Prices and sensitivities of barrier, first touch digital and double barrier options in Lévy-driven models

Mitya Boyarchenko and Sergei Levendorskiǐ

University of Chicago, University of Leicester

Leicester, June 18, 2009
Aims of the talk: to explain

I. different types of errors arising in applications of FFT and iFFT,
   ▶ why in many important cases, it is impossible to make all of them small simultaneously unless grids with too many points are chosen
   ▶ how to modify FFT-iFFT technique to alleviate this problem

II. how to use these modifications to overcome well-known difficulties in numerical realizations of
   ▶ explicit general formulas for factors in the Wiener-Hopf factorization formula
   ▶ action of the corresponding convolution operators which appear in the solution of the Wiener-Hopf equation

III. how to apply Carr’s randomization and efficient realizations of WHF to barrier options with continuous monitoring,
   ▶ why Carr’s randomization + WHF can be expected to perform better than other approaches, and
   ▶ illustrate the last point with numerical examples
Objects of study
Knock-out options with one or two barriers, which for brevity will simply be called “barrier options.” They are among the most popular OTC options currently traded on financial markets.

Our goal
To present fast and accurate algorithms for calculating the prices and sensitivities of these options in a wide class of asset pricing models.

Advantages of our approach
Our methods are very efficient on the one hand, and easy to implement in practice on the other hand.
Strong Points of Our Approach

- We allow rather general terminal payoff functions.
- Option prices are calculated for a (rather fine) uniformly spaced grid of initial log-spot prices (as opposed to one initial spot price).
- This allows us to calculate the deltas and gammas of the option at the points of the same grid using numerical differentiation.
- The prices and sensitivities corresponding to log-spot prices that do not lie on the grid are found using interpolation (the additional computational cost of interpolation is negligible).
- High accuracy is maintained even in the regions where the initial spot price of the underlying is very close to the barrier(s).
Outline

1. Lévy processes: general definitions and examples
2. Barrier options with continuous monitoring
3. Carr’s randomization
4. EPV-operators and Wiener-Hopf factorization
5. Perpetual knock-out options with one and two barriers
6. General formulas for WHF factors and EPV-operators
7. Fourier Transforms and FFT
8. Refinement and enhancement of FFT-iFFT technique
9. Efficient realization of EPV-operators as convolution operators
10. Example: pricing down-and-out barrier options, and their sensitivities
11. Example: pricing double barrier options
Lévy processes: general definitions

$L$, infinitesimal generator, and $\psi$, characteristic exponent of $X = (X_t)$:

$$E\left[e^{i\xi X_t}\right] = e^{-t\psi(\xi)}, \quad L e^{i\xi x} = -\psi(\xi) e^{i\xi x}$$

Hence, $L = -\psi(D)$ is the PDO with the symbol $-\psi(\xi)$. In 1D,

$$Lu(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi}(-\psi(\xi))\hat{u}(\xi)d\xi.$$ 

Explicit formulas (Lévy-Khintchine) are (also in 1D)

$$\psi(\xi) = \frac{\sigma^2}{2} \xi^2 - i\mu \xi + \int_{\mathbb{R}\setminus 0} (1 - e^{iy\xi} + iy\xi 1_{|y|<1}(y))F(dy),$$

$$Lu(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x) + \int_{\mathbb{R}\setminus 0} (u(x+y) - 1_{|y|<1}(y)yu'(x) - u(x))F(dy),$$

where $F(dy)$, the Lévy density, satisfies

$$\int_{\mathbb{R}\setminus 0} \min\{|y|^2, 1\} F(dy) < \infty.$$
Some examples

1. **Brownian motion** (no jumps)

2. **Double-exponential model** (Kou (2002)): sizes of jumps are exponentially distributed on each half-axis

\[ \psi(\xi) = \frac{\sigma^2}{2} \xi^2 - i\mu \xi + \frac{i c_+ \xi}{\lambda^+ + i\xi} + \frac{i c_- \xi}{\lambda^- + i\xi}, \]

where \( \sigma > 0, \mu \in \mathbb{R}, c_\pm > 0, \text{ and } \lambda^- < 0 < \lambda^+. \)

3. **Extended Koponen’s model** Boyarchenko and Levendorskiĭ (1999); a.k.a. CGMY-model (Carr, Madan, Geman, Yor (2002)) and KoBoL model. The Lévy density behaves as \( c|y|^{-\nu-1} \) near 0 and exponentially decays at infinity. For \( \nu \in (0, 2), \nu \neq 1, \)

\[ \psi(\xi) = -i\mu \xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \]

where \( c > 0, \mu \in \mathbb{R}, \text{ and } \lambda_- < 0 < \lambda_+. \)
Analytical expressions for the price of a barrier option with continuous monitoring

i in diffusion models: in many cases, in particular, in the Brownian Motion (BM) model and exponential BM model, explicit analytical formulas are available

ii for certain classes of diffusions with embedded compound Poisson jumps of a relatively simple structure, explicit formulas are derived by several authors starting with Lipton (2001) and Kou (2002), different probabilistic methods being used

iii for wide classes of exponential Lévy models containing the models above, general analytic formulas were derived by Boyarchenko and Levendorskiǐ (2000) using the Wiener-Hopf technique. An important ingredient was the derivation of the generalization of the Black-Scholes equation
The generalized Black-Scholes equation for barrier options in Lévy models (Boyarchenko and Levendorskiǐ (2002))

