∞-Operads & Analytic Monads

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These are notes of a mini-course given by Rune Haugseng at the 16th YaMCATS (Yorkshire and Midlands Category Theory Seminar) meeting at the University of Leeds.[1] The focus of the mini-course was the theory of ∞-operads and analytic monads.

In the first lecture, the theory of ∞-categories is introduced, while the second deals with the theory of ∞-operads. The final lecture then explores some recent work of Gepner–Haugseng–Kock on the theory of analytic monads and their relation to ∞-operads.[2]

Contents

[1]The organisers gratefully acknowledge the support of an LMS Scheme 3 grant.
[2]Notes taken and typeset by Scott Balchin, Jordan Williamson and Raffael Stenzel.

1 ∞-Categories

Question 1.1. *What are ∞-categories?*

We pose two answers to this question:

1. A homotopical approach to higher categories.
2. A good language for working with homotopical structures.

The goal of this lecture is to explore both of these answers in turn.

1.1 Higher categories and the homotopy hypothesis

An (n, k)-category is a structure with

- objects;
- morphisms;
- 2-morphisms;
- $\dots$;
- n-morphisms;

along with composition and identities, where we require that all i-morphisms are invertible for $i > k$.

Examples 1.2.

- $(0, 0)$*-categories = sets;*
- $(1, 0)$*-categories = groupoids;*
- $(1, 1)$*-categories = categories;*
- $(n, 0)$*-categories =* n*-groupoids;*
- (n, n)*-categories =* n*-categories.*

We shall regularly abuse notation, and use the term ∞-*category* to mean an $(\infty, 1)$-category.

Problem: To get interesting examples, composition can't be strictly associative. That is, we cannot require that we have an identity $h(gf) = (hg)f$, but rather we need to specify an invertible 2-morphism between these two morphisms, and then for compositions of four morphisms we must specify an invertible 3-morphism between different ways of applying these 2-morphisms, and so forth. Let us demonstrate why one might want this more copmlicated data. Let X be a topological space. We want a *fundamental* n*-groupoid* $\Pi_{\leq n}X$ with:

- objects = points of X;
- morphisms = paths in X;
- 2-morphisms = homotopies of paths
- 3-morphisms = homotopies of homotopies;
- $\dots$;

such that the n-truncation

$$\tau_{\leq n}X = \text{truncation to an } n\text{-type} = \text{killing all } \pi_i X \text{ for } i > n$$

can be recovered from $\Pi_{\leq n}X$.

However, Simpson showed it is impossible to model $\tau_{\leq 3}S^2$ using strict 3-groupoids. Therefore, we need this weaker notion of composition.

The Homotopy Hypothesis (Grothendieck)**.**

- $(n, 0)$*-categories/*n*-groupoids are the same as* n*-types.*
- $(\infty, 0)$*-categories/*∞*-groupoids are the same as homotopy types of spaces.*

Idea: Define ∞-groupoids to be topological spaces or some other model for homotopy types, and then build on that to define $(\infty, 1)$-categories.

First Guess: Take $(\infty, 1)$-categories to be topological categories (that is, categories enriched in topological spaces).

1.2 Abstract homotopy theory and simplicial sets

What is a homotopy theory? Some examples are:

- Topological spaces and weak homotopy equivalences;
- Chain complexes and quasi-isomorphisms;
- Categories and equivalences.

Informally, it should be a category with a notion of equivalence which is weaker than isomorphism.

Definition 1.3. *A* relative category *is a category* $\mathcal{C}$ *equipped with a class* $\mathcal{W}$ *of morphisms called "weak equivalences" such that* $\mathcal{W}$ *is closed under composition and contains all isomorphisms.*

Given $(\mathcal{C}, \mathcal{W})$ we can extract a homotopy category $h\mathcal{C}$ such that any functor $\mathcal{C} \to \mathcal{D}$ that takes weak equivalences to isomorphisms factors uniquely through $h\mathcal{C}$.

Problem: This loses too much information.

Let Δ be the category whose objects are non-empty finite ordered sets (or $[n] = \{0, 1, ..., n\}$), with order-preserving morphisms between these. The category of *simplicial sets* is $\mathbf{Set}_\Delta := \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$.

