Vortex-speed selection within the boundary-layer flow over a rotating sphere placed in an enforced axial flow

S.J. Garrett*
Department of Mathematics, University of Leicester, Leicester, LE1 7RH, UK

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ABSTRACT
This paper furthers existing work into the instability mechanisms within the boundary-layer flow over a rotating sphere through the study of amplification rates within the convectively-unstable region. The onset of convective instability is associated with the experimentally observed onset of spiral vortices reported in the literature. Axial flow is found to stabilize the boundary layer by both delaying the onset of convective instability at all latitudes and also by significantly reducing the spatial amplification rates. We find that the type II (streamline curvature) mode becomes increasingly amplified with respect to the type I (crossflow) mode and is therefore likely to be selected in practice for sufficiently high axial flow rates. Furthermore, in experiments where special care is taken to remove all surface roughness, we predict that vortices will rotate at around 75% of the local surface speed. This is consistent with the experimental observations of Kobayashi & Arai who note a speed of around 76% under particular experimental conditions. These predictions are entirely consistent with related work on the rotating-disk and cone boundary layers.

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1. Introduction

In this paper we revisit the transition characteristics of the boundary-layer flow over rotating spheres, as first considered in detail by Garrett & Peake [9,10]. There, convective and absolute instability analyses were presented with the sphere rotating both in and out of enforced axial flows. The studies focused on the calculation of the critical Reynolds numbers at the onset of instabilities at each latitude. The work was a natural extension of the previous theoretical work into the basic-flow profiles within the boundary layer due to Manohar [20] and Banks [12] and the stability analyses of the rotating-disk flow due to, for example, Malik [18] and Lingwood [17]. Indeed the experimental studies of Sawatzki [22]; Kohama & Kobayashi [15] and Kobayashi & Arai [14] showed transition behaviour similar to that over rotating disks which prompted the study. The transition characteristics are summarized as follows: when the sphere is spun, the induced boundary-layer flow remains laminar around the pole, with co-rotating spiral vortices (characteristic of crossflow instabilities) appearing at a higher latitude, a turbulent region then occurs at a higher latitude still. Garrett & Peake found that local convective instability analyses could be used to successfully predict the onset of vortices at each latitude, and that the onset of absolute instability appears to be related to the onset of turbulence (at least for low to moderate latitudes). As with rotating-disks and broad cones, modes of type I (crossflow) and II (streamline curvature) were found to dictate the convective instability at all latitudes over the sphere, and modes of type I and III the absolute instability. Further information on these instability modes within the related boundary layers can be found in Lingwood [17]; Garrett [6]; Garrett & Peake [11] and Garrett et al. [78].

An interesting observation made by Kobayashi & Arai was that the co-rotating spiral vortices were fixed on the sphere surface when the rotation rate was large (and transition occurred at low to moderate latitudes), whilst they moved relative to the sphere surface when the rotation rate was smaller (and transition occurred at high latitudes). The relative speed of the slow vortices was always around 76% of the local surface speed of the sphere. This observation was unique to this set of experiments, i.e. slow vortices had not been observed by Sawatzki or Kohama & Kobayashi, and also had not been observed in rotating-disk and cone experiments (see Kobayashi et al. [13]; Kobayashi & Izumi [12] and Kohama [16], for example). This current paper clarifies the vortex-speed selection process within the rotating sphere flow in the light of related analyses of the rotating-disk and cone boundary layers due to...
Garrett [5,6]. The paper focuses on the behaviour of the spiral vortices and is therefore restricted to the local convectively-unstable parameter ranges at each latitude.

Garrett & Peake originally attempted to clarify the appearance of slow vortices using a method of investigation since denoted method-1 by Garrett [5]. This method involves calculating the critical parameters of a set of neutral curves, each pertaining to a different fixed azimuthal wavenumber of disturbance; the azimuthal phase velocity of disturbances, \( c \), is then calculated from the globally-critical parameters and associated with the vortex speed. They predicted stationary vortices (the globally-critical parameters and associated with the vortex azimuthal phase velocity of disturbances, \( \Omega \), and the azimuthal wavenumber. Garrett & Peake concluded that the onset of the slow vortices was related to the dominance of the type II mode and the appearance of the inflection point, although they found in the distribution of critical Reynolds number with azimuthal wavenumber. Garrett & Peake observed mode and the appearance of the inflection point, although they were unable to determine a theoretical speed to compare to the observed \( c = 0.76 \). By artificially fixing \( c = 0.76 \) using method-2 as denoted in Garrett [5], they were able to correctly predict properties of the slow vortices (i.e. critical Reynolds number, number and angle of orientation).