The stock price $S_t = \exp X_t$, where $X_t$ is a Lévy process under $\mathbb{Q}$. The boundary problem for the down-and-out call option:

$$(r - \partial_t - L)V(t, x) = 0, \quad x > h := \log H, \quad t < T;$$

$$V(T, x) = (e^x - K)_+, \quad x \geq h := \log H;$$

$$V(t, x) = G_b, \quad x \leq h, \quad t \leq T,$$

plus the natural condition on the rate of growth as $x \to +\infty$

**GBS equation:** in the sense of the theory of generalized functions

$X$ satisfies **(ACT)-property:** transition densities exist

$\psi$ admits the analytic continuation into the strip $\text{Im} \xi \in [-1, 0]$

Generalizations for Lévy processes in $\mathbb{R}^n$ are also proved
Numerical calculations in the non-gaussian case:

a. General formulas (Boyarchenko and Levendorskiĭ (2000)) are not computationally efficient

b. but help to derive the asymptotics of the price near the barrier: needed to understand why naive approximations cannot work

c. Explicit formulas obtained by probabilistic methods are applicable to jump-diffusion processes which are not observed in the real markets

d. If these processes are used as approximations to more realistic ones (e.g., Cont-Volchkova method; Pistorius with different co-authors), sizable errors result near the barrier

e. Monte-Carlo methods also give sizable errors near the barrier

Most efficient method for calculation of prices of barrier options and American options with finite time horizon $T$: discretize time $(0 =) t_0 < t_1 < \cdots < t_N (= T)$ but not the space variable.

The equivalent probabilistic version: **Carr’s randomization**

For **American options**: put, BM - Carr (1998); Lévy processes and general payoff functions - Boyarchenko and Levendorskiĭ (2002); proof of convergence for American options - Bouchard, El Karoui, Touzi (2006)


For the barrier option with payoff $G(X_T)^+$, barrier $h$ and no rebate, the result is a sequence of stationary boundary problems, equivalently, a sequence of perpetual barrier options with the same barrier.
Carr’s randomization: down-and-out option

Calculate the sequence of approximations to the option value:
\[ v_s(x) = V(t_s, x), \ s = 0, 1, \ldots, N - 1. \]

1. Set \( v_N(x) = G(X_T)_+ \)

2. For \( s = N - 1, N - 2, \ldots, 0 \), set \( \Delta_s = t_{s+1} - t_s \), \( q^s = r + \Delta_s^{-1} \), and calculate

\[
v_s(x) = \mathbb{E}^x \left[ \int_0^\tau e^{-q^s t} \Delta_s^{-1} v_{s+1}(X_t) dt \right],
\]

where \( \tau \) is the hitting time of \(( -\infty, h)\).

Standard approach: use the Wiener-Hopf factorization
(Ir)regularity of the price near barrier, and why Carr’s randomization + WHF can be expected to perform better than other methods

A typical case of a pure jump Lévy process with the Lévy density which

- exponentially decays at infinity (sufficiently fast)
- behaves as $|x|^{-\nu-1}$ at origin

For simplicity, let either $\nu > 1$ or the “drift” is zero then

1. the exact down-and-out option price behaves as $|x - h|^{\nu/2}$ near the log-barrier $h$ (Boyarchenko and Levendorskiĭ 2002)
2. the same is true of Carr’s randomization + WHF
   Boyarchenko and Levendorskiĭ (2002)
3. the other methods give $C^2$ up to the boundary
### Supremum and infimum processes

- \( \overline{X}_t = \sup_{0 \leq s \leq t} X_s \) - the supremum process
- \( \underline{X}_t = \inf_{0 \leq s \leq t} X_s \) - the infimum process

### Normalized EPV operators under \( X, \overline{X}, \) and \( \underline{X} \):

- \((\mathcal{E}_q g)(x) := q \mathbb{E}^x \left[ \int_0^{+\infty} e^{-qt} g(X_t) dt \right] \)
- \( (\mathcal{E}_q^+ g)(x) := q \mathbb{E}^x \left[ \int_0^{+\infty} e^{-qt} g(\overline{X}_t) dt \right] \)
- \( (\mathcal{E}_q^- g)(x) := q \mathbb{E}^x \left[ \int_0^{+\infty} e^{-qt} g(\underline{X}_t) dt \right] \)
Example 1: Brownian Motion. EPV-operators are of the form

$$E_q^+ u(x) = \beta^+ \int_0^{+\infty} e^{-\beta^+ y} u(x + y) dy,$$

$$E_q^- u(x) = (-\beta^-) \int_{-\infty}^{0} e^{-\beta^- y} u(x + y) dy,$$

where $\beta^- < 0 < \beta^+$ are the roots of $q + \psi(-i\beta) = 0$.

Example 2: Kou’s model. EPV-operators are of the form

$$E_q^+ u(x) = \sum_{j=1,2} a^+_j \beta^+_j \int_0^{+\infty} e^{-\beta^+_j y} u(x + y) dy,$$

$$E_q^- u(x) = \sum_{j=1,2} a^-_j (-\beta^-_j) \int_{-\infty}^{0} e^{-\beta^-_j y} u(x + y) dy,$$

where $\beta^-_2 < \lambda^- < \beta^-_1 < 0 < \beta^+_1 < \lambda^+ < \beta^+_2$ are the roots of the “characteristic equation” $q + \psi(-i\beta) = 0$, and $a^\pm_j > 0$ are constants.
Wiener-Hopf factorization formula

Three versions:

1. Let $T \sim \text{Exp}(q)$ be the exponential random variable of mean $q^{-1}$, independent of process $X$. For $\xi \in \mathbb{R}$,

$$\mathbb{E}[e^{i\xi X_T}] = \mathbb{E}[e^{i\xi \bar{X}_T}] \mathbb{E}[e^{i\xi X_T}];$$

2. For $\xi \in \mathbb{R}$,

$$\frac{q}{q + \psi(\xi)} = \phi_q^+(\xi) \phi_q^-(\xi),$$

where $\phi_q^{\pm}(\xi)$ admits the analytic continuation into the corresponding half-plane and does not vanish there

3. $\mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+ = \mathcal{E}_q^+ \mathcal{E}_q^-$.  

