The effects of roughness levels on the instability of the boundary-layer flow over a rotating disk with an enforced axial flow

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M. A. S. Al-Malki,1,a) S. J. Garrett,2,b) S. Camarri,3,c) and Z. Hussain2,d)

AFFILIATIONS
1School of Mathematics and Actuarial Science, University of Leicester, Leicester LE1 7RH, United Kingdom and Department of Mathematics and Statistics, Taif University, P.O. Box 888, Taif, Saudi Arabia
2School of Engineering, University of Leicester, Leicester LE1 7RH, United Kingdom
3Dipartimento di Ingegneria Aerospaziale, Università di Pisa, 56122 Pisa, Italy

a)Author to whom correspondence should be addressed: masam2@leicester.ac.uk
b)stephen.garrett@leicester.ac.uk
c)simone.camarri@unipi.it
d)zahir.hussain@leicester.ac.uk

ABSTRACT
This paper investigates the effects of surface roughness on the convective stability behavior of boundary-layer flow over a rotating disk. An enforced axial flow and the Miklavčič and Wang (MW) model of roughness are applied to this flow. The effects of both anisotropic and isotropic surface roughness on the distinct instability properties of the boundary-layer flow over a rotating disk will also be examined for this model. It is possible to implement these types of roughness on this geometric shape while considering an axial flow. This approach requires a modification for the no-slip condition and that the current boundary conditions are partial-slip conditions. The Navier–Stokes equations are used to obtain the steady mean-flow system, and linear stability equations are then formulated to obtain neutral stability curves while investigating the convective instability behavior for stationary modes. The stability analysis results are then confirmed by the linear convective growth rates for stationary disturbances and the energy analysis. The stability characteristics of the inviscid type I (or cross-flow) instability and the viscous type II instability are examined over a rough, rotating disk within the boundary layer at all axial flow rates considered. Our findings indicate that the radial grooves have a strong destabilizing effect on the type II mode as the axial flow is increased, whereas the concentric grooves and isotropic surface roughness stabilize the boundary-layer flow for the type I mode. It is worth noting that the flows over a concentrically grooved disk with increasing enforced axial flow strength are the most stable for the inviscid type I instability.

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I. INTRODUCTION
There are recent studies concerning the reduction of the skin-friction drag resulting from viscous shear forces that influence the movement of different bodies through a fluid and under various conditions. It is known that the turbulent transition of flow in three-dimensional boundary layers, rather than in the laminar state, causes skin-friction drag, which has become a greater issue. Therefore, the most appealing solution to avoid this problem to date is drag reduction using surface organized roughness to delay the laminar-turbulent transition process of flow under different characteristics. Sirovich and Karlsson1 using a number of experiments on wind tunnel walls showed that certain patterns of roughness, carefully arranged, lead to drag reduction. However, determining the right sort of roughness to achieve the desired drag reduction is the most common issue with this method, as presented in Carpenter.2

When taking into consideration a third component of boundary-layer flow, as found in aircraft wings and rotating disks, this third component is called cross-flow velocity, which is constituted by the centrifugally induced radial flow velocity. The inflection point that appears in the cross-flow velocity profile of the boundary layers with the cross-flow component leads to the flow instability characteristic. This is clearly shown in a series of cross-flow vortices within the transition zone. This type of instability is called the inviscid type I (or cross-flow) instability.

Studies have used the smooth rotating disk for six decades as a model...
for conducting experiments and theories on boundary layers with the overlapping flow component such as those in Refs. 3–5.

The first work concerning a swept wing experimentally was by Gray et al., which generated inviscid spiral vortices using cross-flow instabilities. This work was supported later by Gregory et al., in which authors noted that the cross-flow-induced co-rotating vortices caused imprints on a china clay disk. One of the common differences between the swept wing and the rotating disk boundary layers is that the rotating disk boundary layers are within Coriolis forces. Malik performed a numerical analysis to compute the neutral curves for stationary instability modes on a boundary layer of the rotating disk, which indicated a type of instability. This research is the first modification of the previous fourth-order Orr-Sommerfeld type analyses in favor of a sixth-order system, including centrifugal terms and rotational Coriolis effects.

Hall verified asymptotically that there are two defined branches. The first one is the previously known upper branch associated with cross-flow instability (type I). The second branch is a lower branch associated with streamline curvature and the balancing of Coriolis and viscous forces (type II). This type, however, is difficult to prove experimentally as the type I mode is almost ubiquitously dominant theoretically. According to Fedorov et al., the type II mode has been produced from 14 to 16 vortices at a wave angle of 20°, which is considered a rare observation. Essentially both type I and type II are governed by the crossflow instability. Type I is related to the inviscid Rayleigh criterion, where both the effective velocity and its double derivative with respect to the wall-normal coordinate vanish, whereas type II is due to a balance of viscous and Coriolis forces, which is still governed by the crossflow effects. Physically, type I modes are known to arise from an inflection point in the radial velocity profile and are inviscid in origin. The vortex activity is located in a critical layer at the location of the inflection point and the perturbation wavelengths are scaled on boundary layer thickness. Meanwhile, the type II modes are known to originate from viscous effects close to the disk boundary and correspond to the effective velocity profile exhibiting zero shear at the wall (see Hall).

The early emerging works on linear stability of flows played an important role in recent works. Those previous studies indicated that obtaining an appropriate value of a critical Reynolds number is a beneficial result in determining the stability strength of the flow through a specific point in the transition from the laminar to turbulent regime. Brown disagreed with other experimental studies that ignored curvature effects. Therefore, the authors have taken curvature into consideration. For instance, Malik et al. were successful, placing the critical Reynolds number at 287, which showed a strong agreement in line with the experimental results of Wilkinson and Malik.

Recent studies concentrate almost exclusively on types of roughness patterns by examining a small number of roughness elements to test characteristics of particular modes of disturbance. This is to improve specific control mechanisms such as. Zoueshtiagh et al. worked with more general roughness for rotating disks, where the authors provided limited empirical data for velocity profiles over quartz-grain rough disks that were reviewed by Cooper et al. Moreover, Watanabe et al. discovered a reduction in the number of the vortices associated with the inviscid Type I mode from an initial 32 for a smooth cone to 26 for a rough cone, using a modest roughness level in their experiments. The most important question is whether roughness levels affect the number of spiral vortices and their transitional path so that increasing roughness levels leads to a decreasing number of spiral vortices and vice versa. This would affect various transitional mechanisms of flows over geometric bodies related to different engineering applications.

Early academic studies introduced valuable contributions for the characteristics of laminar-turbulent transition flow over the rough surfaces of bodies. These works have depended on using a small number of roughness elements, in particular, patterns that lead to exciting some particular disturbance modes of the boundary layer flows. In this article, the authors found that wall compliance is effective at stabilizing the inviscid type I mode and destabilizing the viscous type II mode. Hence, it could be seen that increased energy production is a result of viscous forces on the disk surface. Those results were then confirmed by Colley et al. This study was an experimental investigation of Watanabe et al., who considered the laminar-turbulent transition of the boundary layer flow over a rotating cone.

