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On the stability of a heated rotating-disk boundary layer in a temperature-dependent viscosity fluid

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ABSTRACT

The paper presents a linear stability analysis of the temperature-dependent boundary-layer flow over a rotating disk. Gas- and liquid-type responses of the viscosity to temperature are considered, and the disk rotates in both a quiescent and an incident axial flow. Temperature-dependent-viscosity flows are typically found to be less stable than the temperature independent cases, with temperature dependences that produce high wall viscosities yielding the least stable flows. Conversely, increasing the incident axial flow strength produces greater flow stability. Transitional Reynolds numbers for these flows are then approximated through an $e^N$-type analysis and are found to vary in approximate concordance with the critical Reynolds number. Examination of the component energy contributions shows that flow stability is affected exclusively through changes to the mean flow. The results are discussed in the context of chemical vapor deposition reactors.

I. INTRODUCTION

A convectively unstable flow is the one featuring the manifestation of small disturbances in an otherwise laminar state. These convective instabilities develop and grow downstream, eventually gaining sufficient amplitude to trigger non-linear behavior that causes the flow to transition from the laminar to turbulent regime. As such, there is a long history of interest in the mechanisms and control of these instabilities. The effects of small disturbances on the laminar flow over a flat plate were first investigated by Tollmien and Schlichting, who considered the amplification of linear, wave-like disturbances (aptly named Tollmien–Schlichting waves) as the primary mechanism for flow instability. Later, Gregory et al. investigated the formation of co-rotating stationary vortices (crossflow instabilities) on the rotating disk.

The physics of a rotating disk situated in a stationary fluid can be described as a centrifugal pump or fan. As the disk rotates, viscous effects cause the surrounding fluid to rotate with it. The fluid is dragged down toward the disk surface before flowing radially outwards to be cast off at the disk edge. The seminal work of von Kármán found exact similarity solutions to the Navier–Stokes equations for the infinite-plane rotating disk problem, allowing them to be reduced to a set of ordinary differential equations that characterize the mean flow.

Much of the early motivation for the study of the stability of the rotating-disk flow is drawn from aerospace engineering; both the rotating disk and the swept wing exhibit inflectional mean-flow profiles susceptible to crossflow instabilities. These arise from inviscid effects and manifest as spiral vortices as first observed experimentally by Gray for a swept wing and later by Gregory et al. for the rotating disk. The rotating disk, however, proved advantageous for the study of crossflow instabilities due to its axisymmetric geometry and exact similarity solutions of the Navier–Stokes equations allowing for significantly simplified mathematical investigation.
when compared to the swept wing. The study of rotating-disk flows has subsequently developed into its own field, with direct industrial applications as well as remaining a model flow of fundamental interest.

One of the key differences between the swept wing and the rotating disk boundary layers is that the latter is subject to Coriolis forces. Malik discovered an additional instability mechanism arising from this by performing a numerical analysis to compute the neutral curves for stationary instability modes in the rotating-disk boundary layer. This work was the first departure from the previous fourth-order Orr–Sommerfeld type analyses in favor of a sixth-order system that includes rotational Coriolis and viscous terms. The work, also verified asymptotically by Hall, found that two branches exist: the previously known upper branch associated with crossflow instability (type I) and a lower branch associated with streamline curvature (SC) and the balancing of Coriolis and viscous forces (type II). The type II mode has been found in all subsequent studies and is known to be particularly important when traveling modes are permitted; see, for example, the work of Hussain et al. Observing the type II disturbance experimentally has, however, proved difficult as the type I mode is almost always dominant for all practical flows where unavoidable roughness acts to select modes that rotate with the disk surface. The work presented by Fedorov et al. is often pointed to as a rare observation of the type II mode, and more recent experimental work has shown the potential for traveling modes to be selected.

This paper considers the linear convective stability behavior of flows over the rotating disk under a temperature-dependent viscosity model and is closely related to the sister publication Miller et al. concerning the Blasius/Falkner–Skan boundary layer stability under the same viscosity model. There are, however, a number of key physical differences between the stability of the von Kármán and Blasius/Falkner–Skan boundary layers. Notably, while rotation of the disk is sufficient for a potentially unstable boundary layer due to the inviscid nature of the crossflow instability, Tollmien–Schlichting waves are generated through viscous effects and as such external forcing is required to generate a boundary layer (and, therefore, instability) over the flat plate. The two-dimensional nature of the plate problem also renders all instabilities “traveling,” with no direct analog to the stationary vortices formed on the disk. Nevertheless, the methodology presented here is much the same as that presented in the work of Miller et al. and the interested reader is referred to there for an in-depth discussion regarding two-dimensional temperature dependent boundary layer flows.

While normal-mode linear stability analyses typically aim to model the early stages of transitional flows, numerous studies have highlighted the importance of nonlinearity and non-normality in the prediction of transition to turbulence, as well as their role in the formation of primary instabilities. The analysis developed by van Ingen reviewed extensively in van Ingen and utilized in this paper is often used to consolidate nonlinearity within the linear framework, utilizing the linear amplification rates to find empirical agreement with transitional Reynolds numbers observed in experiments. Bertolotti et al. utilize parabolic stability equations to examine both non-parallel and nonlinear effects on the Blasius boundary layer stability, concluding that while both are destabilizing with reference to the linear results, neither is significant enough to account for the discrepancies between neutral curve data obtained from theory and experiments.

Non-normal studies have allowed for the investigation of global instabilities in boundary layers and their role in transition. In response to the assertion of Huerrre and Monkevitz that a linear analysis is sufficient for investigating global behavior, Chomaz identifies that instability behavior becomes rapidly non-linear beyond a global threshold. Lingwoodan provided the benchmark for the absolute instability of the rotating disk boundary layer flow, finding theoretical and experimental agreement of the flow becoming absolutely unstable at a Reynolds number of 510 and transition to turbulence at 513. From this, it was concluded that absolute instability acts as a linear threshold, beyond which non-linear behavior in the flow begins to dominate and develops into fully turbulent flow. Briggs’ pinching criterion states that the absolutely unstable parameters must lie within the range of those that are convectively unstable (that is, the absolute neutral stability curve is contained within the convective neutral curve). To this end, Healey shows that the absolute neutral stability curve associated with the rotating disk flow also exhibits the two branch structure of its convective equivalent. Utilizing front propagation theory and direct numerical simulation (DNS), Cossu et al. examined the nonlinear behavior of disturbances beyond the threshold for global instability, finding that for large-amplitude impulses, the globally unstable behavior of the Blasius boundary layer is decisively nonlinear, yet the wave front speed is identical to that of its linear counterpart. The findings of Healey and Cossu et al. lead to the conclusion that conducting local linear analyses of unstable flows over both the disk and the plate is still of importance when extending to a global stability analysis to inform the nonlinear behavior. While nonlinearity, non-normality, and absolute instability are beyond the scope of the work presented here, it is acknowledged that they form an active area of research that is greatly informed by the behavior of convectively unstable flows. As such, the work presented here and in the work of Miller et al. can be considered as a first step toward understanding the global criteria for instability and transition in modified von Kármán and Blasius/Falkner–Skan flows. The interested reader is directed to Ref. 17 for an extensive review of non-normal, nonlinear, and global studies and their role in boundary layer stability. In addition, the review presented by Lingwood and Alfredsson includes recent developments regarding absolute instability over rotating disk boundary layers.

The significance of viscous effects on flow instability has led to interest in variable viscosity flows. Strong temperature gradients can cause the viscosity of a fluid to vary immensely and, as such, there is a great deal of literature studying temperature dependent viscosity flows. Ling and Dybbs assert that an inverse-linear relationship between temperature and viscosity accurately models the behavior of a range of real fluids. Kafoussias and Williams utilized this relationship in their study of the free-forced vertical flow with temperature dependent viscosity over a flat plate. There, it is found that fluids that become more viscous with increasing temperature create both a narrower temperature profile and a velocity profile, while the effect is reversed for fluid viscosities that decrease with temperature. Wall and Wilson considered the effects of two different exponential-type viscosity temperature relationships on Blasius flow stability. In both models, they noted that for an isothermally heated plate, an exponentially decaying viscosity produces a
destabilized flow, while an exponentially increasing viscosity is more stable. An inverse linear temperature dependent viscosity law is also considered by Jasmine and Gajjar in their study of flow stability over a rotating disk, and similar qualitative effects as cited by Kafousias and Williams are observed, as well as finding that a viscosity that decreases more rapidly with increasing temperature yields a less stable flow. Again, this broadly agrees with the findings of Wall and Wilson. The work presented in this paper considers the inverse-linear relationship between viscosity and temperature utilized by Ling and Dybbs, Kafousias and Williams, and Jasmine and Gajjar, amongst others.

There is a wealth of literature concerning flows over rotating media, most of which set the fluid as otherwise stationary. More recent studies examined the effects of an oncoming flow over rotating media to better understand transitional flows in industrially relevant configurations. Chen and Mortazavi produce a model of a Chemical Vapor Deposition (CVD) reactor featuring a forced flow parameter to represent the injection of gas into the system. Garrett et al. present a linear stability analysis (both asymptotic and numerical) of the enforced axial flow over a rotating cone, where it is found that a stronger axial flow results in delaying the onset of transition in both modes of instability. This is extended to the flow over a rotating disk in the work of Hussain, where a similar result is found.

