Lecture 3: The three Segal-type models and Segalic pseudo-functors

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The three Segal-type models and Segalic pseudo-functors

Definition: Category $\mathsf{Cat}^n_{\mathsf{hd}}$ of homotopically discrete n-fold categories

→ *Definition:* Category $\mathsf{Ta}^n_{\mathsf{wg}}$ of weakly globular Tamsamani n-categories

→ *Definition:* Category $\mathsf{Cat}^n_{\mathsf{wg}}$ of weakly globular n-fold categories

→ *Proposition:* Criterion for an n-fold category to be weakly globular

Definition: Category $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$ of Segalic pseudo-functors

→ *Proposition:* n-Fold categories levelwise equivalent via pseudo-natural transformation to Segalic pseudo-functors are weakly globular

→ *Theorem:* Functor St : $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \to \mathsf{Cat}^n_{\mathsf{wg}}$

Example: weakly globular Tamsamani 3-categories

Definition

A weakly globular Tamsamani 3-category $X \in \mathsf{Ta}^3_{\mathsf{wg}}$ is given by $X \in [\Delta^{op}, \mathsf{Ta}^2_{\mathsf{wg}}]$ such that

- $X_0 \in \mathsf{Cat}^2_{\mathsf{hd}}$.
- For each $k \geq 2$ the induced Segal maps $X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$ are 2-equivalences in $\mathsf{Ta}^2_{\mathsf{wg}}$.

Note that by the closure properties of $\mathsf{Ta}^2_{\mathsf{wg}}$, $X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \in \mathsf{Ta}^2_{\mathsf{wg}}$

Example: weakly globular Tamsamani 3-categories, cont.

- By the definition, for all $k \geq 0$ and $p^{(2)}X \in \mathsf{Ta}^2_{\mathsf{wg}}$.
- By the closure properties of 2-equivalences in $\mathsf{Ta}^2_{\mathsf{wg}}$, the closure properties of $\mathsf{Ta}^3_{\mathsf{wg}}$ hold.

Example: 3-equivalences in $\mathsf{Ta}_{\mathsf{wg}}$

Definition

We define a map $f : X \to Y$ in $\mathsf{Ta}^3_{\mathsf{wg}}$ to be a 3-equivalence if

- For all $a, b \in X^d_0 \quad f(a, b) : X(a, b) \to Y(fa, fb)$ is a 2-equivalence in $\mathsf{Ta}^2_{\mathsf{wg}}$.
- $p^{(2)}f$ is a 2-equivalence in $\mathsf{Ta}^2_{\mathsf{wg}}$.

Note that the closure properties hold for 3-equivalences.

Definition

A Tamsamani 3-category $X \in \mathrm{Ta}^3$ is given by $X \in \mathrm{Ta}^3_{\mathrm{wg}}$ such that X_0 and X_{k0} are discrete for all $k \geq 0$.

Example: Tamsamani 3-categories, cont.

A Tamsamani 3-category $X \in \mathrm{Ta}^3$ consists of $X \in [\Delta^{op}, \mathrm{Ta}^2]$ such that

- X_0 is discrete.
- For each $k \geq 2$ the Segal maps $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are 2-equivalences in Ta^2.

Definition

A weakly globular 3-fold category $X \in \mathsf{Cat}^3_{\mathsf{wg}}$ is given by $X \in \mathsf{Ta}^3_{\mathsf{wg}}$ such that $X \in \mathsf{Cat}^3$ and $p^{(2)}X \in \mathsf{Cat}^2_{\mathsf{wg}}$.