3 is valid in appropriate function spaces, and can be either proved as 1 or deduced from 2 because $\mathcal{E}_q = q(q + \psi(D))^{-1}$, $\mathcal{E}_q^{\pm} = \phi_q^{\pm}(D)$.  

Perpetual barrier down-and-out option

Theorem 1

Let $g$ be a measurable locally bounded function satisfying certain conditions on growth at $\infty$. Then for any $h$,

$$V(x; h) = q^{-1} \mathcal{E}_q^-(1_{(h, +\infty)} \mathcal{E}_q^+ g(x)).$$ (1)

The proof is based on the following result

Lemma 2

Let $X$ and $T$ be as above. Then

(a) the random variables $\overline{X}_T$ and $X_T - \overline{X}_T$ are independent (deep!); and
(b) the random variables $\overline{X}_T$ and $X_T - \overline{X}_T$ are identical in law.
Proof of Theorem 1

Let $T \sim \text{Exp } q$ be independent of $X$. Then $\mathcal{E}g(x) = \mathbb{E}[g(x + X_T)]$,

$$\mathcal{E}^+ g(x) = \mathbb{E}[g(x + \overline{X}_T)], \quad \mathcal{E}^- g(x) = \mathbb{E}[g(x + \underline{X}_T)],$$

and, by definition,

$$V(x; h) := \mathbb{E} \left[ \int_0^{\tau^-_h} e^{-qt} g(x + X_t) dt \right]$$

$$= \mathbb{E} \left[ \int_0^{\infty} 1_{x + X_T > h} e^{-qt} g(x + X_t) dt \right]$$

$$= q^{-1} \mathbb{E}[g(x + X_T) 1_{x + \overline{X}_T > h}].$$

Applying Lemma 2, we continue

$$V(x; h) = q^{-1} \mathbb{E}[g(x + \overline{X}_T + X_T - \underline{X}_T) 1_{x + \overline{X}_T > h}]$$

$$= q^{-1} \mathbb{E}[1_{x + \overline{X}_T > h} \mathcal{E}^+ g(x + \overline{X}_T)]$$

$$= q^{-1} \mathcal{E}^- 1_{(h, +\infty)} \mathcal{E}^+ g(x).$$
Now assume that \(0 < L < U < +\infty\) and write \(h_\pm = \ln L, h_\pm = \ln U\).

Value of a knock-out stream \(\{g(\ln S_t)\}_{t \geq 0}\) with barriers \((L, U)\):

\[
v_{k.o.}(x; q; h_-, h_+; g) = G_0^0(x) - G_1^1(x) - G_1^1(x) + G_2^2(x) + G_2^2(x)
- G_3^3(x) - G_3^3(x) + G_4^4(x) + G_4^4(x) - \cdots
\]

To find the terms on the RHS, first calculate \(G_0^0(x) = q^{-1} \cdot (\mathcal{E}_q g)(x)\).

Next, use the formulas

\[
G_0^0(x) = G_0^0(x)\bigg|_{(h_+, +\infty)}, \quad G_0^0(x) = G_0^0(x)\bigg|_{(-\infty, h_-)},
\]

\[
G_+^n(x) = \mathcal{E}_q^-(\mathbb{1}_{(-\infty, h_-)}(x) \cdot ((\mathcal{E}_q^-)^{-1} G_-^{n-1})(x)) \quad \forall \ n \geq 1,
\]

\[
G_-^n(x) = \mathcal{E}_q^+(\mathbb{1}_{[h_+, +\infty)}(x) \cdot ((\mathcal{E}_q^+)^{-1} G_+^{n-1})(x)) \quad \forall \ n \geq 1.
\]
Integral formulas for the Wiener-Hopf factors

Under certain regularity conditions on the characteristic exponent $\psi(\xi)$,

$$
\phi_{q}^{\pm}(\xi) = \exp \left[ \pm \frac{1}{2\pi i} \int_{\text{Im } \eta = \omega_{\mp}} \frac{\xi \cdot \ln(1 + q^{-1}\psi(\eta))}{\eta(\xi - \eta)} \, d\eta \right],
$$

where $\omega_{-} < 0 < \omega_{+}$ are suitably chosen. Main requirements:

- $\psi(\xi)$ admits analytic continuation into an open strip in $\mathbb{C}$ that contains the closed strip $\{\xi \in \mathbb{C} | \text{Im } \xi \in [\omega_{-}, \omega_{+}]\}$,
- $\Re (q + \psi(\xi)) > 0 \ \forall \ \xi$ in the closed strip.

Practical applications of the above formula

Truncate the integral and apply the trapezoid rule and FFT tools to calculate the values of $\phi_{q}^{\pm}(\xi)$ on a suitable grid $\vec{\xi} = (\xi_{k})_{k=1}^{M}$.