The work presented in this paper has particular relevance to vertical CVD systems where it is important that laminar flow is maintained over a heated substrate rotating in a forced flow. This is necessary to promote regular and cohesive film growth. Longitudinal roll instabilities are well established in buoyancy-driven flows with a free convection element and have been observed in CVD reactors at Rayleigh numbers (the Rayleigh number is the product of the Grashof and Prandtl numbers, i.e., the ratio of buoyancy and viscous forces multiplied by the ratio of momentum and thermal diffusivities) beyond 1708. While there is a great deal of literature regarding solutions to the laminar flow in a CVD reactor, there is (to the authors’ knowledge) no such study on the existence of unstable or transitional flows due to forced convection. Convective instabilities arising from forced flow are perhaps not considered due to a dichotomic view of the flow being laminar or turbulent. Typically, the operating Reynolds number of a vertical CVD reactor is in the range of 1–200, which most would consider to be firmly in the laminar regime. However, modifications to the stability solutions of flows over both the disk and the plate have shown that destabilizing effects are possible, as demonstrated by Wall and Wilson where a critical Reynolds number of approximately 220 is achieved at one extreme of their viscosity temperature dependence parameter. Hence, it is possible that certain destabilizing criteria may be sufficiently influential to produce primary instabilities within the operational parameters of a CVD reactor. It is widely accepted that the steep temperature gradients in a CVD reactor cause variation in all physical properties of the fluid, supporting the need for a temperature dependent viscosity model, and as such this study serves as the first step toward a larger stability analysis of flows in CVD reactors.

In particular, the work presented here considers the convective instability of the boundary layer resulting from the forced flows of fluids with temperature dependent viscosity incident on a heated rotating disk. It proceeds as follows: In Sec. II, we formulate and solve the equations representing the steady, mean flow of the fluid over the disk. The linear convective stability of these flows is then assessed in Sec. III, via the formulation of neutral stability curves and integral energy analyses. Finally, our conclusions are discussed in Sec. IV.

II. PROBLEM FORMULATION AND MEAN-FLOW ANALYSIS

A. Derivation of the mean-flow equations

Consider a heated disk of infinite radius, rotating about its center of axis $z^*$ with angular velocity $\Omega^*$. The disk is situated perpendicularly downstream from an incompressible Newtonian fluid. As the disk rotates, fluid is entrained toward its surface and moves with velocity $\mathbf{U}^* = (u^*, v^*, w^*)$, representing boundary-layer velocities in the radial, azimuthal, and axial directions ($r^*$, $\theta^*$, and $z^*$), respectively. The disk is heated to a temperature $T^*_{\infty}$, while the temperature of the fluid in the free stream is $T^*$; the fluid in the boundary layer has temperature $T^*$. Note that in all that follows, an asterisk indicates a dimensional quantity.

In the rotating frame of reference, the flow can be described by the Navier–Stokes equations in cylindrical polar coordinates,

$$\nabla^* \cdot \mathbf{U}^* = 0,$$

$$\left(\frac{\partial}{\partial t^*} + \mathbf{U}^* \cdot \nabla^*\right) \mathbf{U}^* + \frac{2\Omega^* \times \mathbf{U}^* + \Omega^* \times \mathbf{U}^* \times \mathbf{r}^*}{\kappa^*} = -\frac{1}{\rho^*} \nabla^* p^* + \frac{1}{\rho^*} \nabla^* \cdot \mathbf{r}^*, \quad (1b)$$

$$\left(\frac{\partial}{\partial t^*} + \mathbf{U}^* \cdot \nabla^*\right) T^* = \frac{k^*}{\rho^* C_v^*} \nabla^* v^* T^*, \quad (1c)$$

where $t^*$ is the time, $\Omega^* = (0, 0, 0)$ is the angular velocity vector, $\mathbf{r}^* = (r, 0, 0)$ is the radial position vector, $\rho^*$ is the density, $p^*$ is the pressure, $k^*$ is the thermal diffusion coefficient, and $C_v^*$ is the specific heat of the fluid. The viscous stress tensor is $\mathbf{\tau}^* = \mu^* \mathbf{\dot{\gamma}}^*$, where $\mu^*$ is the viscosity and $\mathbf{\dot{\gamma}}^* = \nabla^* \mathbf{U}^* + \nabla^* (\mathbf{U}^*)^T$ is the rate-of-strain tensor. The terms labeled $R_1$ and $R_2$ are rotational terms arising from Coriolis and centrifugal acceleration forces, respectively. The heat continuity Eq. (1c) is coupled to Eqs. (1a) and (1b) via the imposed temperature dependence of the fluid viscosity

$$\mu^* = \frac{\mu^*_{\infty}}{1 + \epsilon^* (T^* - T^*_{\infty})}.$$

Here, $\mu^*_{\infty}$ is the fluid viscosity in the free stream and $\epsilon^*$ is the temperature dependence constant in inverse temperature units. Note that $\epsilon^*$ is an intrinsic property of the fluid and changing $\epsilon^*$ would represent a fundamentally different fluid. For an isothermally heated disk (as considered here), $\epsilon^* > 0$ represents the viscous behavior of a liquid (a fluid that becomes less viscous with increasing temperature), while $\epsilon^* < 0$ represents that of a gas (a fluid that becomes more viscous with increasing temperature, with appropriate consideration of parameters to ensure positive viscosity). Applying $\epsilon = 0$ returns a constant viscosity and uncouples Eq. (1c) from Eqs. (1a) and (1b).
Ling and Dybbs\textsuperscript{22} cite this expression for $\mu^*$ as an accurate representation of the relationship between viscosity and temperature for a range of real fluids such as water and crude oil; this motivates its use here.

The motion of the fluid is influenced both by the rotation of the disk and the axially enforced toward the disk surface. The enforced axial flow component is derived from the radial pressure balance at the boundary layer edge

$$U_e^* \frac{DU_e^*}{dr^*} = -\frac{1}{\rho^*} \frac{dp^*}{dr^*},$$

where $U_e^* = C^* r^*$ is the flow velocity at the boundary-layer edge. The constant $C^*$ represents the strength of the enforced flow in inverse time units.

We non–dimensionalize \textbf{(1)} by the scaling variables

$$\begin{align*}
U^* &= (u, v, \delta w) U_e^*, \\
(r^*, z^*) &= (r, \delta z) L^*, \\
t^* &= \frac{t L^*}{U_e^*},
\end{align*}$$

$$p^* = p(\frac{U_e^*}{\mu^*})^2, \quad T^* - T^*_{\infty} = T(\frac{T^*_w - T^*_{\infty}}{\Theta}), \quad \epsilon^* = \frac{\epsilon}{T^*_w - T^*_{\infty}},$$

where $U_e^* = L^* \Omega^*$, with $L^*$ defined as a generic length scale. The boundary-layer scaling constant is $\delta$. We note that setting $\epsilon$ as greater or less than 0 remains representative of liquid- or gas-type viscosity behavior, respectively, consistent with the definition of $\epsilon^*$.

The laminar flow is steady and axisymmetric, meaning that derivatives with respect to $t$ and $\Theta$ are neglected. We define the Reynolds number as $Re = U_e^* L^* \rho^* / \mu^* \epsilon$ and, with $Re \gg 1$, determine the governing equations at leading order to be

$$\begin{align*}
1 \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial u}{\partial r} + \frac{w}{r} \frac{\partial u}{\partial z} &= \frac{(r + \epsilon)^2}{r} + \frac{1}{\delta} \frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial z},
\end{align*}$$

$$\begin{align*}
\frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial v}{\partial z} &= \frac{1}{\delta} \frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial z},
\end{align*}$$

$$\begin{align*}
\frac{\partial w}{\partial r} + \frac{w}{r} \frac{\partial w}{\partial z} &= -\frac{1}{\delta} \frac{\partial p}{\partial z} + \frac{1}{\delta} \frac{\partial^2 w}{\partial z^2} + 2 \frac{\partial u}{\partial z} \frac{\partial w}{\partial z},
\frac{\partial T}{\partial r} + \frac{w}{r} \frac{\partial T}{\partial z} &= \frac{1}{\delta^2 \rho \epsilon} \frac{\partial^2 T}{\partial z^2},
\end{align*}$$

where it is concluded that $\delta = \mathcal{O}(Re^{-1/2})$ and the boundary-layer thickness is $\delta^* = \delta L^* = \sqrt{\mu^*/\rho^* \Omega^*}$. Note that $Pr = \frac{C_P \rho^*}{\mu^*}$ is the Prandtl number that is henceforth set to $Pr = 0.72$. This is a suitable value for various fluids including air.

Imposing the similarity solutions,

$$u = rU(z), \quad v = rV(z), \quad w = W(z), \quad p = P(z), \quad T = \Theta(z),$$

leads to a modified form of the von Kármán equations

$$2U^* + W^* = 0,$$

$$U^* + U^* W = (V^* + 1)^2 - (\frac{\mu}{\rho} U^*)^2 - T^*_w = 0,$$  \hspace{1cm} (5b)

$$2U^* (V^* + 1) + V^* W - (\frac{\mu}{\rho} V^*)^2 = 0,$$  \hspace{1cm} (5c)

$$WW' + P^* - \frac{\mu}{\rho} W'' = 2\mu^* W^* = 0,$$  \hspace{1cm} (5d)

$$Pr W \Theta' - \Theta'' = 0,$$  \hspace{1cm} (5e)

where $\mu = 1/(1 + r \Theta)$ and primes indicate differentiation with respect to $z$. We note that \textbf{(5d)} is obtained from a leading order expansion of $P(z)$ in terms of $\Theta$. At leading order, one finds that $P(z = 0) = P_0$. The solution at the next order is then determined from \textbf{(5d)}. The system is subject to the boundary conditions

$$U(0) = V(0) = W(0) = P(0) = \Theta(0) = 1 = 0,$$  \hspace{1cm} (6a)

$$U(z \rightarrow \infty) \rightarrow T_{\infty}, \quad V(z \rightarrow \infty) \rightarrow -1, \quad \Theta(z \rightarrow \infty) \rightarrow 0,$$  \hspace{1cm} (6b)

which reflect the no-slip criteria and Coriolis force balance conditions in the rotating frame. Note that the condition for the outer radial velocity is modified according to the pressure balance condition at the boundary-layer edge due to the enforced axial flow. The conditions for $\Theta(0)$ and $\Theta(z \rightarrow \infty)$ are indicative of a heated disk.

