American options in the Heston model with stochastic interest rate

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Outline

- Heston model with CIR stochastic interest rate
- Numerical example: comparison with Medevedev-Scaillet method (2007) and Longstaff-Schwartz method
- Outline of our approach to jump-diffusion models with regime-switching, stochastic volatility and/or stochastic interest rates
American put in Heston model with CIR interest rates

Non-dividend paying stock

The stock dynamics, $S_t$, stock volatility, $\hat{\nu}_t$, and the riskless interest rate, $r_t$, follow the system of SDE

\[
\frac{dS_t}{S_t} = r_t dt + \sqrt{\hat{\nu}_t} d\hat{W}_{1,t}, \\
d\hat{\nu}_t = \hat{\kappa}_v (\hat{\theta}_v - \hat{\nu}_t) dt + \hat{\sigma}_v \sqrt{\hat{\nu}_t} dW_{2,t}, \\
dr_t = \kappa_r (\theta_r - r_t) dt + \sigma_r \sqrt{r_t} dW_{3,t},
\]

where $\hat{W}_{1,t}$, $W_{2,t}$, $W_{3,t}$ are components of the Brownian motion in 3D, with unit variances. For simplicity, only $\hat{W}_{1,t}$ and $W_{2,t}$ are correlated, the correlation coefficient being $\rho$. The coefficients $\hat{\kappa}_v$, $\kappa_r$, $\hat{\theta}_v$, $\theta_r$, $\hat{\sigma}_v$ and $\sigma_r$ are positive.

- M&S: “Up to now, there exist no feasible methodology for pricing American options when volatility and interest rates are stochastic.”

- M&S develop an asymptotic method based on a certain suboptimal exercise rule, which can be regarded as intuitively natural.

- M&S produce a numerical example of Heston model with CIR interest rates, and compare the results with the results obtained by Longstaff-Schwarz algorithm with 200,000 sample paths, 500 time steps and 50 exercise dates.
We consider the same example.

The parameters are

$\hat{\kappa}_v = 1.58, \hat{\theta}_v = 0.03, \hat{\sigma}_v = 0.2, \kappa_r = 0.26, \theta_r = 0.04, \sigma_r = 0.08, \rho = -0.26$, the spot value of the riskless rate is 0.04, the spot stock price is $S = 100$, the spot volatility $v$ and strike $K$ vary: $v = 0.04, 0.09, 0.16$, $K = 90, 100, 110$, and time to expiry is $T = 1/12, 0.25, 0.5$. 
American put in Heston model with CIR interest rates

- (AP): prices of the American put calculated using discretization of $(t, v, r)$–space and the reduction to a regime-switching model;
- (MC): prices of the American put calculated in Medvedev and Scaillet (2007) using the Longstaff-Schwarz algorithm with 200,000 sample paths, 500 time steps and 50 exercise dates;
- (MS): prices of the American put, which Medvedev and Scaillet (2007) calculated using their asymptotic method with 5 terms of the asymptotic expansion.
American put in Heston model with CIR interest rates

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<th>$T = 1/12$</th>
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American put in Heston model with CIR interest rates

(continuation)

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Early exercise threshold
General set-up:

- regime-switching model, finite number of states
- in each state $j$, a 2-3 factor process $(X_t^{(j)}, Y_t^{(j)})$
- $Y_t^{(j)}$ is a mean-reverting process, which defines the dynamics of the interest rate and/or volatility
- jump-diffusion innovations in $X_t^{(j)}$ are independent of innovations in $Y_t^{(j)}$, and both are independent of the modulating Markov chain
- drift of $X_t^{(j)}$ depends on $Y_t^{(j)}$
- log-stock price $\log S_t^{(j)} = X_t^{(j)} + \text{affine or quadratic function of } Y_t^{(j)}$
Basic examples:

- $Y_t^{(j)}$ (or its components) are of the OU-type or CIR-type, with embedded jumps.
- Interest rate and/or volatility are affine or quadratic function of $Y_t^{(j)}$.
- SDE for $X_t^{(j)}$ is of the form

$$dX_t^{(j)} = \mu^{(j)}(Y_t^{(j)}) \, dt + \sigma(Y_t^{(j)}) \, dZ_t^{(j)}$$

where $Z_t^{(j)}$ is a Lévy process.
General scheme of solution. Part I.