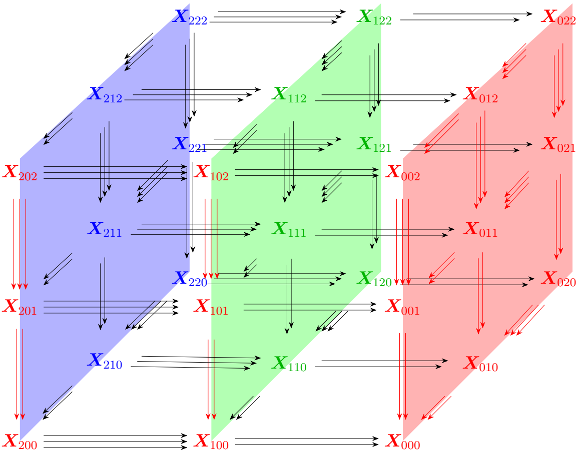
A weakly globular 3-fold category $X \in \mathrm{Cat}^3_{\mathrm{wg}}$ consists of $X \in [\Delta^{op}, \mathrm{Cat}^2_{\mathrm{wg}}]$ such that

- $X_0 \in \mathrm{Cat}^2_{\mathrm{hd}}$.
- For each $k \geq 2$, $X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$.
- For each $k \geq 2$ the induced Segal maps $X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$ are 2-equivalences in $\mathrm{Cat}^2_{\mathrm{wg}}$.
- For each $X \in \mathrm{Cat}^3_{\mathrm{wg}}$, $p^{(2)} X \in \mathrm{Cat}^2_{\mathrm{wg}}$.

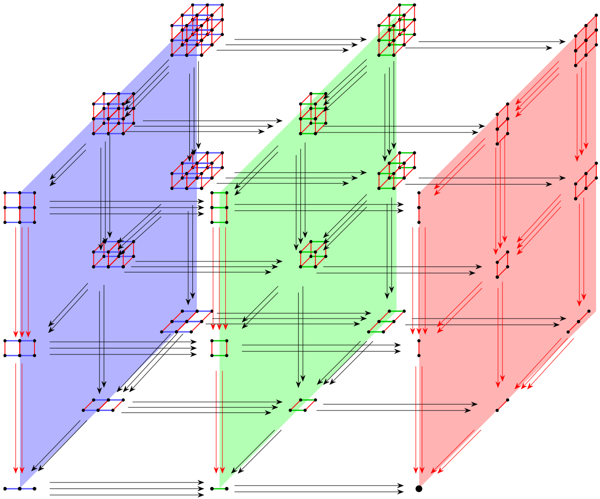
Corner of the multinerve of a weakly globular 3-fold category

In the following picture, for all $i, j, k \in \Delta^{op}$

$X_{2jk} \cong X_{1jk} \times_{X_{0jk}} X_{1jk}, \; X_{i2k} \cong X_{i1k} \times_{X_{i0k}} X_{i1k}, \; X_{ij2} \cong X_{ij1} \times_{X_{ij0}} X_{ij1}$.



Geometric picture of the corner of the multinerve of a weakly globular 3-fold category



- Inductive multi-simplicial definition.
- Closure properties.
- Weak globularity condition.
- Functor $p^{(n-1)} : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$.
- n-Equivalences.
- $(n-1)$-equivalences of the induced Segal maps

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

A remark about closure properties

- The closure properties of objects of $\mathrm{Ta}_{\mathrm{wg}}^{\mathrm{n}-1}$ ensure that $X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \in \mathrm{Ta}_{\mathrm{wg}}^{\mathrm{n}-1}$ and $X(a, b) \in \mathrm{Ta}_{\mathrm{wg}}^{\mathrm{n}-1}$.
  This is needed to formulate the induced Segal maps condition and to define n-equivalences.

- The closure properties of $(n-1)$-equivalences in $\mathrm{Ta}_{\mathrm{wg}}^{\mathrm{n}-1}$ ensure that the closure properties of objects and n-equivalences hold at step n, completing the inductive step.

Closure properties

- Closure properties of $\mathrm{Ta}^{n}_{\mathrm{wg}} \subset [\Delta^{n^{op}}, \mathrm{Set}]$.

C1 Repletion under isomorphisms; that is, if $A \cong B$ in $[\Delta^{n^{op}}, \mathrm{Set}]$ and $A \in \mathrm{Ta}^{n}_{\mathrm{wg}}$ then $B \in \mathrm{Ta}^{n}_{\mathrm{wg}}$.

C2 Closure under finite products.