The functor $\Delta \to \mathbf{Top}$ given by

$$[n] \mapsto |\Delta^n| := \Big\{(x_0, ..., x_n) \mid x_i \geq 0, \sum x_i = 1\Big\} \subset \mathbb{R}^{n+1}$$

extends to a unique colimit-preserving functor $|-|\colon \mathbf{Set}_\Delta \to \mathbf{Top}$ called the *geometric realization.* A map $f\colon X \to Y$ of simplicial sets is a weak equivalence if $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence.

Theorem 1.4 (Kan-Quillen)**.** *The geometric realization* $|-|\colon \Delta \to \boldsymbol{Top}$ *induces an equivalence of homotopy categories* $h(\boldsymbol{Set}_\Delta) \xrightarrow{\sim} h(\boldsymbol{Top})$.

Theorem 1.5 (Dwyer-Kan)**.** *For a relative category* $(\mathcal{C}, \mathcal{W})$ *we can extract a simplicial category* $\mathcal{C}[\mathcal{W}^{-1}]$ *which universally inverts* $\mathcal{W}$ *up to homotopy.*

Altogether, this shows that simplicial categories can also be used as a model for $(\infty, 1)$-categories. Usually one would prefer to work with simplicial sets over topological spaces as their combinatorial structure is easier to deal with in this context.

Problem: If we have a functor of simplicial categories $F\colon \mathcal{C} \to \mathcal{D}$, and for all $c \in \mathcal{C}$ an equivalence $F(c) \to X_c$ in $\mathcal{D}$, then we would expect there to exist a new functor F' taking c to X_c, and a natural equivalence $F \Rightarrow F'$, but this is **false**. Note that by an equivalence in $\mathcal{D}$, we mean that $\mathcal{D}(X_c, d) \to \mathcal{D}(F(c), d)$ is a weak equivalence of simplicial sets for all $d \in \mathcal{D}$.

The way around this developed by Boardman-Vogt is to use *homotopy coherent diagrams*. For a pair of composable arrows $c \xrightarrow{f} c' \xrightarrow{g} c''$ we don't have $F(gf) = F(g)F(f)$ but we have an edge $F(gf) \to F(g)F(f)$ in $\mathcal{D}(c, c'')$.

And then for a triple of composable arrows $c \xrightarrow{f} c' \xrightarrow{g} c'' \xrightarrow{h} c'''$ we have two 2-simplices

$$\begin{array}{ccc} F(hgf) & \longrightarrow & F(h)F(gf) \\ \downarrow & \searrow & \downarrow \\ F(hg)F(f) & \longrightarrow & F(h)F(g)F(f) \end{array}$$

in $\mathcal{D}(c, c''')$.

Cordier defined a "coherent nerve" functor N from simplicial categories to simplicial sets and observed that a homotopy coherent diagram of shape $\mathcal{C}$ in $\mathcal{D}$ is precisely a morphism of simplicial sets $N\mathcal{C} \to N\mathcal{D}$. This suggests that we can get a better-behaved model for ∞-categories as certain simplicial sets.

1.3 Quasicategories

The simplicial set Δ^n is represented by $[n]$ i.e., $(\Delta^n)_k = \mathrm{Hom}_\Delta([k],[n])$. The *horn* $\Lambda^n_k \subset \Delta^n$ is obtained from Δ^n by removing the interior and the face opposite the vertex k.

Λ^2_0 Λ^2_1

Definition 1.6. *A simplicial set X is a* Kan complex *if*

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

for all $0 \leq k \leq n$.

Definition 1.7. *A simplicial set X is a* quasicategory *if*

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

for all $0 < k < n$.

For $\mathcal{C}$ a quasicategory, we should think of the 0-simplices as objects, the 1-simplices as morphisms and the 2-simplices as homotopies

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & \simeq & \overset{g}{\searrow} & \\ x & & \xrightarrow[h]{} & & z. \end{array}$$

In other words, this 2-simplex presents h as *a* composite of g and f.

Comparing this picture to the depiction of Λ^2_1 above, we see that the data of the inner horn filler allows us to fill in a composite of f and g. In general, quasicategories are characterized by the property that composites exist and the space of choices of composites is contractible — more precisely, a simplicial set X is a quasicategory if and only if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is a trivial fibration.