In each state of the modulating Markov chain

- discretize the state space for and the infinitesimal generator of $Y_t^{(j)}$
- freeze the riskless rate and coefficients in SDE for $X_t^{(j)}$ at points of the discretized state space
- truncate the discretized state space for $Y_t^{(j)}$
- impose appropriate boundary conditions so that the discretized part of the infinitesimal generator can be interpreted as the infinitesimal generator of a Markov chain
General scheme of solution. Part II.

- Apply to the resulting regime-switching Lévy model (with a larger modulating Markov chain) a natural generalization of the method of lines (equivalently, Carr’s randomization method).

- Carr’s randomization reduces the pricing of an American option with finite time horizon to a sequence of perpetual options in regime-switching models.

- Apply to each perpetual option in the sequence an iteration procedure.

- The main building block of the iteration procedure is a general optimal stopping theorem for American and real options in Lévy models based on the Wiener-Hopf method.
## Non-switching Lévy models

### Notation

- $q > 0$ – the riskless rate
- $X = \{X_t\}_{t \geq 0}$ – a Lévy process
- $\Psi$ – Lévy exponent of $X$
- $L$ – infinitesimal generator of $X$
- $g(X_t)$ – payoff stream
- $G(X_t)$ – instantaneous payoff
- $x$ – the current realization of $X$
- $E^x[f(X_t)] := E[f(X_t) | X_0 = x]$
Non-switching Lévy models

**Moment generating function of a Lévy process**

\[
E \left[ e^{zX_t} \right] = e^{t\Psi(z)}
\]

\[
Le^{zx} = \Psi(z)e^{zx}
\]

**Lévy process with jump component of finite variation: infinitesimal generator and Lévy Khintchine formula.**

- \( b \) – drift, \( \sigma^2 \) – variance,
- \( F(dy) \) – density of jumps

\[
Lu(x) = \frac{\sigma^2}{2} u''(x) + bu'(x) + \int_{-\infty}^{+\infty} (u(x + y) - u(x)) F(dy)
\]

\[
E \left[ e^{zX_t} \right] = \exp \left[ t \left( \frac{\sigma^2}{2} z^2 + bz + \int_{-\infty}^{+\infty} (e^y - 1) F(dy) \right) \right]
\]
Perpetual American options

Find the optimal stopping time $\tau$ which maximizes

$$V(x; \tau) = E^x \left[ \int_0^{\tau} e^{-qt} g(X_t) dt \right] + E^x \left[ e^{-r\tau} G(X_\tau) \right].$$

- Can be reduced to the case $G = 0$.
- In many cases, $\tau$ is the hitting time of a semi-infinite interval.
- Explicit formula for $V(x; \tau)$ and equation for the optimal exercise threshold are available provided $g$ is monotone and changes sign (after reduction to the case $G = 0$).
- At each step of the backward induction, each step of the iteration procedure, and for each state, we will solve an optimal stopping problem of this form.
Regime-switching models

General set-up and notation

- finite state Markov chain
- \( \lambda_{jk} \) - transition rates; \( \Lambda_j = \sum_{k \neq j} \lambda_{jk} \)
- \( X_t^{(j)} \) - Lévy process in state \( j \)
- \( \Psi_j \) and \( L_j \): Lévy exponent and infinitesimal generator of \( X_t^{(j)} \)
- \( q_j \) - state-\( j \) riskless rate
- \( g_j \) - payoff stream in state \( j \)
- \( G_j \) - instantaneous payoff in state \( j \)
Regime-switching models. Finite time horizon

Assumptions: for \( j = 1, \ldots, m \),

(i) the payoff functions \( G_j \) decrease and change sign;

(ii) \( L_j G_j \) is well-defined;

