# Lecture 3: Rigidification of weakly globular Tamsamani $n$-categories

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# Overall Summary

**The three Segal-type models and Segalic pseudo-functors**

*Definition:* $\mathrm{Cat}^n_{hd}$
*Definition:* $\mathrm{Ta}^n_{wg}$
*Definition:* $\mathrm{Cat}^n_{wg}$
*Definition:* $\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

*Theorem*
$St : \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^n_{wg}$

**Rigidification of weakly globular Tamsamani *n*-categories**

*Definition:* $\mathrm{LTa}^n_{wg}$

*Theorem*
$Tr_n : \mathrm{LTa}^n_{wg} \to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

*Theorem:* Rigidification functor
$Q_n : \mathrm{Ta}^n_{wg} \xrightarrow{P_n} \mathrm{LTa}^n_{wg} \xrightarrow{Tr_n}$
$\to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^n_{wg}$

**Weakly globular *n*-fold categories as a model of weak *n*-categories**

*Definition:* $\mathrm{FCat}^n_{wg}$
*Definitions*: $\mathrm{GCat}^n_{wg}$, $\mathrm{GTa}^n_{wg}$, $\mathrm{GTa}^n$

*Theorem* : Discretization functor
$Disc_n : \mathrm{Cat}^n_{wg} \to \mathrm{Ta}^n$

*Theorem*: $\mathrm{Ta}^n / \sim^n \; \simeq \; \mathrm{Cat}^n_{wg} / \sim^n$

*Theorem:* $\mathrm{GCat}^n_{wg} / \sim^n \; \simeq \; \mathrm{Ho}(n\text{-types})$

# The construction of the rigidification functor $Q_n$

*Definitions:* Categories
$\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$, $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

*Propositions:* Functors
$q^{(n-1)} : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$
$q^{(n-1)} : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$
and their properties

*Definition:* The category $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:* Approximating $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ with $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:*
$Tr_n : \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

*Theorem:*
$St : \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:* Rigidification functor
$Q_2 : \mathrm{Ta}^{2}_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{2}_{\mathrm{wg}}$

For $n > 2$

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$$

We identified a subcategory

$$\mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \subset \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}]$$

of Segalic pseudo-functors such that $St$ restricts to

$$\mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \subset [\Delta^{{n-1}^{op}}, \mathrm{Cat}].$$

The rigidification functor factors as

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

# Building pseudo-functors

- There is a general technique to build pseudo-functors called transport of structure along an adjunction.

Given a functor $G \in [\mathcal{C}, \mathrm{Cat}]$ and adjoint equivalences of categories

$$G(c) \simeq F(c)$$

for all $c \in ob(\mathcal{C})$, then $F$ can be lifted to a pseudo-functor

$$F \in \mathrm{Ps}[\mathcal{C}, \mathrm{Cat}]$$

and there is a pseudo-natural transformation from $G$ to $F$.

# Methodology

- We can use the above technique in a straighforward way to produce pseudo-functors from $\mathrm{Ta}^2_{\mathrm{wg}}$.
- When $n > 2$, the above technique cannot be applied directly to $\mathrm{Ta}^n_{\mathrm{wg}}$ and we need the intermediate category $\mathrm{LTa}^n_{\mathrm{wg}}$ to produce pseudo-functors using transport of structure.
- This construction is one of the main divides in the theory between the case $n = 2$ and $n > 2$.

# From $\mathrm{Ta}^2_{\mathrm{wg}}$ to pseudo-functors.

- Recall $X \in \mathrm{Ta}^2_{\mathrm{wg}}$ if $X \in [\Delta^{op}, \mathrm{Cat}]$ is such that

$$X_0 \in \mathrm{Cat}_{\mathrm{hd}}, \qquad X_k \simeq X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \quad k \geq 2.$$

- Let

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1 \end{cases}$$

Then $X_k \simeq (Tr_2 X)_k$ for all $k$.

