On the Transfer of Multiplicative Structure

Joint work with Samson Saneblidze

Ron Umble
Millersville University of PA

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Goal of the Talk

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- **Theorem.** *If A is an A_∞-structure over a field $\mathbf{k}$, there is an induced A_∞-structure on $H(A;\mathbf{k})$.*

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- Our method relaxes the conditions under which the transfer of A_∞-algebra structure occurs, and transfers A_∞-bialgebra structure as well

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- An A_∞-*algebra* is a DGM (A, d) together with a family of operations

$$\left\{\mu^n \in Hom^{n-2}\left(A^{\otimes n}, A\right)\right\}_{n\geq 2}$$

and a map of non-Σ operads $\varphi : \mathcal{A}_\infty \to \left\{Hom\left(A^{\otimes n}, A\right)\right\}_{n\geq 1}$

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and a map of matrads $\varphi : \mathcal{H}_\infty \to End\,(TH)$

Introduction and Overview

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- However, the converse holds whenever B is free

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- Our algorithm requires neither freeness nor a homotopy operator
- $B = H(A)$ is an important special case of interest in this talk

Key Points in the Talk

- **Corollary 1.** *Let A be an A_∞-algebra such that $A = H \oplus X$. If $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is an induced A_∞-algebra structure on H.*

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- **Corollary 2.** *Given an A_∞-bialgebra A over a field $\mathbf{k}$, there is an induced A_∞-bialgebra structure on H*

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- **Example 2.** *Let X be a space. There is an induced A_∞-bialgebra structure on $H_*(\Omega X; \mathbf{k})$*

A–infinity Maps and Multiplihedra

- Given A_∞-algebras A and B, and a chain map $g : B \to A$, a morphism

$$G = g + g_2 + g_3 + \cdots : B \Rightarrow A$$

is defined in terms of parameter spaces $\{J_n\}_{n\geq 1}$, called *multiplihedra*

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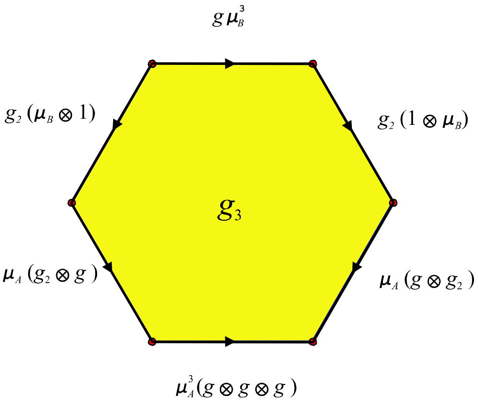
- The endpoints of J_2 are identified with components of the coboundary

$$\nabla g_2 = \mu_A (g \otimes g) - g\mu_B$$

A–infinity Maps and Multiplihedra

- J_3 is an hexagonal plane region identified with $g_3 \in Hom^2\left(B^{\otimes 3}, A\right)$



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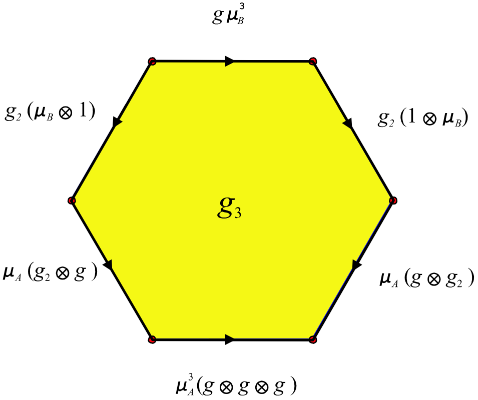

The Multiplihedron J_3

- $\nabla g_3 = \mu_A^3 g^{\otimes 3} + \mu_A\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_B \otimes 1 - 1 \otimes \mu_B\right) - g\mu_B^3$

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- Let $\ominus_n$ denote the cochain identified with $\partial J_n - \text{int}\, Q^{n-2}$

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- Then ∂Q^{n-2} is identified with the coboundary $\nabla \ominus_n$

First Transfer Theorem

- **Transfer Problem 1:** *Let A be an A_∞-algebra, let B be a* DGM*,and let $g : B \to A$ be a chain map. Given $\{\mu_B^i, g_i\}_{2\leq i<n}$ construct μ_B^n and g_n so that*

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- Transfer Problem 1 has a solution whenever $\bar{g}$ is a quasi-isomorphism
- **Theorem 1**. *If $\bar{g}$ is a quasi-isomorphism, then*

  (i) g transfers the A_∞-algebra structure from A to B; the induced structure on B is unique up to automorphism.