The effects of distributed surface roughness on the laminar-turbulent transition of the von Kármán flow can be used as the first step toward explaining Watanbe’s results. Two alternative models for surface roughness in the von Kármán flow exist in the literature. The first model comes from Yoon et al., referred to as the HYP model. This model is limited to a particular case of anisotropic roughness, namely, concentric grooves (radially anisotropic roughness), which is modeled by imposing a particular surface distribution as a function of radial position only and assuming a rotational symmetry. The second approach is the MW model developed by Miklavčič and Wang. The MW method empirically models surface roughness by converting the usual no-slip boundary conditions to partial-slip conditions at the rotating disk surface without imposing a particular mathematical form. Subsequently, using this method depends on separately modifying the boundary conditions in the radial and azimuthal directions, which helps in modeling independent levels of roughness in these directions. The MW approach can therefore model all variations of distributed roughness that will be used throughout this paper. These include two forms of anisotropic roughness, concentric grooves (radially anisotropic) and radial grooves (azimuthally anisotropic), and isotropic roughness (uniform in both directions).

Cooper et al. applied the MW model on the von Kármán flow. This study found that a stabilization of the type I mode in terms of increased critical Reynolds number is achieved by introducing surface roughness. Alveroglu et al. discussed isotropic surface roughness, which is the most significant effect on delaying the onset of convective instability at all values of the Rossby number. The study indicated a passive-drag reduction mechanism for the entire BEK system of flows, which are observed in rotor-stator-type engineering applications. Alqarni et al. examined the stability of the boundary-layer flows for non-Newtonian fluids over rough, rotating disks. The authors considered both isotropic and anisotropic surface roughnesses and found stabilization in the flows of the non-Newtonian fluids with a carefully designed surface roughness. In contrast, the work of Garrett et al. found a significant destabilization; hence, the type II mode can be the most dangerous mode for a critical Reynolds number of higher levels of roughness. This study is parallel to the results for the type II mode from Garrett et al. for radial grooves.

Our study aims to broaden the theoretical study of flows within axial flow over smooth disks found in Hussian et al. to flows over
rough disks in an enforced axial flow. The principle objective to analyze the effects of distributed surface roughness on the characteristics of enforced axial flows over a rotating disk in terms of the response of the convective instability properties. Thereby, a linear convective instability analysis will be applied to those flows in order to identify curves of neutral stability and produce convective growth rates. Moreover, the effect of increased surface roughness on delaying of the branch exchange will also be investigated due to the absolute instability mechanism, which limits the computation of the convective growth rate curves in certain flows. This is because of the branch exchange issue resulting from the coalescence of the type I and type II modes as investigated by Lingwood and Garrett.29 It should be noted that this study will be confined to stationary disturbances of the convective stability, the type II mode is investigated due to the absolute instability mechanism behind the effects of roughness on the stability of the flows. Our results are summarized and discussed in Sec. VII.

II. MEAN FLOW OF A ROUGH DISK

A. Formulation

We consider a rotating disk of infinite radius, rotating about its axis \( z^* \) with angular velocity \( \Omega^* \) as shown in Fig. 1 in a fluid of kinematic viscosity \( \nu^* \). The disk is situated downstream from an incompressible Newtonian fluid. As the disk rotates, the fluid is entrained toward its surface and moves with velocity \( \mathbf{U}^* = (\nu^*, \nu^*, w^*) \), with each component representing velocities in the radial, azimuthal, and axial directions \( r^*, \theta^*, \) and \( z^* \), respectively. Note that in all that follows, an asterisk indicates a dimensional quantity. In the rotating frame of reference, depending on the Navier–Stokes equations in cylindrical polar coordinates for the flow external to the boundary layer, we have

\[
\nabla^* \cdot \mathbf{U}^* = 0, \quad \left( \frac{\partial}{\partial r^*} + \mathbf{U}^* \cdot \nabla^* \right) \mathbf{U}^* + \frac{2 \Omega^* \times \mathbf{U}^*}{\hat{r}_1} + \frac{\Omega^* \times \Omega^* \times \mathbf{r}^*}{\hat{r}_2} = -\frac{1}{\rho^*} \nabla^* p^* + \frac{1}{\rho^*} \nabla^* \cdot \mathbf{r}^*,
\]

where \( t^* \) is the time, \( \Omega^* = (0, 0, \Omega^*) \) is angular velocity vector, \( \mathbf{r}^* = (r^*, \theta^*, 0) \) is the radial position vector, \( \rho^* \) is the density, and \( p^* \) is pressure. The viscous stress tensor is given by \( \tau^* = \mu^* \nabla^* \cdot \mathbf{r}^* \), where \( \mu^* \) is the constant viscosity and \( \dot{\gamma} = \nabla^* \mathbf{U}^* + \nabla^* (\mathbf{U}^*)^T \) is the rate-of-strain tensor. The terms labeled \( \hat{r}_1 \) and \( \hat{r}_2 \) are rotational terms arising from Coriolis and centrifugal acceleration, respectively.

The surface roughness can be approximately represented by modifying the no-slip conditions at the rotating disk surface, \( z^* = 0 \). In particular, the MW model considers partial-slip conditions at the disk surface, whereas the type of the boundary conditions is identical to the smooth disk formulation. A Navier–Stokes generalization of partial-slip conditions is given in the radial direction.
\[
\begin{align*}
\nu'(0) &= N_1 \rho' \mu \frac{\partial \nu'}{\partial z}(0), \\
\nu'(0) &= N_2 \rho' \mu \frac{\partial \nu'}{\partial z}(0).
\end{align*}
\]

and in the azimuthal direction,

\[
\frac{\partial \nu}{\partial r} + \frac{w \nu}{r} = 2u = \frac{1}{\delta R e} \left( \mu \frac{\partial^2 \nu}{\partial z^2} + \frac{\partial \nu \partial u}{\partial z} \right),
\]

Let \( \lambda = N_1 \rho' \sqrt{\mu' \Omega} \), \( \eta = N_2 \rho' \sqrt{\mu' \Omega} \), where \( N_1, N_2 \) are the respective slip coefficients to be determined by experiments, \( \mu' \) is kinematic viscosity, and \( \rho' \) is the density. Physically, these are linked to the roughness heights. The MW model is to study the flow due to a rough rotating disk. Generally, the principle directions of the roughness are radial and azimuthal, e.g., concentric grooves of a disk such as a phonograph record or a laser-etched disk. The results of this study can also be applied to the special case of randomly rough disk as presented by Miklavčič and Wang. ²³ Therefore, we assume here that the surface is grooved by the form shown in Fig. 1, where \( \lambda \) and \( \eta \) are independent of \( r' \) and \( \theta \). This means that the slip boundary condition will be independently wavy in the \( r' \)- and \( \theta \)-directions, respectively.

However, Smith ¹ indicated through the triple-deck theory that a displacement function is produced to the main deck by the lower deck, and this function shows a wavy nature due to the wavy surface. It is therefore easy to show that if the roughness height is sufficiently small, the boundary condition should be \( \nu'(0) = \lambda \frac{\partial \nu'}{\partial z}(0) \), and \( \lambda = -h f(r', \theta) \), where \( h \) is the roughness height, and \( f(r', \theta) \) is the surface shape. Then we need to solve the displacement function using numerical approaches when \( h \) is of \( O(Re^{-5/8}r') \), where the lower deck is nonlinear.