By transport of structure $Tr_2 X \in [ob(\Delta^{op}), \mathrm{Cat}]$ lifts to a pseudo-functor $Tr_2 X \in \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$, which is Segalic.

# The rigidification functor, case $n = 2$

### Definition

Let $Q_2$ be the composite

$$Q_2 : \mathrm{Ta}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}$$

# From $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ to pseudo-functors.

- The case $n > 2$ is more complex, since the induced Segal maps of $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ are $(n-1)$-equivalences but not, in general, levelwise equivalences of categories.

We identify a subcategory $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$ and functors

$$\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \; .$$

# The rigidification functor, case $n > 2$

## Definition

Define $Q_n$ for $n > 2$ to be the composite

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

# The construction of the rigidification functor $Q_n$

*Definitions:* Categories
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$q^{(n-1)} : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$
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*Definition:* The category $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$

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*Theorem:*
$Tr_n : \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

*Theorem:*
$St : \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:* Rigidification functor
$Q_2 : \mathrm{Ta}^{2}_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{2}_{\mathrm{wg}}$
For $n > 2$
$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

# Notation

- If $\{r\} \in \Sigma_n$ denotes the $r$-cycle $(1, 2, \ldots, r)$ (for $1 \leq r \leq n$) denote, for each $(k_1, \ldots, k_n) \in \Delta^{n^{op}}$

$$X^{\{r\}}_{k_1 \ldots k_n} = \begin{cases} X_{k_2 k_3 \ldots k_r k_1 k_{r+1} \ldots k_n}, & \text{if} \quad 1 \leq r < n \\ X_{k_2 k_3 \ldots k_{n-1} k_1}. & \text{if} \quad r = n\,. \end{cases} \qquad (1)$$

- Note that $X^{\{1\}} = X$ .

# Notation, cont.

- Given $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ and $1 \leq r < n$, $X_k^{\{r\}} \in [\Delta^{n-2^{op}}, \mathrm{Cat}]$.
- There is a map $X_0^{\{r\}} \to p^{(n-2)} X_0^{\{r\}}$ in $[\Delta^{n-2^{op}}, \mathrm{Cat}]$ and therefore a corresponding induced Segal map in $[\Delta^{n-2^{op}}, \mathrm{Cat}]$ for $k \geq 2$

$$v_k^{\{r\}} : X_k^{\{r\}} \to X_1^{\{r\}} \times_{p^{(n-2)} X_0^{\{r\}}} \overset{k}{\cdots} \times_{p^{(n-2)} X_0^{\{r\}}} X_1^{\{r\}} \ .$$

## Lemma

*Let* $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$. *Then*

a) $p^{(n-1)}X \in \mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$.

b) *For all* $1 \leq r < n$ *the induced Segal map in* $[\Delta^{n-2^{op}}, \mathrm{Cat}]$ *for* $k \geq 2$

$$v_k^{\{r\}} : X_k^{\{r\}} \to X_1^{\{r\}} \times_{p^{(n-2)}X_0^{\{r\}}} \overset{k}{\cdots} \times_{p^{(n-2)}X_0^{\{r\}}} X_1^{\{r\}}$$

*is a levelwise equivalence of categories.*

## Sketch of proof

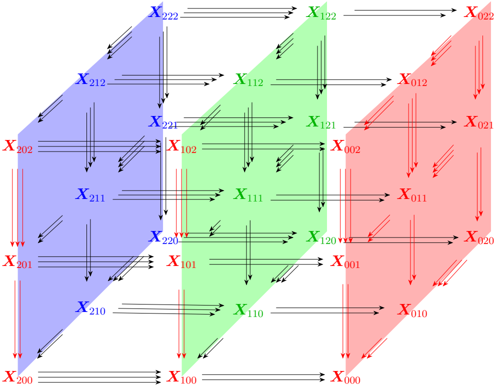
- Condition b) in this lemma is equivalent to requiring that, if $\underline{k} = (k_2, \ldots, k_{n-1}) \in \Delta^{n-2^{op}}$, then the functor $X^{\{r\}}(\text{-}, \underline{k}) : \Delta^{op} \to \mathsf{Cat}$ is in $\mathsf{Cat}^2_{\mathsf{wg}}$.