The integrand does not decay fast $\Rightarrow$ standard FFT does not suffice
The EPV-operators as PDO

For suitably chosen integration contours

\[ \mathcal{E}_q^+ u(x) = (2\pi)^{-1} \int e^{ix\xi} \phi_q^+(\xi) \hat{u}(\xi) d\xi \]

\[ \mathcal{E}_q^- u(x) = (2\pi)^{-1} \int e^{ix\xi} \phi_q^-(\xi) \hat{u}(\xi) d\xi \]

To apply the FTT-technique, horizontal lines are preferred, and the integrals must be truncated. For \( \mathcal{E}_q^+ \),

\[ \mathcal{E}_q^+ u(x) = (2\pi)^{-1} \int_{-\Lambda+i\omega}^{\Lambda+i\omega} e^{ix\xi} \phi_q^+(\xi) \hat{u}(\xi) d\xi, \]

where \( \omega > \omega_- \), and \( \Lambda \) must be large because

For important classes of Lévy processes, one of \( \phi_q^\pm(\xi) \) or both decay slowly at infinity, as well as the Fourier images of Carr’s randomized approximations to prices of barrier options
Fourier Transforms and FFT

Fourier transforms on the real line

\[ \hat{f}(\xi) = (\mathcal{F}f)(\xi) = (\mathcal{F}_{x\to \xi}f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \, dx \]

\[ (\mathcal{F}^{-1}g)(\xi) = (\mathcal{F}_{\xi\to x}^{-1}g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} g(\xi) \, d\xi \]

Fast Fourier transforms

Consider uniformly spaced grids \( \vec{x} = (x_j)_{j=1}^M \) and \( \vec{\xi} = (\xi_k)_{k=1}^M \) with mesh \( \Delta \) and \( \zeta \), respectively. Replace \( (\mathcal{F}_{x\to \xi}f)(\xi) \) and \( (\mathcal{F}_{\xi\to x}^{-1}g)(x) \) with

\[ (\mathcal{F}_{\text{fast}}f)(\xi) = \Delta \cdot \sum_{j=1}^{M} f(x_j) e^{-i\xi x_j}, \quad (\mathcal{F}_{\text{fast}}^{-1}g)(x) = \frac{\zeta}{2\pi} \cdot \sum_{k=1}^{M} g(\xi_k) e^{i\xi k x}. \]
Let $\vec{f} = (f_j)^M_{j=1}$ be an array of complex numbers. Set

$$\text{fft}(\vec{f})_k = \sum_{j=1}^{M} f_j \cdot e^{-2\pi i (j-1)(k-1)/M}, \quad 1 \leq k \leq M.$$ 

Standard FFT algorithms are designed for fast calculation of the vector $\text{fft}(\vec{f})$ (“fast” means $O(M \cdot \ln M)$ arithmetic operations).

$F_{\text{fast}}^{\pm 1}$ can be expressed in terms of $\text{fft}$ provided the identity $\Delta \cdot \zeta = 2\pi / M$ holds (“Nyquist relation” or “uncertainty principle”).

$\text{fft}$ can also be used for very fast calculation of sums of the form $h_k = \sum_{j=1}^{M} f_j g_{k-j} \quad (1 \leq k \leq M)$, where $\vec{f} = (f_j)^M_{j=1}$ and $\vec{g} = (g_\ell)^{M-1}_{\ell=1-M}$ are arrays of complex numbers.
Difficulties for efficient realizations using FFT/iFFT

1. For accurate calculation of the WHF factors, very long and fine grids in \( \xi \)-space are needed: \( \zeta \) must be small, \( \Lambda = M \zeta / 2 \) must be large.

2. For efficient realizations of \( \mathcal{E}^\pm = \phi^\pm_q(D) \), somewhat larger \( \zeta \) and smaller \( \Lambda \) typically suffice.

3. For typical applications in finance, the length of the grid \( M \Delta \) in \( x \)-space is less than 1 (e.g., from 20% below to 20% above the strike), and very small \( \Delta \) is not needed.

4. If the grids in \( x \)- and \( \xi \)-spaces of the same length are chosen, the “uncertainty principle” gives \( \Lambda = \pi / \Delta \).

- **Slow decay at infinity**\( \Rightarrow \) to decrease truncation error, large \( \Lambda \) is needed \( \Rightarrow \) unnecessary fine grid in \( x \)-space

- **To decrease the discretization error**, small \( \zeta \) is needed \( \Rightarrow \)
  \( M = \Lambda / (2 \zeta) \) very large \( \Rightarrow \) unnecessary long grid in \( x \)-space
allows one to compute $\mathcal{F}_{\text{fast}} f$ and $\mathcal{F}_{\text{fast}}^{-1} g$ even when the Nyquist relation is not satisfied.

Calculating $\mathcal{F}_{\text{fast}} f$ using fractional FFT involves three applications of ordinary FFT to arrays with $2M$ elements.

By comparison, when the Nyquist relation is satisfied, then a single application of FFT to an array with $M$ elements suffices $\Rightarrow$ when $M$ is large, using fractional FFT decreases the computational speed significantly.

Our approach to improving FFT was partially inspired by fractional FFT; however, the underlying ideas are quite different. In particular, our method minimizes the number of arithmetic operations that must be performed.
Explaining Our Idea Using a Simple Example

• Suppose we are given uniformly spaced grids $\vec{x} = (x_j)_{j=1}^M$ and $\vec{\xi} = (\xi_k)_{k=1}^M$ of mesh $\Delta$ and $\zeta$; and the relation $\Delta \zeta = 2\pi/M$ holds.

