Lecture 5: Weakly globular n-fold categories as a model of weak n-categories

Simona Paoli

Department of Mathematics
University of Leicester

University of Vienna

Overall Summary

The three Segal-type models and Segalic pseudo-functors

Definition: Cat^n_{hd}

Definition: Ta^n_{wg}

Definition: Cat^n_{wg}

Definition: $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

Theorem

$St : \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \to \mathsf{Cat}^n_{wg}$

Rigidification of weakly globular Tamsamani *n*-categories

Definition: LTa^n_{wg}

Theorem

$Tr_n : \mathsf{LTa}^n_{wg} \to \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

Theorem: Rigidification functor

$Q_n : \mathsf{Ta}^n_{wg} \xrightarrow{P_n} \mathsf{LTa}^n_{wg} \xrightarrow{Tr_n} \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^n_{wg}$

Weakly globular *n*-fold categories as a model of weak *n*-categories

Definition: FCat^n_{wg}

Definitions: GCat^n_{wg}, GTa^n_{wg}, GTa^n

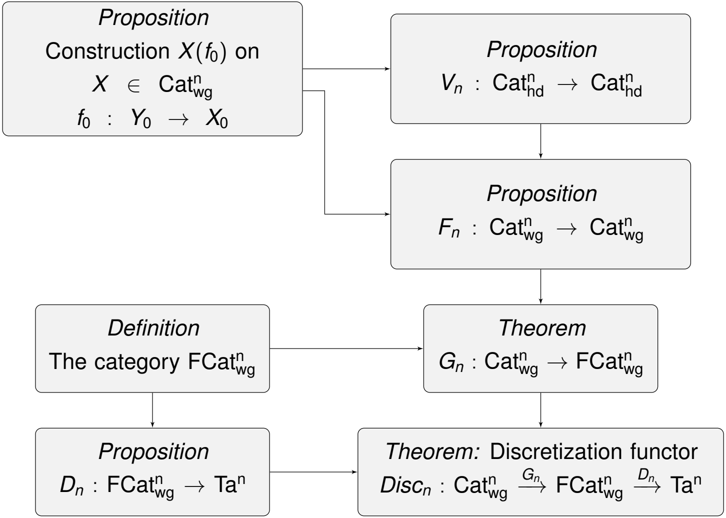
Theorem : Discretization functor

$Disc_n : \mathsf{Cat}^n_{wg} \to \mathsf{Ta}^n$

Theorem: $\mathsf{Ta}^n/\sim^n \;\simeq\; \mathsf{Cat}^n_{wg}/\sim^n$

Theorem: $\mathsf{GCat}^n_{wg}/\sim^n \;\simeq\; \mathsf{Ho}(n\text{-types})$

Construction of the discretization functor



The idea of the discretization functor.

- The idea of $Disc_n$ is to replace the homotopically discrete sub-structures in $\mathsf{Cat}^n_{\mathsf{wg}}$ by their discretizations.
- This recovers the globularity condition, but at the expenses of the Segal maps, which from being isomorphisms become $(n-1)$-equivalences.

The case $n = 2$.

- Let $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, then $X_0 \in \mathrm{Cat}_{\mathrm{hd}}$.
- Choose a section $\gamma' : X_0^d \to X_0$ of $\gamma : X_0 \to X_0^d$.

Let $D_0 X \in [\Delta^{op}, \mathrm{Cat}]$ be given by

$$\cdots \quad X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\xrightarrow{\gamma\partial_0}\\ \xrightarrow{\gamma\partial_1}\\ \xleftarrow[\sigma\gamma']{}}\; X_0^d$$

The Segal maps of $D_0 X$

$$X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

are equivalences of categories and X_0^d is discrete. Thus $D_0 X \in \mathrm{Ta}^2$.

The case $n = 2$, cont.

- Given a map $f : X \to Y$ in $\mathrm{Cat}^2_{\mathrm{wg}}$, we have a pseudo-commuting diagram

$$\begin{array}{ccc} X_0^d & \longrightarrow & Y_0^d \\ {\scriptstyle \gamma'_{X_0}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma'_{Y_0}} \\ X_0 & \longrightarrow & Y_0 \end{array}$$

  for given choices of sections γ'_{X_0}, γ'_{Y_0}.