- This holds since $X^{\{r\}}(\text{-}, \underline{k}) \in \mathsf{Cat}^2$, $X^{\{r\}}(0, \underline{k}) \in \mathsf{Cat}_{\mathsf{hd}}$ and $p^{(1)} X^{\{r\}}(\text{-}, \underline{k}) \in \mathsf{Cat}$.

# Corner of the multinerve of a weakly globular 3-fold category

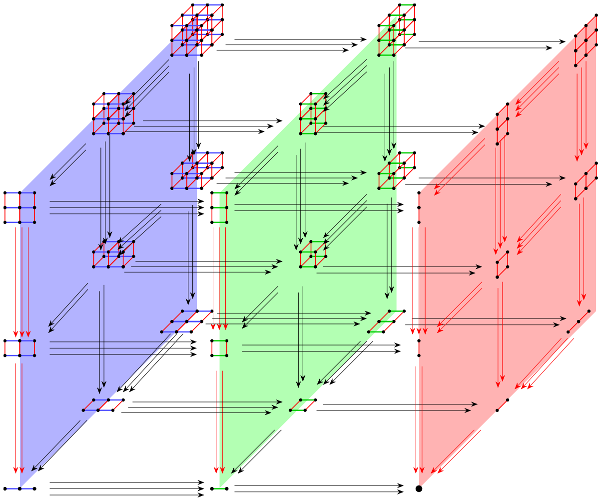
In the following picture, for all $i, j, k \in \Delta^{op}$

$X_{2jk} \cong X_{1jk} \times_{X_{0jk}} X_{1jk}$, $X_{i2k} \cong X_{i1k} \times_{X_{i0k}} X_{i1k}$, $X_{ij2} \cong X_{ij1} \times_{X_{ij0}} X_{ij1}$ .




Fact: $X_{*k*} \in \mathsf{Cat}^2_{\mathrm{wg}}$ for all $k$.

# Geometric picture of the corner of the multinerve of a weakly globular 3-fold category



# The idea of the category $\mathrm{LTa}^n_{wg}$

- The idea of the category $\mathrm{LTa}^n_{wg}$ is to produce a generalization of the category $\mathrm{Cat}^n_{wg}$ so that the properties a) and b) hold and there are embeddings

$$\mathrm{Cat}^n_{wg} \subset \mathrm{LTa}^n_{wg} \subset \mathrm{Ta}^n_{wg} \ .$$

- The properties a) and b) are the key to construct Segalic pseudo-functors from the category $\mathrm{LTa}^n_{wg}$.

### Definition

$\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$ is the full subcategory of $\mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$ whose objects $X$ are such that

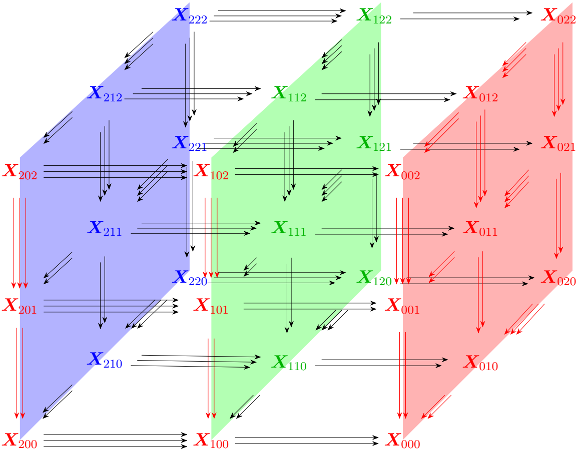
a) $p^{(n-1)}X \in \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}$.

b) For all $1 \leq r < n$ the induced Segal map in $[\Delta^{n-2^{op}}, \mathsf{Cat}]$ for $k \geq 2$

$$v_k^{\{r\}} : X_k^{\{r\}} \to X_1^{\{r\}} \times_{p^{(n-2)}X_0^{\{r\}}} \overset{k}{\cdots} \times_{p^{(n-2)}X_0^{\{r\}}} X_1^{\{r\}}$$

is a levelwise equivalence of categories.