The familiar temperature-independent problem is returned when $\epsilon = 0$, and the enforced axial flow is eliminated when $T_{\infty} = 0$. Setting $\epsilon = 0$, returns the standard von Kármán equations.

With the exception of \textbf{(5d)}, the system \textbf{(5)} is solved utilizing a Newton–Raphson searching routine to find a suitable condition for the unknowns $U^*$ and $V^*$, alongside a double precision, fourth-order Runge–Kutta integration scheme to solve the mean flow functions through the boundary layer. Convergence is reached at $z_{\text{max}} = 20$ to a tolerance of $10^{-7}$, which can be considered representative of the free stream although plots will be truncated to $z = 10$ to better visualize the effects of varying the free parameters. Note that \textbf{(5d)} is independently solvable for the temperature independent case, i.e., $\epsilon = 0$,

$$P - P_0 = -\left(2U^* + \frac{W^*}{2}\right),$$

with the stipulation that $P(z = 0) = P_0$. However, the introduction of a temperature dependent viscosity leads to an additional term owing to the viscous stress tensor, $2\mu^* W^*$, and an incomplete solution that requires additional numerical integration is obtained

$$P = -\left(2\mu^* U^* + \frac{W^*}{2}\right) + \int \mu^* W^* dz,$$

in this case, the $P_0$ term is absent because, as noted above, it is obtained in the leading order expansion of $P(z)$.

B. Mean-flow profiles

1. Temperature-dependent viscosity (no axial flow)

Figure 1 shows the resulting mean-flow profiles for a range of $\epsilon$ values when $T_{\infty} = 0$. We see that $\mu^*$ is uniformly 1 for the temperature independent case of $\epsilon = 0$. For $\epsilon \neq 0$, the viscosity converges on $\mu^*(z \rightarrow \infty) = 1$; this is due to the uniform temperature in the

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free stream. Prior to this convergence, the viscosity is everywhere greater or less than one, i.e., the viscosity is increased or decreased throughout the boundary layer, for $\epsilon < 0$ and $\epsilon > 0$, respectively.

Increasing the value of $\epsilon$ is found to move the radial velocity profile, $U(z)$, closer to the disk surface. This can be interpreted as a narrowing of the fluid boundary layer. Increasing $\epsilon$ also reduces the viscosity at the wall, reducing the wall shear stress and the viscous interaction between the disk and the fluid. Hence, the ability of the disk to accelerate the fluid is diminished, resulting in a reduced $U(z)$ profile jet.

The azimuthal profile, $V(z)$, reaches convergence at a reduced value of $z$ as $\epsilon$ is increased and is further evidence of the narrowing effect. This is consistent with the absolute value of $V'(z)$ at the wall increasing with $\epsilon$, showing that $V(z)$ grows more significantly with an increasing distance from the disk surface. An interesting detail of the $V'(z)$ profile is that it becomes inflectional for $\epsilon < 0$, meaning that there is a small region for which the azimuthal velocity increases linearly through the boundary layer, sharply increasing close to the disk surface and more steadily at a greater distance. This is of particular relevance to the results presented in Sec. III E and are discussed in further detail there.

Figure 1 also demonstrates that increases in $\epsilon$ reduces the vertical acceleration of $W(z)$ in the region close to the disk, as well as the converging outflow velocity closer to the free stream. The continuity equation suggests that this axial response is related to the effects of $\epsilon$ on the radial profile. We interpret this physically as a result of the reduced viscosity inhibiting the centrifugal pump effect.

In contrast to the behavior of the velocity profiles, the mean temperature profile, $\Theta(z)$, moves away from the disk surface with $\epsilon$. Mathematically, this is a consequence of the $W(z)$-dependent term in (5e): as $\epsilon$ is increased, the term is decreased due to the reduction in $W(z)$. In turn, $\Theta'(z)$ is reduced, resulting in a shallower gradient in the $\Theta(z)$ profile. As such, while the momentum boundary layer is narrowed, increasing $\epsilon$ results in a broadening of the thermal boundary layer, meaning the reduction in outflow must be favorable to the transferral of heat from the disk to the fluid.

Where values of $\epsilon$ match, the solutions presented here are in exact agreement with those of Jasmine and Gajjar.25

2. Enforced axial flow (no temperature dependence)

Figure 2 shows mean-flow solutions for a range of $T_s$ values for $\epsilon = 0$. The prominent feature of the radial velocity profile, $U(z)$, is the free stream convergence to the applied value of $T_s$ according to the boundary condition outlined in (6). The maximum value of $U(z)$ increases with increasing axial flow strength although the characteristic inflection is suppressed for higher axial flow strengths. This maximum also occurs at a moderately increased distance from the disk surface as the axial flow strength is increased. The profiles then converge to the boundary condition at a reduced distance, which is interpreted as a gentle narrowing of the
FIG. 2. Mean-flow profiles for the range $0 \leq T_s \leq 0.3$ in 0.05 increments. The black lines indicate the zero axial enforcement ($T_s = 0$) solution.

boundary layer (consistent with the observed effects of temperature dependence).

The azimuthal velocity profile, $V(z)$, moves closer to the disk surface with increasing axial flow strength. Recall that the axial flow parameter $T_s$ represents the ratio of forced flow to the rotational speed of the disk. As such, increasing $T_s$ inhibits the influence of the disk’s rotation on the flow, meaning that the fluid converges to zero rotational velocity [$V(z) \to 0$] at a reduced distance from the disk surface.

The axial flow profiles, $W(z)$, exhibit a change in shape for $T_s \neq 0$ and become unbounded with increasing the distance from the disk. While acceleration in the axial direction is to be expected with enforced axial flow, the outflow velocity should still, in reality, converge at a certain distance from the disk surface. This unphysical unbounded acceleration is instead a result of the boundary condition imposed at $U(z \to \infty)$. However, the physical element of increased outflow does support the narrowing of the boundary layer as fluid is ejected from the disk surface more rapidly. This is discussed further by Hussain et al. and, where parameter values match, the solutions presented here are in exact agreement with those presented there.

3. Combined effects

Figure 3 shows the combined effects of a temperature dependent viscosity and enforced axial flow on the mean flow over the rotating disk.

The interaction of the two effects is most clearly observed in the viscosity profiles. As the axial flow strength is increased, the viscosity profiles for the cases when $\epsilon \neq 0$ move closer to the disk surface. The change in viscous behavior then becomes localized to the region very close to the disk surface, and the viscosity converges to that of the free stream over a reduced distance. This is consistent with the narrowing of the boundary layer seen from increasing $T_s$ in Fig. 2 (where $\epsilon = 0$).

Increasing the values of $\epsilon$ and $T_s$ individually, both result in the radial velocity profile, $U(z)$, converging to the free stream boundary condition at a reduced distance from the disk surface. When $T_s \neq 0$, we see that increasing $\epsilon$ results in similar movement of the radial profile toward the disk surface. Likewise, when $\epsilon \neq 0$, the effect of increasing $T_s$ results in the profile maximum moving away from the wall, but the overall convergence of the profile occurs closer to the disk surface.
FIG. 3. Mean-flow profiles for a range of temperature dependencies and axial flow strengths. Solid lines (–) $T_s = 0$; dashed lines (−−) $T_s = 0.15$; dashed–dotted lines (−⋅) $T_s = 0.3$. Line colors represent $\epsilon = -0.75$ (blue), 0 (red), and 0.75 (yellow).

Similarly, the azimuthal profiles reveal that increasing both $T_s$ and $\epsilon$ produce effects consistent with those observed in the individual cases, i.e., increasing both parameters shifts the $V(z)$ profile toward the disk surface. In summary, the individually observed effects of increasing $\epsilon$ and $T_s$ combine to result in further narrowing of the boundary layer.

Interestingly, the effect of varying the temperature dependence on the axial and temperature profiles appears to be impacted by the strength of enforced axial flow. When $T_s = 0.15$, varying $\epsilon$ has a negligible impact on $W(z)$, while at $T_s = 0.3$, the effect of increasing $\epsilon$ is the opposite of that seen when $T_s = 0$, i.e., the outflow acceleration is increased. Similarly, $\Theta(z)$ has a negligible variation with $\epsilon$ when $T_s = 0.15$, while in the $T_s = 0.3$ case, the $\Theta(z)$ profile narrows with increasing $\epsilon$ as opposed to the broadening effect seen when $T_s = 0$.

Recalling the continuity equation, the effects on both $W(z)$ and $\Theta(z)$ can be attributed to the boundary condition for $U(z)$ when $T_s \neq 0$.

