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Some flow visualisations

Rotating bodies

Swept wings

(Kohama 2000)

(Bippes 1999)

The rotating-disk flow

- no intrinsic geometrical length scale
- self-similar basic flow
- boundary layer thickness $\delta = \sqrt{\frac{\nu}{\omega_{\text{disk}}}}$

$$U(r, z) = \begin{pmatrix} rU(z) \\ rV(z) \\ W(z) \end{pmatrix}$$

(Kohama 1984)

laminar flow

turbulent régime

transition at $r \simeq 500\delta$
Typical 3D boundary layers

**Common features:**
- crossflow component near the wall
- strong inviscid instability
  \( \rightsquigarrow \) growth and saturation of crossflow vortices
- inflection point
- secondary instabilities
  \( \rightsquigarrow \) transition

**Known features and questions**

- Laminar–turbulent transition, near \( R \approx 500 \)
- \( R < R^{sc} \approx 280 \) stability: all perturbations are damped
- \( R^{sc} < R < R^{ca} \approx 500 \) convective instability:
  spatial exponential amplification of external perturbations
- \( R > R^{ca} \approx 500 \) absolute instability: perturbations grow at fixed \( r \)
  \( \rightsquigarrow \) self-sustained finite-amplitude fluctuations
  \( \rightsquigarrow \) nonlinear wavetrains \( \rightsquigarrow \) secondary instabilities

- Precise dynamics in transition region?
- Spatial distribution of weakly and fully nonlinear fluctuations?
- Characteristic frequencies?
- Sensitivity to external noise, to disk roughness?
- Controllability of natural dynamics?
Self-similar laminar basic flow

no characteristic length scale

\[ U(r, z) = \begin{pmatrix} rU(z) \\ rV(z) \\ W(z) \end{pmatrix} \]

constant boundary layer thickness \( \delta = \sqrt{\frac{\nu}{\Omega}} \)

Weak radial development of basic flow

\[ \frac{1}{R} = \frac{\text{boundary layer thickness}}{\text{typical radii of interest}} \ll 1 \]

Transition occurs at \( R \sim 500 \)

Local properties at given radial location \( R \) are then derived from

\[ U(z; R) = \begin{pmatrix} R U(z) \\ R V(z) \\ W(z) \end{pmatrix} \]

- 3D flow, homogenous in \( \theta \) and \( r \)
- \( R \) effective local Reynolds number
Local linear dispersion relation

Separate total flow fields into basic and perturbation quantities as

$$U(z; R) + u(r, \theta, z, t).$$

Linearize governing equations and write small-amplitude perturbations in normal-mode form as

$$u(r, \theta, z, t) = u^\ell(z) \exp(\alpha r + \beta \theta - \omega t).$$

Solution of eigenvalue problem leads to

$$\omega = \Omega^\ell(\alpha, \beta; R)$$

with $\omega$, $\alpha$ complex, $\beta$ integer, $R$ real.

Temporal growth rate

$\alpha$ real

Isolevels $\Omega^\ell_i = 0, 0.5, \ldots, 3.5$

Instability whenever $\Omega^\ell_i > 0$
Local absolute instability analysis

(Lingwood 1995)

Absolute frequency $\omega_0$ is defined as the frequency observed at a fixed spatial location in the long-time linear response to an initial impulse.

Vanishing radial group velocity condition:

$$\omega_0(\beta; R) = \Omega^\ell(\alpha_0, \beta; R) \quad \text{with} \quad \frac{\partial \Omega^\ell}{\partial \alpha} = 0$$

$\omega_{0,i} < 0$ convective instability $\leadsto$ perturbations propagate radially outwards

$\omega_{0,i} > 0$ absolute instability $\leadsto$ perturbations grow at fixed radial location

Onset of absolute instability at $R^{ca} = 507.4$ and $\beta^{ca} = 68$.

Critical absolute frequency $\omega^{ca} = 50.5$. 
Response to localized impulse

- amplified wavepacket is advected outwards in a spiralling trajectory
- inner boundary of wavepacket asymptotically reaches a critical radius
- turbulent régime prevails beyond this radius without ext. perturbation

However...