In order to clarify this issue, Chicchiero et al. ²⁶ presented a comparison of the mean-flow profiles between the triple-deck theory (TDT) and direct numerical simulations (DNS) which depend on a slight modification of the von Kármán flow similar to what proposed by Miklavčič and Wang. ²³ The interesting result is that the time average velocity field becomes independent of the groove wavelength for values greater than 4 and \( h = 0.1 \). The authors indicated that there is a good agreement between these two approaches.

The motion of the fluid is influenced both by the rotation of the disk and the axially enforced element toward the disk surface. The enforced azimuthal component is derived from the radial pressure balance at the boundary layer edge,

\[
U_r^* \frac{\partial U_r^*}{\partial r} = -\frac{1}{\rho' \partial r}.
\]

Here, \( U_r^* = C' r^* \) is the flow velocity at the boundary-layer edge. The constant \( C' \) represents the strength of the enforced flow in inverse time units, \( \rho' \) is the density, and \( \rho' \) is the pressure. The governing equations are expressed in cylindrical polar coordinates and non-dimensionalised by the scaling variables,

\[
U^* = (u, v, w)U^*_\infty, \quad (r^*, z^*) = (r, \delta z) L^*, \quad t^* = t L^*/U^*_\infty, \quad p^* = p(U^*_\infty)^2 \rho'.
\]

Here, \( U^*_\infty = L^* \Omega^* \) is the velocity of the free stream, \( L^* \) is a generic length scale, and \( \delta \) is the boundary layer thickness. The laminar flow is steady and axisymmetric, leading to neglecting all derivatives with respect to \( t \) and \( r \). Reynolds number is here defined as \( Re = U^*_\infty L^* \rho' / \mu^*_\infty \), where \( \mu^*_\infty \) is the fluid viscosity in the free stream, with \( Re \gg 1 \). The governing equations are then expressed in the following advanced order:

\[
\begin{align*}
1 \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} + \frac{v^2}{r} &= \frac{1}{\delta R e} \left( \mu \frac{\partial^2 u}{\partial z^2} + \frac{\partial u \partial w}{\partial z} \right), \\
\frac{\partial w}{\partial r} + \frac{w \partial w}{r} + 2u &= \frac{1}{\delta R e} \left( \mu \frac{\partial^2 w}{\partial z^2} + \frac{\partial w \partial w}{\partial z} \right), \\
\frac{\partial w}{\partial r} + \frac{w \partial w}{r} &= \frac{1}{\delta R e} \left( \mu \frac{\partial^2 w}{\partial z^2} + 2 \frac{\partial w \partial w}{\partial z} \right).
\end{align*}
\]

Note that the enforced axial flow is eliminated when \( T_i = 0 \).

The fluid is moved by both the rotation of the disk and an axially enforced element, wherein an external source forces the fluid toward the disk surface. The enforced axial flow component is derived from the radial pressure balance at the boundary-layer edge. The mean flow equations can now be derived through the introduction of the following self-similar solutions:

\[
\begin{align*}
u &= r U(z), \quad v = r V(z), \quad w = W(z), \quad p = P(z).
\end{align*}
\]

The formulation of the steady flow is entirely consistent with the related paper of Hussain,²⁸ and full details of data and derivation of the governing equations can be found there. Here, it is sufficient to state that we worked in a cylindrical polar coordinate system of the rotating disk in an enforced axial flow. The steady mean-flow profiles can be written by the ODE system

\[
\begin{align*}
2U + W' &= 0, \\
U^2 + U' W - (V + 1)^2 - T_i^* &= 0, \\
2U (V + 1) + U' W - V^* &= 0, \\
WW' + P' - W^* &= 0,
\end{align*}
\]

where primes indicate differentiation with respect to \( z \).

The modified wall boundary conditions of the ODE system (11)–(14), based on Eqs. (3) and (4), are expressed as

\[
\begin{align*}
U(0) &= \lambda U'(0), \quad V(0) = \eta V'(0), \quad W(0) = P(0) = 0, \\
U(z \to \infty) &= 0, \quad T_i[z \to \infty] = 0.
\end{align*}
\]

where \( \lambda = N_1 \rho' \sqrt{\mu' \Omega}, \eta = N_2 \rho' \sqrt{\mu' \Omega} \). The primes denote differentiation with respect to \( z \), and the two parameters \( \lambda \) and \( \eta \) are related empirically (e.g., by the experiments to the roughness in the radial and azimuthal directions, respectively. However, these boundary conditions reduce to the no-slip criteria of a smooth disk when \( \lambda = \eta = 0 \). This means that the fluid moves exactly as the disk at its surface. Both cases \( \lambda > 0, \eta > 0 \), and \( \lambda = 0, \eta > 0 \) give anisotropic roughness, which are named concentric grooves or radial grooves,
respectively, and the case $\lambda = \eta > 0$ gives isotropic roughness. The conditions for $U(z \to \infty)$ and $V(z \to \infty)$ are the Coriolis force balance conditions in the rotating frame, where the condition for $U(z \to \infty)$ has been modified according to the pressure balance condition at the boundary-layer edge due to enforced axial flow recall Eq. (14). Equations (11)–(13) are solved utilizing a Newton-Raphson searching routine to find a suitable condition for the unknowns $U'(0)$ and $V'(0)$, alongside a double precision, fourth-order Runge–Kutta integration scheme to solve the mean-flow functions through the boundary layer. Convergence is reached at $z_{\text{max}} = 20$ to a tolerance of $10^{-8}$, which can be considered representative of the free stream, although plots are truncated to $z = 10$ to better visualize the effects of varying parameters. It is noted here that Eq. (14) is independently solvable,

$$P - P_0 = -\left(2U + \frac{W^2}{2}\right),$$

with the stipulation that with $P(z = 0) = P_0$. Hence, an additional numerical integration is required for the final term.

**B. The steady-flow results**

The plots in Fig. 2 are the mean-flow solutions for a range of $T_s$ values with anisotropic roughness and isotropic roughness, where the effects of enforced axial flow are observed. Figure 2(a) shows the effects of various axial flows over a rough rotating disk for anisotropic roughness $\lambda > 0$, $\eta = 0$ (concentric grooves). These plots reveal that the maximum of $U(z)$ profile is increased as axial-flow strength $T_s$ and $\lambda$ are increased. However, the characteristic inflection is suppressed for more highly radial jets obtained with increasing both $\lambda$ and axial flow strengths. We also see that the behavior of the component $U(z)$ moves upward in the axial direction, where the maximum value of $U(z)$ occurs at the wall when increasing the distance from the disk surface as $T_s$ and $\lambda$ are increased. The profiles then converge to the boundary condition at a reduced distance; hence, this is interpreted as a gentle narrowing of the boundary layer for effects of roughness. Compared to radially anisotropic roughness (concentric grooves), we see in Fig. 2(b) that there is a slight change of the boundary-layer thickness and a reduction in the radial jets as a result of decreasing axial-flow strength $T_s$ and increasing the roughness parameter $\eta$. There is a slight change of the $U(z)$ profiles over an isotropically rough disk as observed in Fig. 2(c). Here, the results clearly predict a moderate increase in the radial wall jet and a general thinning effect.