  (ii) g extends to a map $G : B \Rightarrow A$ of A_∞-algebras.

Key Propositions

- **Proposition 1.** *Let A be an A_∞-algebra such that $A = H \oplus X$. If $H^* Hom\left(H^{\otimes k}, X\right) = 0$ for $k \geq 2$, there is a cycle-selecting homomorphism $g : H \to A$ such that $\bar{g}$ is a quasi-isomorphism.*

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Computing the Transfer: The Fundamental Cocycle

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- $\nabla \ominus_2 = d\mu(g \otimes g) = \mu(dg \otimes g + g \otimes dg) = 0$

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- We refer to $\ominus_n$ as the *fundamental n-cocycle*

Computing the Transfer: The Induction

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- Solve the linear system $dx = \ominus_n - g\mu_H^n$
- Choose a particular solution g_n
- Then $g + g_2 + \cdots + g_n$ is an A_n-map

- **Example 1.** Consider the DGM

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- $A = T^a M \,/\, \left(a_2^2 + a_4,\, a_3^2,\, a_4 a_3 + a_3 a_4,\, (a_2 a_3 + a_3 a_2)^2,\, a_i b_2,\, b_2 a_i,\, b_2^2\right)$

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- $A = T^a M \,/\, \left(a_2^2 + a_4, a_3^2, a_4 a_3 + a_3 a_4, (a_2 a_3 + a_3 a_2)^2, a_i b_2, b_2 a_i, b_2^2\right)$

- A has no Hodge decomposition since $\mathbb{Z}_4$ contains a non-cycle a_3 and a boundary $2a_3$. Hence $\mathbb{Z}_4$ does not split as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

- **Example 1.** Consider the DGM

$$\begin{array}{ccccccccccccc} M^0 & \to & 0 & \to & M^2 & \to & M^3 & \to & M^4 & \to & 0 & \to & \cdots \\ \\ \mathbb{Z} & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & & \mathbb{Z}_4 & & \mathbb{Z}_2 & & & & \\ & & & & (a_2, b_2) & \mapsto & (0, 2a_3) & & & & & & \\ & & & & & & a_3 & \mapsto & a_4 & & & & \end{array}$$

- $A = T^a M \,/\, (a_2^2 + a_4, a_3^2, a_4 a_3 + a_3 a_4, (a_2 a_3 + a_3 a_2)^2, a_i b_2, b_2 a_i, b_2^2)$

- A has no Hodge decomposition since $\mathbb{Z}_4$ contains a non-cycle a_3 and a boundary $2a_3$. Hence $\mathbb{Z}_4$ does not split as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

- $H^n(A) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

A Cycle-Selecting Homomorphism g

- Denote the module generators of $H = H(A)$ by

$$\begin{aligned} 1 &= [1] \in H^0 \\ u &= [a_2] \in H^2 \\ v &= [a_2 a_3 + a_3 a_2] \in H^5 \\ w &= [a_2(a_2 a_3 + a_3 a_2)] \in H^7 \end{aligned}$$

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$$\begin{aligned} g(1) &= 1 \\ g(u) &= a_2 \\ g(v) &= a_2 a_3 + a_3 a_2 \\ g(w) &= a_2 (a_2 a_3 + a_3 a_2) \end{aligned}$$

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- Transfer of structure in Example 1 cannot be computed using standard techniques

Induced DGA Structure

- $\bar{g} : Hom\left(H^{\otimes 2}, H\right) \to Hom\left(H^{\otimes 2}, A\right)$ is a quasi-isomorphism

	2	4	5	7	9	10	12	14
H	u		v	w				
$H \otimes H$		$u\|u$		$u\|v,\ v\|u$	$u\|w,\ w\|u$	$v\|v$	$v\|w,\ w\|v$	$w\|w$

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$$u|v \overset{f_{u|v}}{\mapsto} w \overset{g}{\mapsto} a_2\left(a_2 a_3 + a_3 a_2\right) = \ominus_2\left(u|v\right)$$
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- Define $\mu_H = \left(\bar{g}_*\right)^{-1}[\ominus_2] = f_{u|v} + f_{v|u}$; then $uv = vu = w$