C3 Closure under small coproducts.

C4 If the small coproduct $\coprod_i A_i$ is in $\mathrm{Ta}^{n}_{\mathrm{wg}}$, then each $A_i \in \mathrm{Ta}^{n}_{\mathrm{wg}}$.

Closure properties, cont.

- $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \subset [\Delta^{n^{op}}, \mathrm{Set}]$, $\mathcal{W}_n$ class of morphisms in $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$.

E1 $\mathcal{W}_n$ is closed under composition with isomorphisms.

E2 $\mathcal{W}_n$ is closed under finite products.

E3 $\mathcal{W}_n$ is closed under small colimits.

E4 If the small colimit $\coprod_i f_i$ of maps in $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ is in $\mathcal{W}_n$, then each $f_i \in \mathcal{W}_n$.

The formal definition of $\mathrm{Ta}^{n}_{\mathrm{wg}}$: inductive hypothesis

- $n = 1$ $\quad \mathrm{Ta}^{1}_{\mathrm{wg}} = \mathrm{Cat}$, $\quad$ 1-equivalences = equiv. of categories.
- Suppose, inductively, that we defined for each $1 \leq k \leq n-1$ a subcategory

$$\mathrm{Ta}^{k}_{\mathrm{wg}} \subset [\Delta^{{k-1}^{op}}, \mathrm{Cat}] \subset [\Delta^{{k}^{op}}, \mathrm{Set}]$$

containing the terminal object and a class $\mathcal{W}_k$ of maps in $\mathrm{Ta}^{k}_{\mathrm{wg}}$ (called k-equivalences) such that

I1) $\mathrm{Ta}^{k}_{\mathrm{wg}}$ satisfies the closure properties **C1-C4**.

I2) The functor $p^{(k-1)} : [\Delta^{{k}^{op}}, \mathrm{Set}] \to [\Delta^{{k-1}^{op}}, \mathrm{Set}]$ restricts to a functor $p^{(k-1)} : \mathrm{Ta}^{k}_{\mathrm{wg}} \to \mathrm{Ta}^{k-1}_{\mathrm{wg}}$ sending k-equivalences to $(k-1)$-equivalences.

I3) $\mathcal{W}_k$ satisfies the closure properties **E1-E4**.

The formal definition of $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$: step n

- An object X of $[\Delta^{n-1^{op}}, \mathrm{Cat}] \subset [\Delta^{n^{op}}, \mathrm{Set}]$ is a weakly globular Tamsamani n-category if:

- *Weak globularity condition*: $X_0 \in \mathrm{Cat}^{\mathrm{n}-1}_{\mathrm{hd}}$.
- $X_k \in \mathrm{Ta}^{\mathrm{n}-1}_{\mathrm{wg}}$ for all $k > 0$.
- *Induced Segal maps condition*. For all $s \geq 2$ the induced Segal maps

$$X_s \to X_1 \times_{X_0^d} \overset{s}{\cdots} \times_{X_0^d} X_1$$

(induced by the map $\gamma : X_0 \to X_0^d$) are $(n-1)$-equivalences.

The formal definition of $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$: defining n-equivalences

- Given $a, b \in X_0^d$, denote by $X(a, b)$ the fiber at (a, b) of the map

$$X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0 \xrightarrow{\gamma\times\gamma} X_0^d \times X_0^d \; .$$

- Since, by inductive hypothesis, $\mathrm{Ta}^{\mathrm{n}-1}_{\mathrm{wg}}$ satisfies **C1-C4**, then $X(a, b) \in \mathrm{Ta}^{\mathrm{n}-1}_{\mathrm{wg}}$.

Think of $X(a, b) \in \mathrm{Ta}^{\mathrm{n}-1}_{\mathrm{wg}}$ as a hom-$(n-1)$-category.

The formal definition of $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$: defining *n*-equivalences, cont.

- We define a map $f : X \to Y$ in $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ to be an *n*-equivalence if
  - For all $a, b \in X^d_0$ $\;f(a,b) : X(a,b) \to Y(fa, fb)$ is an $(n-1)$-equivalence.
  - $p^{(n-1)} f$ is an $(n-1)$-equivalence.