Figure 3(a) shows the azimuthal velocity $V(z)$, where the $V(z)$ profiles are closer to surface disk as radial grooves and $T_s$ are increased. Furthermore, the boundary-layer flows are greatly thinner when radial grooves and $T_s$ increase. Figure 3(b) displays a reduction in the wall value of the azimuthal velocity as $T_s$ and $\eta$ are increased (a direct result of the partial-slip condition). This reduction in the wall value of the $V(z)$ slightly increases with increasing isotropic roughness as noted in Fig. 3(c).

The axial flow profiles, $W(z)$, as seen in Fig. 4, show a great modification in shape for $T_s \neq 0$. Subsequently, the effects of roughness are consistent with this change. In Fig. 4(a), the axial profile moves to the surface of the disk as $T_s$ and $\lambda$ are larger. The convergence at a certain distance from the disk is as a result of the boundary condition imposed in (15). Figure 4(b) exhibits a reverse movement for that shown in Fig. 4(a). We note that the axial flow profiles become unbounded due to the increasing distance from the disk with rising $\eta$ (radial grooves). In Fig. 4(c), the isotropic roughness results in a combined change between both two previous cases in Figs. 4(a) and 4(b). Overall, our findings are consistent with Cooper et al.\textsuperscript{20} in terms of the effects of ruggedness, and are also consistent with Hussain et al.\textsuperscript{28} for increasing axial flow rate.

**III. LINEAR CONVECTIVE STABILITY**

**A. Formulation**

The mean flow is added to a small perturbation quantity for obtaining the linear stability equations. The velocity and pressure are then

![FIG. 2. Mean flow profiles for a range of $T_s$, light blue-solid lines (-- $T_s = 0$; red dashed lines (----- $T_s = 0.1$; green dotted lines (· · ·) $T_s = 0.2$; dark blue dashed-dotted lines (—) $T_s = 0.3$ in the cases of various roughnesses, radial velocity $U(z)$. (a) $\lambda > 0$, $\eta = 0$ (concentric grooves), (b) $\lambda = 0$, $\eta > 0$ (radial grooves), and (c) isotropic roughness $\lambda = \eta$.](image-url)
The covering equations are non-dimensionalized under the new length, velocity, time, and pressure scales,

\[
\begin{align*}
u(r, \theta, z, t) &= \frac{r}{R} V(z) + \hat{v}(r, \theta, z, t), \\
w(r, \theta, z, t) &= \frac{1}{R} W(z) + \hat{w}(r, \theta, z, t), \\
p(r, \theta, z, t) &= \frac{1}{R} P(z) + \hat{p}(r, \theta, z, t).
\end{align*}
\]

The covering equations are non-dimensionalized under the new length, velocity, time, and pressure scales,

\[
U^* = \left( u, v, w \right) r^* \Omega^*, \quad (r^*, z^*) = (r, z) \delta^*,
\]

\[
t^* = t \frac{L^*}{r^* \Omega^*}, \quad p^* = p(r^* \Omega^*)^2 \rho^*.
\]

where \( r^* \) is the local radius of the disk at which instability occurs. The local Reynolds number is then

\[
R = \frac{r^*_s \Omega \delta \rho^*}{\mu_\infty} = \frac{r^*_s}{\delta} = r_s,
\]

which is a non-dimensional radial distance of the Reynolds number along the disk.

The linear stability equations are then formulated with respect to perturbation quantities as follows:

\[
\frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{1}{r} \frac{\partial \hat{v}}{\partial \theta} + \frac{\partial \hat{w}}{\partial z} = 0,
\]

FIG. 3. Mean flow profiles for a range of \( T_s \), light blue solid lines \((- - -) T_s = 0\); red dashed lines \((-- --) T_s = 0.1\); green dotted lines \((--- --) T_s = 0.2\); dark blue dashed-dotted lines \((-----) T_s = 0.3\) in the cases of various roughnesses, azimuthal velocity \( V(z) \). (a) \( \lambda > 0, \eta = 0 \) (concentric grooves), (b) \( \lambda = 0, \eta > 0 \) (radial grooves), and (c) isotropic roughness \( \lambda = \eta > 0 \).

FIG. 4. Mean flow profiles for a range of \( T_s \), light blue solid lines \((- - -) T_s = 0\); red dashed lines \((-- --) T_s = 0.1\); green dotted lines \((--- --) T_s = 0.2\); dark blue dashed-dotted lines \((-----) T_s = 0.3\) in the cases of various roughnesses, axial velocity \( W(z) \). (a) \( \lambda > 0, \eta = 0 \) (concentric grooves), (b) \( \lambda = 0, \eta > 0 \) (radial grooves), and (c) isotropic roughness \( \lambda = \eta > 0 \).
\[
\begin{aligned}
\frac{\partial \bar{u}}{\partial t} &+ rU \frac{\partial \bar{u}}{\partial r} + \frac{U \bar{u}}{R} + \frac{V \partial \bar{u}}{R \partial z} + \frac{W \bar{u}^*'}{R} + rU \bar{w}' - \frac{2(V+1)\bar{v}}{R} \\
&= -\frac{\partial \bar{p}}{\partial R} + \frac{1}{R} \left( \frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{v}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{v}}{\partial \theta^2} + \frac{\bar{u}''}{2} - \frac{20\bar{v}}{r^2} \right), \quad (22)
\end{aligned}
\]

The normal mode form of the perturbation quantities represents the stability equations to be separable in \( r, \theta, \) and \( t \), giving

\[
(\bar{u}, \bar{v}, \bar{w}, \bar{p}) = (\bar{u}, \bar{v}, \bar{w}, \bar{p}) e^{i(ax \pm yt \pm oz)},
\]

where the perturbation eigenfunctions depend on \( z \) are expressed by the variables marked with a tilde. Here, \( x = \alpha x + \beta z \) is the complex radial wavenumber. The azimuthal wavenumber, \( n = \beta R \), is the number of vortices on the disk and is interpreted as an integer quantity. The frequency of the disturbances is represented by \( \omega \).

Following the substitution, in Eqs. (21)–(24), we have the perturbation quantities in normal mode form and with neglecting all \( O(R^{-1}) \) terms,

\[
\begin{aligned}
t^a \bar{u} + \frac{\bar{u}}{R} + i\beta \bar{v} + \bar{w}' &= 0, \quad (25) \\
-\lambda \bar{u} + i\alpha \bar{u} &= \frac{U \bar{u}}{R} + i\beta \bar{v} + \bar{w}' - \frac{2(V+1)\bar{v}}{R} \\
&= -i\lambda \bar{p} + \frac{1}{R} \left( -\lambda^2 \bar{u} - \beta^2 \bar{u} + \bar{u}' \right), \quad (26)
\end{aligned}
\]

The system (25)–(28) can be solved as a quadratic eigenvalue problem of the form \( (A_2 x^2 + A_1 x + A_0)Q = 0 \), where \( Q = (\bar{u}, \bar{v}, \bar{w}, \bar{p}) \) is the vector of eigenfunctions and the quantities \( A_0 \) are matrices containing the coefficients of the \( O(x^2) \) terms. As we are interested in stationary vortices which rotate with the rough disk in the rotating reference frame, we set \( \omega = 0 \).