\( R^{ca} \) closely corresponds to experimentally observed transition radius

However:

Fully linearized dynamics does not display global instability
- Analytic derivation, Peake & Garrett (2002)

\( \sim \) Nonlinear régime must be accounted for to understand self-sustained behaviour
The rotating disk configuration displays all the desirable features required by the theory of “elephant” nonlinear global modes (Pier, Huerre & Chomaz, 2001)
“Elephant” nonlinear global mode

Basic advection

Stationary front at transition station CU/AU
- generates downstream propagating nonlinear wavetrain
- tunes entire system to global frequency

\[ \omega^c_0 = \omega_0(R^c) \text{ with } \omega_0, i(R^c) = 0 \]

- AU region is a sufficient condition for nonlinear global instability

Predicted behaviour of the rotating disk flow

- in central CU region \( R < R^c \)
  unperturbed basic flow
- at marginal AU station \( R = R^c \simeq 507.4 \)
  stationary front of frequency \( \omega^c_0 \simeq 50.5 \)
- in outer AU region \( R > R^c \)
  finite-amplitude saturated spiral vortices

Questions:
- Do saturated waves exist in this configuration?
- Why is a turbulent state observed instead?
- What about secondary instabilities of the saturated vortices?
Local saturated crossflow vortices

Whenever $\Omega^\ell_i(\alpha, \beta; R) > 0$

$\sim$ linear exponential temporal growth of spatially periodic perturbations

$\sim$ nonlinear quadratic interactions produce higher harmonics

$\sim$ amplitude saturation, finite-amplitude periodic wavetrain of the form

$$u(r, \theta, z, t) = \sum_n u_n(z) \exp ni(\alpha r + \beta \theta - \omega t)$$

real nonlinear frequency $\omega$

- **Nonlinear dispersion relation**

$$\omega = \Omega^{n\ell}(\alpha, \beta; R)$$

Nonlinear waves at $\beta = 68$
Structure of finite-amplitude waves

At $R = 500$, $\beta = 68$, $\alpha = 0.35 \Rightarrow \omega = 50.5$

• new inflection points
• further instabilities

Secondary stability analysis

• Primary finite-amplitude equilibrium solution
  
  $u(r, \theta, z, t) = \sum u_n(z) \exp i(\alpha r + \beta \theta - \omega t)$

  is $2\pi$-periodic in $\phi \equiv \alpha r + \beta \theta - \omega t$.

• Secondary (in)stability by Floquet theory
  perturbation in normal-mode form

  $\hat{u}(r, \theta, z, t) = \left( \sum \hat{u}_n(z) \exp i(\alpha r + \beta \theta - \omega t) \right) \exp i(\hat{\alpha} r + \hat{\beta} \theta - \hat{\omega} t)$

  $\hat{\alpha}$ secondary radial wavenumber (complex)
  $\hat{\beta}$ secondary azimuthal modenumber (integer)

• Secondary dispersion relation:
  
  $\hat{\omega} = \hat{\Omega}^\ell(\hat{\alpha}, \hat{\beta}; \alpha, \beta; R)$
Secondary absolute instability

Primary nonlinear structure affected by secondary disturbances only for secondary absolute instability.

Secondary absolute frequency $\hat{\omega}_0$ and wave number $\hat{\alpha}_0$ are obtained by a pinching condition in the complex $\hat{\alpha}$-plane (Brevdo & Bridges 1996):

$$\hat{\omega}_0(\hat{\beta}; \alpha, \beta; R) \equiv \hat{\Omega}^\ell(\hat{\alpha}_0, \hat{\beta}; \alpha, \beta; R)$$

with $\hat{\alpha}_0$ defined by

$$\frac{\partial \hat{\Omega}^\ell}{\partial \hat{\alpha}}(\hat{\alpha}_0, \hat{\beta}; \alpha, \beta; R) = 0$$

Secondary absolute growth rate

$$\hat{\omega}_{0,i} \equiv \text{Im} \, \hat{\omega}_0(\hat{\beta}; \alpha, \beta; R)$$

At onset of primary nonlinearity: $R = 508, \alpha = 0.35, \beta = 68$:

![Graph showing strong secondary absolute instability](image-url)
Revised self-sustained behaviour

- in central CU region $R < R^{ca}$
  unperturbed basic flow
- at marginal AU station $R^{ca} \approx 507$
  stationary front
- in outer AU region $R > R^{ca}$
  front generates nonlinear spiral vortices
  that are already absolutely unstable
- Transition to turbulent state
  immediately occurs at $R = R^{ca}$

Open-loop control

Goal and method

Delay transition to larger radius $\rightarrow$ reduce energy losses, noise. . .