The above is a higher dimensional generalization of a functor which is fully faithful and essentially surjective on objects.

The formal definition of $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$: completing the inductive step

- It is easily shown that $p^{(n-1)}X \in \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$ for $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$.
- The fact that $p^{(n-1)}$ sends n-equivalences to $(n-1)$-equivalences is part of the definition of n-equivalences in $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$, so property I2) holds at step n.
- Properties I1) and I3) at step n are checked using the closure properties **E1-E4** of $\mathcal{W}_{n-1}$.

Definition

- For each $n \geq 0$ the category Ta^n of Tamsamani n-categories is the full subcategory of Ta^n_{wg} whose objects X are such that X_0 and $X_{k_1 \ldots k_r 0}$ are discrete for all $(k_1 \ldots k_r) \in \Delta^{r^{op}}$, $1 \leq r \leq n-2$.

- A morphism in Ta^n is an n-equivalence if it is an n-equivalence as a morphism in Ta^n_{wg}.

The three Segal-type models and Segalic pseudo-functors

Definition: Category $\mathsf{Cat}^n_{\mathsf{hd}}$ of homotopically discrete n-fold categories

Definition: Category $\mathsf{Ta}^n_{\mathsf{wg}}$ of weakly globular Tamsamani n-categories

Definition: Category $\mathsf{Cat}^n_{\mathsf{wg}}$ of weakly globular n-fold categories

Proposition: Criterion for an n-fold category to be weakly globular

Definition: Category $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$ of Segalic pseudo-functors

Proposition: n-Fold categories levelwise equivalent via pseudo-natural transformation to Segalic pseudo-functors are weakly globular

Theorem: Functor St : $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \to \mathsf{Cat}^n_{\mathsf{wg}}$

The idea of weakly globular n-fold category

- We have $\mathsf{Cat}^n_{\mathsf{wg}} \subset \mathsf{Cat}^n$ and $\mathsf{Cat}^n_{\mathsf{wg}} \subset \mathsf{Ta}^n_{\mathsf{wg}}$.
- We require the functor $p^{(n-1)} : \mathsf{Ta}^n_{\mathsf{wg}} \to \mathsf{Ta}^{n-1}_{\mathsf{wg}}$ to restrict to

$$p^{(n-1)} : \mathsf{Cat}^n_{\mathsf{wg}} \to \mathsf{Cat}^{n-1}_{\mathsf{wg}} \ .$$

The formal definition of the category Cat^n_{wg}

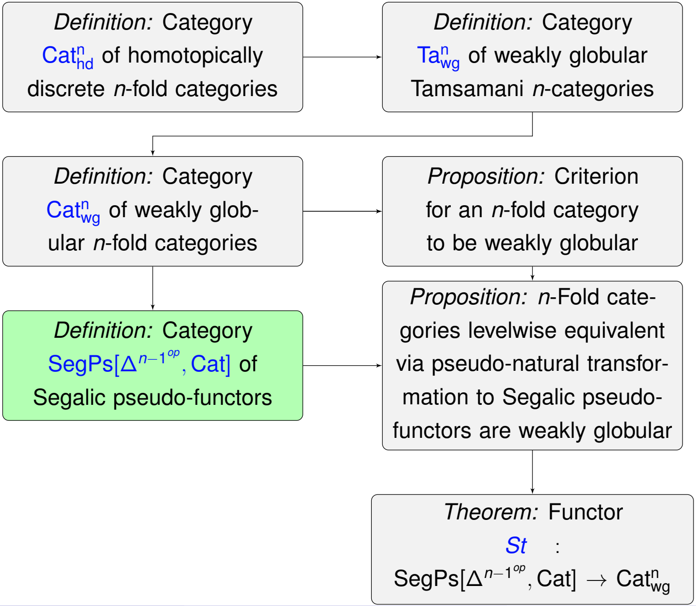
Definition

We say that an n-fold category $X \in \mathrm{Cat}^n$ is truncatable if for all $0 \leq r < n$, $p^{(r)}X \in [\Delta^{r^{op}}, \mathrm{Set}]$ is an r-fold category. We denote by Cat^n_t the full subcategory of Cat^n consisting of truncatable n-fold categories, with $\mathrm{Cat}^0_t = \mathrm{Set}$.