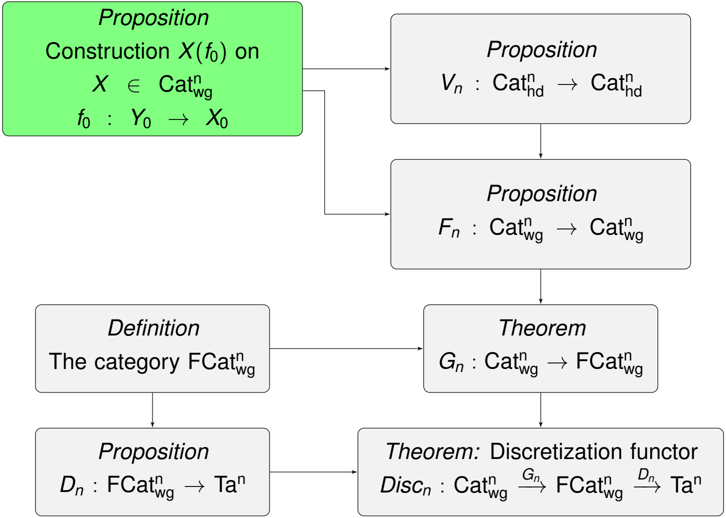
- Therefore the corresponding map $D_0 X \to D_0 Y$ is in $\mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$. That is

$$D_0 : \mathrm{Cat}^2_{\mathrm{wg}} \to (\mathrm{Ta}^2)_{\mathrm{ps}} \ .$$

Overall strategy

- To remedy this problem we introduce the category FCat^n_{wg} which exhibits functorial sections to the discretization maps of the homotopically discrete substructures in Cat^n_{wg}.
- Because of this property of FCat^n_{wg}, the discretization process can be done functorially, using an iteration of the above idea, via a functor $D_n : \mathsf{FCat}^n_{wg} \to \mathsf{Ta}^n$.
- We show that we can approximate any object of Cat^n_{wg} with an n-equivalent object of FCat^n_{wg} via a functor $G_n : \mathsf{Cat}^n_{wg} \to \mathsf{FCat}^n_{wg}$.
- *$Disc_n$* is defined as the composite $\mathsf{Cat}^n_{wg} \xrightarrow{G_n} \mathsf{FCat}^n_{wg} \xrightarrow{D_n} \mathsf{Ta}^n$.

Construction of the discretization functor



A general construction on internal categories.

$X \in \mathrm{Cat}\,\mathcal{C}$, $f_0 : Y_0 \to X_0$. Then $X(f_0) \in \mathrm{Cat}\,\mathcal{C}$ with $X(f_0)_0 = Y_0$ and $X(f_0)_k$ for $k \geq 2$ given by the pullbacks

$$\begin{array}{ccc} X(f_0)_1 & \longrightarrow & Y_0 \times Y_0 \\ \downarrow & & \downarrow{\scriptstyle f_0 \times f_0} \\ X_1 & \longrightarrow & X_0 \times X_0 \end{array} \qquad \begin{array}{ccc} X(f_0)_k & \longrightarrow & Y_0 \times \overset{k+1}{\cdots} \times Y_0 \\ \downarrow & & \downarrow{\scriptstyle f_0 \times \overset{k+1}{\cdots} \times f_0} \\ X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 & \longrightarrow & X_1 \times \overset{k+1}{\cdots} \times X_1 \end{array} \qquad k \geq 2$$

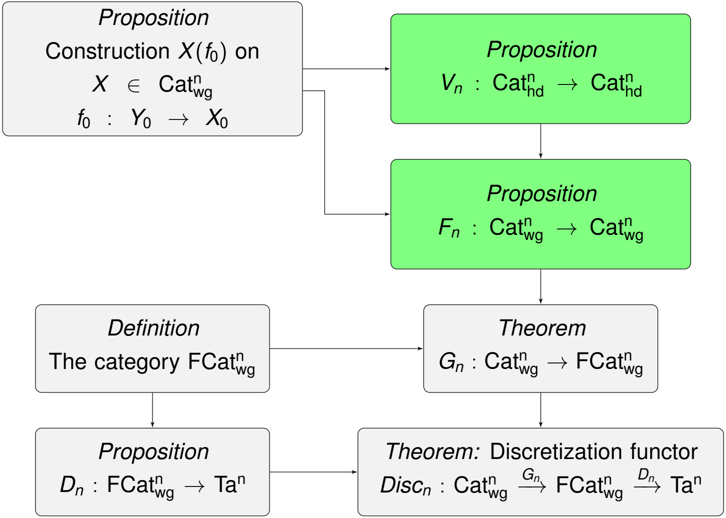
Proposition

$X \in \mathsf{Cat}^n_{wg}$, $f_0 : Y_0 \to X_0$ morphism in Cat^{n-1}_{hd} such that f_0 is a levelwise isofibration in Cat *surjective on objects. Then*

a) $X(f_0) \in \mathsf{Cat}^n_{wg}$.