Low-tech method usable in practical situations:
- No real-time computations
- No real-time measurements
- No closed-loop control methods

Apply localized perturbations in the BL upstream of transition
$\rightarrow$ Modify the naturally selected flow behaviour

Take advantage of primary BL instabilities
  to control secondary instabilities
Spatial response to localized harmonic forcing

One-dimensional spatially developing system (e.g., CGL equation)

\[ X < X_f \quad \text{upstream exponential decay} \]

\[ X = X_f \quad \text{ localized forcing of frequency } \omega_f \]

\[ X_f < X < X_{nl} \quad \text{downstream exponential growth} \]

\[ X = X_{nl} \quad \text{response reaches finite amplitude} \]

\[ X_{nl} < X < X_n \quad \text{finite-amplitude response} \]

Forced response vs. self-sustained oscillations

source of self-sustained oscillations located at \( X^{ca} \)

front tunes entire system
Control of natural oscillations

Numerical simulation of 1D complex Ginzburg–Landau equation with spatially varying coefficients

- Natural oscillations at $\omega_0^{ca} = 0.4$
- Forcing at $\omega_f = 1$ switched on at $t = 0$

Effective control:
When forced response reaches finite amplitude upstream of the front, this front is overwhelmed and the intrinsic oscillations are replaced by the forced response.

Control strategy applied to the rotating-disk flow

Modify primary nonlinear state to avoid secondary instability.

The transition station $R^{ca} = 507$ acts as a source for the naturally selected spiral vortices.

- Perturb this source to replace the natural state by spiral vortices of our own choice.
- Apply localized periodic forcing at $R_f$ in the convect. unstable region:
  - linear spatial response for $R_f < R < R_{nl}$,
  - nonlinear spatial response for $R > R_{nl}$,
  - control dominates whenever $R_{nl} < R^{ca}$.

Available control parameters: frequency $\omega_f$ and azimuthal modenumber $\beta_f$.

Careful choice of $\omega_f$ and $\beta_f$ delays secondary instability.
Secondary absolute instability features

Secondary absolute growth rate at $R = 500$ by Floquet analysis of primary NL spiral vortex $[\alpha$: radial wavenumber $\beta$: azimuthal modenumber]

naturally selected vortices: $\beta = 68$, $\alpha = 0.35$, $\omega = 50.5$
$\hat{\omega}_{0,i} = 0.55$

control at: $\beta = 34$, $\alpha = 0.57$, $\omega = 46$
$\hat{\omega}_{0,i} = -1.06$

Transition delay

Primary finite-amplitude spiral waves follow the nonlinear spatial branch $\alpha^{nl}(R; \omega_f, \beta_f)$, determined by forcing frequency $\omega_f$ and modenumber $\beta_f$. Evolution of secondary absolute growth rate $\hat{\omega}_{0,i}^{\text{max}}$ along these branches

Onset of sec. absolute instability may be postponed to nearly $R = 600$.
Transition delay

Apply harm. forcing at $R_f$ to replace intrinsic behaviour by spatial response.

Choose forcing characteristics to delay sec. inst. (and transition) to $R^*$.

Conclusions

- Complete map obtained of primary and secondary stability properties
- Control strategy:
  Apply localized harmonic forcing upstream of transition
  $\rightarrow$ natural behaviour suppressed, entire system tuned to ext. forcing
  $\rightarrow$ secondary instability delayed by about 100 boundary layer units
- Low-resources control:
  control parameters determined only once, no real-time measurements or computations
- Low-cost control:
  by exponential growth of spatial response in central region