The results of this current study suggest that the observation of slow vortices at the position of streamline curvature dominance was merely coincidental, and slow vortices would be selected at all latitudes under specific experimental conditions. Furthermore, we explain the appearance of the apparent inflection point in the distribution of critical Reynolds numbers found by Garrett & Peake.

Section 2 summaries the formulation of the problem, which is entirely consistent with Garrett & Peake’s. In §3 the sphere is assumed to rotate in an otherwise still fluid, and in §4 a non-zero axial flow is introduced. In each case we study the linear convective growth rates of disturbances traveling at different azimuthal phase speeds, formulated under method-2. Conclusions are drawn in §5.

## 2. Formulation

The formulation of the problem is exactly as described by Garrett & Peake [9,10] and is summarized here for completeness.

Consider spherical polar coordinates fixed in space with origin located at the centre of the sphere. The radius of the sphere is \( a^* \), and it rotates at a constant angular frequency \( \Omega^* \) in a uniform axial flow with free-stream velocity \( U_0^* \). The distance \( r^* \) is measured radially from the centre of the sphere, \( \theta \) is the angle of latitude measured from the axis of rotation and \( \psi \) is the angle of azimuth (asterisks indicate dimensional quantities). The form of the surface velocity distribution \( U_0^* \) at the edge of the boundary layer is taken from the empirical results of Fage [4], who fitted the curve

\[
U_0^*(\theta) = U_0^* \left( 1.5 \theta_{\text{rad}} - 0.4371 \theta_{\text{rad}}^3 + 0.1481 \theta_{\text{rad}}^5 - 0.0423 \theta_{\text{rad}}^7 \right),
\]

for \( 0 < \theta < 85^\circ \), where \( \theta_{\text{rad}} \) denotes \( \theta \) measured in radians. This is considered to be a better approximation than the inviscid solution owing to the boundary-layer separation from the sphere surface close to the equator. This choice of slip velocity leads to a favourable pressure gradient at latitudes below \( \theta = 74^\circ \), and so increasing the free-stream velocity has a stabilising effect. Indeed, this was demonstrated by Garrett & Peake [10] through the increase in critical Reynolds numbers for the onset of both convective and absolute instabilities at each latitude.

The slip velocity \( U_0^* \) is scaled on the free-stream velocity \( U_0^* \), while other steady velocities are scaled using \( \Omega^* a^* \). The non-dimensional steady flow variables are then defined as

\[
U(\eta, \theta) = U_0^* \frac{\Omega^* a^*}{c^*} V(\eta, \theta) = V_0^* \frac{\Omega^* a^*}{c^*} W(\eta, \theta) = W_0^* \frac{\Omega^* a^*}{c^*},
\]

where \( U, V \) and \( W \) are the non-dimensional velocities in the \( \theta \), and \( r^* \) directions respectively and \( \eta = (\Omega^*/a^*)^{1/2} (r^* - a^*) \) is the non-dimensional distance from the sphere surface in the radial direction. Note that \( \eta \) is scaled on the boundary-layer thickness \( \delta^* = (a^*/\Omega^*)^{1/2} \), where \( \delta^* \) is the coefficient of kinematic viscosity.

The equations that govern the steady flow in the boundary layer are stated by Mangler [19] and are scaled as in (2) leading to

\[
U_0^* \frac{\partial U}{\partial \eta} + W_0^* U = V_0^* \frac{\partial V}{\partial \eta} - V^2 \cot \theta = T U_0^* \frac{\partial U_0}{\partial \eta} \frac{\partial^2 U}{\partial \eta^2},
\]

\[
U_0^* \frac{\partial W}{\partial \eta} + V_0^* \frac{\partial V}{\partial \eta} + U_0^* V \cot \theta = \frac{\partial^2 V}{\partial \eta^2},
\]

\[
\frac{\partial W}{\partial \eta} + U_0^* \cot \theta + \frac{\partial U}{\partial \theta} = 0,
\]

where \( T = U_0^*/a^* \Omega^* > 0 \) is the non-dimensional axial flow parameter, which is the ratio of the free-stream flow speed to the speed of the points on the sphere equator. The choice of scalings leads to \( T \) appearing in the basic-flow equations only, i.e. the unsteady perturbation equations are independent of the incident axial flow. This formulation means that the effect of non-zero axial flow is simply to change the basic-flow profiles upon which the stability analyses are performed.

