(x), e^{-r\Delta} \mathbb{E}_{j\Delta, x}[f^{j+1}(X_{(j+1)\Delta})] \right), \quad 0 \leq j < N$$

$$V^0(S_0) = f^0(\ln(S_0/K))$$
Direct implementation

- Knowing the transition density $p_\Delta(\cdot)$

\[
\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] = \int_{\mathbb{R}} f^{j+1}(y)p_\Delta(y - x)dy
\]

When discretized, the convolution becomes a **Toeplitz matrix vector multiplication**, can be implemented using FFT (**Eydeland 94**)

- Density may not be available, polynomial convergence

- **Double exponential fast Gauss transform** (**Broadie & Yamamoto 05**), limited to the Black-Schoels-Merton model and Merton’s model

- For general Lévy models: **COS method** (**Fang & Oosterlee 08**)
Dampening for integrability

- For $\alpha > 0$ (for puts), define

$$f_j^\alpha(x) = e^{\alpha x} f^j(x), \quad g_\alpha(x) = e^{\alpha x} g(x)$$

- Backward induction

$$f_N^\alpha(x) = g_\alpha(x)$$

$$f_j^\alpha(x) = \max \left( g_\alpha(x), e^{\alpha x - r \Delta} \mathbb{E}_{x, \Delta} \left[ e^{-\alpha X_{(j+1)\Delta}} f_{\alpha}^{j+1}(X_{(j+1)\Delta}) \right] \right),$$

$$V^0(S_0) = e^{-\alpha \ln(S_0/K)} f_\alpha^0(\ln(S_0/K))$$
Esscher transform

- Define new measure $\mathbb{P}^\alpha$ via

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{e^{-\alpha X_t}}{\phi_t(i\alpha)}$$

- Esscher transformed Lévy process is still a Lévy process

$$\phi^\alpha_t(\xi) = \frac{\phi_t(\xi + i\alpha)}{\phi_t(i\alpha)}$$

- Applying property $\mathbb{E}^\alpha_s[Y_t] = \mathbb{E}_s[Y_t Z_t/Z_s]$

$$\mathbb{E}_{j\Delta,x}[e^{-\alpha X_{(j+1)\Delta}} f_{\alpha}^{j+1}(X_{(j+1)\Delta})] = e^{-\alpha x - \Delta \psi(i\alpha)} \mathbb{E}^\alpha_{j\Delta,x}[f_{\alpha}^{j+1}(X_{(j+1)\Delta})]$$
Dampened backward induction

Dampened backward induction

\[ f^{N}_{\alpha}(x) = g_{\alpha}(x) \]

For \( 0 \leq j < N \)

\[
\begin{align*}
f^{j}_{\alpha}(x) &= \max \left( g_{\alpha}(x), e^{-(r+\psi(i\alpha))\Delta} \mathbb{E}_{j\Delta, x}^{\alpha} [f^{j+1}_{\alpha}(X_{(j+1)\Delta})] \right) \\
&= g_{\alpha}(x) \mathbf{1}_{(-\infty, x^{*}_{j})}(x) \\
& \quad + e^{-(r+\psi(i\alpha))\Delta} \mathbb{E}_{j\Delta, x}^{\alpha} [f^{j+1}_{\alpha}(X_{(j+1)\Delta})] \mathbf{1}_{[x^{*}_{j}, \infty)}(x)
\end{align*}
\]

where \( x^{*}_{j} \) is the early exercise boundary at time \( j\Delta \)
Backward induction in Fourier space

- By convolution theorem, Fourier transform of

\[ \mathbb{E}^{\alpha}_{j\Delta,x} \left[ f_{\alpha}^{j+1}(X_{(j+1)\Delta}) \right] = \int_{\mathbb{R}} f_{\alpha}^{j+1}(y)p_{\Delta}^{\alpha}(y-x) \, dy \]

is a product \( \hat{f}_{\alpha}^{j+1}(\xi) \phi_{\Delta}^{\alpha}(-\xi) \)