(iii) \( \tilde{g}_j = \sum_{k \neq j} \lambda_{jk} G_k - (q_j + \Lambda_j - L_j) G_j \) does not decrease, and \( \tilde{g}_j(-\infty) < 0 \)

Example: for \( G_j(x) = K_j - B_j e^x \)

(i) \( B_j, K_j > 0 \);

(ii) \( E \left[ \exp(X_1^{(j)}) \right] < \infty \);

(iii) \( \sum_{k \neq j} \lambda_{jk} (K_k/K_j - 1) - q_j < 0 \leq \sum_{k \neq j} \lambda_{jk} (1 - B_k/B_j) + (q_j - \Psi_j(1)) \)

The RHS: minus the rate of instantaneous discounted gains
First step of Algorithm

1. Choose \((0 =) t_0 < t_1 < \cdots < t_N (= T)\).

2. For \(s = N - 1, N - 2, \ldots\), set \(\Delta_s = t_{s+1} - t_s\), \(q^s_j = q_j + \Lambda_j + \Delta_s^{-1}\).

3. Set \(v_j^N(x) = G_j(x)_+, j = 1, 2, \ldots, m\).

4. For \(s = N - 1, N - 2, \ldots\), denote by \(v_j^s\) and \(h_j^s\) the Carr’s randomization approximations to state-\(j\) option value and exercise boundary for interval \([t_s, t_{s+1})\).
Backward induction and iteration procedure

- for \( s = l < N \), calculate the (approximations to the) exercise boundaries \( h_{j,*}^s \) and option values \( v_{j,*}^s \), \( j = 1, 2, \ldots, m \), assuming that for \( l + 1 \leq s \leq N \) and \( j = 1, 2, \ldots, m \), \( h_{j,*}^s \) and \( v_{j,*}^s \) are known
- for each \( s = N - 1, N - 2, \ldots \) and \( j = 1, 2, \ldots, m \), construct sequences \( \{ h_{j}^{s,n} \}_{n=0}^{\infty} \) and \( \{ v_{j}^{s,n} \}_{n=0}^{\infty} \), s.t.

\[
 h_{j,*}^s = \lim_{n \to +\infty} h_{j}^{s,n}, \quad v_{j,*}^s = \lim_{n \to +\infty} v_{j}^{s,n} \tag{1}
\]

Thus, for each \( s \), we need to introduce an additional cycle in \( n \); and inside the cycle in \( n \), we use additional cycles in \( j = 1, 2, \ldots, m \).
At step $s = N - 1, N - 2, \ldots$

- set $v_j^{s0} = 0, h_j^{s0} = +\infty, j = 1, \ldots, m,$
- for $n = 1, 2, \ldots,$ and $j = 1, 2, \ldots, m,$ find an optimal $\tau_{j}^{sn,-},$ which maximizes

$$v_j^{sn}(x) = E^{j,x} \left[ \int_0^{\tau_{j}^{sn,-}} e^{-q_j^s t} (\Delta_{s-1}^{s+1} X_t^{(j)}) + \sum_{k \neq j} \lambda_{jk} v_k^{s,n-1}(X_t^{(j)}) ) dt \right]$$

$$+ E^{j,x} \left[ e^{-q_j^s \tau_{j}^{sn,-}} G_j(X_{\tau_{j}^{sn,-}}) \right].$$

We find $\tau_{j}^{sn,-}$ as the hitting time of the unique interval of the form $(-\infty, h_j^{sn}]$ solving the problem in non-regime-switching Lévy model.
Main technical tools

- expected present value operators (EPV-operators) under a Lévy process and its supremum and infimum processes
- Wiener-Hopf factorization
- general theorem about optimal timing to abandon a monotone stream that is a monotone function of a Lévy process
Wiener-Hopf factorization

Supremum and infimum processes

- $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ - the supremum process
- $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ - the infimum process

Normalized EPV operators under $X$, $\bar{X}$, and $\underline{X}$:

1. $(\mathcal{E} g)(x) := qE^x \left[ \int_{0}^{+\infty} e^{-qt} g(X_t) \, dt \right]$