Definition 1.8. *The* nerve *functor is the functor $N\colon \mathbf{Cat} \to \mathbf{Set}_\Delta$ defined by $(N\mathcal{C})_k = Hom_{\mathbf{Cat}}([k], \mathcal{C})$.*

Lemma 1.9. *N is fully faithful, and X is in the essential image if and only if for all $\Lambda^n_k \to X$ with $0 < k < n$ there exists a* unique *filler. That is, nerves of 1-categories are the precisely the quasicategories which have a unique composition.*

Lemma 1.10 (Cordier)**.** *If $\mathcal{C}$ is a simplicial category such that $\mathcal{C}(c, c')$ are all Kan complexes, then $N_\Delta\mathcal{C}$ is a quasicategory, where N_Δ is the homotopy coherent nerve.*

Idea: (Joyal) Develop analogues of category theory in the world of quasicategories.

1.4 Fibrations

The language of fibrations gives a way of constructing functors between ∞-categories. This is important because it is generally impossible to just “write down” such functors.

Definition 1.11. *Let $F\colon \mathcal{E} \to \mathcal{B}$ be a functor of ∞-categories (if we think of $\mathcal{E}$ and $\mathcal{B}$ as quasicategories, then this is just a morphism of simplicial sets). A morphism $f\colon x \to y$ in $\mathcal{E}$ is said to be F-*cocartesian *if for every $z \in \mathcal{E}$ the square*

$$\begin{array}{ccc} Map_{\mathcal{E}}(y, z) & \xrightarrow{f^*} & Map_{\mathcal{E}}(x, z) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ Map_{\mathcal{B}}(Fy, Fz) & \xrightarrow[(Ff)^*]{} & Map_{\mathcal{B}}(Fx, Fz) \end{array}$$

is a pullback square in the ∞-category $\mathcal{S}$ of spaces.

Here, given an ∞-category $\mathcal{C}$ and objects x, y in $\mathcal{C}$, the mapping spaces in the above diagram are defined as the fibers

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \{(x, y)\} & \longrightarrow & \mathcal{C} \times \mathcal{C}. \end{array}$$

Further, “square” means a functor $\Delta^1 \times \Delta^1 \to \mathcal{S}$ of the form

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \searrow \simeq \; \simeq & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

and pullbacks have an ∞-categorical universal property of the form

$$\left\{ \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array} \right\} \simeq \{X \to Y \times_W Z\}$$

in $\mathcal{S}$.

Definition 1.12. *A* cocartesian fibration *is a fibration $F\colon \mathcal{E} \to \mathcal{B}$ such that for every object $e \in \mathcal{E}$ and every morphism $\varphi\colon Fe \to b$ there is a cocartesian morphism $\overline{\varphi}\colon e \to e'$ such that $F(\overline{\varphi}) \simeq \varphi$.*

Theorem 1.13 (Straightening Theorem, Lurie)**.** *There is an equivalence of ∞-categories*

$\{$*(Cocartesian fibrations)/$\mathcal{B}$ + functors preserving cocartesian morphisms*$\} \simeq \{$*Functors $\mathcal{B} \to$ **Cat**$_\infty$*$\}$

Example 1.14. *The codomain map $ev_1\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$ is always a cocartesian fibration. It induces a functor $\mathcal{C} \to$ **Cat**$_\infty$ given by*

$$x \mapsto \mathcal{C}/x$$
$$(x \xrightarrow{f} y) \mapsto (\mathcal{C}/x \to \mathcal{C}/y) \text{ composition with } f.$$

*If $\mathcal{C}$ has pullbacks, the map ev_1 is also a cartesian fibration. It induces a functor $\mathcal{C}^{op} \to$ **Cat**$_\infty$ given by*

$$x \mapsto \mathcal{C}/x$$
$$(x \xrightarrow{f} y) \mapsto (\mathcal{C}/y \to \mathcal{C}/x) \text{ pullback along } f.$$

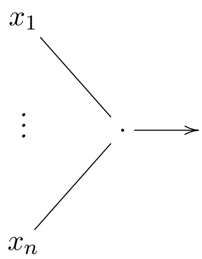
Note that the theorem and this example are stated in a fairly model independent way. However, to actually prove the theorem one has to work in some specific model (such as quasicategories). For example, to construct a functor from a cocartesian fibration we might want to start by taking certain pullbacks — but to obtain a pullback *functor* for an ∞-category we have to use unstraightening!