- Backward induction in Fourier space

\[
\hat{f}_{\alpha}^{N}(\xi) = \hat{g}_{\alpha}(\xi)
\]

\[
\hat{f}_{\alpha}^{j}(\xi) = \mathcal{F}(g_{\alpha} \mathbf{1}_{(-\infty,x^*_j)})(\xi) + e^{-(r+\psi(i\alpha))\Delta} \left( \frac{1}{2} \hat{f}_{\alpha}^{j+1}(\xi) \phi_{\Delta}^{\alpha}(-\xi) 
\right.
\]

\[
+ \left. \frac{i}{2} e^{i\xi x^*_j} \mathcal{H} \left( e^{-i\eta x^*_j} \hat{f}_{\alpha}^{j+1}(\eta) \phi_{\Delta}^{\alpha}(-\eta) \right) (\xi) \right)
\]
Early exercise boundary

- Early exercise boundary solves

\[ g_\alpha(x) = e^{-(r+\psi(i\alpha))\Delta} \mathbb{E}_{j\Delta,x}^{\alpha} \left[ f_{\alpha}^{j+1}(X_{(j+1)\Delta}) \right] \]

- Using Fourier inverse representation

\[ g_\alpha(x) = \frac{1}{2\pi} e^{-(r+\psi(i\alpha))\Delta} \int_{\mathbb{R}} e^{-i\xi x} \hat{f}_{\alpha}^{j+1}(\xi) \phi_{\Delta}(\xi) d\xi \]

- \( x^*_N = K \). To solve for \( x^*_j \), use Newton-Raphson, with starting point \( x_{j+1}^* \)
Algorithm summarized

- Start with Fourier transform of dampened payoff $\hat{f}_N = \hat{g}_\alpha$
- At time $j\Delta$, with $\hat{f}_{j+1}$, compute early exercise boundary $x_j^*$ using Newton Raphson
- Compute $\hat{f}_j$ from $\hat{f}_{j+1}$ and $x_j^*$ (Hilbert transform)
- With $\hat{f}_1$, option value at time 0

$$f_0^0(x) = \max \left( g_\alpha(x), \frac{1}{2\pi} e^{-(r+\psi(i\alpha))\Delta} \int_{\mathbb{R}} e^{-i\xi x} \hat{f}_1(\xi) \phi_\Delta(\xi) d\xi \right)$$
Discrete approximation

- Need to repeatedly evaluate a Fourier inverse integral and $H\varphi(\xi)$
- Trapezoidal rule for Fourier inverse integral, truncate infinite series with truncation level $M$, computational cost $O(M)$
- Replace $H\varphi$ by discrete Hilbert transform, truncate resulting infinite series

$$H\varphi(\xi) \leftarrow \sum_{m=-M}^{M} \varphi(mh) \frac{1 - \cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h}$$
Error estimate

- Discretization error $\sim O(\exp(-\pi d/h))$
- With $\phi_t(\xi) \sim \exp(-ct|\xi|^\nu)$, truncation error is essentially $O(\exp(-\Delta c(Mh)^\nu))$
- Select $h = h(M)$ according to
  
  \[ h(M) = \left( \frac{\pi d}{\Delta c} \right)^{\frac{1}{1+\nu}} M^{-\frac{\nu}{1+\nu}} \]

- Total error: $O(\exp(-CM^{\frac{\nu}{1+\nu}}))$
Evaluate

\[ \mathcal{H}\psi(\xi) \leftarrow \sum_{m=-M}^{M} \psi(mh) \frac{1 - \cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h} \]

for \( \xi = -Mh, \cdots, Mh \)

- Correspond to Toeplitz matrix vector multiplication
- FFT based method for such multiplications: \( O(M \log(M)) \)
- Total computational cost of the method: \( O(NM \log(M)) \)
Introduction
Bermudan vanilla
Bermudan barrier/look
Backward induction in state space
Backward induction in Fourier space
Discrete approximation

NIG model

- Pricing Bermudan put option in NIG
- NIG process: time changed Brownian motion, with an inverse Gaussian process as the stochastic time
- Characteristic exponent

\[ \psi(\xi) = -i\mu\xi + \delta_{\text{NIG}}(\sqrt{\alpha_{\text{NIG}}^2 - (\beta_{\text{NIG}} + i\xi)^2} - \sqrt{\alpha_{\text{NIG}}^2 - \beta_{\text{NIG}}^2}) \]