b) *$V(X) : X(f_0) \to X$ is an n-equivalence.*

Construction of the discretization functor



The idea of the functor F_n

- When $n = 2$ we define $F_2 X = X(u_{X_0})$ where $u_{X_0} : \mathrm{Dec}\, X_0 \to X_0$ is an isofibration surjective on objects.
- This replaces X with a 2-equivalent $F_2 X$ in which $(F_2 X)_0 = \mathrm{Dec}\, X_0$ is homotopically discrete and has a functorial section to the discretization map.
- When $n > 1$ the definition of $F_n X$ is again based on the construction $X(f_0)$ of the previous Proposition but for a more complex choice of the map $f_0 : Y_0 \to X_0$ requiring a functor $V_n : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}} \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}}$ and a map $v_n(X) : V_n X \to X$.

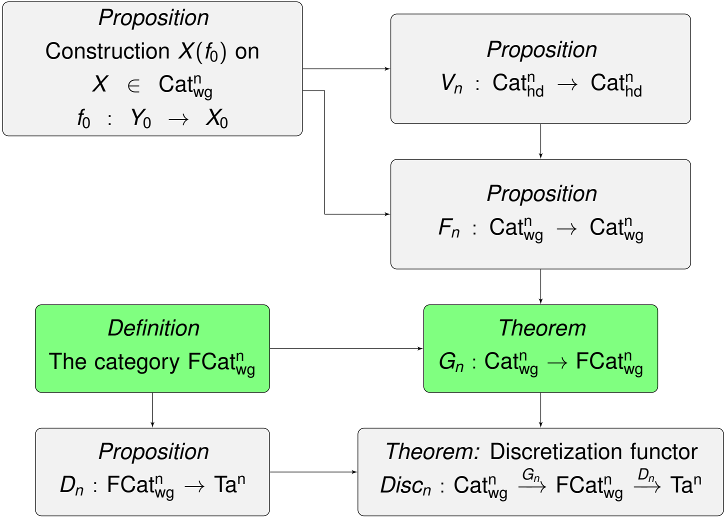
- When $n = 2$ we define $F_2 X = X(u_{X_0})$ where $u_{X_0} : \operatorname{Dec} X_0 \to X_0$.
- We define $F_n : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ by

$$F_n X = X(v_{n-1}(X_0))$$

for $v_{n-1}(X_0) : V_{n-1} X_0 \to X_0$.

- The functor $F_n : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ replaces $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ with an n-equivalent $F_n X$ where $(F_n X)_0$ admits a functorial section to the discretization map.

Construction of the discretization functor



The category $\mathsf{FCat}^2_{\mathsf{wg}}$

Let $\mathsf{FCat}^2_{\mathsf{wg}}$ have the following objects and morphisms:

- Objects of $\mathsf{FCat}^2_{\mathsf{wg}}$ are objects of $X \in \mathsf{Cat}^2_{\mathsf{wg}}$ such that the discretization map $\gamma : X_0 \to X_0^d$ has a section $\gamma\,'$ which is natural in X.

- A morphism $F : X \to Y$ in $\mathsf{FCat}^2_{\mathsf{wg}}$ is a morphism in $\mathsf{Cat}^2_{\mathsf{wg}}$ such that the following diagram commutes

$$\begin{array}{ccc} X_0 & \xrightarrow{F_0} & Y_0 \\ \uparrow{\scriptstyle \gamma\,'(X_0)} & & \uparrow{\scriptstyle \gamma\,'(Y_0)} \\ X_0^d & \xrightarrow[F_0^d]{} & Y_0^d \end{array} \qquad (1)$$

where $\gamma\,'(X_0)$ and $\gamma\,'(Y_0)$ are sections to the discretization maps.

The functor G_2

- By definition of $\mathrm{FCat}^2_{\mathrm{wg}}$, the functor $F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{Cat}^2_{\mathrm{wg}}$ is in fact a functor $F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{FCat}^2_{\mathrm{wg}}$.
- We define $G_2 = F_2$.