We see that the radial profile converges to zero when $T_s = 0$, while increasing $\epsilon$ moves the radial profile toward the disk surface. As such, the area under the $U(z)$ curve decreases with increasing $\epsilon$, which decreases the magnitude of $W(z)$ throughout the boundary layer. However, when $T_s \neq 0$, the radial profile converges to $T_s$, which drastically increases the area under $U(z)$ and results in the unbound acceleration observed in the $W(z)$ profile. For sufficient axial flow, the narrowing effect of increasing $\epsilon$ further increases the area under the $U(z)$ curve. Hence, the axial outflow velocity is increased throughout the boundary layer as can be observed when $T_s = 0.3$. Due to its dependence on $W(z)$, the $\Theta(z)$ profile is moved toward the disk surface when $T_s = 0.3$, narrowing the thermal boundary layer (as opposed to the thickening effect seen when varying $\epsilon$ at $T_s = 0$).

Summary data for the mean-flow profiles for a range of flow parameters are shown in Table I.

III. LINEAR CONVECTIVE INSTABILITY ANALYSIS

In this section, we derive the perturbation equations for the linear instability of the flow and proceed to present a full stability analysis under various parameter regimes.

A. Derivation of the perturbation equations

The mean-flow solutions are now subject to small perturbation quantities leading to

$$u(r, \theta, z, t) = \frac{r}{R} U(z) + \hat{u}(r, \theta, z, t),$$

$$v(r, \theta, z, t) = \frac{r}{R} V(z) + \hat{v}(r, \theta, z, t),$$

$$w(r, \theta, z, t) = \frac{1}{R} W(z) + \hat{w}(r, \theta, z, t).$$
TABLE I. Mean-flow data for a range of flow parameters.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$U_{\text{max}}$</th>
<th>$U'(0)$</th>
<th>$V'(0)$</th>
<th>$\Theta'(0)$</th>
<th>$W(z \to \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_s = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.75$</td>
<td>0.1884</td>
<td>0.2282</td>
<td>-0.2216</td>
<td>-0.3279</td>
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<tr>
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<td>-0.3334</td>
<td>-0.9951</td>
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<tr>
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<td>-0.9377</td>
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<td>-0.7511</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.2365</td>
<td>-0.2340</td>
<td>-0.3588</td>
<td>...</td>
</tr>
<tr>
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<td>-0.4117</td>
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<td>0.7136</td>
<td>-0.9168</td>
<td>-0.3844</td>
<td>...</td>
</tr>
</tbody>
</table>

| $T_s = 0.3$ |                 |         |         |              |                 |
| $-0.75$    | 0.3005          | 0.2640  | -0.2545 | -0.3991      | ...             |
| $-0.50$    | 0.3003          | 0.4187  | -0.4518 | -0.4192      | ...             |
| $-0.25$    | 0.3001          | 0.5311  | -0.6076 | -0.4309      | ...             |
| $0.00$     | 0.3000          | 0.6219  | -0.7358 | -0.4389      | ...             |
| $0.25$     | 0.3000          | 0.6997  | -0.8451 | -0.4444      | ...             |
| $0.50$     | 0.3000          | 0.7689  | -0.9409 | -0.4496      | ...             |
| $0.75$     | 0.3000          | 0.8318  | -1.0265 | -0.4536      | ...             |

| $T_s = 0.3$ |                 |         |         |              |                 |
| $\epsilon = -0.75$ |         |         |         |              |                 |
| $0.00$     | 0.1884          | 0.2282  | -0.2216 | -0.3279      | -0.3279         |
| $0.05$     | 0.1927          | 0.2288  | -0.2245 | -0.3366      | -0.3471         |
| $0.10$     | 0.2015          | 0.2316  | -0.2287 | -0.3471      | -0.3588         |
| $0.15$     | 0.2153          | 0.2365  | -0.2340 | -0.3588      | -0.3611         |
| $0.20$     | 0.2352          | 0.2436  | -0.2402 | -0.3716      | -0.3851         |
| $0.25$     | 0.2626          | 0.2528  | -0.2471 | -0.3851      | -0.3991         |
| $0.30$     | 0.3005          | 0.2640  | -0.2545 | -0.3991      | -0.4076         |

| $\epsilon = 0$ |                 |         |         |              |                 |
| $0.00$     | 0.1808          | 0.5102  | -0.6159 | -0.3286      | -0.3286         |
| $0.05$     | 0.1847          | 0.5133  | -0.6269 | -0.3447      | -0.3621         |
| $0.10$     | 0.1937          | 0.5225  | -0.6429 | -0.3621      | -0.3806         |
| $0.15$     | 0.2083          | 0.5381  | -0.6626 | -0.3806      | -0.3806         |
| $0.20$     | 0.2296          | 0.5599  | -0.6851 | -0.3997      | -0.4192         |
| $0.25$     | 0.2592          | 0.5879  | -0.7097 | -0.4192      | -0.4389         |
| $0.30$     | 0.3000          | 0.6219  | -0.7358 | -0.4389      | -0.4389         |

TABLE I. (Continued.)

<table>
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<th>$\epsilon$</th>
<th>$U_{\text{max}}$</th>
<th>$U'(0)$</th>
<th>$V'(0)$</th>
<th>$\Theta'(0)$</th>
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<td>0.1779</td>
<td>0.6753</td>
<td>-0.8444</td>
<td>-0.3135</td>
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<tr>
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<td>0.6791</td>
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<td>0.6918</td>
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<td>-0.3611</td>
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</tr>
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<tr>
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<td>0.3000</td>
<td>0.8318</td>
<td>-1.0265</td>
<td>-0.4536</td>
</tr>
</tbody>
</table>

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This system is now subject to a parallel-flow approximation, wherein all \( r \) coefficients are replaced by \( R \). This approximation is representative of a constant boundary-layer thickness across the disk surface. Physically, the rotating-disk boundary-layer does not exhibit radial growth and the terminology is borrowed from the approximation’s use in growing boundary layers, such as Blasius or Falkner–Skan flows. Following this substitution, the stability equations become separable in \( r, \theta, \) and \( t \), allowing the perturbation quantities to be expressed in normal mode form,

\[
\boldsymbol{\hat{U}}(\hat{\alpha}, \hat{v}, \hat{w}, \hat{p}, \hat{\Theta}) = (\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\Theta}) e^{(i \alpha(n+\beta) - \omega t)},
\]

where the variables marked with a tilde are the perturbation eigenfunctions dependent on \( z \). Here, \( \alpha = \alpha_{\tau} + i \alpha_{\sigma} \) is the complex radial wavenumber such that \( \alpha_{\tau} < 0 \) denotes convectively unstable disturbances that grow radially. The azimuthal wavenumber, \( n = \beta R \), is an integer quantity representing the number of vortices present on the disk, and \( \omega \) is the frequency of the disturbance. Disturbances that rotate with the disk surface are represented by \( \omega = 0 \) in this frame of reference and are henceforth referred to as stationary.

The perturbation quantities in (9) are now expressed in the normal mode form, and all \( \mathcal{O}(R^{-3}) \) terms are considered negligible,

\[
\mathcal{D}\hat{\alpha} + \frac{W\hat{\alpha}}{R} + \hat{U}\hat{\alpha} + \frac{U\hat{\alpha}}{R} = -i\alpha\hat{\alpha} + \hat{\mu}(\hat{\alpha} - \beta\hat{\alpha} + \hat{\alpha}^{'}),
\]

where

\[
\mathcal{D}\hat{\alpha} + \frac{W\hat{\alpha}}{R} + \hat{U}\hat{\alpha} + \frac{U\hat{\alpha}}{R} = -i\alpha\hat{\alpha} + \hat{\mu}(\hat{\alpha} - \beta\hat{\alpha} + \hat{\alpha}^{'}),
\]

which states the governing perturbation equations as a quadratic eigenvalue problem of the form \( A\alpha^{2} + A_{1}\alpha + A_{0} = 0 \), where \( \alpha_{1}, \alpha_{2} \) is the vector of eigenfunctions, and the quantities \( A_{1}, A_{0} \) are matrices containing the coefficients of the \( \mathcal{O}(R^{2}) \) terms. The eigenfunctions are then computed according to the boundary conditions that constrain the perturbations to reside within the boundary layer

\[
\hat{\alpha}(y = 0) = \hat{\alpha}(y = 0) = \hat{\alpha}'(y = 0) = \alpha(y = 0) = 0,
\]

\[
\hat{\alpha}(y \to \infty) = \hat{\alpha}(y \to \infty) = \hat{\alpha}'(y \to \infty) = \alpha(y \to \infty) \to 0.
\]

The neutral temporal and spatial stability solutions are obtained via a Chebyshev polynomial discretization method. An exponential map is used to transform the Gauss–Lobatto collocation points into the physical domain. The stability equations are then solved as primitive variables over 100 collocation points distributed between the upper and lower boundaries, with the exception of the conditions described in (11) which are imposed at \( z = 0 \) and \( z = z_{\text{max}} = 20 \). Further details of the numerical method employed here can be found in the work of Alveroglu,\(^{7}\) and the code used here is based on that developed by Alveroglu during that related work.

### B. Neutral stability curves

We now proceed to compute the neutral stability curves for various parameter combinations of the mean flow. In all cases, the stability of the flow will be examined by plotting neutral points (\( \alpha_{\tau} = 0 \)) in the \((R, n)\)- and \((R, \psi)\)-planes, where \( \psi = \tan^{-1}(\beta/\alpha_{\sigma}) \) is the orientation angle of spiral vortices relative to a circle concentric to the disk. Both \( n \) and \( \psi \) are physical, measurable quantities. While \( n \) must be interpreted as an integer, it is permitted to take any value during the mathematical analysis.