From Eq. (25), the first derivative of \( \bar{w} \) should also be zero at the disk surface. We set all perturbation quantities to zero at the far end of the physical domain to be ensured that the disturbances are contained within the boundary layer. All these conditions are consistent with the analyses of Cooper et al.\textsuperscript{25}

We have then computed the eigenfunctions according to the boundary conditions constrained small perturbations for residing within the boundary layer,

\[
\begin{aligned}
\bar{u}(z = 0) = \bar{v}(z = 0) = \bar{w}(z = 0) = \bar{w}'(z = 0) = 0, \quad (29)
\end{aligned}
\]

\[
\begin{aligned}
\bar{u}(z \to \infty) = \bar{v}(z \to \infty) = \bar{w}(z \to \infty) = \bar{p}(z \to \infty) = 0. \quad (30)
\end{aligned}
\]

We obtain the neutral temporal and spatial stability solutions with a Chebyshev polynomial discretization method. An exponential map is used to transform the Gauss–Lobatto collocation points into the physical domain. We solve then the stability equations as primitive variables over a number of the collocation points distributed between the upper and lower boundaries, with the exception of the boundary conditions in Eqs. (29)–(30), which are imposed at \( z = 0 \) and \( z = \zeta_{\text{max}} \). Further details of the numerical method employed here can be found in Alverghiu.\textsuperscript{33}

### B. Neutral stability curves results

All the points on the neutral curves identify the solutions of the system (25)–(28), when \( x = 0 \). The unstable points of the flow must be found in the region enclosed by the neutral curves, in this case \( x < 0 \). The region outside the curves is significantly stable when \( x > 0 \). Generally, the critical Reynolds number interprets the stability of the flow when it is increasing, whereas the destabilizing effects can be seen with the minimum values. The linear stability Eqs. (25)–(28) of the eigenfunctions are then computed according to the boundary conditions (29) and (30). The stability of the flow is examined by plotting neutral points \( (x = 0) \) in \((R, n), (R, \beta), \) and \((R, \zeta)\) planes, where \( \phi = \tan^{-1} (\beta / \zeta) \) is the orientation angle of spiral vortices which are related to a circle concentric to the disk. Both \( n \) and \( \beta \) are physical, measurable quantities.

The plots in Fig. 5 indicate neutral stability curves of vortex-number \( n \) for various axial flows \( T_1 = 0.1, T_2 = 0.2 \), and a range of roughness. In Figs. 5(a) and 5(d), we note that the critical Reynolds number is increasing rapidly as \( \lambda \) takes higher values. This means that the effect of increasing anisotropic roughness in a concentrically grooved disk shows a strongly stabilizing effect on both the type I and type II modes when increasing the axial flow strength as shown in Figs. 5(a) and 5(d). In this case, we observe that the type II lobe is completely eliminated with increasing relatively modest levels of roughness. Furthermore, the value of \( n \) also increases significantly, but the range of unstable values \( n \) decreases as concentric grooves and \( T_1 \) are increased although the unstable values of \( n \) are enclosed by the successive curves. As might be expected, the critical wave angle will follow the behavior of the critical value of \( n \) as noted in Figs. 6(a) and 6(d) for neutral stability curves of wavevanes \( \phi \). For further data, see Tables I–III.

Predicted critical values of the wavevane in the \( r \)-direction are shown in Fig. 7 for a range of axial flow \( T_1 = 0.1, T_2 = 0.2 \) and Tables I–III. The data demonstrate that increasing axial flow strength acts to increase the value of \( \zeta_n \), whereas all surface roughness levels act to gradually decrease the value of \( \zeta_n \).

On the contrary, the radially grooved disks have an increasingly destabilizing effect on the type II mode, as evidenced by the lowering of the critical Reynolds number. Therefore, this leads to a widening region of instability and the vanishing of the type I mode as seen in Figs. 5(b) and 5(c) as \( T_1 \) is increased and the type II mode becoming the dominant form of instability.

For isotropic roughness, Figs. 5(c) and 5(f) shows a stronger stabilizing effect for the type I mode of a range of axial flow than the case of radial roughness, but not as strong as the stabilizing effect of the
FIG. 5. Neutral stability curves of vortex-number for a range of axial flow ($T_s = 0.1, 0.2$) and a range of roughness, respectively. (a) and (d) Neutral curves of anisotropic $\lambda > 0, \eta = 0$ (concentric grooves). (b) and (e) Neutral curves of anisotropic $\lambda = 0, \eta > 0$ (radial grooves). (c) and (f) Neutral curves of isotropic $\lambda = \eta > 0$. 
**FIG. 6.** Angle neutral stability curves at a fixed value of axial flow ($T_s = 0.1, 0.2$) and a range of roughness. (a) and (d) Neural curves of anisotropic $\lambda > 0, \eta = 0$ (concentric grooves). (b) and (e) Neural curves of anisotropic $\lambda = 0, \eta > 0$ (radial grooves). (c) and (f) Neural curves of isotropic $\lambda = \eta > 0$. 
concentric grooves. It is worth noting that a rotating concentrically grooved disk has the most stabilizing effect on type I mode as the axial flow strength is increased. In contrast, the flows over a radially grooved disk are the most unstable on type II instability.

In this study, we note that the governing receptivity mechanism is quite clear, where the roughness wavelength is a dominating factor to generate instability with the same wavelength. This regime is due to zero frequency disturbances, resulting in stationary instability modes. This is seen qualitatively in our findings for type I modes, for example, for concentric grooves in Fig. 5(a), increasing the roughness parameter \( \lambda \) leads to increased slip, which is achieved via an increased number of roughness elements. This corresponds to a shorter roughness wavelength. If we compare the effect of increasing \( \lambda \) on the instability, then we observe that the vortex-number \( n \) also increases, which represents a shorter wavelength instability. Therefore, we observe that the instability wavelength responds to the governing receptivity mechanism, namely the roughness wavelength of the concentric grooves. Similar behavior is observed for radial and isotropic grooves in Figs. 5(b) and 5(c).