The non-slip boundary condition on the surface of the sphere scales to

\[
U = W = V - \sin \theta = 0 \quad \text{on} \quad \eta = 0,
\]

and the condition at the edge of the boundary layer becomes

\[
V = U - T U_0 = 0 \quad \text{as} \quad \eta \rightarrow \infty.
\]

The solution of equations (3)–(5) subject to conditions (6) and (7) follows the method outlined by Garrett & Peake. The steady flow profiles are shown in Fig. 1 of Garrett & Peake [9] (for \( T = 0 \)) and Figs. 1–3 of Garrett & Peake [10] (for \( T \neq 0 \)).

The stability analysis conducted at a particular latitude involves imposing small perturbations on the steady basic flow in the boundary layer at that latitude. The dimensional perturbation variables (denoted by lower case hatted quantities) are assumed to have the normal-mode form

\[
(\hat{u}^*, \hat{v}^*, \hat{w}^*, \hat{p}^*) = \left( u^*(r^*), v^*(r^*), w^*(r^*), p^*(r^*) \right) e^{i(\sigma^* a^* \theta + \beta^* a^* \sin \theta - \gamma^* r^*)}.
\]

The distance measured over the surface of the sphere from the pole to the latitude under consideration is \( a^* \theta \), and the dimensional wavenumber in this direction is \( \beta^* \). The distance measured along a circular cross section of the sphere by a plane perpendicular to the axis of rotation is \( a^* \sin \theta \), and the dimensional wavenumber in this direction is \( \gamma^* \).

The perturbing quantities are scaled on the typical length, velocity, time and pressure scales: \( \delta^* \), \( a^* \Omega^* \), \( \delta^*/a^* \Omega^* \) and
\[ \rho^*(a^*\Omega^* u^*)^2, \]

respectively. This leads to the Reynolds number
\[ R = \delta^* a^* \Omega^* \frac{u^*}{\nu^*} = a^* f / \delta^*. \]

The Reynolds number can therefore be interpreted as the non-dimensional radius of the sphere, however it is more useful to interpret it as a measure of the rotation rate, for fixed values of the other parameters. This is in contrast to the rotating-disk and cone analyses, due to Lingwood and Garrett & co-workers, where the Reynolds number is interpreted as the location of the local analysis along the surface of the body. It is clear that in the sphere analysis the location is set by the latitude.

The resulting unsteady perturbation equations are identical to equations (2.13)–(2.18) of Garrett & Peake [9], but are included here in the Appendix for completeness. Details of the derivation of the perturbation equations can be found in Garrett & Peake [9], which includes a discussion of the parallel-flow type approximation made. Here it is sufficient to understand that the approximation limits the analysis to a local analysis at each value of \( \theta \) and means that the governing equations are not entirely rigorous at \( \Omega R^{-1} \).

The wavenumber in the \( \theta \)-direction is a complex quantity, \( \alpha = \alpha_n + i \alpha_c \), as required by the spatial analyses presented later. The assumed form of the disturbances, given by equation (8), means that \( -\alpha_n \) is interpreted as the spatial growth rate. The frequency, \( \gamma \), and the azimuthal wavenumber, \( \beta \), are real. Under method-2 it is necessary to insist that the vortices rotate at some fixed multiple of the sphere surface velocity, thereby fixing the ratio \( \gamma / \beta \). The non-dimensional speed of the sphere surface is \( \sin \theta \), and equating the relevant multiple of this with the disturbance phase velocity in the same direction, \( \gamma / \beta \), leads to \( \gamma = c \sin \theta \). This relationship must be satisfied with \( c = 1.0 \) if the vortices are to rotate with the sphere, and \( c = 0.76 \) if the vortices are those reported by Kobayashi & Arai [14], for example.