• Given a function $f(x)$, we can (quickly) calculate $(F_{\text{fast}} f)(\xi_k)$ for all $k$ using standard FFT techniques.

• Now suppose we wish to halve the mesh of the $\xi$-grid and double the number of points in it, while leaving the $x$-grid intact.

• Call the new grid $\vec{\xi}' = (\xi'_k)_{k=1}^{2M}$. It has mesh equal to $\zeta/2$.

• The points $\{\xi'_1, \xi'_3, \xi'_5, \ldots, \xi'_{2M-1}\}$ and $\{\xi'_2, \xi'_4, \xi'_6, \ldots, \xi'_{2M}\}$ form two uniformly spaced grids with mesh $\zeta$.

• Apply the standard FFT technique twice, and we are in good shape.
An Improved Setup for FFT and Inverse FFT

**Input:** a uniformly spaced grid \( \vec{x} = (x_j)_{j=1}^M, x_j = x_1 + (j - 1)\Delta \in \mathbb{R}; \)

\( M \) and \( \Delta > 0 \) are fixed

- Choose two positive integers, \( M_2 \) and \( M_3 \)
  - \( M_2 \): responsible for **refinement**
  - \( M_3 \): responsible for **stretching** the \( \xi \)-grid.

- Choose \( \xi_1 \in \mathbb{C} \), the desired initial point of the \( \xi \)-grid.

- \( M_1 := MM_2 M_3 \) is the total number of points in the \( \xi \)-grid

- Set \( \zeta = 2\pi/(M\Delta) \).

- \( \zeta_1 = \zeta/M_2 \) is the mesh of the \( \xi \)-grid

- \( M_3 \cdot (2\pi/\Delta) \) is the length of the \( \xi \)-grid

Explicitly, the \( \xi \)-grid is given by

\[
\vec{\xi} = (\xi_k)_{k=1}^{M_1}, \quad \xi_k = \xi_1 + (k - 1)\zeta_1 = \xi_1 + (k - 1) \cdot \frac{\zeta}{M_2}.
\]
Implementing FFT in the New Setup

Need to calculate $\mathcal{F}_{\text{fast}} f$ at all points of the grid $\vec{\xi}$.

The best one can hope for is to reduce the calculation to $M_2 \cdot M_3$ applications of FFT for arrays of length $M$ (input array: length $M$; output array: length $M \cdot M_2 \cdot M_3$).

Represent the grid $\vec{\xi} = (\xi_k)_{k=1}^{M_1}$ as a disjoint union of $M_2 \cdot M_3$ grids, each of which has $M$ points and mesh $\zeta$, and apply ordinary FFT to each of them:

$$\left(\xi_{M_2 \cdot (k-1)+1}\right)_{k=1}^{M}, \left(\xi_{M_2 \cdot (k-1)+2}\right)_{k=1}^{M}, \ldots, \left(\xi_{M_2 \cdot k}\right)_{k=1}^{M},$$

$$\left(\xi_{M_2 \cdot (k-1+M)+1}\right)_{k=1}^{M}, \left(\xi_{M_2 \cdot (k-1+M)+2}\right)_{k=1}^{M}, \ldots, \left(\xi_{M_2 \cdot (k+M)}\right)_{k=1}^{M},$$

$$\ldots,$$

$$\left(\xi_{M_2 \cdot (k-1+(M_3-1)M)+1}\right)_{k=1}^{M}, \ldots, \left(\xi_{M_2 \cdot (k+(M_3-1)M)}\right)_{k=1}^{M}.$$
Implementing Inverse FFT in the New Setup

- Given function \( g(\xi) \) whose domain contains the grid \( \xi \)
- Need to calculate the values of the function \( \mathcal{F}_{\text{fast}}^{-1} g \) on the grid \( \vec{x} \).

To this end,

- for each \( 1 \leq j \leq M_3 \) and each \( 1 \leq \ell \leq M_2 \), let \( g_{j,\ell} \) be the restriction of \( g \) to the sub-grid \( \vec{\xi}(j, \ell) = (\xi_{M_2 \cdot (k-1+(j-1)M)+\ell})_{k=1}^M \)
- values of \( \mathcal{F}_{\text{fast}}^{-1} g_{j,\ell} \) on the grid \( \vec{x} \) can be calculated using standard FFT
- \( \forall (j, \ell) \), need to calculate a single FFT for a vector of length \( M \)
- it follows immediately from the definitions that

\[
\mathcal{F}_{\text{fast}}^{-1} g = \frac{1}{M_2} \sum_{j=1}^{M_3} \sum_{\ell=1}^{M_2} \mathcal{F}_{\text{fast}}^{-1}(g_{j,\ell}).
\]

- This method of calculating \( \mathcal{F}_{\text{fast}}^{-1} g \) requires only \( O(M_1 \cdot \ln M) \)
  arithmetic operations, which, again, is the best one can hope for.
Consider a grid $\tilde{x} = (x_j)_{j=1}^M$, where $x_j = x_1 + (j - 1)\Delta$ for all $1 \leq j \leq M$, and $\Delta > 0$ is fixed. Approximating $f$ with a piecewise linear function yields

$$
\hat{f}(\xi) \approx (F_{\text{enh}} f)(\xi) = \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} \cdot (F_{\text{fast}} f)(\xi)
$$

$$
+ \frac{1 + i\xi\Delta - e^{i\xi\Delta}}{(i\xi\Delta)^2} \cdot \Delta \cdot f_1 \cdot e^{-i\xi x_1}
$$

$$
+ \frac{1 - i\xi\Delta - e^{-i\xi\Delta}}{(i\xi\Delta)^2} \cdot \Delta \cdot f_M \cdot e^{-i\xi x_M}.
$$

