Lecture 2: The Three Segal-type models

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Segal-type models: common features

- We discuss three Segal-type models of weak n-categories, collectively denoted $\mathsf{Seg_n}$

$$\mathsf{Ta^n} \hookrightarrow \mathsf{Ta^n_{wg}} \hookleftarrow \mathsf{Cat^n_{wg}}$$

- We have $\mathsf{Seg_n} \subset [\Delta^{{n-1}^{op}}, \mathsf{Cat}]$, and the multi-simplicial structure encodes the composition of higher cells via the Segal maps.

Building on dimensions.

- Recall that in a weak n-category we want to have k-cells with source and target being $(k-1)$-cells for $1 \leq k \leq n$.
- $\mathrm{Seg_n}$ is built by induction on n starting with $\mathrm{Seg_1} = \mathrm{Cat}$.

For each $n > 1$:

$$\mathrm{Seg_n} \hookrightarrow [\Delta^{op}, \mathrm{Seg_{n-1}}]$$

Closure properties

- The subcategory $\mathrm{Seg_n} \hookrightarrow [\Delta^{n^{op}}, \mathrm{Set}]$ contains the terminal object and has the following closure properties:

  i) It is replete under isomorphisms.

  ii) It is closed under finite products.

  iii) It is closed under small coproducts.

  iv) If the small coproduct $\coprod\limits_i A_i$ is in $\mathrm{Seg_n}$, each $A_i \in \mathrm{Seg_n}$.

- Similar closure properties hold for the class of weak equivalences in $\mathrm{Seg_n}$.

Thus all (co)limits of interest in $\mathrm{Seg_n}$ are computed as they are in $[\Delta^{n^{op}}, \mathrm{Set}]$.

Closure properties, cont.

- $X \in \mathrm{Seg_n}$ is discrete if, viewed as an object of $[\Delta^{n^{op}}, \mathrm{Set}]$ it is a constant functor.
- From the closure properties, if $A \to X \leftarrow B$ is a diagram in $\mathrm{Seg_n}$ with X discrete, then $A \times_X B \in \mathrm{Seg_n}$.

Encoding the sets of cells: weak globularity condition

We encode in two ways the sets of cells of $X \in \text{Seg}_n$

i) Globularity condition:

$$X_0, \quad X_{\underset{k_1 \dots k_r 0\text{-}}{r}} \qquad 1 \leq r < n-1 \quad \text{discrete}$$

ii) Weak globularity condition:

$$X_0, \quad X_{\underset{k_1 \dots k_r 0\text{-}}{r}} \qquad 1 \leq r < n-1 \quad \text{homotopically discrete}$$

Let $X \in \text{Seg}_n \subset [\Delta^{op}, \text{Seg}_{n-1}]$ to be such that X_0 satisfies i) or ii).

Equivalence relations.

- An equivalence relation is a groupoid with no non-trivial loops.
- Such a groupoid is categorically equivalent to the discrete category on its sets of connected components.

Homotopically discrete *n*-fold categories are an iteration of the notion of internal equivalence relation.

Lemma

Given $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}}$*, there is a map* $\gamma : X \to X^d$ *where* X^d *is discrete, and this is a n-equivalence*

The truncation functor $p^{(n-1)}$.

- There is a functor $p^{(n-1)} : \mathsf{Seg_n} \to \mathsf{Seg_{n-1}}$ obtained by applying levelwise the isomorphism classes of objects functor $p : \mathsf{Cat} \to \mathsf{Set}$.

The functor $p^{(n-1)}$ divides out by the highest dimensional invertible cells.

- Let $X \in \mathrm{Seg_n}$, so $X_0 \in \mathrm{Cat_{hd}^{n-1}}$, $\gamma : X \to X_0^d$.
- For each $a, b \in X_0^d$, let $X(a,b)$ be the fiber at (a,b) of

$$X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0 \xrightarrow{\gamma\times\gamma} X_0^d \times X_0^d \ .$$

- Since $X_1 = \coprod_{a,b\in X_0^d} X(a,b)$, by the closure properties above, $X(a,b) \in \mathrm{Seg_{n-1}}$.

Think of $X(a,b)$ as hom $(n-1)$-category.

n-Equivalences in $\mathrm{Seg_n}$.

Definition

Define *n-equivalences* in $\mathrm{Seg_n}$ by induction on n.

1-equivalences are equivalences of categories.

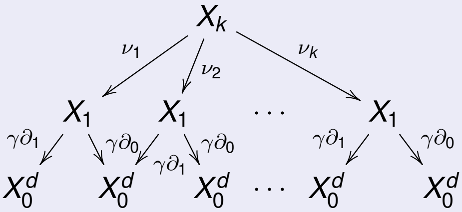
$f : X \to Y$ in $\mathrm{Seg_n}$ is a n-equivalence if and only if

a) $f(a,b) : X(a,b) \to Y(fa, fb)$ is a $(n-1)$-equivalence for all $a, b \in X_0^d$.

b) $p^{(n-1)}f$ is a $(n-1)$-equivalence.

This definition is a higher dimensional generalization of a functor which is fully faithful and essentially surjective on objects.