The angle that the phase fronts make with a circle parallel to the equator is denoted \( \epsilon \), and is found from \( \epsilon = \arctan(\beta / \alpha) \). The integer number of complete cycles of the disturbance round the azimuth is \( n = \beta R \sin \theta \). Later we will identify \( \epsilon \) and \( n \) as being the angle and number of spiral vortices on the sphere surface, respectively.

3. Quiescent fluid

In this section we assume that the sphere rotates in an otherwise still fluid and set \( T = 0 \). We begin by considering the growth rates of stationary disturbances in \( 3.3 \) and then move onto the growth rates of traveling disturbances in \( 3.4 \), where the vortex-speed selection process is investigated.

3.1. Stationary disturbances

In practical applications roughness elements on the rotating sphere will act to select stationary vortices and we begin by explicitly assuming this. In particular, we solve the dispersion relation \( D(\alpha, \beta, \gamma = c \beta \sin \theta; R, \theta) = D(\alpha, \beta, R, \theta, c) = 0 \) with the constraint that \( \epsilon = 1 \) using the numerical methods described in the Appendix. The analysis involves solving \( D \) for \( \alpha \) whilst marching through values of \( \beta \) at a particular set of \( R, c, \) and \( \theta \). Physically this can be interpreted as sampling the stability (as determined by the spatial growth rate) of a range of disturbance waves at a particular location (determined by \( \theta \)) and rotation rate (determined by \( R \)).

Garrett & Peake [9] have shown that the rotating sphere boundary layer is absolutely unstable at particular \( R = R_a \), and beyond this the flow is known to become turbulent (although transition may already have occurred by this point at higher latitudes through an alternative mechanism). In our analysis of the spiral vortices it is therefore only necessary to consider the convective growth rates through the convectively-unstable region bounded by \( R_a \). Table 1 gives the critical Reynolds numbers for the onset of convective instability, \( R_c \), and absolute instability at each latitude.

3.2. Traveling disturbances and vortex-speed selection

Disturbance speeds have been considered in the range \( c = 0.5 \pm 8 \) at each location and neutral curves computed. Figs. 2 and 3 show the neutral curves at \( \theta = 10^\circ \) and \( \theta = 70^\circ \), respectively, in terms of \( \alpha_n, \beta, n, \) and \( \epsilon \) for \( c = 0.7, 0.8, 1 \) and 2. Note that \( n \) and \( \epsilon \) are observable quantities in experiments which motivates their use here. Recall that \( c = 0.8 \) corresponds to disturbances traveling at 80% of the local surface speed, and \( c = 2 \) corresponds to disturbances traveling at twice the local surface speed.

At all latitudes below \( \theta = 66^\circ \) we find that the critical Reynolds number for the type I mode is minimized with \( c = 1 \). This is consistent with the appearance of the minimum at \( c = 1 \) in the method-1 analysis presented by Garrett & Peake [9]. In addition, we find that the lobe arising from the type II mode is sensitive to the disturbance speed; in particular, the type II lobe is quickly eliminated for \( c < 1 \) and exaggerated for \( c > 1 \). Although not shown here, the type II lobe appears to limit to a particular shape for increasingly large values of \( c \), whilst the type I lobe remains dependent on the value of \( c \). Similar behaviour is also found within the rotating disk and cone boundary layers, Garrett [5,6].

At latitudes above \( \theta = 66^\circ \) we find that the type II lobe is the most dangerous for all \( c \geq 1 \), and increasing \( c \) acts to significantly broaden the range of unstable parameters and reduce the critical Reynolds number. This explains the lack of the minimum in the method-1 analysis of Garrett & Peake [9] at these locations which led to their apparent “point of inflection”.