The main advantage of using $(F_{\text{enh}} f)(\xi)$, as opposed to $(F_{\text{fast}} f)(\xi)$, as an approximation to $\hat{f}(\xi)$, stems from the fact that $|\hat{f}(\xi) - (F_{\text{enh}} f)(\xi)|$, the error of the former approximation, can be estimated \emph{independently of the size of $\xi$}. The analogous statement is \emph{false} for $(F_{\text{fast}} f)(\xi)$. 
Normalized EPV Operators and Fourier Transforms

- $X = \{X_t\}_{t \geq 0}$ a Lévy process with characteristic exponent $\psi(\xi)$
- fix $q > 0$ and let $\phi_q^\pm(\xi)$ be the Wiener-Hopf factors of $q \cdot (q + \psi(\xi))^{-1}$
- PDO realization of the normalized EPV operators of $X$:
  $$(\mathcal{E}_q^\pm f)(x) = \mathcal{F}_{\xi \to x}^{-1}(\phi_q^\pm(\xi) \cdot \hat{f}(\xi))$$
- convolution realization of the normalized EPV operators:
  $$(\mathcal{E}_q^+ f)(x) = \int_0^{+\infty} f(x + y) p^+_q(dy), \quad (\mathcal{E}_q^- f)(x) = \int_{-\infty}^0 f(x + y) p^-_q(dy),$$

  where $p^\pm_q(dy)$ are Borel probability measures on $\mathbb{R}$ supported on the positive and the negative half axis, respectively
- the Fourier transforms of $p^\pm_q$ are given by $\hat{p}_q^\pm(\xi) = \phi_q^\pm(\xi)$
Given a function $f(x)$ and a uniformly spaced grid $\bar{x} = (x_j)_{j=1}^M$ of points in $\mathbb{R}$, we approximate $f(x)$ with a linear function on each subinterval $[x_j, x_{j+1}]$, and approximate $f(x)$ with 0 outside of $[x_1, x_M]$.

Now we must calculate the action of $\mathcal{E}_q^\pm$ on a function of the form
\[
(f_j + \Delta^{-1} \cdot (f_{j+1} - f_j) \cdot (x - x_j)) \cdot 1_{[x_j, x_{j+1}]}(x).
\]
This is done using the convolution realization of $\mathcal{E}_q^\pm$, described on the previous slide.

The answer can be expressed in terms of $\phi_q^\pm(\xi)$ and the inverse Fourier transforms of certain auxiliary functions.

The resulting explicit formulas (see the next two slides) can be realized efficiently in practice using fast discrete convolution.
Consider a grid $\mathbf{x} = (x_j)_{j=1}^M$, where $x_j = x_1 + (j - 1)\Delta$ for all $1 \leq j \leq M$, and $\Delta > 0$ is fixed. Approximating $f$ with a piecewise linear function yields

$$(E_q^+ f)(x_k) \approx -d_k^+ \cdot f_M + \sum_{j=k}^{M} c_{k-j}^+ \cdot f_j \quad (1 \leq k \leq M),$$

where $f_j = f(x_j)$ for $1 \leq j \leq M$,

$$d_k^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-M)\Delta \xi} \cdot \phi_q^+(\xi) \cdot \frac{e^{-i\xi\Delta} + i\xi\Delta - 1}{(i\xi\Delta)^2} \, d\xi$$

for $1 \leq k \leq M$,

$$c_{\ell}^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta \xi} \cdot \phi_q^+(\xi) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} \, d\xi$$

for $1 - M \leq \ell \leq -1$, and

$$c_0^+ = 1 - \sum_{1-M \leq \ell \leq -1} c_{\ell}^+.$$
Similar formulas for $\mathcal{E}_q^-$:

$$(\mathcal{E}_q^- f)(x_k) \approx -d_k^- \cdot f_1 + \sum_{j=1}^{k} c_{k-j}^- \cdot f_j \quad (1 \leq k \leq M),$$

where $f_j = f(x_j)$ for $1 \leq j \leq M$,

$$d_k^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-1)\Delta \xi} \cdot \phi_q^-(\xi) \cdot \frac{e^{i\Delta \xi} - i\xi \Delta - 1}{(i\xi \Delta)^2} \, d\xi,$$

for $1 \leq k \leq M$,

$$c_\ell^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta \xi} \cdot \phi_q^-(\xi) \cdot \frac{e^{i\Delta \xi} + e^{-i\Delta \xi} - 2}{(i\xi \Delta)^2} \, d\xi$$

for $1 \leq \ell \leq M - 1$, and

$$c_0^- = 1 - \sum_{1 \leq \ell \leq M-1} c_\ell^-.$$
Table: Prices and sensitivities of a down-and-out barrier put option in the NIG model: comparison with the results of M. Jeannin and M. Pistorius