Definition

We say that $X \in [\Delta^{n^{op}}, \mathrm{Set}]$ is a weakly globular n-fold category if $X \in \mathrm{Cat}^n_t$ and $X \in \mathrm{Ta}^n_{wg}$. We denote by Cat^n_{wg} the category of weakly globular n-fold categories.

The three Segal-type models and Segalic pseudo-functors



Pseudo-functors.

- Recall the category $\mathrm{Ps}[\mathcal{C}, \mathrm{Cat}]$ of pseudo-functors and pseudo-natural transformations.

Theorem (Power; Lack...)

There is a strictification functor

$$St : \mathrm{Ps}[\mathcal{C}, \mathrm{Cat}] \to [\mathcal{C}, \mathrm{Cat}]$$

left adjoint to the inclusion and such that the components of the unit are equivalences in $\mathrm{Ps}[\mathcal{C}, \mathrm{Cat}]$.

We identify a subcategory

$$\mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \subset \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}]$$

of Segalic pseudo-functors such that St restricts to

$$\mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \subset [\Delta^{{n-1}^{op}}, \mathrm{Cat}].$$

We will then build

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

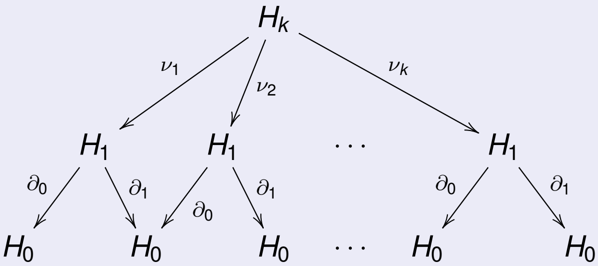
The idea of Segalic pseudo-functor

- $X \in \text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ consists of categories $X_{\underline{k}}$ for each object $\underline{k}$ of $\Delta^{n^{op}}$ with multi-simplicial face and degeneracy maps satisfying the multi-simplicial identities not as equalities but as isomorphisms, and these isomorphisms satisfy coherence axioms.

- Guided by this intuition, we generalize to certain pseudo-functors the multi-simplicial notion of Segal map.

- The additional conditions for a pseudo-functor to be Segalic use these Segal maps as well the the functor $p^{(n)} : \text{Ps}[\Delta^{n^{op}}, \text{Cat}] \to [\Delta^{n^{op}}, \text{Set}]$ obtained by applying p levelwise.

Let $H \in \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$ be such that H_0 is discrete. The following diagram in Cat commutes



Hence there is a unique Segal map for all $k \geq 2$

$$H_k \to H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Definition

Define $H \in \text{SegPs}[\Delta^{op}, \text{Cat}] \subset \text{Ps}[\Delta^{op}, \text{Cat}]$ if

i) H_0 is discrete.

ii) All Segal maps are isomorphisms.

Note: Since p commutes with pullbacks over discrete objects, there is a functor

$$p^{(1)} : \text{SegPs}[\Delta^{op}, \text{Cat}] \to \text{Cat}\ ,$$
$$(p^{(1)}X)_k = pX_k\ .$$

Segal maps for pseudo-functors.