In the following picture, $X_{*k*} \in \mathsf{Ta}^2_{\mathsf{wg}}$ for all $k$.



# Remarks

- We have $\mathrm{LTa}^{n}_{\mathrm{wg}} = \mathrm{Ta}^{n}_{\mathrm{wg}}$ for $n = 0, 1, 2$.
- Condition b) can be re-written as stating that, for all $\underline{k} = (k_1, \ldots, k_{n-1}) \in \Delta^{n-1^{op}}$, $1 \leq r \leq n-1$, the induced Segal maps in Cat for $k_r \geq 2$

$$X_{k_r} \to X_{\underline{k}(1,r)} \times_{X^d_{\underline{k}(0,r)}} \overset{k_r}{\cdots} \times_{X^d_{\underline{k}(0,r)}} X_{\underline{k}(1,r)}$$

are equivalences of categories, where

$\underline{k} = (k_1, \ldots, k_n) \in \Delta^{n^{op}}$, $1 \leq i \leq n$

$\underline{k}(1, i) = (k_1, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_n)$

$\underline{k}(0, i) = (k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n)$

# Properties of the category $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$

The following is an inductive description of the category $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$

## Proposition

*Let $X \in \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$. Then $X \in \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$ if and only if*

a) $p^{(n-1)}X \in \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}$.

b) *For each $k \geq 2$ the induced Segal map in $[\Delta^{n-2^{op}}, \mathsf{Cat}]$*

$$v_k : X_k \to X_1 \times_{p^{(n-2)}X_0} \overset{k}{\cdots} \times_{p^{(n-2)}X_0} X_1$$

*is a levelwise equivalence of categories.*

c) *For all $k \in \Delta^{op}$, $X_k \in \mathsf{LTa}^{\mathsf{n-1}}_{\mathsf{wg}}$.*

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$\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

# From $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$ to pseudo-functors

**Theorem**

*There is a functor*

$$Tr_n : \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$$

*together with a pseudo-natural transformation*

$$t_n(X) : Tr_n X \to X$$

*for each* $X \in \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$ *which is a levelwise equivalence of categories.*

The construction of the functor $Tr_n$ uses the defining properties a) and b) of $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$.

# The idea of the functor $Tr_n$

We build a diagram $Tr_n X \in [ob(\Delta^{{n-1}^{op}}), \mathsf{Cat}]$ in which

- For all $\underline{k} \in \Delta^{{n-1}^{op}}$, $1 \leq i \leq n-1$

$$(Tr_n X)_0^{\{i\}} = (X_0^{\{i\}})^d$$

  is discrete .

- $(Tr_n X)_{\underset{1 \ldots 1}{n-1}} = X_{\underset{1 \ldots 1}{n-1}}$ .

- For $k_1 \geq 2$, $\underline{s} = (k_2, \ldots, k_{n-1})$, $\underline{k} = (k_1, \underline{s})$,

$$(Tr_n X)_{\underline{k}} = (Tr_{n-1} X_1)_{\underline{s}} \times_{X^d_{\underline{k}(0,1)}} \overset{k_1}{\cdots} \times_{X^d_{\underline{k}(0,1)}} (Tr_{n-1} X_1)_{\underline{s}}$$

# The functor $Tr_3$.

- When $n = 3$ we set

$$
\begin{aligned}
(Tr_3 X)_{k_1 k_2} &= X^d_{k_1 k_2} \quad \text{if } k_1 = 0 \text{ or } k_2 = 0 \\
(Tr_3 X)_{11} &= X_{11} \\
(Tr_3 X)_{k_1 1} &= X_{11} \times_{X^d_{01}} \overset{k_1}{\cdots} \times_{X^d_{01}} X_{11} \quad \text{if } k_1 \geq 2 \\
(Tr_3 X)_{1 k_2} &= X_{11} \times_{X^d_{10}} \overset{k_2}{\cdots} \times_{X^d_{10}} X_{11} \quad \text{if } k_2 \geq 2\,.
\end{aligned}
$$