Figure 8 shows a range of roughness values with their associated type I critical Reynolds numbers at various \( T_s \) values for each curve, which represents a different fixed-axial flow. Note that all Figs. 8(a)–8(c) exhibit high values of the critical Reynolds number as all roughness levels and axial flow strength are increased. However, we see that increasing concentrically anisotropic grooves of curves in

### TABLE I. Critical data for anisotropic (concentric grooves \( \lambda \) and radial grooves \( \eta \)) and isotropic roughness \( \lambda = \eta \) at a fixed \( T_s = 0.1 \). Type I and (type II). The most dangerous mode is indicated as bold text in terms of critical Reynolds number.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( R_c )</th>
<th>( n_c )</th>
<th>( \phi_c )</th>
<th>( x_{c,c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>339 (443)</td>
<td>39 (28)</td>
<td>14.4 (21.4)</td>
<td>0.449 (0.161)</td>
</tr>
<tr>
<td>0.25</td>
<td>448 (-)</td>
<td>60 (-)</td>
<td>18.6 (-)</td>
<td>0.397 (-)</td>
</tr>
<tr>
<td>0.5</td>
<td>642 (-)</td>
<td>94 (-)</td>
<td>23.2 (-)</td>
<td>0.341 (-)</td>
</tr>
<tr>
<td>0.75</td>
<td>910 (-)</td>
<td>142 (-)</td>
<td>27.3 (-)</td>
<td>0.302 (-)</td>
</tr>
<tr>
<td>1</td>
<td>1249 (-)</td>
<td>202 (-)</td>
<td>31.3 (-)</td>
<td>0.266 (-)</td>
</tr>
</tbody>
</table>

### TABLE II. Critical data for anisotropic (concentric grooves \( \lambda \) and radial grooves \( \eta \)) and isotropic roughness \( \lambda = \eta \) at a fixed \( T_s = 0.2 \). Type I and (type II). The most dangerous mode is indicated as bold text in terms of critical Reynolds number.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( R_c )</th>
<th>( n_c )</th>
<th>( \phi_c )</th>
<th>( x_{c,c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>426 (485)</td>
<td>71 (40)</td>
<td>18.1 (24.2)</td>
<td>0.507 (0.185)</td>
</tr>
<tr>
<td>0.25</td>
<td>568 (-)</td>
<td>102 (-)</td>
<td>23 (-)</td>
<td>0.422 (-)</td>
</tr>
<tr>
<td>0.5</td>
<td>867 (-)</td>
<td>161 (-)</td>
<td>28.2 (-)</td>
<td>0.347 (-)</td>
</tr>
<tr>
<td>0.75</td>
<td>1312 (-)</td>
<td>250 (-)</td>
<td>33.2 (-)</td>
<td>0.292 (-)</td>
</tr>
<tr>
<td>1</td>
<td>1882 (-)</td>
<td>367 (-)</td>
<td>37.7 (-)</td>
<td>0.252 (-)</td>
</tr>
</tbody>
</table>

Fig. 8(a) result in greater values of the critical Reynolds number as the enforced axial flow is increased, whereas the plots in Fig. 8(b) are progressively less stable while increasing both radial grooves and \( T_s \). Furthermore, increasing both isotropic roughness and \( T_s \) show a lower value of the critical Reynolds number than those seen in Figs. 8(a) and 8(b). Hence, this case results in flows less stable than flows of the radially anisotropic case for type I mode as seen in Table II.

### IV. LINEAR AMPLIFICATION RATE

As Hussain et al. investigated, the type II mode is reduced relative to the type I mode when increasing the axial flow strength. However, adding roughness levels on a rotating disk in axial flows extinguishes the type II mode and vice versa. We consider the linear growth rates of both modes through the region of convective instability for various surface roughness and at a fixed \( T_s = 0.1 \) at a fixed extent in \( R_c \).

Figure 9 presents the growth rates of both modes for various surface roughness at a fixed \( T_s = 0.1 \) in the rotating frame for showing the amplification rates of stationary modes. We see that the amplification rates are significantly reduced as axial flow and concentric grooves are increased as shown in Fig. 9(b). Furthermore, increasing both \( \lambda \) and \( T_s \) leads to eliminating the viscous type II instability. These results confirm the stabilizing effect of radially anisotropic roughness presented in Sec. III, whereas Fig. 9(c) shows that increasing both \( \eta \) and \( T_s \) reduces the amplification rates and the type II mode becomes increasingly larger than the type I mode, leading to the disappearance of this
In Fig. 9(d), the isotropic roughness shows the same behavior of con-
tinuous grooves (λ) and axial flow on spatial growth rates, uc, for
the steady-mean flow, we apply the partial-slip condition to
and isotropic roughness
The most
dangerous mode is indicated as bold text in terms of critical Reynolds number.

<table>
<thead>
<tr>
<th>λ = η</th>
<th>Rc</th>
<th>nc</th>
<th>ϕc</th>
<th>xec</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>536 (560)</td>
<td>122 (59)</td>
<td>22.2 (27.2)</td>
<td>0.556 (0.206)</td>
</tr>
<tr>
<td>0.25</td>
<td>737 (-)</td>
<td>167 (-)</td>
<td>27.8 (-)</td>
<td>0.428 (-)</td>
</tr>
<tr>
<td>0.5</td>
<td>1244 (-)</td>
<td>279 (-)</td>
<td>34 (-)</td>
<td>0.332 (-)</td>
</tr>
<tr>
<td>0.75</td>
<td>2033 (-)</td>
<td>455 (-)</td>
<td>40 (-)</td>
<td>0.267 (-)</td>
</tr>
</tbody>
</table>

Table III. Critical data for anisotropic (concentric grooves λ) and radial grooves η) and isotropic roughness λ = η at a fixed T* = 0.3. Type I and (type II). The most dangerous mode is indicated as bold text in terms of critical Reynolds number.

Based on this approach, we find that either the results are very close to each other or the same results in many cases for both methods as seen in Fig. 11. However, the difference between both values (for two methods) of the critical Reynolds number slightly increases when increasing concentric grooves (λ) and axial flow. Furthermore, the results of the new method lead to more stabilizing effects and less desta-

<table>
<thead>
<tr>
<th>T* = 0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 536 (560)</td>
</tr>
<tr>
<td>0.25 737 (319)</td>
</tr>
<tr>
<td>0.5 (244)</td>
</tr>
</tbody>
</table>

VI. ENERGY ANALYSIS

Another approach given in the work of Cooper and Carpenter conform-
s the stability analysis of an initial perturbation by measuring its
kinetic energy in the volume of the boundary layer. An integral energy
balance equation for three-dimensional disturbances (u, v, w, p) to
the undisturbed three-dimensional boundary-layer flow (U, V, W) is derived from the governing equations to ensure that all flows in an en
forced axial flow over a rough, rotating disk. Examining the energy input and
output of a disturbance to the mean flow is considered in this section.