- \( \phi_t(\xi) \) has exponential tails with \( \nu = 1 \), error estimate in \( M \):
  \[ O(e^{-C\sqrt{M}}) \]
Bermudan put in the NIG model

Figure: $T = 1, N = 252, S_0 = 100, K = 110, r = 0.1, q = 0, \alpha_{NIG} = 25, \beta_{NIG} = -5, \delta_{NIG} = 0.5$, implemented in Matlab using a Laptop with CPU 2GHz, RAM 1G, last node takes 2.9s
Bermudan barrier options

- Bermudan down-and-out put with lower barrier \( L, l = \ln(L/K) \)
- Backward induction

\[
f^N(x) = g(x)1_{(l, \infty)}(x)
\]

For \( 0 \leq j < N \),

\[
f^j(x) = 1_{(l, \infty)}(x) \max(g(x), e^{-r\Delta}E_{j\Delta, x}[f^{j+1}(X_{(j+1)\Delta})])
\]

\[
= g(x)1_{(l, \infty)}(x)1_{(-\infty, x^*_j)}(x)
\]

\[
+ e^{-r\Delta}E_{j\Delta, x}[f^{j+1}(X_{(j+1)\Delta})]1_{(l, \infty)}(x)1_{(x^*_j, \infty)}(x)
\]
Pricing Bermudan down-and-out put in CGMY

Levy density

\[ \pi(x) = \frac{Ce^{Gx}}{|x|^{1+Y}} 1_{\{x<0\}} + \frac{Ce^{-Mx}}{|x|^{1+Y}} 1_{\{x>0\}} \]

Characteristic exponent

\[ \psi(\xi) = -i\mu\xi - CG(\nu - Y)((M - i\xi)^Y - M^Y + (G + i\xi)^Y - G^Y) \]

Characteristic function has exponential tails with \( \nu = Y \), error estimate in \( M \): \( O(\exp(-CM^Y/(1+Y))) \)
Figure: $T = 1, N = 252, S_0 = 100, K = 110, r = 0.1, q = 0, C = 3, G = 22, M = 28, Y = 0.8$, implemented in Matlab using a Laptop with CPU 2GHz, RAM 1G, last node takes 2.0s
Bermudan floating strike lookbacks

- Standard backward induction involves two state variables: asset price, maximum asset price
- Can be reduced to one state variable, maximum asset price/asset price

\[ f^j(y) = \max(e^y - 1, e^{-q\Delta} \mathbb{E}^*_{j\Delta, y}[f^{j+1}(e^{Y(j+1)\Delta})]) \]

- Double exponential fast Gauss transform method (Yamamoto 05), limited to BSM and Merton’s models
Kou’s model

- Pricing Bermudan floating strike lookback put
- Log asset price follows

\[ X_t = \mu t + \sigma B_t + \sum_{n=1}^{N_t} Z_n, \quad Z_n \sim p \eta_1 e^{-\eta_1 x} 1_{\{x>0\}} + (1-p) \eta_2 e^{\eta_2 x} 1_{\{x<0\}} \]

- Characteristic exponent

\[ \psi(\xi) = -i\mu \xi + \frac{1}{2} \sigma^2 \xi^2 + \lambda \left( 1 - \frac{p \eta_1}{\eta_1 - i\xi} - \frac{(1-p) \eta_2}{\eta_2 + i\xi} \right) \]

- Characteristic function has exponential tails with \( \nu = 2 \), error estimate \( O(\exp(-CM^{2/3})) \)
Bermudan floating strike lookback put in Kou’s model

Figure: $T = 1$, $N = 252$, $S_0 = 100$, $r = 0.1$, $q = 0$, $\sigma = 0.25$, $\lambda = 3$, $p = 0.3$, $\eta_1 = 17$, $\eta_2 = 12$, implemented in Matlab using a Laptop with CPU 2GHz, RAM 1G, last node takes 1.1s
Summary

- Hilbert transform method for pricing Bermudan style options in Lévy process models
- Very accurate with exponentially decay errors
- Fast with computational cost $O(NM \log(M))$
- European vanilla, barrier, lookback, defaultable bonds, Bermudan vanilla, barrier, floating strike lookback, monte carlo simulation etc.