All neutral curves obtained are found to have a two-lobed structure: the upper representing the type I crossflow instability and the lower representing the type II viscous instability. This is consistent with all previous work on the rotating disk boundary layer, as discussed in Sec. I. Our discussions focus on the critical Reynolds number, which is typically obtained from a non-dimensional analysis.
number, $R_c$, which is the lowest Reynolds number that permits an unstable mode.

**1. Temperature-dependent viscosity (no axial flow)**

Figure 4 shows neutral stability curves for stationary disturbances ($\omega = 0$) for a range of temperature dependencies. We see that $\epsilon = -0.75$ is the most unstable case for both modes by a significant margin. Examining the type I mode, we find that increasing the temperature dependence from $\epsilon = -0.75$ results in a sharp increase in the critical Reynolds number. This stabilizing behavior continues through the gas-type regime, to the temperature independent case, and into the liquid-type regime to $\epsilon = 0.25$. However, a further increase in the value of $\epsilon$ results in a reduction in the critical Reynolds number, such that the case when $\epsilon = 0.75$ is in fact less stable than the temperature independent case. It can, therefore, be inferred that a turning point in the stabilizing behavior of the type I mode exists in the range $0 \leq \epsilon \leq 0.5$.

The type II mode exhibits similar behavior although it appears that the maximum of the type II critical Reynolds number occurs when $\epsilon = 0$. It is possible that any temperature dependence leads to destabilization of the type II mode. The type II mode is also significantly less stable for cases when $\epsilon < 0$, such that when $\epsilon = -0.5$, the two modes cannot be visually distinguished, while for the case when $\epsilon = -0.75$, the type II mode is dominant.

In spite of its non-monotonic relationship with the critical Reynolds number, increasing $\epsilon$ appears to consistently increase the critical number of vortices $n_c$, as well as the range of unstable $n$ values. We see that the critical wave angles for both modes follow the same trends with changing $\epsilon$; with the $\epsilon = -0.75$ neutral curve encompassing the largest range of unstable wave angles.

**2. Axial flow (no temperature dependence)**

Figure 5 shows the stationary mode neutral stability curves for a range of axial flow strengths for the case when $\epsilon = 0$. As $T_s$ is increased, the critical Reynolds number is significantly increased, stabilizing the type I mode. The value of $n_c$ also increases significantly, as well as the range of unstable values of $n$ enclosed by the successive curves. Conversely, the type II mode is initially
FIG. 6. Vortex-number and -angle neutral stability curves for a range of axial flow strengths and temperature dependencies. Curves show $-0.75 \leq \epsilon \leq 0.75$ in increments of 0.25 for the indicated $T_s$ value or $0 \leq T_s \leq 0.3$ in increments of 0.05 for the indicated $\epsilon$ value. The black curves are provided as a reference to the temperature independent, zero axial enforcement ($\epsilon = T_s = 0$) case.
destabilized for small values of \( T_s \), before steadily stabilizing as \( T_s \) is increased further. From this, it can be inferred that there is a minimum stability of the type II mode in the region \( 0 < T_s < 0.1 \).

For larger values of \( T_s \), although both modes are shifted along the \( R \)-axis, the type II mode becomes more prominent as the type I mode undergoes greater stabilization. It is likely that further increasing \( T_s \) will lead to the type II mode becoming the dominant form of instability.

As might be expected, the behavior of the critical wave angle follows that of \( n_r \). The range of unstable wave angles is decreased as \( T_s \) is increased although the unstable angles enclosed by the curves are significantly increased.

### 3. Combined effects

Figure 6 depicts the neutral curves for a range of temperature dependences at a constant, non-zero enforced axial flow and vice versa. We see that the effect of increasing \( \epsilon \) is consistent with that seen in Fig. 4, where increasing \( \epsilon \) results in increased flow stability limited to some positive value of \( \epsilon \), beyond which flows are less stable. At \( T_s = 0.15 \), the turning point for stabilizing behavior of increasing \( \epsilon \) appears in the range \( 0 \leq \epsilon \leq 0.5 \), which is consistent with the unenforced case (i.e., \( T_s = 0 \)). When \( T_s = 0.3 \), the stabilizing effect is extended and the point of maximum stability occurs in the range \( 0.25 \leq \epsilon \leq 0.75 \). The wave angles enclosed by the neutral curves increase significantly for all \( \epsilon \) values when \( T_s = 0.15 \), increasing further still when \( T_s = 0.3 \).

Examining the type II mode behavior, the lower branch critical Reynolds number increases as \( \epsilon \) is increased; an effect consistent with that seen in Fig. 4 for the range \( \epsilon \leq 0 \). Examining the individual curves in this range shows that a flow with \( \epsilon = -0.5 \) is type I dominant when \( T_s = 0.15 \) and \( T_s = 0.3 \), unlike for the case when \( T_s = 0 \) where there is no obvious distinction between the branches. However, for the cases when \( \epsilon > 0 \), there does not appear to be an immediately discernible relationship between the type II critical Reynolds number and the value of \( \epsilon \).

Increasing \( T_s \) yields effects consistent with that seen in Fig. 5, in that both modes are stabilized (the type I mode more significantly), the range of vortex values enclosed by the neutral curves increases, and the enclosed range of wave angles decreases.

When \( \epsilon = -0.75 \), we find that the curves are significantly less stable than those in Fig. 4, where the type II mode is dominant for all \( T_s \) values. As \( T_s \) is increased, the type I mode is appreciably stabilized. This further enforces the dominance of the type II mode caused by the temperature dependence (in spite of it also being stabilized).

When \( \epsilon = 0.75 \), we observe neutral curves very similar to those of the temperature-independent case presented in Fig. 5. The curves for values of \( T_s \geq 0.1 \) are slightly stabilized for this value of temperature dependence, while the cases when \( T_s = 0 \) and \( T_s = 0.05 \), both yield a reduced critical Reynolds number when compared to the temperature-independent problem.

Figure 7 plots a range of \( \epsilon \) values against their associated type I critical Reynolds numbers at various \( T_s \) values; each curve, therefore, represents a different fixed axial flow strength. Note that the range of \( \epsilon \) values reported on is extended beyond that used for the neutral curves. In each case, an optimal value of \( \epsilon > 0 \) (with respect to the maximum value of the critical Reynolds number) is determined such that further increases beyond this point result in flows that are progressively less stable. We find that the most stable flow is produced for an increasingly greater value of \( \epsilon \) as \( T_s \) is increased. Furthermore, the range of \( \epsilon \) values that results in flows that are more stable than the temperature independent case is increased with \( T_s \). Critical data for the range of \( \epsilon \) and \( T_s \) values considered are presented in Table II.

### C. Growth rates and \( e^N \) analysis

Using an \( e^N \) type analysis, as outlined by van Ingen,\(^{14}\) we are able to investigate the evolution of the complex radial wave number for a range of axial flow strengths and temperature dependencies. First, the amplitude of a disturbance at radial positions \( r \) and \( r + dr \) is considered,

\[
a(r) = |(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\Theta})| = |(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\Theta})|e^{-\alpha r},
\]

\[
a(r + dr) = |(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\Theta})|e^{-\alpha (r + dr)}.
\]

This leads to the ratio of amplitudes

\[
\frac{a(r + dr)}{a(r)} = e^{-\alpha dr}.
\]

Recalling the definition of the Reynolds number (8), we replace \( r \) with \( R \) and integrate from some initial position \( R_0 \) (interpreted as the smallest Reynolds number, at a fixed value of \( n_r \), for which instability is observed) to the location under consideration, \( R \). This leads to the amplitude ratio

\[
N = \ln \left( \frac{a}{a_0} \right) = - \int_{R_0}^R \alpha dR.
\]
This quantity can be computed for a range of fixed vortex numbers $n$ from which an enveloping curve can be fitted. The enveloping curve can then be used to approximate a transition region by comparison to empirical transition data. Cooper and Carpenter\textsuperscript{33} utilized this method for transition prediction of the flow over a rotating disk with a compliant surface, and the in-depth review by van Ingen\textsuperscript{14} highlights a number of comparisons between $\epsilon$ predictions and stability experiments. Malik et al.\textsuperscript{34} determined that $N = 10.7$ according to a transitional Reynolds number of 513, later updating this to be in the region $9 \leq N \leq 10$. It is noted here that modifications to the flow may result in a variation of the transitional $N$-factor although for the purposes of this qualitative study, $N = 9$ will be utilized as the reference amplitude ratio required for the prediction of transitional flows.

Figure 8 displays the enveloping $N$-curve formed from a range of amplitude ratios from the temperature independent, zero enforced flow case (i.e., $\epsilon = T_s = 0$). Here, the curve crosses at $N = 9$

![Figure 8](https://example.com/figure8.png)

**FIG. 8.** Amplitude ratio curve for $\epsilon = T_s = 0$. The dashed curves (---) represent amplitude ratio curves at fixed $n$; the solid curve is the enveloping $N$-factor curve used to determine a transitional Reynolds number.
when \( R = 521 \), showing good agreement with the results presented by Wilkinson and Malik.\(^5\) Figure 9 shows the \( N \)-curves for a range of temperature dependencies and axial flow strengths. It can be seen that by varying \( \epsilon \) for fixed \( T_s \), the temperature independent case has the largest transitional Reynolds number. It is likely that this behavior mimics that of the neutral curves, where a maximum transitional Reynolds number exists. Interestingly, for the case when \( T_s = 0.3 \), the transitional Reynolds number for flows with \( \epsilon = 0.75 \) is lower than the temperature-independent case, in spite of the latter having a larger \( R \), than the former. This indicates that the relationship between the critical and transitional Reynolds numbers is not strictly monotonic.