- If both $k_1 \geq 2$ and $k_2 \geq 2$, we set

$$(Tr_3 X)_{k_1 k_2} = (Tr_2 X_1)_{1k_2} \times_{X^d_{0k_2}} \overset{k_1}{\cdots} \times_{X^d_{0k_2}} (Tr_2 X_1)_{1k_2} =$$

$$= (X_{11} \times_{X^d_{10}} \overset{k_2}{\cdots} \times_{X^d_{10}} X_{11}) \times_{(X^d_{01} \times_{X^d_{00}} \overset{k_2}{\cdots} \times_{X^d_{00}} X^d_{01})} \overset{k_1}{\cdots}$$

$$\overset{k_1}{\cdots} \times_{(X^d_{01} \times_{X^d_{00}} \overset{k_2}{\cdots} \times_{X^d_{00}} X^d_{01})} (X_{11} \times_{X^d_{10}} \overset{k_2}{\cdots} \times_{X^d_{10}} X_{11})$$

where we used the fact that, since $X_0 \in \mathrm{Cat}^2_{\mathrm{hd}}$,

$$X^d_{0k_2} \cong X^d_{01} \times_{X^d_{00}} \overset{k_2}{\cdots} \times_{X^d_{00}} X^d_{01} \; .$$

# The idea of the functor $Tr_n$, cont.

- We show that, for each $\underline{k} \in \Delta^{{n-1}^{op}}$, there is an equivalence of categories $(Tr_n X)_{\underline{k}} \simeq X_{\underline{k}}$
- Using 'transport of structure', we lift $Tr_n X \in [ob(\Delta^{{n-1}^{op}}), \mathrm{Cat}]$ to a pseudo-functor $Tr_n X \in \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}]$.
- We show that this is in fact a Segalic pseudo-functor.

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$\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{n}_{\mathrm{wg}}$

# The functor $q^{(n)}$

## Proposition

- *The functor* $q^{(n-1)} : [\Delta^{n^{op}}, \text{Set}] \to [\Delta^{n-1^{op}}, \text{Set}]$ *restricts to a functor* $q^{(n-1)} : \text{Ta}^{\text{n}}_{\text{wg}} \to \text{Ta}^{\text{n}-1}_{\text{wg}}$.
- *For each* $X \in \text{Ta}^{\text{n}}_{\text{wg}}$, *there is a map natural in* $X$

$$\gamma^{(n-1)} : X \to q^{(n-1)} X \, .$$

Think of $q^{(n)}$ as a 'categorical Postnikov functor'.

# Properties of the functor $q^{(n)}$

## Proposition

- *$q^{(n-1)}$ sends n-equivalences to $(n-1)$-equivalences and preserves pullbacks over discrete objects.*
- *If $X \in \mathsf{Cat}^{\mathrm{n}}_{\mathrm{hd}}$, then $q^{(n-1)}X = p^{(n-1)}X$.*

# Strategy in building the rigidification functor $Q_n$

- We show that, if $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ is such that $q^{(n-1)}X$ can be approximated up to $(n-1)$-equivalence with an object of $\mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$, then $X$ can be approximated up to an $n$-equivalence with an object of $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$.