2 ∞-Operads

Definition 2.1. *An* operad $\mathcal{O}$ *(sometimes called a colored operad) in* ***Set*** *consists of*

- *a set of objects;*

- *for all $x_1, \dots, x_n, y$ in $\mathcal{O}$ a set $\mathcal{O}(x_1, \dots, x_n; y)$ of multimorphisms*



y equipped with a Σ_n-action permuting the inputs $(x_i)_{i \leq n}$,

- *identities $x \to x$;*
- *an equivariant, associative and unital composition;*

Many algebraic structures can be described using operads. In homotopy theory, we want notions to be invariant under homotopy $\rightsquigarrow$ ∞-operads are a homotopy-coherent analogue of operads.

Example 2.2. *Symmetric monoidal categories are homotopy-coherent commutative algebras in* ***Cat****.*

In the following, we use the description of Lurie via categories of operators as it is the most developed one.

2.1 Operads via categories of operators (May-Thomason)

Notation. *$\mathbb{F}_*$ denotes the category of finite pointed sets (or sets $\langle n \rangle = (\{*, 1, \dots, n\}, *)$). A map $f\colon \langle n \rangle \to \langle m \rangle$ is* active *if $f^{-1}(*) = \{*\}$. Moreover, we say that f is* inert *if $|f^{-1}(i)| = 1$ for all $i \neq *$.*

These two classes of maps form a factorization system on $\mathbb{F}_*$. We further have maps

$$\rho_i\colon \langle n \rangle \to \langle 1 \rangle, j \mapsto \begin{cases} 1, & j = i \\ *, & \text{otherwise.} \end{cases}$$

Given an operad $\mathcal{O}$, define a category $\mathcal{O}^{\otimes}$ over $\mathbb{F}_*$ as follows:

- objects are sequences $(\langle n \rangle \in \mathbb{F}_*, x_1, \dots, x_n \in \mathcal{O})$;
- a morphism $(\langle n \rangle, x_1, \dots, x_n) \to (\langle m \rangle, y_1, \dots, y_m)$ is given by a map $\varphi\colon \langle n \rangle \to \langle m \rangle$ in $\mathbb{F}_*$ and multimorphisms $(x_j)_{\varphi(j)=i} \to y_i$ for $i = 1, \dots, m$.

In fact, we can characterize categories over $\mathbb{F}_*$ which come from an operad in the following way.

Proposition 2.3. *A category $\mathcal{C} \xrightarrow{\pi} \mathbb{F}_*$ is equivalent to the category of operators of an operad if and only if:*

1. *for all $c \in \mathcal{C}$ and inert maps $\pi c \xrightarrow{\varphi} \langle n \rangle$ there is a π-cocartesian morphism $c \to \varphi_1 c$ over φ;*
2. *$\mathcal{C}_{\langle n \rangle} \xrightarrow{(\rho_i)_{i \leq n}} \prod_{i=1}^{n} \mathcal{C}_{\langle 1 \rangle}$ is an equivalence of categories, where $\mathcal{C}_{\langle m \rangle}$ denotes the fiber of π over $\langle m \rangle$ and the functors come from the cocartesian morphisms;*
3. *for all $\varphi\colon \langle n \rangle \to \langle m \rangle$, $x \in \mathcal{C}_{\langle x \rangle}$, $y \in \mathcal{C}_{\langle m \rangle}$ and π-cocartesian morphisms $y \to y_i$ over ρ_i the map*
$$\mathcal{C}(x, y)_{\varphi} \to \prod_{i=1}^{n} \mathcal{C}(x, y_i)_{\rho_i \varphi}$$
is an isomorphism. Here, $\mathcal{C}(x, y)_{\varphi}$ denotes the fiber at φ of $\mathcal{C}(x, y) \to \mathbb{F}_*(\langle n \rangle, \langle m \rangle)$.

Idea of proof. Condition 2 implies that objects of $\mathcal{C}$ are sequences $(x_1, \dots, x_n)$ for $x_i \in \mathcal{C}_{\langle 1 \rangle}$. Condition 3 implies that the morphisms are determined by maps $(x_1, \dots, x_n) \to y$ over active maps $\langle n \rangle \to \langle 1 \rangle$. □

If we start with a given operad $\mathcal{O}$ and consider its category $\mathcal{O}^{\otimes} \to \mathbb{F}_*$ of operators, the proof reconstructs the operad $\mathcal{O}$ that we started with. Moreover, functors of operads $\mathcal{O} \to \mathcal{P}$ correspond to diagrams

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{F} & \mathcal{P}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_* & \end{array}$$

such that F preserves cocartesian morphisms over inert maps.