The idea of the functor G_n

- When $n > 2$, by definition objects of $\mathsf{FCat}^n_{\mathsf{wg}}$ are such that the homotopically discrete substructures have functorial sections to the discretization maps, as well as other functoriality properties.
- The idea of the functor G_n is to inductively apply F_s to every sub-simplicial dimension s.

Definition

Define inductively $G_n : \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \to [\Delta^{op}, \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}]$ by

- $G_2 = F_2$.
- Given $G_{n-1} : \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}} \to \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}$ let G_n be the composite

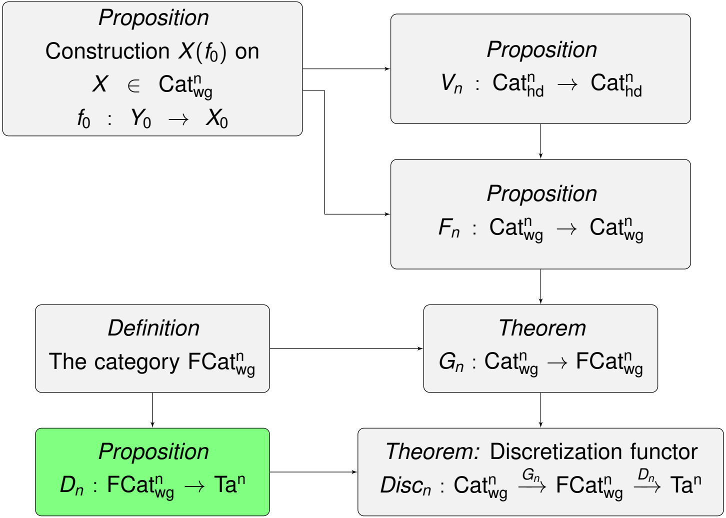
$$G_n = \overline{G}_{n-1} \circ F_n : \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \xrightarrow{F_n} \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \hookrightarrow [\Delta^{op}, \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}] \xrightarrow{G_{n-1}} [\Delta^{op}, \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}]$$

that is, $(G_n X)_k = G_{n-1}(F_n X)_k$.

Theorem

- $G_n : \mathsf{Cat}^{n}_{\mathsf{wg}} \to \mathsf{FCat}^{n}_{\mathsf{wg}}$.
- *There is an n-equivalence* $g_n(X) : G_n X \to X$ *natural in* $X \in \mathsf{Cat}^{n}_{\mathsf{wg}}$

Construction of the discretization functor



The idea of the functor D_n.

- The idea of the functor $D_n : \mathrm{FCat}^n_{\mathrm{wg}} \to \mathrm{Ta}^n$ is to replace the homotopically discrete sub-structures in $X \in \mathrm{FCat}^n_{\mathrm{wg}}$ by their discretizations, thus recovering the globularity condition.

- From the definition of $\mathrm{FCat}^n_{\mathrm{wg}}$, this can be done in a functorial way.

- This discretization process goes at the expenses of the Segal maps, which from being isomorphisms in $\mathrm{FCat}^n_{\mathrm{wg}}$ become higher categorical equivalences, so we obtain objects of Ta^n.

The functor R_0

- To define D_n we first discretize the structure at level 0 via a functor $R_0 : \mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}} \to [\Delta^{op}, \mathsf{FCat}^{\mathsf{n-1}}_{\mathsf{wg}}]$ such that $(R_0 X)_0$ is discrete for all $X \in \mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}}$.

- This is based on the following general construction of $R_0 : [\Delta^{op}, \mathcal{C}] \to [\Delta^{op}, \mathcal{C}]$.

Given a category $\mathcal{C}$ with finite limits, let $Y \in [\Delta^{op}, \mathcal{C}]$, $Y_0^d \in \mathcal{C}$ and suppose there are maps in $\mathcal{C}$

$$\gamma(Y_0) : Y_0 \to Y_0^d \qquad \gamma'(Y_0) : Y_0^d \to Y_0$$

natural in Y, such that $\gamma(Y_0)\gamma'(Y_0) = \mathrm{Id}$ and such that any morphism $F : Y \to Z$ in $[\Delta^{op}, \mathcal{C}]$ induces commuting diagrams

$$\begin{array}{ccc} Y_0 & \xrightarrow{F_0} & Z_0 \\ {\scriptstyle \gamma(Y_0)} \downarrow & & \downarrow {\scriptstyle \gamma(Z_0)} \\ Y_0^d & \xrightarrow[F_0^d]{} & Z_0^d \end{array} \qquad \begin{array}{ccc} Y_0^d & \xrightarrow{F_0^d} & Z_0^d \\ {\scriptstyle \gamma'(Y_0)} \downarrow & & \downarrow {\scriptstyle \gamma'(Z_0)} \\ Y_0 & \xrightarrow[F_0]{} & Z_0 \end{array} \qquad (2)$$

A general construction cont.