- This property is used to construct the functor $P_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$ from which the rigidification functor $Q_n$ is built as composite

$$\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

### Proposition

*Let*

$$A \xrightarrow{f} q^{(n-1)}C \xleftarrow{\gamma^{(n-1)}} C$$

*be a diagram in* $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ *where* $f : A \to q^{(n-1)}C$ *is a morphism in* $\mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$ *and consider the pullback in* $[\Delta^{n-1^{op}}, \mathrm{Cat}]$

$$\begin{array}{ccc} P & \xrightarrow{w} & C \\ \downarrow & & \downarrow \gamma_C^{(n-1)} \\ A & \xrightarrow[f]{} & q^{(n-1)}C \end{array}$$

a) *Then* $P \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ *and if* $f$ *is an* $(n-1)$*-equivalence,* $P \xrightarrow{w} C$ *is an* $n$*-equivalence.*

# The main construction to approximate $\mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$ with $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$

- The basic construction is the pullback in $[\Delta^{{n-1}^{op}}, \mathsf{Cat}]$

$$\begin{array}{ccc} P & \xrightarrow{\ w\ } & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \gamma^{(n-1)}} \\ Z & \xrightarrow[\ r\ ]{} & q^{(n-1)}X \end{array}$$

with $X \in \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$, $Z \in \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}$ and $r$ an $(n-1)$-equivalence.

- The goal is to show that $P \in \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$ and $w$ is an $n$-equivalence.

## Proposition

*Let* $f : X \to Y$ *be a morphism in* $\mathsf{Ta}^n_{\mathsf{wg}}$ *with* $n \geq 2$*, such that*

- $f$ *is a n-equivalence,*
- $p^{(n-2)} X_0 \cong p^{(n-2)} Y_0$*,*
- *For each* $1 \leq r < n-1$ *and all* $k_1, \dots, k_r \geq 0$*,*

$$p^{(n-r-2)} X_{k_1,\dots,k_r,0} \cong p^{(n-r-2)} Y_{k_1,\dots,k_r,0}\ .$$

*Then* $f$ *is a levelwise equivalence of categories.*

We use this to check condition b) in the definition of $\mathsf{LTa}^n_{\mathsf{wg}}$.

- From the properties of $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ discussed earlier, $P \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$.
- To prove that $P \in \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$, we show that the map

$$P_k \to P_1 \times_{p^{(n-2)} P_0} \overset{k}{\cdots} \times_{p^{(n-2)} P_0} P_1$$

satisfies the hypotheses of the above criterion.

- Consider the case $k = 2$.

- Since $p$ commutes with pullbacks over discrete objects

$$p^{(n-3)}P_{k0} = Z_{k0}\ .$$

Thus, for instance

$$\begin{aligned} & p^{(n-3)}(P_1 \times_{p^{(n-2)}P_0} P_1)_0 = p^{(n-3)}(P_{10} \times_{p^{(n-3)}P_{00}} P_{10}) = \\ = \ & p^{(n-3)}P_{10} \times_{p^{(n-3)}P_{00}} p^{(n-3)}P_{10} = Z_{10} \times_{Z_{00}} Z_{10}\ . \end{aligned}$$

- Since $Z \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$, $Z_{20} \cong Z_{10} \times_{Z_{00}} Z_{10}$ so that

$$p^{(n-3)}P_{20} \cong p^{(n-3)}(P_1 \times_{p^{(n-2)}P_0} P_1)_0\ .$$

- In conclusion, the hypotheses of the criterion hold for the map $P_2 \to P_1 \times_{p^{(n-2)}P_0} P_1$.
- Similarly for the maps $P_k \to P_1 \times_{p^{(n-2)}P_0} \overset{k}{\cdots} \times_{p^{(n-2)}P_0} P_1$ , which are therefore levelwise equivalences of categories.

- It follows that

$$\begin{aligned} p^{(n-2)}P_k &\cong p^{(n-2)}(P_1 \times_{p^{(n-2)}P_0} \overset{k}{\cdots} \times_{p^{(n-2)}P_0} P_1) \cong \\ &\cong p^{(n-2)}P_1 \times_{p^{(n-2)}P_0} \overset{k}{\cdots} \times_{p^{(n-2)}P_0} p^{(n-2)}P_1 \ . \end{aligned}$$

  So the Segal maps of $p^{(n-1)}P$ are isomorphisms and one can check that $p^{(n-1)}P \in \mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$.