Hence, we are led to define the notion of ∞-operads in the following way.

Definition 2.4. *An* ∞-operad *is a functor of ∞-categories $\mathcal{O} \xrightarrow{\pi} \mathbb{F}_*$ such that π satisfies natural analogues of Conditions* 1-3 *from Proposition 2.3. A morphism of ∞-operads is a diagram*

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{F} & \mathcal{P}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_* & \end{array}$$

such that F preserves cocartesian morphisms over inert maps. This is also called an $\mathcal{O}$-algebra in $\mathcal{P}$, so we have $\boldsymbol{Alg}_{\mathcal{O}}(\mathcal{P}) \subset \boldsymbol{Fun}_{\mathbb{F}_*}(\mathcal{O}, \mathcal{P})$, the full subcategory of $\mathcal{O}$-algebras.

2.2 Symmetric monoidal ∞-categories

Every symmetric monoidal category $\mathcal{V}$ induces an operad $\mathcal{V}$ with

$$\mathcal{V}(x_1, \dots, x_n; y) := \mathcal{V}(x_1 \otimes \cdots \otimes x_n, y).$$

Conversely, an operad $\mathcal{O}$ arises from a symmetric monoidal category in this way if and only if the category of operators $\mathcal{O}^{\otimes} \to \mathbb{F}_*$ is a cocartesian fibration (i.e., a Grothendieck opfibration).

Lemma 2.5. *Symmetric monoidal categories correspond to cocartesian fibrations $\mathcal{C} \to \mathbb{F}_*$ such that $\mathcal{C}_{\langle n \rangle} \to \prod_{i=1}^{n} \mathcal{C}_{\langle 1 \rangle}$ is an equivalence for all n.*

This motivates the following definition.

Definition 2.6. *A symmetric monoidal ∞-category is a cocartesian fibration $\mathcal{V}^{\otimes} \to \mathbb{F}_*$ such that $\mathcal{V}_{\langle n \rangle} \to \prod_{i=1}^{n} \mathcal{V}_{\langle 1 \rangle}$ is an equivalence of ∞-categories for all n.*

To suggest that this is a reasonable choice, let us recall Segal's definition of homotopy-coherent commutative monoids in ∞-categorical language:

Definition 2.7 (Segal)**.** *A commutative monoid in an ∞-category $\mathcal{C}$ with finite products is a functor $M \colon \mathbb{F}_* \to \mathcal{C}$ such that the map $M(\langle n \rangle) \xrightarrow{(M(\rho_i))_{i \le n}} \prod_{i=1}^{n} M(\langle 1 \rangle)$ is an equivalence in $\mathcal{C}$.*

Indeed, given a commutative monoid M in $\mathcal{C}$, writing $M_n := M(\langle n\rangle)$, the active map $\langle 2\rangle \to \langle 1\rangle$ yields a product $\times^M$ via $M_1 \times M_1 \xleftarrow{\sim} M_2 \to M_1$. Together with the map swap: $\langle 2\rangle \xrightarrow{\sim} \langle 2\rangle$ we obtain a commutative diagram

$$\begin{array}{ccccc} M_1 \times M_1 & \xleftarrow{\sim} & M_2 & \longrightarrow & M_1 \\ {\scriptstyle \text{swap}}\downarrow & & \downarrow & \nearrow & \\ M_1 \times M_1 & \xleftarrow{\sim} & M_2 & & \end{array}$$

in $\mathcal{C}$ that imples commutativity of the product $\times^M$. The active map $\langle 0\rangle \to \langle 1\rangle$ yields a unit $* \xleftarrow{\sim} M_0 \to M_1$ for $\times^M$.

So, for the ∞-category $\mathcal{C} = \mathbf{Cat}_\infty$, the Straightening Theorem gives an equivalence

$$\{\text{Symmetric monoidal } \infty\text{-categories}\} \simeq \{\text{Commutative monoids in } \mathbf{Cat}_\infty\}.$$

This approach yields a concise definition of symmetric monoidal category objects however complex your surrounding category theory is.