- Define $R_0 Y$ as follows:

$$(R_0 Y)_k = \begin{cases} Y_0^d, & k = 0 \\ Y_k, & k > 0 \end{cases}$$

- Let $F : Y \to Z$ be a map in $[\Delta^{op}, \mathcal{C}]$. This induces a map in $[\Delta^{op}, \mathcal{C}]$, $R_0 F : R_0 Y \to R_0 Z$.

- Thus R_0 is a functor $R_0 : [\Delta^{op}, \mathcal{C}] \to [\Delta^{op}, \mathcal{C}]$.

The functor D_n.

- We apply this construction to the case where $Y = N^{(1)}X$ with $X \in \mathrm{FCat}^{\mathrm{n}}_{\mathrm{wg}}$, $\gamma : X_0 \to X_0^d$ is the discretization map and $\gamma' = X_0^d \to X_0$ a functorial section.

- We define inductively $D_n : \mathrm{FCat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n}}$ by

$$D_2 = R_0, \qquad D_n = \overline{D}_{n-1} R_0$$

Given $X \in \mathrm{FCat}^{\mathrm{n}}_{\mathrm{wg}}$, $D_n X \in [\Delta^{op}, \mathrm{Ta}^{\mathrm{n-1}}]$ has the form

$$\cdots \quad D_{n-1}X_2 \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \ D_{n-1}X_1 \ \substack{\longrightarrow\\ \longrightarrow} \ X_0^d$$

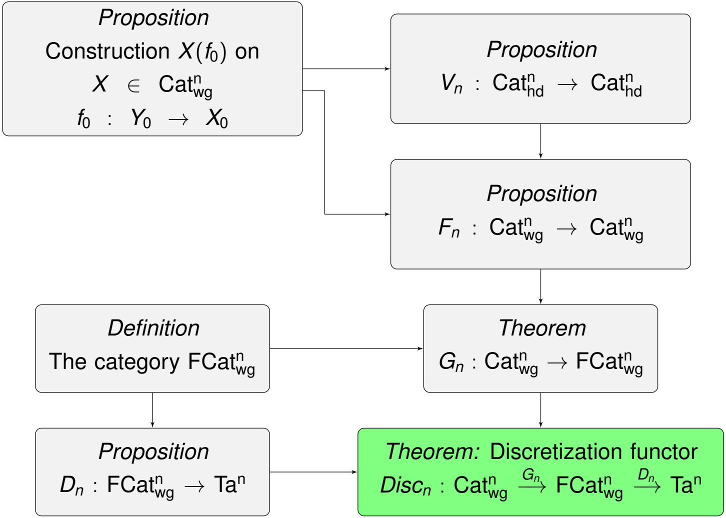
Corner of $X \in \mathsf{FCat}^3_{\mathsf{wg}} \subset [\Delta^{2^{op}}, \mathsf{Cat}]$

$$
\begin{array}{ccccc}
 & \cdots & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & X_{11} \times_{X_{10}} X_{11} & \rightrightarrows & X_{01} \times_{X_{00}} X_{01} \\
 & \Big\downarrow\Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow\Big\downarrow \\
\cdots & X_{11} \times_{X_{01}} X_{11} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & X_{11} & \rightrightarrows & X_{01} \\
 & \Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow \\
\cdots & X_{10} \times_{X_{00}} X_{10} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & X_{10} & \rightrightarrows & X_{00}
\end{array}
$$

Corner of $D_3 X \in [\Delta^{2^{op}}, \mathrm{Cat}]$ for $X \in \mathsf{FCat}^3_{wg}$

$$
\begin{array}{ccccc}
\cdots & \Rrightarrow & X_{11} \times_{X_{10}} X_{11} & \rightrightarrows & X_0^d \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots \quad X_{11} \times_{X_{01}} X_{11} & \Rrightarrow & X_{11} & \rightrightarrows & X_0^d \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\cdots \quad (X_{10} \times_{X_{00}} X_{10})^d & \Rrightarrow & X_{10}^d & \rightrightarrows & X_0^d
\end{array}
$$