<table>
<thead>
<tr>
<th>Spot price</th>
<th>Option price</th>
<th>Delta</th>
<th>Gamma</th>
<th>Theta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BL</td>
<td>JP</td>
<td>BL</td>
<td>JP</td>
</tr>
<tr>
<td>64%</td>
<td>507.0212</td>
<td>486.8291</td>
<td>0.929</td>
<td>0.907</td>
</tr>
<tr>
<td>66%</td>
<td>554.2226</td>
<td>532.6638</td>
<td>0.443</td>
<td>0.437</td>
</tr>
<tr>
<td>68%</td>
<td>573.7149</td>
<td>551.8006</td>
<td>0.123</td>
<td>0.127</td>
</tr>
<tr>
<td>70%</td>
<td>574.2871</td>
<td>552.6467</td>
<td>-0.102</td>
<td>-0.092</td>
</tr>
<tr>
<td>72%</td>
<td>561.3327</td>
<td>540.3645</td>
<td>-0.265</td>
<td>-0.251</td>
</tr>
<tr>
<td>74%</td>
<td>538.5875</td>
<td>518.5689</td>
<td>-0.383</td>
<td>-0.366</td>
</tr>
<tr>
<td>76%</td>
<td>508.8566</td>
<td>490.0120</td>
<td>-0.465</td>
<td>-0.445</td>
</tr>
<tr>
<td>78%</td>
<td>474.3919</td>
<td>456.9306</td>
<td>-0.518</td>
<td>-0.496</td>
</tr>
<tr>
<td>80%</td>
<td>437.1020</td>
<td>421.2329</td>
<td>-0.545</td>
<td>-0.521</td>
</tr>
</tbody>
</table>

Spot prices are reported as percentages of the strike price $K = 3500$.

Other parameters: $H = 2100$ (barrier), $r = 0.03$ (riskless rate), $T = 1$ (maturity).
Some Comments on the Previous Table

- NIG parameters: $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$.
- Jeannin and Pistorius report computational times of 55 sec. per option price on a 2GHz machine. Our computational time: $\approx 8$ sec. for all prices and sensitivities in the last table (also on a 2GHz PC).
- J&P use approximation of NIG by a HEJD process, which changes the asymptotics of the value function of the option near the barrier. More precisely, it leads to underpricing of the option near the barrier.
- Carr’s randomization approximation + WHF does not change the order of the asymptotics, so it is expected to be more accurate on theoretical grounds.
- All our numerical tests are consistent with this theoretical expectation.
Example: pricing double barrier options

Figure: Prices of a knock-out double barrier put option (left panel) and of a double-no-touch option (right panel) in the KoBoL model. Solid lines represent the results obtained using our algorithm. Crosses represent the results obtained using Monte-Carlo simulations. KoBoL parameters: $\nu = 0.5, c = 1, \lambda_+ = 9, \lambda_- = -8, \mu \approx -0.0423$. Option parameters: $K = 3500, H_- = 2800, H_+ = 4200, r = 0.03, T = 0.1$. Algorithm parameters: $n = 812$ (number of points on the “main” $x$-grid), $\Delta = \frac{\ln H_+ - \ln H_-}{n-1} \approx 0.005$, $M = 4096$, $M_2 = 4$, $M_3 = 16$, $\zeta_1 \approx 0.767$, $m = 8$ (for the calculation of the Wiener-Hopf factors), $N = 80$ (number of time steps), $\epsilon = 10^{-7}$ (error tolerance for the iterative procedure).
Table: Prices of a knock-out double barrier put option and of a double-no-touch option in the KoBoL model