3 Analytic monads

3.1 Operads via symmetric sequences

A *symmetric sequence of sets* is a sequence $(X(0), X(1), \dots)$ of sets together with a group-action of the symmetric group Σ_n on $X(n)$. Such sequences correspond to functors $\mathbb{F}^\simeq \to \mathbf{Set}$ where $\mathbb{F}^\simeq \cong \coprod_n B\Sigma_n$ denotes the groupoid of finite sets. These functors have a composition product of the form

$$(X \circ Y)(n) = \coprod_k X(k) \times_{\Sigma_k} \left(\coprod_{i_1+\cdots+i_k=n} (Y(i_1) \times \cdots \times Y(i_k)) \times_{\Sigma_1 \times \dots \Sigma_k} \Sigma_n \right).$$

One-object operads are precisely monoids in $(\mathbf{Fun}(\mathbb{F}^\simeq, \mathbf{Set}), \circ)$: a multiplication $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ precisely corresponds to the (equivariant) operad composition maps

$$\mathcal{O}(k) \times \mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_k) \to \mathcal{O}(i_1 + \cdots + i_k).$$

Theorem 3.1 (Joyal)**.** *The left Kan extension*

$$\boldsymbol{Fun}(\mathbb{F}^\simeq, \boldsymbol{Set}) \to \boldsymbol{Fun}(\boldsymbol{Set}, \boldsymbol{Set})$$

is faithful with image the subcategory of analytic functors (via $X \mapsto \coprod_n A(n) \times_{\Sigma_n} X^n$) with weak cartesian transformations as arrows. The composition product $\circ$ is mapped to composition of endofunctors.

Corollary 3.2 (Joyal, Weber)**.** *One-object operads in Set are equivalent to analytic monads in **Set**.*

3.2 Polynomial functors

Polynomial functors are a categorification of power series, e.g., functors of the form $F\colon \mathbf{Set} \to \mathbf{Set}$, $X \mapsto \coprod_n A_n \times X^n$ are polynomial. In this example, if we let $A = \coprod_n A_n$, $E = \coprod A_n \times \underline{n}$ and $p\colon E \to A$ the projection, then we can reformulate

$$F(X) \cong \coprod_{a \in A} X^{E_a} \cong \mathrm{colim}_{a \in A} \mathrm{lim}_{Ea} X.$$

The projection $p\colon E \to A$ induces an adjoint triple

$$\begin{array}{c} \xrightarrow{\quad p_! \quad} \\ \mathbf{Set}/E \xleftarrow{\quad p^* \quad} \mathbf{Set}/A \\ \xrightarrow[\quad p_* \quad]{} \end{array}$$

such that $F \cong t_! p_* s^*$ for

$$* \xleftarrow{s} E \xrightarrow{p} A \xrightarrow{t} *.$$

In general, a polynomial functor

$$\mathbf{Set}/I \to \mathbf{Set}/J$$

is a composition $t_1 p_* s^*$ for some sequence of maps

$$I \xleftarrow{s} E \xrightarrow{p} A \xrightarrow{t} J$$

in **Set**. These are closed under composition. A good notion of morphism between polynomial functors is given by cartesian natural transformations, which correspond to diagrams of the form

$$\begin{array}{ccccc} & E & \longrightarrow & A & \\ I \swarrow & \downarrow & & \downarrow & \searrow J. \\ I \nwarrow & E' & \longrightarrow & A' & \nearrow J \end{array}$$

where the square is cartesian.

A *polynomial monad* on $\mathbf{Set}/I$ is a monad with an underlying polynomial endofunctor and a cartesian natural transformation for multiplication and unit.

- With a view towards Corollary 3.2, from polynomial functors on **Set** we only get operads with Σ-free actions.
- When you consider the projection $p\colon E \to A$ to be a fibration of groupoids with finite discrete fibers, you in fact obtain all operads (Weber).