- Notation:
  $\underline{k} = (k_1, \dots, k_n) \in \Delta^{n^{op}}$, $1 \leq i \leq n$

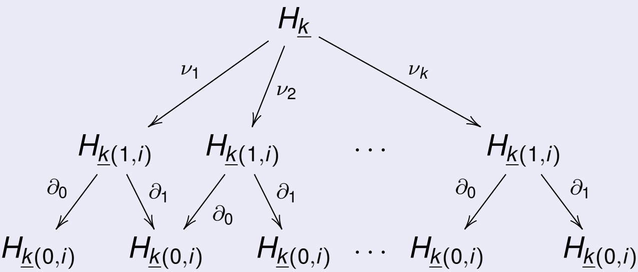
  $\underline{k}(1,i) = (k_1, \dots, k_{i-1}, 1, k_{i+1}, \dots, k_n)$

  $\underline{k}(0,i) = (k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n)$

- Let $H \in \text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ be such that $H_{\underline{k}(0,i)}$ is discrete for all $\underline{k} \in \Delta^{n^{op}}$ and all $1 \leq i \leq n$.

For pseudo-functors H satisfying this condition we can define Segal maps as follows.

The following diagram in Cat commutes



Hence there is a unique Segal map for all $k_i \geq 0$

$$H_{\underline{k}} \to H_{\underline{k}(1,i)} \times_{H_{\underline{k}(0,i)}} \overset{k_i}{\cdots} \times_{H_{\underline{k}(0,i)}} H_{\underline{k}(1,i)} \ .$$

Definition

We denote by

$$p^{(n)} : \text{Ps}[\Delta^{n^{op}}, \text{Cat}] \to [\Delta^{n^{op}}, \text{Set}]$$

the functor $(p^{(n)}X)_{\underline{k}} = pX_{\underline{k}}$ for $X \in \text{Ps}[\Delta^{n^{op}}, \text{Cat}]$ and $\underline{k} \in \Delta^{n^{op}}$.

Definition

Define $H \in \mathrm{SegPs}[\Delta^{n^{op}}, \mathrm{Cat}] \subset \mathrm{Ps}[\Delta^{n^{op}}, \mathrm{Cat}]$ if

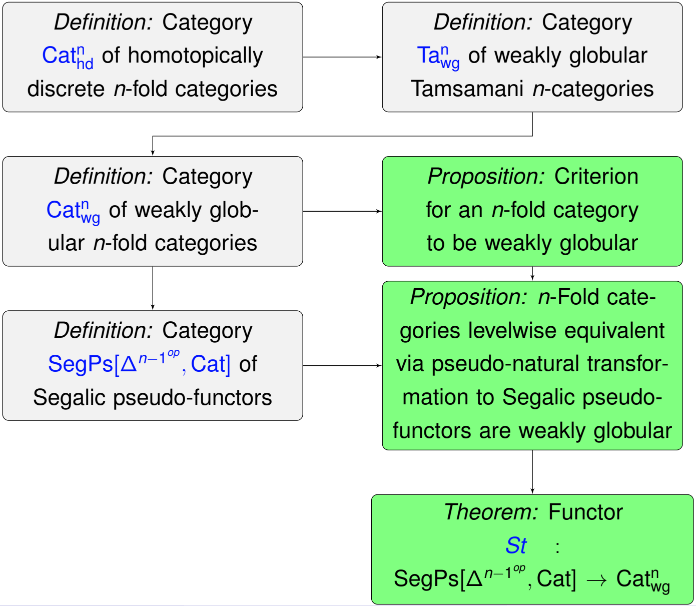
i) $H_{\underline{k}(0,i)}$ is discrete for all $\underline{k} \in \Delta^{n^{op}}$ and $1 \leq i \leq n$.

ii) All Segal maps are isomorphisms.

iii) The functor $p^{(n)} : \mathrm{Ps}[\Delta^{n^{op}}, \mathrm{Cat}] \to [\Delta^{n^{op}}, \mathrm{Set}]$ restricts to a functor

$$p^{(n)} : \mathrm{SegPs}[\Delta^{n^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \ .$$

The three Segal-type models and Segalic pseudo-functors



From Segalic pseudo-functors to weakly globular n-fold categories

Theorem

The strictification functor

$$St : \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \to [\Delta^{{n-1}^{op}}, \mathrm{Cat}]$$

restricts to a functor

$$St : \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$$

Strategy of proof of the theorem

a) Sufficient criterion for an n-fold category to be weakly globular.