<table>
<thead>
<tr>
<th>Spot price</th>
<th>Knock-out double barrier put price</th>
<th>Our price</th>
<th>MC price</th>
<th>MC error</th>
<th>Double-no-touch option</th>
<th>Our price</th>
<th>MC price</th>
<th>MC error</th>
</tr>
</thead>
<tbody>
<tr>
<td>81%</td>
<td>222.5256</td>
<td>222.5448</td>
<td>0.0001</td>
<td>0.4194</td>
<td>0.4192</td>
<td>-0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>82%</td>
<td>302.2846</td>
<td>301.8782</td>
<td>-0.0013</td>
<td>0.5778</td>
<td>0.5768</td>
<td>-0.0017</td>
<td></td>
<td></td>
</tr>
<tr>
<td>83%</td>
<td>344.5279</td>
<td>344.6685</td>
<td>0.0004</td>
<td>0.6799</td>
<td>0.6799</td>
<td>0.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>84%</td>
<td>364.5539</td>
<td>364.1193</td>
<td>-0.0012</td>
<td>0.7504</td>
<td>0.7500</td>
<td>-0.0004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>85%</td>
<td>370.3752</td>
<td>370.9815</td>
<td>0.0016</td>
<td>0.8009</td>
<td>0.8004</td>
<td>-0.0007</td>
<td></td>
<td></td>
</tr>
<tr>
<td>86%</td>
<td>366.6665</td>
<td>367.2755</td>
<td>0.0017</td>
<td>0.8383</td>
<td>0.8394</td>
<td>0.0013</td>
<td></td>
<td></td>
</tr>
<tr>
<td>87%</td>
<td>356.3501</td>
<td>356.6483</td>
<td>0.0008</td>
<td>0.8665</td>
<td>0.8670</td>
<td>0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>88%</td>
<td>341.3540</td>
<td>341.7309</td>
<td>0.0011</td>
<td>0.8881</td>
<td>0.8887</td>
<td>0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>89%</td>
<td>323.0086</td>
<td>323.8335</td>
<td>0.0027</td>
<td>0.9048</td>
<td>0.9046</td>
<td>-0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90%</td>
<td>302.2708</td>
<td>303.3371</td>
<td>0.0035</td>
<td>0.9179</td>
<td>0.9179</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>91%</td>
<td>279.8571</td>
<td>280.7147</td>
<td>0.0031</td>
<td>0.9280</td>
<td>0.9288</td>
<td>0.0008</td>
<td></td>
<td></td>
</tr>
<tr>
<td>92%</td>
<td>256.3280</td>
<td>257.1714</td>
<td>0.0033</td>
<td>0.9358</td>
<td>0.9355</td>
<td>-0.0004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>93%</td>
<td>232.1444</td>
<td>232.8017</td>
<td>0.0028</td>
<td>0.9419</td>
<td>0.9415</td>
<td>-0.0004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>94%</td>
<td>207.7075</td>
<td>208.4755</td>
<td>0.0037</td>
<td>0.9464</td>
<td>0.9463</td>
<td>0.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>95%</td>
<td>183.3905</td>
<td>183.8869</td>
<td>0.0027</td>
<td>0.9496</td>
<td>0.9501</td>
<td>0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>96%</td>
<td>159.5662</td>
<td>160.1593</td>
<td>0.0037</td>
<td>0.9517</td>
<td>0.9522</td>
<td>0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>97%</td>
<td>136.6340</td>
<td>137.2298</td>
<td>0.0044</td>
<td>0.9527</td>
<td>0.9531</td>
<td>0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>98%</td>
<td>115.0501</td>
<td>115.5238</td>
<td>0.0041</td>
<td>0.9529</td>
<td>0.9534</td>
<td>0.0005</td>
<td></td>
<td></td>
</tr>
<tr>
<td>99%</td>
<td>95.3578</td>
<td>95.7482</td>
<td>0.0041</td>
<td>0.9522</td>
<td>0.9524</td>
<td>0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100%</td>
<td>78.1900</td>
<td>78.6724</td>
<td>0.0062</td>
<td>0.9506</td>
<td>0.9508</td>
<td>0.0002</td>
<td></td>
<td></td>
</tr>
<tr>
<td>101%</td>
<td>64.0702</td>
<td>64.5565</td>
<td>0.0076</td>
<td>0.9481</td>
<td>0.9487</td>
<td>0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>102%</td>
<td>52.8802</td>
<td>53.2834</td>
<td>0.0076</td>
<td>0.9448</td>
<td>0.9451</td>
<td>0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>103%</td>
<td>44.0286</td>
<td>44.2211</td>
<td>0.0044</td>
<td>0.9404</td>
<td>0.9413</td>
<td>0.0009</td>
<td></td>
<td></td>
</tr>
<tr>
<td>104%</td>
<td>36.9574</td>
<td>37.1324</td>
<td>0.0047</td>
<td>0.9351</td>
<td>0.9357</td>
<td>0.0007</td>
<td></td>
<td></td>
</tr>
<tr>
<td>105%</td>
<td>31.2445</td>
<td>31.3603</td>
<td>0.0037</td>
<td>0.9285</td>
<td>0.9294</td>
<td>0.0010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>106%</td>
<td>26.5798</td>
<td>26.9603</td>
<td>0.0143</td>
<td>0.9206</td>
<td>0.9207</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>107%</td>
<td>22.7348</td>
<td>22.7653</td>
<td>0.0013</td>
<td>0.9112</td>
<td>0.9115</td>
<td>0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>108%</td>
<td>10.5381</td>
<td>10.5361</td>
<td>0.0001</td>
<td>0.8990</td>
<td>0.8995</td>
<td>0.0005</td>
<td></td>
<td></td>
</tr>
<tr>
<td>109%</td>
<td>6.4015</td>
<td>6.3761</td>
<td>-0.0021</td>
<td>0.8742</td>
<td>0.8755</td>
<td>0.0013</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110%</td>
<td>4.0101</td>
<td>4.0000</td>
<td>0.0012</td>
<td>0.8427</td>
<td>0.8428</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>111%</td>
<td>2.6714</td>
<td>2.6710</td>
<td>-0.0004</td>
<td>0.8182</td>
<td>0.8182</td>
<td>0.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>112%</td>
<td>1.7183</td>
<td>1.7205</td>
<td>0.0022</td>
<td>0.7908</td>
<td>0.7913</td>
<td>0.0005</td>
<td></td>
<td></td>
</tr>
<tr>
<td>113%</td>
<td>1.1454</td>
<td>1.1507</td>
<td>0.0055</td>
<td>0.7647</td>
<td>0.7653</td>
<td>0.0006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>114%</td>
<td>0.8033</td>
<td>0.8072</td>
<td>0.0040</td>
<td>0.7361</td>
<td>0.7361</td>
<td>0.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>115%</td>
<td>0.6020</td>
<td>0.6050</td>
<td>0.0030</td>
<td>0.7105</td>
<td>0.7106</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>116%</td>
<td>0.4214</td>
<td>0.4248</td>
<td>0.0034</td>
<td>0.6864</td>
<td>0.6865</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>117%</td>
<td>0.3080</td>
<td>0.3109</td>
<td>0.0029</td>
<td>0.6590</td>
<td>0.6592</td>
<td>0.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>118%</td>
<td>0.2219</td>
<td>0.2248</td>
<td>0.0030</td>
<td>0.6337</td>
<td>0.6339</td>
<td>0.0002</td>
<td></td>
<td></td>
</tr>
<tr>
<td>119%</td>
<td>0.1581</td>
<td>0.1609</td>
<td>0.0025</td>
<td>0.6093</td>
<td>0.6094</td>
<td>0.0001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>