3.3 Polynomial functors for ∞-groupoids

In the following, let $\mathcal{S}$ denote the ∞-category of spaces. A map $f\colon X \to Y$ in $\mathcal{S}$ gives an adjoint triple

$$\begin{array}{c} \xrightarrow{\quad f_! \quad} \\ \mathcal{S}/X \xleftarrow{\quad f^* \quad} \mathcal{S}/Y \\ \xrightarrow[\quad f_* \quad]{} \end{array}$$

where we have $\mathcal{S}/X \simeq \mathbf{Fun}(X, \mathcal{S})$. Hence, from a sequence

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

we get a functor

$$t_! p_* s^* \colon \mathcal{S}/J \to \mathcal{S}/J.$$

These are the *polynomial functors.*

Proposition 3.3. *A functor $F \colon \mathcal{S}/I \to \mathcal{S}/J$ is polynomial if and only if it preserves κ-filtered colimits for some κ (i.e., it is accessible) and weakly contractible limits.*

Definition 3.4. *If $p \colon E \to B$ has finite discrete fibers, then we say that the functor $t_! p_* s^*$ is* analytic.

Proposition 3.5. *A functor $F \colon \mathcal{S}/I \to \mathcal{S}/J$ is analytic if and only if it preserves sifted colimits (i.e., filtered colimits and simplicial colimits) and weakly contractible limits.*

We can show that every analytic functor $\mathcal{S} \to \mathcal{S}$ is of the form

$$X \mapsto \coprod_n B(n) \times_{\Sigma_n} X^n.$$

Let $\mathbf{An}(I, J)$ denote the ∞-category of analytic functors $\mathcal{S}/I \to \mathcal{S}/J$ with cartesian natural transformations. Then $\mathbf{An}(*, *)$ is equivalent to the ∞-category of maps $E \to B$ with finite discrete fibers and cartesian squares between them. This category has a terminal object

$$\coprod_n \underline{n}_{h\Sigma_n} \to \coprod_n B\Sigma_n,$$

since this is the universal morphism with finite discrete fibres. Thus $\coprod_n B\Sigma_n$ classifies maps $E \to B$ with discrete fibers, i.e., for every such map there is a cartesian square

$$\begin{array}{ccc} E & \longrightarrow & \coprod_n \underline{n}_{h\Sigma_n} \\ \downarrow & & \downarrow \\ B & \longrightarrow & \coprod_n B\Sigma_n \end{array}$$

in $\mathcal{S}$. Hence, by the Straightening Theorem and as $\mathbb{F}^{\simeq} \simeq \coprod_n B\Sigma_n$ holds, we have

$$\mathbf{An}(*, *) \simeq \mathcal{S}/\coprod_n B\Sigma_n \simeq \mathbf{Fun}(\mathbb{F}^{\simeq}, \mathcal{S}).$$

For $I \in \mathcal{S}$, the ∞-category $\mathbf{AnEnd}(I) := \mathbf{An}(I, I)$ has a monoidal structure $\circ$ given by composition, and we define $\mathbf{AnMnd}(I) := \mathbf{Alg}(\mathbf{AnEnd}(I))$ with respect to $\circ$.

Theorem 3.6 (Gepner-Haugseng-Kock)**.** *There is a map between cartesian fibrations*

$$\begin{array}{ccc} \boldsymbol{AnMnd} & \xrightarrow{U} & \boldsymbol{AnEnd} \\ & \searrow \quad \swarrow & \\ & \mathcal{S} & \end{array}$$

such that U preserves cartesian morphisms. Over I, the map U is given by the forgetful functor $\boldsymbol{AnMnd}(I) \to \boldsymbol{AnEnd}(I)$.

Theorem 3.7 (Gepner-Haugseng-Kock)**.** *U has a left adjoint F and the adjunction $F \dashv U$ is monadic.*

3.4 Analytic functors and trees

We can view a tree as an analytic endofunctor in the following sense. Given a tree (V,E) with V the set of vertices and E the set of edges, define $V_* := \{(v,e) \mid v \in V, e \in E \text{ is an incoming edge at } v\}$ and consider the polynomial functor

$$\begin{gathered} E \leftarrow V_* \to V \to E \\ e \mapsfrom (v,e) \mapsto e \mapsto \text{ outgoing edge.} \end{gathered}$$