It is important to note that the range of waveangles and vortex numbers predicted to be unstable to quickly traveling modes is extremely narrow at all locations. In a sense this is a stabilizing effect because only a very narrow range of vortex parameters can be selected. We therefore appear to have two competing factors in the vortex-speed selection process: the critical Reynolds numbers for the onset of the type II mode reduce with increased \( c \), but the range of parameter values that the corresponding vortices can exist at

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( R_c )</th>
<th>( R_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1600</td>
<td>2880</td>
</tr>
<tr>
<td>20</td>
<td>777</td>
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<td>70</td>
<td>92</td>
<td>240</td>
</tr>
</tbody>
</table>
becomes increasingly narrow, thereby prohibiting selection. In order to further clarify the process we consider the linear growth rates for traveling modes through the region of convective instability bounded by $R_A$.

The results of this are given in Figs. 4 and 5 where plots of the spatial branches at different values of $c$ are presented in order to visualize the growth rates of both modes at $\theta = 10^\circ$ and $\theta = 70^\circ$, respectively. At all latitudes we see that the growth rates within the

![Figure 1. Linear convective growth rates for stationary disturbances through the convectively-unstable region at various $\theta$.](image1)

![Figure 2. Neutral curves for traveling disturbances with $c = 0.7$ (- -), $c = 0.8$ (--), $c = 1$ (-) and $c = 2$ (-x) at $\theta = 10^\circ$.](image2)
Fig. 3. Neutral curves for traveling disturbances with $c = 0.7$ (···), $c = 0.8$ (−·), $c = 1$ (−) and $c = 2$ (−x) at $\theta = 70^\circ$.

Fig. 4. Linear convective growth rates for traveling disturbances with $c = 0.7$ at $\theta = 10^\circ$.
type II lobe increase relative to the type I mode as $c$ increases. However, more importantly, we note that the globally maximum growth rates are for the type I mode, and these peak between $c = 0.7$ and $c = 0.8$.

Fig. 6 demonstrates that the growth rate is maximized for $c = 0.75$. Although the figure demonstrates this at a single Reynolds number, this is found to be true at all Reynolds numbers in the unstable region at all latitudes. It is therefore most likely that $c = 0.75$ is the preferred vortex speed over perfectly smooth rotating spheres at all rotation rates.

It is very interesting to note that this is entirely consistent with the results of Garrett [5,6] where traveling modes of type I with $c = 0.75$ are found to be the most amplified for rotating-disks and cones of all half-angles.

4. Enforced axial flow

In this section we enforce an axial flow onto the rotating sphere by setting $T > 0$. As in §3, we begin by considering the growth rates of stationary disturbances (§4.1) and then move onto the growth rates of traveling disturbances in order to investigate the vortex-speed selection process in the presence of an axial flow (§4.2).

4.1. Stationary disturbances

Garrett & Peake [10] found that an enforced axial flow had a stabilizing effect on both the stationary-convective modes and absolute instabilities within the boundary layer, as demonstrated by an increase in the critical Reynolds numbers at each latitude with increased $T$. They also demonstrated that the type II mode becomes the most dangerous mode for sufficiently high $T$ at each latitude. Exactly how this behaviour affects the dominant instability mode in each case is unknown without considering the amplification rates as we do now.

In order to properly compare amplifications rates under different axial flows, it is necessary to consider an equal extent of convective instability in each case. At each latitude we take this to be the region determined by $R_A - R_C$ in the quiescent fluid case (as given by Table 1). Fig. 7 shows plots of the spatial branches of the type I and II modes at a latitude of 10° through a region of around 1200 in $R$ for various axial flow rates. We see that the growth rates of each mode are significantly reduced with increased axial flow. In additional, the amplification rate of the type II mode increases relative to the type I mode with increased axial flow. Similar behaviour is found at all latitudes (although plots are not shown here) and we can conclude that axial flow acts to stabilize the boundary layer to convective instabilities.
both through their delayed onset and also through a reduction in amplification rates.

Furthermore, for a sufficiently high axial flow rate, it is expected that the onset of the type II mode would become the most dominant mode at each latitude and is likely to be selected. However, for sufficiently high rotation rates such that transition is seen to originate at \( \theta = 10 \), an axial flow rate of \( T > 0.25 \) would be required to select the type II mode. It is clear that the required axial flow rate reduces with \( \theta \) because of the increased importance of the type II mode at higher latitudes.