Construction of the discretization functor



The discretization functor

Definition

Define the discretization functor

$$Disc_n : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n}}$$

to be the composite

$$\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{G_n} \mathrm{FCat}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{D_n} \mathrm{Ta}^{\mathrm{n}} .$$

Thus $Disc_n = \overline{D}_{n-1} R_0 \circ G_n$

Given $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$, $Disc_n X \in [\Delta^{op}, \mathrm{Ta}^{\mathrm{n-1}}]$ has the form

$$\cdots \quad D_{n-1}(G_n X)_2 \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} D_{n-1}(G_n X)_1 \rightrightarrows (G_n X)_0^d$$

Theorem

- *$Disc_n$ is identity on discrete objects and commutes with pullbacks over discrete objects.*
- *For each $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ there is a zig-zag of n-equivalences in $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ between X and $Disc_n X$.*

Proposition

For each $X \in \mathrm{FCat}^{\mathrm{n}}_{\mathrm{wg}}$, $Q_n D_n X = Q_n X$.

- When applied to $G_n X$ (for $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$) this fact implies

$$Q_n Disc_n X = Q_n D_n G_n X = Q_n G_n X.$$

- The required zig-zag is then obtained using the maps $s_n(Disc_n X)$, $s_n(G_n X)$, $g_n(X)$ as follows

$$Disc_n X \xleftarrow{s_n(Disc_n X)} Q_n Disc_n X = Q_n G_n X \xrightarrow{s_n(G_n X)} G_n X \xrightarrow{g_n(X)} X$$

$\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ as a model of weak n-categories

Theorem

$Q_n \ : \ \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}} \ \to \ \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$

Theorem

$Disc_n \ : \ \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \ \to \ \mathsf{Ta}^{\mathsf{n}}$

Theorem

$\mathsf{Ta}^{\mathsf{n}}/ \sim^n \ \simeq \ \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}/ \sim^n$

Definitions: Categories

$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}, \quad \mathsf{GTa}^{\mathsf{n}}, \quad \mathsf{GTa}^{\mathsf{n}}_{\mathsf{wg}}$

Theorem

$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/ \sim^n \simeq \mathrm{Ho}(n\text{-types})$

Corollary

$n\text{-types} \xrightarrow{\mathcal{H}_n} \mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}} \xrightarrow{j} \mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$

$\widetilde{B} \ : \ \mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}} \ \to \ n\text{-types}$

$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/ \sim^n \ \simeq \ \mathrm{Ho}(n\text{-types})$

The main comparison result

Theorem

The functors

$$Q_n : \mathsf{Ta}^{\mathsf{n}} \leftrightarrows \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} : Disc_n$$

induce an equivalence of categories after localization with respect to the n-equivalences

$$\mathsf{Ta}^{\mathsf{n}}/\sim^n \simeq \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n$$

- Let $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$. There are n-equivalences

$$Q_n Disc_n X = Q_n D_n G_n X = Q_n G_n X \to G_n X \to X \ .$$

- Hence there is an n-equivalence in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$

$$\beta_X : Q_n Disc_n X \to X \ .$$

- It follows that $Q_n Disc_n X \cong X$ in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n$.

Proof of the main comparison result, cont.

- Let $Y \in \text{Ta}^{\text{n}}$.There are n-equivalences in $\text{Ta}^{\text{n}}_{\text{wg}}$

$$Disc_n Q_n Y \xleftarrow{s_n(Disc_n Q_n Y)} Q_n Disc_n Q_n Y \xrightarrow{\beta_{Q_n Y}} Q_n Y \xrightarrow{s_n(Y)} Y\ .$$

- Composing this with the n-equivalence

$$Z = G_n Q_n Disc_n Q_n Y \xrightarrow{g_n(Q_n Disc_n Q_n Y)} Q_n Disc_n Q_n Y$$

  we obtain n-equivalences in $\text{Ta}^{\text{n}}_{\text{wg}}$ $Disc_n Q_n Y \xleftarrow{a} Z \xrightarrow{b} Y$.

- Since $Z \in \text{FCat}^{\text{n}}_{\text{wg}}$, $Disc_n Q_n Y \in \text{Ta}^{\text{n}}$ and $Y \in \text{Ta}^{\text{n}}$, we obtain a zig-zag of n-equivalences in Ta^{n} $Disc_n Q_n Y \leftarrow D_n Z \rightarrow Y$.