We note that the gradient of the \( N \)-curve is reduced with increasing \( T_s \). This suggests that a stronger axial flow reduces the disturbance growth rate and delays the onset of transition. This effect is consistent for all \( \epsilon \) values considered and is most likely the reason for the previously discussed increase in the transitional Reynolds number in spite of a reduced critical Reynolds number. Additional \( N \)-curve data are presented in Table III.

### D. Eigenfunctions

It is useful to examine the type I eigenfunctions which will be used to conduct an energy balance analysis in Sec. III E. Here, all eigenfunctions are assessed at \( R = R_t + 200 \) (i.e., well into the unstable regime), and the value of \( \alpha \) is chosen such that the most amplified disturbance, at this particular Reynolds number, is selected.

The plots depicted in Fig. 10 are the magnitudes of the perturbation eigenfunctions for \( \epsilon \) in the range \(-0.75 \leq \epsilon \leq 2 \). We see that, as \( \epsilon \) is increased, the eigenfunction profiles are narrowed. This effect can largely be attributed to a mirroring of the mean-flow profiles.

![Amplitude ratio curves for a range of \( \epsilon \) and \( T_s \) values.](image)

**FIG. 9.** Amplitude ratio curves for a range of \( \epsilon \) and \( T_s \) values. Solid lines (-) represent \( T_s = 0 \); dashed lines (--) represent \( T_s = 0.15 \); dashed–dotted lines (--) represent \( T_s = 0.3 \). Line colors blue, red, and yellow represent \( \epsilon = -0.75 \), 0, and 0.75, respectively. The black, dotted line (-----) indicates \( N = 9 \), i.e., the transitional \( N \)-factor.

A notable exception is seen where the \( |\bar{\Theta}(z)| \) profile narrows as \( \epsilon \) is increased, as opposed to the broadening of the mean-flow profile depicted in Fig. 1. We further notice that for positive values of \( \epsilon \), the two maxima of the \([\bar{u}(z)]\) profile become more prominent as \( \epsilon \) is increased and occur closer to the disk surface. Increasing \( \epsilon \) in the range \(-0.75 \leq \epsilon \leq -0.5 \) decreases the maximum of the \([\bar{v}(z)]\) profile. Increasing \( \epsilon \) beyond this leads to an increase in the maximum of the \([\bar{v}(z)]\) profile. Similarly, increasing \( \epsilon \) initially also decreases the maximum of the \([\bar{w}(z)]\) profile. The limit of this behavior occurs in the range \( 0 \leq \epsilon \leq 0.5 \), where an increase in the temperature dependence parameter thereafter results in the profile maximum increasing.

Figure 11 depicts the magnitudes of the perturbation eigenfunctions for a range of enforced axial flow strengths. We see that the velocity perturbation profiles narrow as \( T_s \) is increased, decaying closer to the disk surface. Again, this can be seen as a mirroring of the mean-flow profiles in Fig. 2. For small values of \( T_s \), the two maxima of the radial eigenfunction \([\bar{u}(z)]\) are both reduced. The maximum located further from the disk surface continues to decrease for greater values of \( T_s \), eventually becoming entirely suppressed in the case when \( T_s = 0.3 \). It is inferred that this is a direct response to the suppression of the mean radial velocity inflection seen in Fig. 2. We also observe that the maximum of the azimuthal eigenfunction is increased as the axial flow strength is increased, whereas the maximum of the \([\bar{w}(z)]\) profile is reduced.

Although not shown here, the plots for the combined effects of temperature dependence and enforced axial flow on the perturbation eigenfunctions are such that increasing \( \epsilon \) produces a narrowing effect consistent with that seen in Fig. 10 and this effect is strengthened by the presence of enforced axial flow. Enforced axial flow also acts to amplify the primary maximum of the \([\bar{u}(z)]\) profile, as well as the maxima of both the \([\bar{v}(z)]\) and \([\bar{\Theta}(z)]\) profiles. Conversely, the maximum of the axial profile is significantly reduced for all \( \epsilon \) values considered here. Other effects such as the variation in profile maxima with increasing \( \epsilon \) are largely consistent with the \( T_s = 0 \) case. Similarly, the effect of increasing \( T_s \) is largely consistent with the \( \epsilon = 0 \) case. The primary maximum of the \([\bar{u}(z)]\) profile and the maxima of the \([\bar{v}(z)], [\bar{w}(z)], \text{and } [\bar{\Theta}(z)]\) profiles are all increased for all \( T_s \) values considered.

### Table III. Transitional Reynolds numbers, \( R(N = 9) \), for a range of temperature dependencies and axial flow strengths.

<table>
<thead>
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<th>( T_s )</th>
<th>( \epsilon )</th>
<th>( R(N = 9) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>329</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>521</td>
</tr>
<tr>
<td>0</td>
<td>0.75</td>
<td>503</td>
</tr>
<tr>
<td>0.15</td>
<td>-0.75</td>
<td>500</td>
</tr>
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<td>0</td>
<td>751</td>
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<tr>
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<td>758</td>
</tr>
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<td>0</td>
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</tr>
<tr>
<td>0.3</td>
<td>0.75</td>
<td>1053</td>
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</table>
E. Energy balance analysis

It is possible to conduct an analysis of the energy balance within the system to better understand the underlying mechanisms of the flow instability. Following Cooper and Carpenter, an energy balance equation is now derived from the eigenfunctions to examine the energetic input and output of a disturbance to the mean flow.

To begin, the momentum stability Eqs. (9b)–(9d) are multiplied by their corresponding velocity component and summed

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{w}}{\partial t} + \frac{U \partial \bar{u}}{\partial r} + \frac{V \partial \bar{v}}{\partial \theta} + \frac{W \partial \bar{w}}{\partial z} + & U \left( \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial z} \right) + \frac{V}{R} \left( \frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \theta} \right) + \frac{W}{R} \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial z} \right) + U' \bar{u} \bar{w} + V' \bar{v} \bar{w} + W' \bar{w}^2 R \\
= - & \left( \frac{\partial \bar{p}}{\partial t} + \frac{\bar{v}}{R} \frac{\partial \bar{p}}{\partial \theta} + \frac{\bar{u}}{R} \frac{\partial \bar{\sigma}_d}{\partial r} \right) + \frac{\bar{p}'}{R} \left( \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial z} \right) + \frac{1}{R} \left( \frac{\partial U' \bar{u}}{\partial r} + \frac{\partial (V' \bar{v})}{\partial z} + \frac{\partial (W' \bar{w})}{\partial z} \right) + \bar{w} \left( \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial z} \right) + \bar{w} \left( \frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial z} \right) \left( \frac{\partial \bar{u}}{\partial \theta} \right) + \bar{w} \left( \frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial z} \right) \left( \frac{\partial \bar{w}}{\partial \theta} \right)
\end{align*}
\]  
(12)
Here, $\sigma_{ij}$ represents the viscous terms

$$\sigma_{11} = \frac{\partial \hat{u}}{\partial r}, \quad \sigma_{12} = \frac{1}{R} \frac{\partial \hat{u}}{\partial \theta}, \quad \sigma_{13} = \frac{\partial \hat{u}}{\partial z},$$

$$\sigma_{21} = \frac{\partial \hat{v}}{\partial r}, \quad \sigma_{22} = \frac{1}{R} \frac{\partial \hat{v}}{\partial \theta}, \quad \sigma_{23} = \frac{\partial \hat{v}}{\partial z},$$

$$\sigma_{31} = \frac{\partial \hat{w}}{\partial r}, \quad \sigma_{32} = \frac{1}{R} \frac{\partial \hat{w}}{\partial \theta}, \quad \sigma_{33} = \frac{\partial \hat{w}}{\partial z}.$$

Introducing a new kinetic energy variable, $\hat{e} = \left( \hat{u}^2 + \hat{v}^2 + \hat{w}^2 \right)/2$, leads to (12) being re-written in the following form:

$$\frac{\partial \hat{e}}{\partial t} + \frac{\partial (U\hat{e})}{\partial r} + \frac{1}{R} \frac{\partial (V\hat{e})}{\partial \theta} + \frac{1}{R} \frac{\partial (W\hat{e})}{\partial z} + \frac{U(\hat{u}^2 + \hat{v}^2)}{R} + U'\hat{w}\hat{v} + V'\hat{v}\hat{w} + W'\hat{w}^2 - \hat{e}$$

$$= \left[ \frac{\partial (\hat{u}\hat{\hat{u}})}{\partial r} + \frac{1}{R} \frac{\partial (\hat{v}\hat{\hat{v}})}{\partial \theta} + \frac{1}{R} \frac{\partial (\hat{w}\hat{\hat{w}})}{\partial z} \right] + \left[ \frac{\partial (\hat{\mu}\hat{\hat{u}})}{\partial x_1} - \hat{\mu} \frac{\partial \hat{u}}{\partial x_1} \right]$$

$$+ \left[ \frac{\partial (\hat{\mu}'\hat{\hat{u}})}{\partial r} + \frac{1}{R} \frac{\partial (\hat{\mu}'\hat{\hat{v}})}{\partial \theta} + \frac{1}{R} \frac{\partial (\hat{\mu}'\hat{\hat{w}})}{\partial z} - \hat{\mu}' \hat{w}^2 \right] + \left[ \frac{\partial (\hat{\mu}''\hat{\hat{u}})}{\partial z} - \hat{\mu}'' \hat{e} \right]$$

$$+ \left[ \frac{1}{R} \frac{\partial (\hat{U}\hat{\hat{u}})}{\partial r} + \frac{1}{R} \frac{\partial (\hat{V}\hat{\hat{w}})}{\partial \theta} + \frac{1}{R} \frac{\partial (\hat{W}\hat{\hat{w}})}{\partial z} - U'\hat{u}\hat{\hat{w}} - V'\hat{v}\hat{\hat{w}} - W'\hat{w}\hat{\hat{w}} \right]$$

$$+ \left[ \frac{1}{R} \frac{\partial (U'\hat{\hat{u}})}{\partial r} + \frac{1}{R} \frac{\partial (V'\hat{\hat{v}})}{\partial \theta} + \frac{1}{R} \frac{\partial (W'\hat{\hat{w}})}{\partial z} - U''\hat{u}\hat{\hat{w}} - V''\hat{v}\hat{\hat{w}} - W''\hat{w}\hat{\hat{w}} \right].$$