### 4.2. Traveling disturbances and vortex-speed selection

As in the quiescent fluid case, we find that the critical Reynolds number for the type I mode is minimised with \( c = 1 \) for all values of \( T \) considered at latitudes lower than \( \theta_1 \) (defined to be the latitude below which two distinct lobes exist in the neutral curve). The type II lobe in the neutral curve is again exaggerated for \( c > 1 \) and diminished for \( c < 1 \) at all such latitudes. At latitudes above \( \theta_1 \), we find that increasing \( c \) acts to significantly broaden the range of unstable parameters and reduce the critical Reynolds numbers. As reported in Garrett & Peake [10], the value of \( \theta_1 \) increases slightly with axial flow, for example, from 66 to 71 in increments of 1 as \( T \) increases from 0 to 0.25 in steps of 0.05.

Although axial flow acts to reduce the growth rates, it is still the case that the dominant growth rate for all values of \( T \) considered is given by the type I mode with \( c < 1 \); this is true at all latitudes and Reynolds numbers. This is demonstrated for \( \theta = 10 \) in Fig. 8, where the maximum growth rate of the type I mode is plotted against disturbance speed at a Reynolds number of 1200\(^1\). The sharp peak in the \( T = 0 \) curve is a consequence of the spatial branch approaching a pinch point under the Briggs-Bers criterion.

\(^1\) Note that this places the analysis just prior to the critical Reynolds number for the onset of absolute instability for \( T = 0 \) but has no special significance for \( T > 0 \). The steep peak in the \( T = 0 \) curve is a consequence of the spatial branch approaching a pinch point under the Briggs-Bers criterion.

Fig. 7. Linear convective growth rates for stationary disturbances at \( \theta = 10 \) with \( T = 0-0.25 \).

Fig. 8. Maximum linear convective growth rates approximately 1,200R into the convectively-unstable region. We see that that as \( T \) increases the most amplified vortex speed is increased slightly, for example, to \( c = 0.85 \) when \( T = 0.20 \). We also see that the region of instability is contained within a finite region of \( c \) which decreases with increased \( T \). This is in contrast to the quiescent fluid case where Fig. 6 appears to show the maximum growth rate asymptoting to some value.

As reported in \(|4.1|\), for sufficiently high values of \( T \) the stationary type II mode is expected to become the most amplified at all latitudes. Higher axial flow rates therefore have implications for the conclusions of this traveling mode analysis: eventually it is expected that the type II mode will dominate over the slowly traveling type I mode and the vortex-speed selection process described here would break down. 
5. Conclusion

In this paper we have shown that slowly rotating vortices are the most amplified and would be selected in experiments where perfectly smooth spheres are used. Vortices that rotate at around 75% of the sphere surface speed are predicted when \( T = 0 \), and this speed increases slightly with incident axial flow rates at least up to 25% of the equatorial rotation speed. Although this is unlikely to be the case in most practical applications where surface roughness would be unavoidable and stationary vortices selected, the discovery could explain the unusual observation by Kobayashi & Arai [14] of vortices traveling at 76% of the sphere surface in particular conditions. It is suggested that their observations are consistent with a perfectly smooth surface region beyond a latitude of around \( \theta = 66^\circ \).

It is interesting to note that the same vortex-speed selection process has been demonstrated in the boundary-layer flow over rotating-disks and cones by Garrett [5,6]. Although experiments on smooth rotating-disks have been reported by Corke & Kniaziak [3] and Othman & Corke [21] where non-stationary modes have been observed, the results cannot be interpreted in terms of a particular vortex speed. Further experiments similar to Kobayashi & Arai [14] are therefore required over all these geometries to test the theoretical results.

The paper has extended the results of Garrett & Peake [9,10] through a full analysis of the relative amplification rates of the type I (crossflow) and type II (streamline curvature) modes at each latitude in practical applications were stationary modes are known to be selected. From these results we conclude that the type I mode is most likely to be selected when the sphere is rotating in quiescent fluid. However, as the axial flow rate is increased it is likely that the type II mode would eventually be selected at each latitude. This is due to the increase in relative amplification rate of the type II mode with both increased latitude and axial flow rate.

It is acknowledged that an approximation similar to the parallel-flow approximation was made in the derivation of the governing equations. This approximation is found in many other boundary-layer investigations and means that the perturbation equations solved here are not rigorous at \( O(1/R) \). Although it is clear that the approximation will lead to inaccuracies in the predicted critical Reynolds numbers, it is the author’s opinion that these will be small and will not affect the conclusions of this work. This is discussed in detail by Garrett & Peake [9].