- In conclusion, $P \in \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$.

# The construction of the rigidification functor $Q_n$

*Definitions:* Categories
$\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$, $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

*Propositions:* Functors
$q^{(n-1)} : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$
$q^{(n-1)} : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$
and their properties

*Definition:* The category $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:* Approximating $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ with $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:*
$Tr_n : \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

*Theorem:*
$St : \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

*Theorem:* Rigidification functor

$$Q_2 : \mathrm{Ta}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}$$

For $n > 2$

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$$

### Theorem

*There is a functor, called rigidification,*

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$$

*and for each* $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ *a morphism in* $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$

$$s_n(X) : Q_n X \to X$$

*natural in* $X$*, such that* $(s_n(X))_k$ *is a* $(n-1)$*-equivalence for all* $k \geq 0$*. In particular,* $s_n(X)$ *is an n-equivalence.*

# The rigidification functor: case n=2

- Let $Q_2$ be the composite

$$Q_2 : \mathrm{Ta}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}} \ .$$

- There is a morphism

$$t_2(X) : Tr_2 X \to JX \ .$$

Recall the adjunction

$$St \ : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \rightleftarrows [\Delta^{op}, \mathrm{Cat}] : J \ .$$

- By the adjunction $St \dashv J$, the morphism $t_2(X)$ corresponds to a morphism in $[\Delta^{op}, \mathrm{Cat}]$

$$Q_2 X = St\, Tr_2 X \xrightarrow{s_2(X)} X$$

  Making the following diagram commute

$$\begin{array}{ccc} Tr_2 X & \xrightarrow{\eta} & JSt\, Tr_2 X \\ & {\scriptstyle t_2(X)} \searrow & \downarrow {\scriptstyle Js_2(X)} \\ & & JX \end{array}$$

- Since $\eta$ and $t_2(X)$ are levelwise equivalences of categories, such is $Js_2(X)$.

- Suppose, inductively, that we defined $Q_{n-1}$ and $s_{n-1}(X) : Q_{n-1}X \to X$ for each $X \in \mathrm{Ta}_{\mathrm{wg}}^{\mathrm{n-1}}$.

- Define the functor $P_n : \mathrm{Ta}_{\mathrm{wg}}^{\mathrm{n}} \to \mathrm{LTa}_{\mathrm{wg}}^{\mathrm{n}}$ by the diagram

$$\begin{array}{ccc} P_n X & \xrightarrow{\;w_n(X)\;} & X \\ \big\downarrow & & \big\downarrow {\scriptstyle \gamma^{(n-2)}} \\ Q_{n-1} q^{(n-1)} X & \xrightarrow[\;s_{n-1}(q^{(n-1)}X)\;]{} & q^{(n-1)} X \end{array}$$

# The rigidification functor: the general case, cont.

- Let $Q_n$ (when $n > 2$) be given by the composite

$$\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

- Let $s_n(X) : Q_n X \to X$ be the composite

$$s_n(X) : Q_n X \xrightarrow{h_n(P_n X)} P_n X \xrightarrow{w_n(X)} X\ .$$

# The rigidification functor: the general case, cont.

- By the adjunction $St \dashv J$, the morphism in $[\Delta^{n-1^{op}}, \mathsf{Cat}]$

$$Q_n X = St\, Tr_n P_n X \xrightarrow{h_n(P_n X)} P_n X$$

corresponds by adjunction to the morphism in $\mathsf{Ps}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

$$Tr_n P_n X \xrightarrow{t_n(P_n X)} JP_n X$$

such that the following diagram commutes

$$\begin{array}{ccc} Tr_n P_n X & \xrightarrow{\eta} & JSt\, Tr_n P_n X = JQ_n X \\ & \searrow^{t_n(P_n X)} & \downarrow{\scriptstyle Jh_n(P_n X)} \\ & & JP_n X \end{array}$$

- It can be shown that $s_n(X)$ is a levelwise $(n-1)$-equivalence, so in particular an $n$-equivalence.