The energy equation is obtained by multiplying the linearized momentum
equations (27) and (28) by u, v, and w, respectively. The combined
equations are then summed for obtaining a kinetic energy
equation of the disturbances as follows:

\[
\frac{\partial}{\partial t} + \frac{U}{\partial r} + \frac{V}{\partial \theta} + W + \frac{\partial}{\partial z} + \frac{\partial}{\partial r} \right) K
\]

\[
= -\frac{\partial}{\partial r} \left[ \frac{\partial U}{\partial u} + \frac{\partial V}{\partial v} + \frac{\partial W}{\partial w} + \frac{\partial P}{\partial r} \right] + \frac{\partial}{\partial r} \left[ \frac{\partial (\sigma_1 u_1 + \sigma_2 v_2 + \sigma_3 w_3)}{\partial r} \right],
\]

where \[ K = \frac{1}{2} (u^2 + v^2 + w^2) \] and \[ \sigma_1 \] are the viscous stress terms,
\[ \sigma_1 = \frac{1}{r} \left( \frac{\partial u}{\partial r} - \frac{\partial v}{\partial \theta} \right) \]

Repeated suffixes in Eq. (33) indicate summation from 1 to 3. All O(1/r) viscous terms in order to be consistent with neglect of O(1/R^2) terms can be omitted in the governing stability equations. If an average of the perturbations is calculated over a single time period
and azimuthal mode, followed by integration throughout the bound-

\[
\int_0^\infty \left[ U \frac{\partial K}{\partial r} + \frac{\partial (\sigma_1 u_1 + \sigma_2 v_2 + \sigma_3 w_3)}{\partial r} \right] dz
\]

\[
= \int_0^\infty \left[ -\frac{\partial u}{\partial u} + \frac{\partial v}{\partial v} + \frac{\partial w}{\partial w} \right] dz + \left( \frac{\partial P}{\partial r} \right)_w
\]

\[
- \int_0^\infty \left[ \frac{\partial \sigma_1 u_1}{\partial x} + \frac{\partial \sigma_2 v_2}{\partial y} + \frac{\partial \sigma_3 w_3}{\partial z} \right] dz
\]

\[
- \int_0^\infty \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] dz + \left( \frac{\partial P}{\partial r} \right)_w
\]

\[
= \int_0^\infty \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz - \int_0^\infty \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz + \left( \frac{\partial P}{\partial r} \right)_w
\]

\[
= \int_0^\infty \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz + \left( \frac{\partial P}{\partial r} \right)_w
\]

\[
= \int_0^\infty \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz + \left( \frac{\partial P}{\partial r} \right)_w
\]

\[
= \int_0^\infty \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz + \left( \frac{\partial P}{\partial r} \right)_w
\]
FIG. 7. The neutral curves of the wave-number in the r-direction at a fixed value of axial flow ($T_s = 0.1, 0.2$) and a range of roughness. (a) and (d) Neural curves of anisotropic $k > 0, \eta = 0$ (concentric grooves). (b) and (e) Neural curves of anisotropic $k = 0, \eta > 0$ (radial grooves). (c) and (f) Neural curves of isotropic $k = \eta > 0$. 
Here, overbars denote a period-averaged quantity, for instance, $\bar{u} = u - u'$ ($'$ indicates the complex conjugate), and $w$ subscripts denote quantities evaluated at the wall. Physical interpretations of the terms on the left side can be given as follows:

(a) the average kinetic energy convected by the radial mean flow,
(b) the work done by the perturbation pressure,
(c) the work done by viscous stresses across internal boundary-layer flows of the fluid.

The terms on the right side can also be interpreted physically as follows:

(I) the Reynolds stress energy production terms $P_i$,
(II) the viscous dissipation energy removal terms $EDV$,
(III) pressure work terms $PW$,
(IV) contributions from work done on the wall by viscous stresses $S_i$,
(V) terms arising from streamline curvature effects and the three-dimensionality of the mean flow, $D_i$.

FIG. 9. Linear convective growth rates of stationary disturbances for various surface roughness at a fixed $T_s = 0.1$ in the rotating frame: (a) $\lambda = \eta = 0$, (b) $\lambda = 0.25$, (c) $\eta = 0.25$, and (d) $\lambda = \eta = 0.25$. 

FIG. 8. Plot of the type I critical Reynolds number arising from each $T_s = 0, 0.1, 0.2, 0.3$ with surface roughness, (a) concentric grooves, (b) radial grooves, and (c) isotropic roughness.
The energy equation is then normalized by the integrated mechanical energy flux to give the Total Mechanical Energy (TME) as
\[
\frac{-2\gamma}{\text{TME}} = \left(\frac{P_1 + P_2 + P_3}{I} + \frac{EDV}{II} + \frac{(PW_1 + PW_2)}{III} + \frac{(S_1 + S_2 + S_3)}{IV} + \frac{(D_1 + D_2 + D_3)}{V}\right).
\]

Any eigenmode can be used for carrying out the energy balance. In Eq. (35), the positive terms contribute to energy, while those which are negative remove energy from the system. Outweighing the energy production more than the energy dissipation in the system acts to spatially amplify of a mode \((\alpha_i < 0)\). The greatest effects of roughness can be identified by calculating all terms in the energy equation (35). The terms \((PW_2, S_1, S_2, S_3)\) in the energy equation are identically zero. Reynolds stress \((P_2)\) and conventional viscous dissipation \((EDV)\) lead to finding the main contributors to the energy production. Terms \(P_1, PW_1,\) and \(D_1\) are negligible, and the geometric terms \(D_2\) and \(D_3\) remove energy from the system and are relatively larger in the type II case. We also analyze the energy balance to reconfirm the findings of the linear stability analysis.

The energy balance calculation is analyzed by calculating the energy contribution of the individual components. This is done for a range of roughness levels and at a fixed axial flow at \(R_c = R_c + 25\) of the type I mode for obtaining eigenfunctions which will be used to conduct an energy balance analysis. Here, all eigenfunctions are assessed at \(R_c = R_c + 25\) (i.e., well into the unstable regime), and the value of \(\alpha\) chosen is the most amplified disturbance at this particular Reynolds number. The plots depicted in Fig. 12 are the magnitudes of the perturbation eigenfunctions for a range of concentric grooves at fixed axial flow \(T_s = 0.1\), assuming example to show the profile of these eigenfunctions. We observe that the eigenfunction profiles are narrowed. This effect can largely be attributed to a mirroring of the mean-flow profiles as \(\lambda\) and \(T_s\) are increased. We further notice that for

![Figure 10](image-url)
concentric grooves the maxima of the $|\bar{u}|$ profile become more prominent as $\lambda$ is increased and occur closer to the disk surface. Increasing $\lambda$ slightly increases the maximum of the $|\bar{v}|$ profile. Similarly, increasing concentric grooves initially also decreases the maximum of the $|\bar{w}|$ profile. We can expect that radial grooves and isotropic roughness have a similar effect for other type I eigenfunctions, and all changes in those functions will be based on the mean-flow profiles. The profiles in Fig. 12 show the perturbation eigenfunctions for both approaches that are presented in Secs. III and V; we note that there is no significant difference in the wall boundary conditions for the perturbation in this case. The reason for this may be that the axial flow and roughness have a similar effect (i.e., both act to increase $R_c$ on the inviscid type I, and decrease $R_c$ on type II mode) for the von Kármán flow.

Figure 13(a) shows the energy balance calculation of the type I mode (or cross-flow) instability for anisotropic roughness $\lambda > 0, \eta = 0$. There is a greater stabilization for each flow in the system caused by a strong decrease in Total Mechanical Energy (TME) of the flows as concentric grooves are increased. This effect arises mainly from the changes in the energy production term $P_2$ and in the energy dissipation term (EDV). We note that both energy production and dissipation decrease for higher values of radially anisotropic roughness parameter when increasing the enforced axial flow.