Extending g to an $A(2)$-map

- The non-trivial values of $\mu\,(g \otimes g) - g\mu_H$ are

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\begin{aligned}
u|u &\mapsto a_2^2\\
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- $\mu^3 = 0$ implies $\ominus_3 = \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_H \otimes 1 - 1 \otimes \mu_H\right)$

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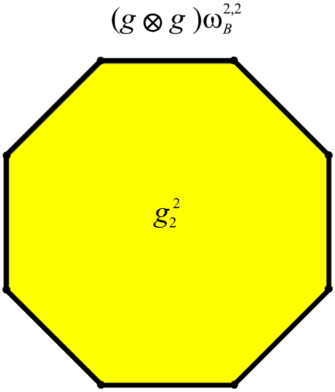
- Theorem 1 extends immediately to A_∞-bialgebras

Generalized Multiplihedra

Transfer is controlled by generalized multiplihedra $\{JJ_{m,n}\}_{m,n\geq 1}$ of which $JJ_{n,1} = JJ_{1,n} = J_n$



The Generalized Multiplihedron $JJ_{2,2}$

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- $\nabla \ominus_m^n \in Hom^{m+n-3}\left(B^{\otimes m}, A^{\otimes n}\right)$ is identified with ∂Q^{m+n-2}
- **Transfer Problem 2:** *Let A be an A_∞-bialgebra, let B be a* DGM,*and let $g: B \to A$ be a chain map. Given $\left\{\omega_B^{j,i},\ g_i^j\right\}_{1\leq i+j<k}$ construct g_m^n* and $\omega_B^{n,m}$ for each (m,n) with $m+n=k$ *so that*

$$\nabla g_m^n = \ominus_m^n - g^{\otimes n}\omega_B^{n,m}$$

Second Transfer Theorem

- A chain map g induces a cochain map

$$\tilde{g} : \left(Hom\left(B^{\otimes m}, B^{\otimes n} \right); \nabla_B \right) \to Hom\left(B^{\otimes m}, A^{\otimes n}; \nabla \right)$$

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- **Theorem 2.** *If $\tilde{g}$ is a quasi-isomorphism, then*

(i) g transfers the A_∞-bialgebra structure from A to B; the induced structure on B is unique up to automorphism

(ii) g extends to a map $G : B \Rightarrow A$ of A_∞-bialgebras

- **Proposition 2.** *Let A* be *an A_∞-bialgebra A over a field* $\mathbf{k}$, and choose a cycle-selecting homomorphism $g : H \longrightarrow A$. *Then $\tilde{g}$ is a quasi-isomorphism.*

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Application: Homology of Loop Spaces

- **Proposition 2.** *Let* A be *an* A_∞*-bialgebra* A *over a field* $\mathbf{k}$, and choose a cycle-selecting homomorphism $g : H \to A$. *Then* $\tilde{g}$ *is a quasi-isomorphism.*

- **Corollary 2.** *If* A *is an* A_∞*-bialgebra over a field* $\mathbf{k}$*, there is an induced* A_∞*-bialgebra structure on* $H(A; \mathbf{k})$.

- **Example 2.** *Let* X *be a space. There is an induced* A_∞*-bialgebra structure on* $H_*(\Omega X; \mathbf{k})$

Rational Cohomology of Loop Spaces

- **Example 3.** Let X be a simply connected space

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- Compatibility of μ with Δ^3 is expressed by the relation

$$\Delta^3\mu = \mu^{\otimes 3}\sigma_{3,2}\left[(\Delta \otimes 1)\,\Delta \otimes \Delta^3 + \Delta^3 \otimes (1 \otimes \Delta)\,\Delta\right],$$

where $\sigma_{p,q} : \left(H^{\otimes p}\right)^{\otimes q} \to \left(H^{\otimes q}\right)^{\otimes p}$ permutes tensor factors

Rational Cohomology of Loop Spaces (continued)

- Compatibility of μ with Δ^n is expressed in terms of the S-U diagonal on cellular chains of associahedra:

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- Then $\xi\left(e^{n-2}\right) = \Delta^n$ and

$$\Delta^n \mu = \mu^{\otimes n} \sigma_{n,2}\left[(\xi \otimes \xi)\, \Delta_K\left(e^{n-2}\right)\right]$$

Thank you!