(13)
which is further averaged over a single time period and azimuthal mode (removing all derivatives with respect to $t$ and $\theta$) and integrated across the boundary layer,

$$\frac{dE}{dr} = -\left\{ \int_0^\infty U'(\hat{u} \hat{w}) + V'(\hat{v} \hat{w}) + \frac{W'}{R}(\hat{w}^2) \, dz \right\}^{\text{EPRS}}$$

$$- \frac{1}{R} \left\{ \int_0^\infty \hat{p} \left( \frac{\partial \hat{u}}{\partial \xi_1} \right) \, dz \right\}^{\text{EDV}} - \frac{1}{R} \left\{ \int_0^\infty \langle \hat{u} \hat{p} \rangle \, dz \right\}^{\text{PW}}$$

$$- \frac{1}{R} \left\{ \int_0^\infty U(\hat{u}' + \hat{\delta}^2) - W'(\hat{z}) \, dz \right\}^{\text{SC}}$$

$$- \frac{1}{R} \left\{ \int_0^\infty \mu''(\hat{\delta}^2) + W''(\hat{\mu} \hat{u}) + U'(\hat{\mu} \hat{\sigma}_{13}) + \hat{\sigma}_{13} \right\}^{\text{AVV}}$$

$$+ V'(\hat{\mu} \hat{\sigma}_{32}) + W'(\hat{\mu} \hat{\sigma}_{33}) \, dz \right\},$$

(14)

where

$$E = \int_0^\infty U(\hat{\delta}) + \langle \hat{u} \hat{p} \rangle - \frac{\hat{p}(\hat{\mu} \hat{\sigma}_{13}) + \langle \hat{\sigma}_{13} \rangle + \langle \hat{\delta} \hat{\hat{u}} \rangle + U'(\hat{\mu} \hat{w}) \, dz.$$

The terms labeled EPRS represent Energy Production terms due to Reynolds Stresses; EDV represent Energy Dissipation terms due to Viscosity; PW represent Pressure Work terms; SC are Streamline Curvature terms arising from the disk’s rotation; and AVV are Additional terms arising from the temperature dependent (Variable) Viscosity. Note that many terms from (13) disappear upon integration due to the mean flow and perturbation boundary conditions. Here, the time-averaged quantities have the form $\langle xy \rangle = x'y '+ xy ' $, where $x''$ indicates the complex conjugate of $x$. The perturbations retain the normal mode form; hence, $r$ and $z$ derivatives can be expressed as $\hat{n} \hat{u}$ and $\hat{u} \hat{z}$, respectively.

The energy contributions of each term in (14) are determined via numerical integration of the eigenfunctions and are shown in Fig. 12 for the temperature independent, zero axial flow case ($\varepsilon = T_z = 0$). We see that many of the integrated terms are negligible and a simplified energy balance equation can then be expressed to give the Total Mechanical Energy (TME) as

$$- 2 \alpha_i \approx \int_0^\infty V'(\hat{u} \hat{w}) \, dz - \frac{1}{R} \int_0^\infty \hat{p} \left( \frac{\partial \hat{u}}{\partial \xi_1} \right) \, dz,$$

(15)

where the terms have been normalized by $E$. The above returns $\alpha_i < 0$ (the criteria for an unstable mode) when EPRS > EDV. That is, a disturbance is amplified when the energy produced by the disturbance outweighs energy dissipated in response. Note that the AVV terms have not been considered here since they are equal to zero for the temperature independent (i.e., $\varepsilon = 0$) case. The relevance of the AVV terms to (15) is discussed overleaf.

**Figure 13(a)** shows the energy integral values split into the physical components for a range of temperature dependencies. The TME curve shows that both positive and negative values of $\varepsilon$ can increase the total mechanical energy of the disturbance. Interestingly, the minimum value of TME does not occur at $\varepsilon = 0$, but rather at $\varepsilon \approx 0.25$, although the TME does not change significantly in the range $0 \leq \varepsilon \leq 0.5$. The location of this minimum aligns with the $\varepsilon$ value responsible for the maximum critical Reynolds number on the $T_z = 0$ curve in Fig. 7. Further comparison between Figs. 7 and 13(a) shows an inverse proportionality between $R_c(\varepsilon)$ and the TME. That is, a reduced critical Reynolds number also results in greater disturbance amplification. It can also be inferred that the flow stability criteria at $R_c$ is consistent with that at $R_z + 200$ for all values of $\varepsilon$ considered here and as such the disturbance growth rate over this range of Reynolds numbers is unaffected by $\varepsilon$. The EDV curve shows that increasing $\varepsilon$ from $\varepsilon = -0.5$ increases the viscous dissipation associated with a disturbance. However, the gradient of the EDV curve is significantly shallower than that of the EPRS curve, leading to an overall increase in the disturbance TME. The minimum dissipation occurs when $\varepsilon = -0.5$, below this value a far greater viscous dissipation is observed. Referring back to Fig. 4, we note that the $\varepsilon = -0.5$ case lacks an inflection between the type I and II branches, leading to a type II dominant instability for the cases when $\varepsilon < -0.5$. As a result, the most unstable vortex number (at which the eigenfunction is assessed) occurs on the type II branch, which may explain the departure from the trend in the otherwise type I dominant flows. We see that the additional terms that arise when $\varepsilon \neq 0$ are mostly negligible for the range of $\varepsilon$ values considered here, only adding a small amount of energy to the system for the cases when $\varepsilon < -0.5$, demonstrating that the normalized relationship in (15) is viable without the inclusion of the variable viscosity terms. The lack of influence of these AVV terms leads to the conclusion that the variations in the energy profiles, and hence the overall flow stability, are affected by changes to the mean flow profiles, rather than the temperature perturbations.
In order to better understand the role of temperature dependent viscosity in the flow energetics, the dissipation term is manipulated to reflect a constant viscosity component ($\bar{\mu} \equiv 1$) and a temperature dependent component,

$$\frac{1}{R} \int_0^\infty (\bar{\mu} - 1) \left( \sigma_{ij} \frac{\partial \tilde{u}_i}{\partial x_j} \right) dz. \quad (16)$$

Here, the term labeled ND represents Newtonian Dissipation, and the term labeled TDD represents Temperature Dependent Dissipation. The terms established in (16) are plotted in Fig. 13(b). As $\varepsilon$ is increased, the Newtonian dissipation briefly reduces, before increasing significantly. Conversely, the dissipation associated with the temperature dependent component is seen to decrease quasi-linearly as $\varepsilon$ increases, where TDD becomes positive when $\varepsilon > 0$, acting as an energy production term and reducing the overall viscous dissipation.

Figure 14 shows the energy integral values for a range of axial flow strengths. Unlike the temperature dependent viscosity, the formulation of enforced axial flow used here does not change the stability equations from the standard von Kármán formulation. Hence, it is known that any variation in the stability of the flow is a result of variations of the mean-flow profiles. We see that as $T_s$ is increased, each of the curves moves toward zero; the energy produced by the disturbance is decreased and the viscous dissipation decreases in turn, reducing the total mechanical energy. The results seen here are consistent with the stabilizing effect of increasing $T_s$ demonstrated in Fig. 5, where higher Reynolds numbers are required to amplify a disturbance sufficiently to cause instability.

Figure 15 shows the energy contribution of each component of the energy balance equation for a range of temperature dependencies at three fixed axial flow strengths. We see that the general trend resulting from increasing $\varepsilon$ remains unchanged when enforced axial flow is applied. In the case when $T_s \approx 0.15$, each curve is notably shallower, which is consistent with the dampening effect of enforced axial flow seen in Fig. 14. The $T_s = 0.3$ curves are shallower still, where overall variation appears to be largely insignificant. However, recalling Fig. 6, we observe that the critical Reynolds number does in fact vary significantly for a range of $\varepsilon$ values when $T_s \neq 0$. As such, enforced axial flow cannot fully suppress the destabilization associated with temperature dependent viscosity flows; it is, however, helpful in damping the growth of the disturbances.
In Sec. II, the formulation and solution of the mean-flow equations were presented for a range of temperature dependent viscosities and enforced axial flow strengths. We found that negative values of the temperature dependence value $\epsilon$ result in a steady convergence of the velocity and temperature profiles to their respective boundary conditions as $z \to \infty$, indicating a broader boundary-layer thickness. Conversely, positive values of $\epsilon$ result in convergence of the profiles at a reduced distance from the disk surface and, therefore, a narrower boundary layer. The physical relation between the temperature dependency and the boundary layer thickness can be described through the viscosity at the wall and its relation to the wall shear, where changes to the viscous forces here affect the impact of the disk’s rotation on the surrounding fluid.