Figure 13(b) presents the energy balance calculation of inviscid type I modes for a range of radial grooves with $T_s = 0.1$. This case shows changes of the energy production terms, the energy dissipation terms, and the total mechanic energy (TME) of the system, where it is noted that the total mechanic energy (TME) is sharply decreased as a result of reducing the energy production and the energy dissipation when increasing $\eta > 0$. However, this case is less stable than the concentric grooves case.

Figure 13(c) displays the energy balance calculation for a range of isotropic roughness $\lambda = \eta > 0$ and axial flow. This case shows also a significant stabilizing effect on the inviscid type I (or cross-flow) instability that mainly comes from the large decline in the energy contribution term $P_i$. We see also that the significant invariance of the energy dissipation of the system (EDV) leads to a clear reduction in the total energy of the system as a result of increasing $\lambda = \eta$ and axial flow.

VII. SUMMARY OF THE CURRENT RESULTS

We have numerically studied the convectively unstable flow over a rotating, rough disk in an enforced axial flow. We summarized and discussed the results of the influences on the stability in axial flows over rough surfaces. Our theoretical analysis depended on the MW approach to modify no-slip conditions to partial slip boundary conditions for modeling a rotating disk with isotropic, radially-, and azimuthally anisotropic surface roughness. In this study, our main interest was the stationary modes of disturbances that rotate with surface of the disk. We also formulated the steady boundary-layer flow over a rotating reference frame attached to the disk. Further, we formulated mean-flow equations for a range of axial flow...
TABLE IV. Critical data of both methods for anisotropic (concentric grooves $\lambda$ and radial grooves $\eta$) and isotropic roughness $\lambda = \eta$ at a fixed $T_R$ value. Type I and type II. The most dangerous mode is indicated as bold text in terms of critical Reynolds number.

<table>
<thead>
<tr>
<th>$T_R = 0.1$</th>
<th>$\lambda$</th>
<th>$R_c^{\text{old}}$</th>
<th>$n_s^{\text{old}}$</th>
<th>$R_c^{\text{new}}$</th>
<th>$n_s^{\text{new}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>339 (443)</td>
<td>39 (28)</td>
<td>339 (443)</td>
<td>39 (28)</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>448 (-)</td>
<td>60 (-)</td>
<td>449 (-)</td>
<td>60 (-)</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>642 (-)</td>
<td>94 (-)</td>
<td>646 (-)</td>
<td>95 (-)</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>910 (-)</td>
<td>142 (-)</td>
<td>925 (-)</td>
<td>145 (-)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1249 (-)</td>
<td>202 (-)</td>
<td>1288 (-)</td>
<td>209 (-)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T_R = 0.1$</th>
<th>$\eta$</th>
<th>$R_c^{\text{old}}$</th>
<th>$n_s^{\text{old}}$</th>
<th>$R_c^{\text{new}}$</th>
<th>$n_s^{\text{new}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>339 (443)</td>
<td>39 (28)</td>
<td>339 (443)</td>
<td>39 (28)</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>403 (325)</td>
<td>38 (17)</td>
<td>403 (325)</td>
<td>38 (18)</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>478 (274)</td>
<td>38 (15)</td>
<td>478 (276)</td>
<td>38 (15)</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>569 (248)</td>
<td>40 (11)</td>
<td>569 (250)</td>
<td>40 (13)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>682 (232)</td>
<td>43 (10)</td>
<td>682 (235)</td>
<td>43 (11)</td>
<td></td>
</tr>
</tbody>
</table>

strengths. The influences of radially anisotropic roughness (concentric grooves, $\eta = 0$) show a more steady state of boundary-layer flows in an enforced axial flow increased via setting $T_R > 0$. The boundary-layer thinning of each flow for all cases can be seen clearly.

The investigations of linear stability analysis based on the steady-mean flow reveal that the stability characteristics of the inviscid type I (or cross-flow) instability result in a stabilizing effect of both anisotropic (concentric grooves and radial grooves) and isotropic (general) roughness as axial flow strength is increased. However, a concentrically grooved disk has the strongest stabilizing effect on the inviscid type I mode with increasing $T_R$, increasing the critical Reynolds number. The case of anisotropic roughness with radial grooves exhibits a significant destabilization for the viscous type II mode instability, whereas both concentric grooves roughness and general isotropic roughness act to eliminate this mode entirely, in particular, with increasing $T_R$. Moreover, it is found that a disk with concentric grooves has a stronger stabilizing effect than a surface with isotropic roughness in an increased axial flow.

In the case of concentric grooves, we have slip in the radial direction, which means the developing vorticity within the boundary layer that leads to the instability forming is effectively swept or slips downstream in the radial direction. This is in-line with the direction of flow entrainment due to the axial flow and centrifugal rotation of the disk surface, and so concentric grooves act to increase this radial flow effect, thereby delaying transition. Physically, the axial flow and concentric grooves combine in a way that convects developing instabilities downstream, and this dual action acts to stabilize the flow. In contrast, the radial grooves correspond to slip in the azimuthal direction, which is not aligned in the radial direction and therefore there is some competing effect for this setup, in particular for the type II mode, where it appears that some developing instabilities are convected azimuthally, and so they may still be able to grow for a given radial location. When combined for the isotropic case, it is clear that the azimuthal slip generated by the radial grooves acts to some extent to counteract the stabilization of the concentric grooves. Therefore, for a given axial flow, isotropic roughness tends to be marginally less stabilizing than the purely concentric case.

The surface roughness leads to reducing the amplification rates of stationary modes, confirming again that combining both the axial flow and various roughness levels result in a strongly stabilizing effect for the type I mode. However, the type II mode is significantly more unstable for azimuthally anisotropic surface roughness and more amplified for all $T_R > 0$ considered here, eventually eliminating entirely the type I mode. Physically, we predict that using modest values of various roughness levels while increasing the axial flow rate forces fluid toward the surface of the disk, leading to development of the destabilizing impact from viscous effects close to the wall location.

The maximum growth rates within the unstable regime are found to gradually reduce with increasing axial flows over a rough disk. In contrast, the radially anisotropic surface roughness strongly causes an unstable state of the viscous type II modes.

In order to confirm our results, we have also found that both anisotropic roughness (concentric grooves and radial grooves) and isotropic roughness act to clearly reduce the total mechanical energy as a result of reducing the energy production and the energy dissipation of the type I mode. This study shows that identifying a suitable sort of roughness, which acts to generate instability with the same wavelength. This means that the governing receptivity mechanism is quite clear in this study.

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beneficial as an effective passive flow-control mechanism for engineering flows over rotating, rough disks as well as other cross-flow dominant flows. Our study also suggests that increasing the axial flow strength over a rough disk is to be preferred as it might act to preserve the laminar flow over a greater region of the support disk. Finally, there are very slight differences in the values of the parameters between the old method which depended on applying roughness only on the boundary conditions of the steady-mean flow and the novel approach which suggests modeling the perturbed flow components as well. However, this difference does not affect the overall course of our results for the von Kármán flow. It is interesting to investigate whether there are different results of the current study between two approaches for other flows such as Bödewadt flow.24

![Type I eigenfunctions of both approaches for a range of concentric grooves with fixed axial flow $T_x = 0.1$.](image-url)
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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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