- It follows that $Disc_n Q_n Y \cong Y$ in $\text{Ta}^{\text{n}}/\sim^n$.

$\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ as a model of weak *n*-categories

Theorem

$Q_n \; : \; \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$

Theorem

$Disc_n \; : \; \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{Ta}^{\mathsf{n}}$

Theorem

$\mathsf{Ta}^{\mathsf{n}}/\sim^n \; \simeq \; \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n$

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Theorem

$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n \simeq \mathrm{Ho}(n\text{-types})$

Corollary

$n\text{-types} \xrightarrow{\mathcal{H}_n} \mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}} \xrightarrow{j} \mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$

$\widetilde{B} \; : \; \mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}} \to n\text{-types}$

$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n \; \simeq \; \mathrm{Ho}(n\text{-types})$

Definition

Define $\mathsf{GCat}^n_{\mathsf{wg}} \subset \mathsf{Cat}^n_{\mathsf{wg}}$ inductively

$n = 1 \quad \mathsf{GCat}^1_{\mathsf{wg}} = \mathsf{Gpd}$

Suppose we defined $\mathsf{GCat}^{n-1}_{\mathsf{wg}}$.

$X \in \mathsf{GCat}^n_{\mathsf{wg}}$ if

i) for all $a, b \in X_0^d$, $X(a,b) \in \mathsf{GCat}^{n-1}_{\mathsf{wg}}$.

ii) $p^{(n-1)}X \in \mathsf{GCat}^{n-1}_{\mathsf{wg}}$.

From the comparison theorem between $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ and Ta^{n} we obtain

Corollary

There is an equivalence of categories

$$\mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}}/\sim^n \;\simeq\; Ho(n\text{-}types)\ .$$

Note: An explicit description of the fundamental *n*-groupoid functor

$$\mathrm{Top} \to \mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}}$$

is given by [P. and Blanc].

$\mathsf{Cat}^n_{\mathsf{wg}}$ as a model of weak n-categories

Theorem

$Q_n \; : \; \mathsf{Ta}^n_{\mathsf{wg}} \to \mathsf{Cat}^n_{\mathsf{wg}}$

Theorem

$Disc_n \; : \; \mathsf{Cat}^n_{\mathsf{wg}} \to \mathsf{Ta}^n$

Theorem

$\mathsf{Ta}^n / \sim^n \; \simeq \; \mathsf{Cat}^n_{\mathsf{wg}} / \sim^n$

Definitions: Categories

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Theorem

$\mathsf{GCat}^n_{\mathsf{wg}} / \sim^n \; \simeq \; \mathrm{Ho}(n\text{-types})$

Corollary

$n\text{-types} \xrightarrow{\mathcal{H}_n} \mathsf{Gpd}^n_{\mathsf{wg}} \xrightarrow{j} \mathsf{GCat}^n_{\mathsf{wg}}$

$\widetilde{B} \; : \; \mathsf{GCat}^n_{\mathsf{wg}} \to n\text{-types}$

$\mathsf{GCat}^n_{\mathsf{wg}} / \sim^n \; \simeq \; \mathrm{Ho}(n\text{-types})$

From spaces to n-fold groupoids

- Fact: the multinerve functor $\mathcal{N}$ has a left adjoint $\mathcal{P}_n$.

Definition

We associate to a space an *n*-fold groupoid with the functor

$$\mathcal{G}_n : \text{Top} \xrightarrow{\mathcal{S}} [\Delta^{op}, \text{Set}] \xrightarrow{or_n^*} [\Delta^{n^{op}}, \text{Set}] \xrightarrow{\mathcal{P}_n} \text{Gpd}n \, .$$

where $\mathcal{S}$ is the singular functor and $or_n : \Delta^n \to \Delta$ is the ordinal sum functor.

Note: $or_n^* \mathcal{S}X$ and $\mathcal{S}X$ have the same homotopy type. Think of $or_n^* \mathcal{S}X$ as an *n*-fold resolution of $\mathcal{S}X$.

Computing $\mathcal{P}_n$

- From the previous slide:

$$\mathcal{G}_n : \text{Top} \xrightarrow{\mathcal{S}} [\Delta^{op}, \text{Set}] \xrightarrow{or_n^*} [\Delta^{n^{op}}, \text{Set}] \xrightarrow{\mathcal{P}_n} \text{Gpd}^n .$$

Proposition (Blanc and P.)