Increasing the axial flow strength results in a reduced convergence distance of the mean flow profiles with the exception of the axial velocity profile, which—as to be expected for forced flow in this direction—exhibits acceleration throughout the boundary layer. An enforced flow results in a gentle narrowing of the boundary layer as the influence of the disk’s rotation is diminished.

When combining the two parameters, the effects they exhibit individually upon the mean flow are generally consistent when applied simultaneously. Notable exceptions are the effects upon the axial velocity and temperature profiles [W(z) and $\Theta(z)$, respectively]. Beyond a certain axial flow strength, the effect of increasing the temperature dependence is reversed. More specifically, when $T_s = 0.3$, the axial velocity is increased with increasing $\epsilon$, and the temperature profile converges at a reduced distance (i.e., the thermal boundary layer is narrowed). This is due to the non-zero convergence of the radial velocity profile when $T_s \neq 0$, which then inhibits the characteristic profile inflection.

Having detailed the physics behind the mean-flow behavior and its variation with temperature dependence and enforced axial flow, it is pertinent to apply the same to the observed effects with regard to flow stability. In Sec. III, we observed that solutions to the stability equations can be plotted in the form of neutral stability curves, where the stability of the flows can be assessed primarily through the location of the critical Reynolds number. It was found that negative values of $\epsilon$ lead to significantly destabilized flows when compared to the temperature independent case. As $\epsilon$ is increased, the type I critical Reynolds number increases and the resulting flows are more stable. When comparing this behavior to that of the mean flow, one might conclude that the narrowing of the boundary layer is favorable to flow stability. However, increasing $\epsilon$ into the positive regime continues this stabilizing trend, up until a point where beyond which the critical Reynolds number reduces, eventually to a value less than that of the temperature independent case ($\epsilon \approx 0.6$ as detailed in Fig. 7). The type II mode appears to be destabilized for all non-zero values of $\epsilon$ considered, suggesting that any variation in the viscosity through the boundary layer will promote this instability.

The stability response to variations in axial flow strength has also been investigated. We find that the type I mode is stabilized significantly as $T_s$ is increased. Again, referring to the mean flow suggests that the narrowing of the boundary layer caused by increasing $T_s$ is favorable to the stability of the type I mode. Referring back to Fig. 2 shows that the characteristic inflection of the radial profile is inhibited as $T_s$ is increased due to the boundary condition on $U(z)$. This inflection is physically responsible for the crossflow instability; hence, if the converging velocity beyond the inflection is closer to the profile maximum, the instabilities arising from the velocity gradient should be less significant. It is also noted in Fig. 2 that, when $T_s = 0.3$, the inflection is completely suppressed. However, Hussain et al. noted that an inflection exists in the resolved velocity profile (the resolved velocity $Q = U \cos \phi + V \sin \phi$, where $\phi$ is the resolution angle from the radial profile in the direction of rotation) for all values of $T_s$, and hence a crossflow instability will always arise.

The type II behavior is more difficult to characterize physically. The eventual stabilizing effect of increasing $T_s$ can be explained as the forced flow suppressing the influence of the disk’s rotation (from which the type II instability arises). However, the small range of $T_s$ values where increasing $T_s$ is destabilizing to the type II mode falls outside of this explanation. Recall that the type II mode is viscous...
in nature; perhaps, this weakly enforced axial flow encourages the viscous interaction between the fluid and the disk.

We have also assessed the radial growth rate of disturbances for a range of temperature dependencies and axial flow strengths, utilizing them to find disturbance amplitude ratios and plotting $N$-factor curves that approximate the location of a transitional Reynolds number. We find that the lowest transitional Reynolds numbers correspond to the $\varepsilon$ values that produce the least stable flows, i.e., $\varepsilon = -0.75$. Increasing the enforced axial flow here significantly increases the transitional Reynolds number, as well as suppressing the growth rate of the $N$-factor curve. The overall behavior largely imitates that of the neutral curves. As such, the physical mechanisms that allow instabilities to form at lower Reynolds numbers are likely also responsible for the development of these instabilities into fully turbulent flow over a reduced distance. A notable result is that enforced axial flow may still delay transition in spite of the destabilizing effect of increasing $\varepsilon$ beyond its most stable value; the reduced gradient of the $N$-factor curves induced by enforced axial flow means that the development of disturbances from instability to turbulence is delayed, and therefore a reduced critical Reynolds number is not necessarily indicative of an earlier onset of transition. Experimental work is required to verify this, but the results presented here imply that, for certain cases, the radial position where crossflow vortices form could occur closer to the disk center, while the turbulent region develops further downstream.

In Sec. III E, the magnitudes of the perturbation eigenfunctions were plotted and an energy balance equation was derived from the stability equations. The total energy of the system follows the trend established between the critical Reynolds number and $\varepsilon$, where the minimum disturbance energy and the maximum critical Reynolds number approximately coincide when $\varepsilon \approx 0.25$. A particularly interesting result is that the additional disturbance terms arising from the inclusion of a temperature dependent viscosity are found to be negligible. We, therefore, conclude that the linear stability characteristics of such flows are predominantly governed by variations in the basic states. It is established in (15) that the energy production term (EPRS) is dependent on $V'(z)$ and the viscous dissipation term (EDV) is dependent on $\dot{\mu}(z; \Theta(z))$. With this in mind, recall the mean flow solutions from Sec. II. As $\varepsilon$ is increased, the gradient of the azimuthal profile in Fig. 1 increases as the profile is narrowed, leading to a significant change in the shape of the $V'(z)$ profile. This change in the $V'(z)$ profile is clearly favorable as $\varepsilon$ is increased in the range $-0.75 \leq \varepsilon \leq 0.25$ (where the upper limit is the point of maximum stability), reducing the energy production term and thereby creating a more stable flow. Perhaps beyond this, the steep velocity gradients through a narrow boundary layer render the flow more susceptible to instability.

It has already been suggested that certain mean flow behaviors—such as the suppression of the radial profile inflection—are in part responsible for the stabilizing effect. It is seen in Fig. 14 that increasing $T_r$ leads to each profile moving toward zero. Examining the $V'(z)$ profile behavior with increasing $T_r$ in Fig. 2 shows that although the absolute value of $V'(0)$ is increasing (see also Table I), the profile is also narrowed such that the area enclosed by the $V'(z)$ curve is reduced. As such, the energy produced by the disturbance is significantly reduced. Figure 15 shows that this effect is consistent for all temperature dependencies considered here.

Enforced axial flow evidently acts as a dampener to the amplification of disturbances.

Where values match, the neutral curve data presented here for varying enforced axial flow are in excellent agreement with the data presented in the work of Hussain et al. An alternative forced flow formulation is presented in the work of Chen and Mortazavi, where the mean flow of a chemical vapor deposition reactor is modeled. The forced flow parameters are expressed as modifications to the mean flow variables $U(z)$ and $W(z)$, where two constants control the ratio of rotational to axial influence on the flow. The formulation utilized by Chen and Mortazavi allows for all flows between the standard, rotation-only von Kármán flow and forced flow-only stagnation flow. The effects observed on the mean flow are consistent with those seen here, where the radial inflection is suppressed due to the imposed boundary condition. As noted by Hussain et al., the advantage of the formulation utilized in this work is that $T_r$ does not appear in the stability equations, while the formulation used by Chen and Mortazavi would modify the stability equations. However, the formulation utilized here is only suitable for enforced axial flow strengths where rotation is dominant. As such, to extend this analysis to account for weak rotational influence on the forced flow over a disk (i.e., $T_r > 1$), a formulation similar to that of Chen and Mortazavi would be more suitable.

We have limited the work presented here to stationary modes of instability that rotate with the disk surface. The most obvious extension of this work would be to investigate traveling modes. Typically, traveling modes yield higher critical Reynolds numbers with reduced amplification rates; hence, stationary modes are often the primary source of linear convective instability. This was first detailed by the visualization experiments of Gregory et al., where the existence of traveling modes was verified theoretically, but stationary vortices were observed on china clay disks. However, experimental work has since shown highlighted the importance of traveling modes over a smooth and clean disk surface.

The behavior of traveling modes in the formulation presented here can be speculated from the results of previous studies. For temperature dependent viscosity flows (without axial forcing), Jasmine and Gajar found that for all values of $\varepsilon$ considered, increasing the frequency of the disturbance is mildly stabilizing to the type I modes, while highly destabilizing with respect to the type II mode. Similarly, for flows with axial forcing (without a temperature dependent viscosity), Hussain et al. found that increasing the disturbance frequency is mildly stabilizing to the type I mode and heavily destabilizing to the type II mode. They also investigate some negative frequencies, which are indicative of a disturbance traveling slower than the disk; these modes are found to be the most amplified although it is suggested that less-amplified near-stationary frequencies are more likely to be selected due to a lower critical Reynolds number. It may, therefore, be expected that traveling modes in this formulation would most likely appear when $\varepsilon \ll 0$ and $T_r \approx 0$ as the stationary modes at these parameter values are the most amplified according to the amplitude ratio curves and integral energy analysis (Figs. 9 and 15, respectively).

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