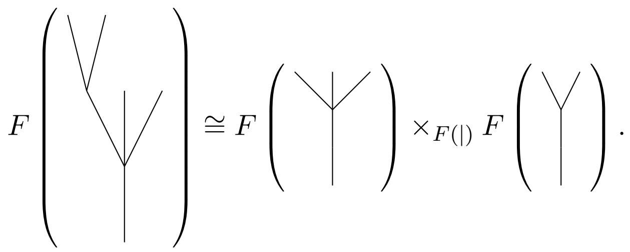
Let $\mathbb{T}$ be the full subcategory of **AnEnd** on trees and $\mathbb{E} \subseteq \mathbb{T}$ be the full subcategory on "elementary trees" η (the tree with one edge and no vertices) and C_n (the n-corolla),

$$\begin{gathered} \eta \colon * \leftarrow \emptyset \to \emptyset \to *, \\ C_n \colon \underline{n+1} \leftarrow \underline{n} \to \underline{1} \to \underline{n+1}. \end{gathered}$$

Proposition 3.8.

$$\boldsymbol{AnEnd} \simeq \boldsymbol{Psh}(\mathbb{E}) = \boldsymbol{Fun}(\mathbb{E}^{op}, \mathcal{S}).$$

Let $\mathbf{Psh}_{\mathrm{Seg}}(\mathbb{T}) \subseteq \mathbf{Psh}(\mathbb{T})$ be the full subcategory of right Kan extensions from $\mathbb{E}$. For $F \in \mathbf{Psh}_{\mathrm{Seg}}(\mathbb{T}) \subseteq \mathbf{Psh}(\mathbb{T})$ the value $F(T)$ decomposes as a limit over the corollas and edges of the tree T; for example, we have



Then the inclusion **AnEnd** $\hookrightarrow$ $\mathbf{Psh}(\mathbb{T})$ is fully faithful with image $\mathbf{Psh}_{\mathrm{Seg}}(\mathbb{T})$.

3.5 Analytic monads and trees

Theorem 3.9 (Gepner-Haugseng-Kock)**.** *Let P be a polynomial functor $I \leftarrow E \to B \to I$. Then, in the notation of Theorem 3.7, $UF(P)$ is given by*

$$I \leftarrow tr_*(P) \to tr(P) \to I$$

where

$$tr(P) = \coprod_{T \in \mathbb{T}} Map_{\boldsymbol{AnEnd}}(T,P)_{hAut(T)} \qquad \text{("trees in } P\text{")},$$

$$tr_*(P) = \coprod_{T \in \mathbb{T}} \{Leaves\ of\ T\} \times Map_{\boldsymbol{AnEnd}}(T,P)_{hAut(T)} \qquad \text{("trees in } P \text{ with marked leaf")}.$$

Let $\Omega \subseteq$ **AnMnd** be the full subcategory of free monads on trees and $j \colon \mathbb{T} \to \Omega$ be the map sending a tree to the free monad over it. We then define $\mathbf{Psh}_{\mathrm{Seg}}(\Omega)$ via the pullback

$$\begin{array}{ccc} \mathbf{Psh}_{\mathrm{Seg}}(\Omega) & \longrightarrow & \mathbf{Psh}(\Omega) \\ \downarrow & & \downarrow {\scriptstyle j^*} \\ \mathbf{Psh}(\mathbb{T}) & \longrightarrow & \mathbf{Psh}(\mathbb{T}) \end{array}$$

Theorem 3.10 (Gepner-Haugseng-Kock)**.** *The functor* $\boldsymbol{AnMnd} \to \boldsymbol{Psh}(\Omega)$ *is fully faithful with image* $\boldsymbol{Psh}_{Seg}(\Omega)$*.*

Theorem 3.11 (Kock)**.** Ω *is the dendroidal category of Moerdijk-Weiss. Hence, the objects of* $\boldsymbol{Psh}_{Seg}(\Omega)$ *are the dendroidal Segal spaces of Cisinski-Moerdijk.*

Theorem 3.12 (Barwick, Chu-Haugseng-Heuts)**.** *There is an equivalence*

$$\{\textit{Complete dendroidal Segal spaces}\} \simeq \{\textit{Lurie's } \infty\textit{-operads}\}.$$

Question 3.13. *Let* $\mathcal{O}$ *be an* ∞*-operad. Then the free* $\mathcal{O}$*-algebra monad, corresponding to the free-forgetful adjunction*

$$\boldsymbol{Alg}_{\mathcal{O}}(\mathcal{S}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \boldsymbol{Fun}(\mathcal{O}^{\simeq}_{\langle 1 \rangle}, \mathcal{S}),$$

is analytic. Could this be used to obtain a direct comparison?