Since $\mathcal{S}X$ is fibrant, it is $\mathcal{G}_n X = \widehat{\pi}_1^{(1)} \widehat{\pi}_1^{(2)} \dots \widehat{\pi}_1^{(n)} or_n^* \mathcal{S}X$ where $\widehat{\pi}_1^{(j)}$ is the fundamental groupoid applied in the j^{th} direction.

- Fact: The functor $\mathcal{G}_n$ in fact lands in $\text{Gpd}^n_{\text{wg}} \subset \text{GCat}^n_{\text{wg}}$.

The décalage functor

- The *décalage functor* $\mathrm{Dec}\,:[\Delta^{op},\mathrm{Set}] \to Aug[\Delta^{op},\mathrm{Set}]$ is obtained by deleting the last face operator:

$$\mathrm{Dec}\,Y: \quad \cdots \quad Y_3 \overset{d_2}{\underset{d_0}{\overset{d_1}{\rightrightarrows\!\!\!\!\!\!\to}}} Y_2 \underset{d_0}{\overset{d_1}{\rightrightarrows}} Y_1 \xrightarrow{d_0} Y_0$$

- This has a right adjoint + which 'forgets the augmentation'. Hence $\mathrm{Dec}^{+}Y$ is the simplicial set:

$$\mathrm{Dec}^{+}Y: \quad \cdots \quad Y_3 \overset{d_2}{\underset{d_0}{\overset{d_1}{\rightrightarrows\!\!\!\!\!\!\to}}} Y_2 \underset{d_0}{\overset{d_1}{\rightrightarrows}} Y_1$$

- $\mathrm{Dec}^{+}Y$ is weakly equivalent to the constant simplicial set on Y_0.

The bisimplicial set $or_2^* X$

- The bisimplicial set $or_2^* Y$, also called *"total Dec"* is given by the comonad resolution of Y for the comonad associated to the adjoint pair $(\mathrm{Dec}\,, +)$.

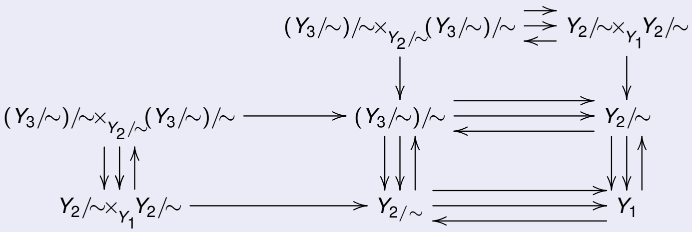
$$
\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots \\
\cdots & Y_5 & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & Y_4 & \substack{\longrightarrow\\\longrightarrow} & Y_3 \\
 & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots & Y_4 & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & Y_3 & \substack{\longrightarrow\\\longrightarrow} & Y_2 \\
 & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\cdots & Y_3 & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & Y_2 & \substack{\longrightarrow\\\longrightarrow} & Y_1
\end{array}
$$

- Note: each row/column of $or_2^* Y$ is simplicially contractible.

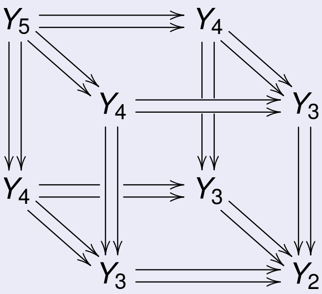
The fundamental weakly globular double groupoid of a space

Space X, $Y = \mathcal{S}X$. The *fundamental weakly globular double groupoid*

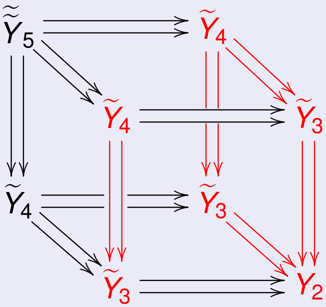
$$\mathcal{G}_2 X = \mathcal{P}_2 or_2^* Y = \widehat{\pi}_1^{(1)} \widehat{\pi}_1^{(2)} or_2^* Y$$



Space X, $SX = Y$



Corner of $or_3^* Y$



$\mathcal{G}_3 X = \hat{\pi}_1^{(1)} \hat{\pi}_1^{(2)} \hat{\pi}_1^{(3)} or_3^* Y$

The structures in red are homotopically discrete.