It is also acknowledged that this work consists of a linear analysis and so would be inaccurate in situations where the growth rates are large and non-linear effects would occur. However, we have predicted the selection of slow vortices from the very onset of convective instability where growth rates are small; it is therefore expected that non-linearity would affect the break down of the slow vortices in situations where the onset of absolute instability is severely delayed (i.e. for larger axial flow rates), but not their initial selection.

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Appendix. The perturbation equations

The perturbation equations can be written as a set of six first-order ordinary differential equations using the transformed dependent variables

\[
\begin{align*}
zm(\eta; \alpha, \beta, \gamma; R, \theta) &= (\alpha - i \cot \theta |R| U) U + \beta V, \\
zd(\eta; \alpha, \beta, \gamma; R, \theta) &= (\alpha - i \cot \theta |R| D) D + \beta D, \\
zg(\eta; \alpha, \beta, \gamma; R, \theta) &= w, \\
zg(\eta; \alpha, \beta, \gamma; R, \theta) &= p, \\
zl(\eta; \alpha, \beta, \gamma; R, \theta) &= (\alpha - i \cot \theta |R| V) V - \beta U, \\
zd(\eta; \alpha, \beta, \gamma; R, \theta) &= (\alpha - i \cot \theta |R| D) V - \beta D,
\end{align*}
\]

where \( D \) represents differentiation with respect to \( \eta \). Writing \( \alpha_1 = \alpha - [i \cot \theta |R|]_0 \), these equations are

\[
\begin{align*}
Dz_1 &= z_2, \\
Dz_2 &= \frac{1}{R} \left( \alpha U' - 2B V - 2U V \right) z_3 + i \left( \alpha U' + B V \right) z_3 + \left( \alpha U + B V \right) z_3, \\
Dz_3 &= -i \eta U - \frac{2}{R} z_3, \\
Dz_4 &= \left[ \frac{1}{R} \right] U + \left[ \frac{1}{R} \right] V + \left[ \frac{2}{R} \right] (U V + V V) - \frac{1}{R} \left( \alpha U + B V - \gamma \right) + D W_3, \\
Dz_5 &= z_6, \\
Dz_6 &= \left[ \frac{1}{R} \right] \left( \alpha U + B V - \gamma \right) z_3 + \left( \frac{1}{R} \right) \left( \alpha U + B V - \gamma \right) z_3 + \left( \frac{1}{R} \right) \left( \alpha U + B V - \gamma \right) z_3 + \left( \frac{1}{R} \right) \left( \alpha U + B V - \gamma \right) z_3.
\end{align*}
\]

The subscripts \( v \) and \( s \) indicate which of the \( O(R^{-1}) \) terms arise from the viscous and streamline curvature effects respectively. Note that the perturbation velocities \( u \) and \( v \) still appear explicitly in the equations, but can be expressed in terms of \( z_3 \) and \( Dz_3 \) via

\[
\begin{align*}
U &= \frac{1}{\alpha_1^2 + \beta^2} (\alpha_1 z_1 - \beta z_5), \\
V &= \frac{1}{\alpha_1^2 + \beta^2} (\alpha_1 z_5 + \beta z_1).
\end{align*}
\]

Applying the homogeneous boundary conditions

\[
\begin{align*}
z_l &= 0, \\
\eta &= 0, \\
z_l &= 0, \\
\eta &\rightarrow \infty,
\end{align*}
\]

where \( i = 1.2 \ldots 6 \), results in an eigenvalue problem that is solved for certain combinations \( \alpha, \beta, \gamma \) at each Reynolds number, \( R \), and for a particular value of \( \theta \). From these we form the dispersion relation, \( D(\alpha, \beta, \gamma; R; \theta) = 0 \), at each \( \theta \), with the aim of studying the occurrence of convective instabilities.

The eigenvalue problem is solved using a double-precision fixed-step-size, fourth-order Runge–Kutta integrator with Gram–Schmidt orthonormalization and a Newton–Raphson linear search procedure. Further details can be found in Garrett & Peake [9,10].
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