2. $(\mathcal{E}^+ g)(x) := qE^x \left[ \int_{0}^{+\infty} e^{-qt} g(\bar{X}_t) \, dt \right],$

3. $(\mathcal{E}^- g)(x) := qE^x \left[ \int_{0}^{+\infty} e^{-qt} g(\underline{X}_t) \, dt \right].$
Wiener-Hopf factorization formula

Three versions:

1. \( T \sim \text{Exp}(q) \) – the exponential random variable of mean \( q^{-1} \), independent of process \( X \). For \( z \in i\mathbb{R} \),

\[
E[e^{zX_T}] = E[e^{z\bar{X}_T}]E[e^{zX_T}];
\]

2. For \( z \in i\mathbb{R} \),

\[
\frac{q}{q - \Psi(z)} = \kappa_+^q(z)\kappa_-^q(z);
\]

3. \( \mathcal{E} = \mathcal{E}^-\mathcal{E}^+ = \mathcal{E}^+\mathcal{E}^- \).

1 and 2 are the same, 3 is valid in the space of locally bounded measurable function which do not grow too fast at infinity.
Example 1: Brownian motion with variance $\sigma^2$ and drift $b$

The Lévy exponent is

$$\Psi(z) = \frac{\sigma^2}{2} z^2 + bz$$

EPV-operators are of the form

$$\mathcal{E}^+ u(x) = \beta^+ \int_0^{+\infty} e^{-\beta^+ y} u(x + y) dy,$$

$$\mathcal{E}^- u(x) = (-\beta^-) \int_{-\infty}^0 e^{-\beta^- y} u(x + y) dy,$$

where $\beta^- < 0 < \beta^+$ are the roots of the “characteristic equation”

$$q - \Psi(\beta) = 0.$$
Example 2: Kou’s model

The Lévy exponent

\[ \Psi(z) = \frac{\sigma^2}{2} z^2 + bz + \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z}. \]

EPV-operators are of the form

\[ \mathcal{E}^+ u(x) = \sum_{j=1,2} a_j^+ \beta_j^+ \int_0^{+\infty} e^{-\beta_j^+ y} u(x+y)dy, \]

\[ \mathcal{E}^- u(x) = \sum_{j=1,2} a_j^- (-\beta_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} u(x+y)dy, \]

where \( \beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+ \) are the roots of the “characteristic equation” \( q - \Psi(\beta) = 0 \), and \( a_j^\pm > 0 \) are constants.
Option to abandon a non-decreasing stream.

Objective:

Find an optimal stopping time \( \tau \)

\[
V(x) = \sup_{\tau} E^x \left[ \int_0^\tau e^{-qt} g(X_t) dt \right]
\]

In many cases, for non-decreasing \( g \), the optimal stopping time is the hitting time of a semi-infinite interval \((-\infty, h]\). Notation: \( \tau_h^- \).
Theorem 1

Let $X$ satisfy the (ACP)-property, and let $g$ be a measurable locally bounded function satisfying certain regularity conditions. Then for any $h$,

$$V(x; h) = q^{-1}E^{-1}(h, +\infty)E^+g(x).$$

(2)

Corollary.

A natural candidate for the optimal exercise boundary is $h_* \text{ s.t.}$

$$E^+g(h_*) = 0.$$ 

(3)
Theorem 2

Assume, in addition, that $g$ is monotone and changes sign. Then

(i) equation $\mathcal{E}^+ g(h_*) = 0$ has a unique solution;

(ii) $\tau_{h_*}^-$ is an optimal stopping time, and $h_*$ is the unique optimal threshold;

(iii) the rational value of the stream (with the option to abandon it) is

$$V_*(x) = q^{-1} \mathcal{E}^- 1_{(h_*,+\infty)} \mathcal{E}^+ g(x);$$

(iv) there exists a non-increasing function $W_*$ which is non-negative below $h_*$ and 0 above $h_*$ such that $V_* = \mathcal{E}(W_* + g)$. 