b) Let $\phi : L \to H$ be a pseudo-natural transformation with $H \in \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$ and $L \in \mathrm{Cat^n}$, such that $\phi_{\underline{k}}$ is an equivalence of categories for all $\underline{k} \in \Delta^{n-1^{op}}$. Then L satisfies the hypotheses of criterion a), so that $L \in \mathrm{Cat^n_{wg}}$.

c) We show that given $X \in \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$, then $St\, X \in \mathrm{Cat^n}$ and there is a pseudo-natural transformation $St\, X \to X$ satisfying the hypothesis of b). Therefore $St\, X \in \mathrm{Cat^n_{wg}}$.

Criterion a) for a double category to be weakly globular

Proposition

Let $X \in \mathrm{Cat}^2$ *be such that*

- $X_0 \in \mathrm{Cat}_{\mathrm{hd}}$.
- $p^{(1)}X \in \mathrm{Cat}$.

Then $X \in \mathrm{Cat}^2_{\mathrm{wg}}$.

- Let $X \in \mathrm{Cat}^2$ be such that $X_0 \in \mathrm{Cat}_{\mathrm{hd}}$ and $p^{(1)}X \in \mathrm{Cat}$.
- Since p commutes with pullbacks over discrete objects,

$$\hat{\mu}_2 : X_1 \times_{X_0} X_1 \to X_1 \times_{X_0^d} X_1$$

is essentially surjective on objects, as $p\hat{\mu}_2$ is an isomorphism.

- This map is also fully faithful since, for each $(a,b),(c,d) \in X_{10} \times_{X_{00}} X_{10}$, we have

$$\begin{aligned} & (X_1 \times_{X_0} X_1)((a,b),(c,d)) \cong X_1(a,c) \times_{X_0(\partial_0 a, \partial_0 c)} X_1(b,d) \cong \\ \cong \quad & X_1(a,c) \times X_1(b,d) \cong (X_1 \times_{X_0^d} X_1)(\hat{\mu}_2(a,b), \hat{\mu}_2(c,d)) \;. \end{aligned}$$

- Thus $\hat{\mu}_2$ is an equivalence of categories. Similarly for $\hat{\mu}_k$, $k \geq 2$, showing that $X \in \mathrm{Cat}^2_{\mathrm{wg}}$.

Proof of property b), case $n = 2$

- Let $\phi : L \to H$ be a pseudo-natural transformation with $H \in \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$ and $L \in \mathrm{Cat}^2$, such that ϕ_k is an equivalence of categories for all $k \in \Delta^{op}$.
- Since $L_0 \simeq H_0$ and H_0 is discrete, $L_0 \in \mathrm{Cat}_{\mathrm{hd}}$.
- Since ϕ is pseudo-natural, $p^{(1)}\phi$ is natural, so there is a commuting diagram

$$\begin{array}{ccc} pL_k & \longrightarrow & pL_1 \times_{pL_0} \overset{k}{\cdots} \times_{pL_0} pL_1 \\ \wr\| & & \wr\| \\ pH_k & \xrightarrow{\cong} & pH_1 \times_{pH_0} \overset{k}{\cdots} \times_{pH_0} pH_1 \end{array}$$

Thus $p^{(1)}L \in \mathrm{Cat}$. By the previous criterion, we conclude that $L \in \mathrm{Cat}^2_{\mathrm{wg}}$.

Strategy of proof of the theorem, again

b) Let $\phi : L \to H$ be a pseudo-natural transformation with $H \in \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$ and $L \in \mathrm{Cat^n}$, such that $\phi_{\underline{k}}$ is an equivalence of categories for all $\underline{k} \in \Delta^{n-1^{op}}$. Then $L \in \mathrm{Cat^n_{wg}}$.

c) We show that given $X \in \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$, then $St\,X \in \mathrm{Cat^n}$ and there is a pseudo-natural transformation $St\,X \to X$ satisfying the hypothesis of b). Therefore $St\,X \in \mathrm{Cat^n_{wg}}$.

The proof of c) relies on some properties of the monad for pseudo-functors.