Given $X \in \mathrm{Seg_n} \subset [\Delta^{op}, \mathrm{Seg_{n-1}}]$, consider the commuting diagram



where $k \geq 2$, $\nu_j = X(r_j)$, $r_j(0) = j - 1$, $r_j(1) = j$. This gives the induced Segal map in $\mathrm{Seg_{n-1}}$

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

The induced Segal maps condition.

To define $X \in \mathrm{Seg_n} \subset [\Delta^{op}, \mathrm{Seg_{n-1}}]$ we require the induced Segal maps

$$X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

to be $(n-1)$-equivalences.

This condition controls the behaviour of the compositions of higher cells.

The functor $q^{(n-1)}$.

- There is a functor $q^{(n-1)} : \mathsf{Seg_n} \rightarrow \mathsf{Seg_{n-1}}$ obtained by applying levelwise the connected components functor $q : \mathsf{Cat} \rightarrow \mathsf{Set}$.

- There is a map, natural in $X \in \mathsf{Seg_{n-1}}$

$$\gamma^{(n-1)} : X \rightarrow q^{(n-1)}X$$

The functor $q^{(n-1)}$ divides out by the highest dimensional cells.

Groupoidal Segal-type models.

Definition

Let $\mathrm{GSeg_1} = \mathrm{Gpd}$. Suppose, inductively, that we defined $\mathrm{GSeg_{n-1}} \subset \mathrm{Seg_{n-1}}$; then $X \in \mathrm{GSeg_n} \subset \mathrm{Seg_n}$ if

i) $X_k \in \mathrm{GSeg_{n-1}}$ for all $k \geq 0$.

ii) $p^{(n-1)}X \in \mathrm{GSeg_{n-1}}$.

Summary of main common features of $\mathsf{Seg_n}$.

- Inductive multi-simplicial definition.
- Closure properties.
- Globularity/weak globularity condition.
- Functors $p^{(r)}, q^{(r)} : \mathsf{Seg_n} \to \mathsf{Seg_r}$.
- n-Equivalences.
- $(n-1)$-equivalences of the induced Segal maps

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

- Groupoidal Segal-type models.

The three models.

Three different models corresponding to different behavior of:

Induced Segal maps $\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$

Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$

	X_0	$\hat{\mu}_k$	η_k
Ta^n	discrete	$(n-1)$-eq	$(n-1)$-eq
Cat^n_{wg}	homotopically discrete	$(n-1)$-eq	isomorphisms
Ta^n_{wg}	homotopically discrete	$(n-1)$-eq	-

Definition

$X \in \mathrm{Ta}^2_{\mathrm{wg}}$ consists of a simplicial object $X \in [\Delta^{op}, \mathrm{Cat}]$ such that

- $X_0 \in \mathrm{Cat}_{\mathrm{hd}}$.
- The induced Segal maps $X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$ for each $k \geq 2$ are equivalences of categories.

- The functor $p^{(1)} : \mathrm{Ta}^2_{\mathrm{wg}} \to \mathrm{Cat} \subset [\Delta^{op}, \mathrm{Set}]$ associates to $X \in \mathrm{Ta}^2_{\mathrm{wg}}$ the simplicial set taking $k \in \Delta^{op}$ to $p(X_k)$.
- The closure properties hold for $\mathrm{Ta}^2_{\mathrm{wg}}$.

Definition

We define a map $f : X \to Y$ in $\mathrm{Ta}^2_{\mathrm{wg}}$ to be an 2-equivalence if

- For all $a, b \in X_0^d \quad f(a,b) : X(a,b) \to Y(fa, fb)$ is an equivalence of categories.
- $p^{(1)} f$ is an equivalence of categories.

Note that the closure properties hold for 2-equivalences.

Example: Tamsamani 2-categories

Definition

A Tamsamani 2-category $X \in \mathrm{Ta}^2$ is given by $X \in \mathrm{Ta}^2_{\mathrm{wg}}$ such that X_0 is discrete.

$X \in \mathrm{Ta}^2$ consists of a simplicial object $X \in [\Delta^{op}, \mathrm{Cat}]$ such that

- X_0 is discrete.
- The Segal maps $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ for each $k \geq 2$ are equivalences of categories.

Example: weakly globular double categories

Definition

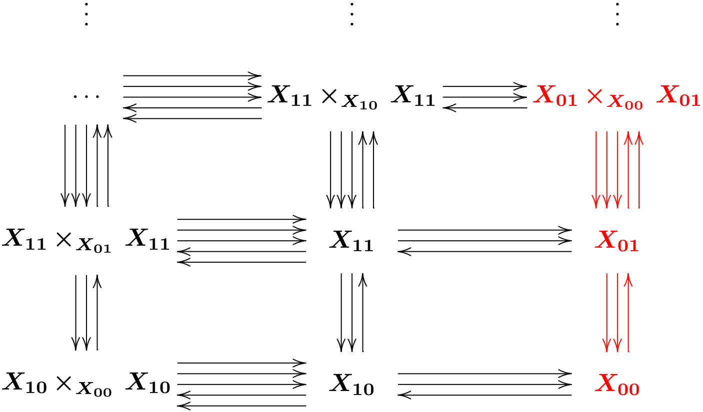
A weakly globular double category $X \in \mathsf{Cat}^2_{\mathsf{wg}}$ is given by $X \in \mathsf{Ta}^2_{\mathsf{wg}}$ such that $X \in \mathsf{Cat}^2$.

Example: weakly globular double categories, cont.

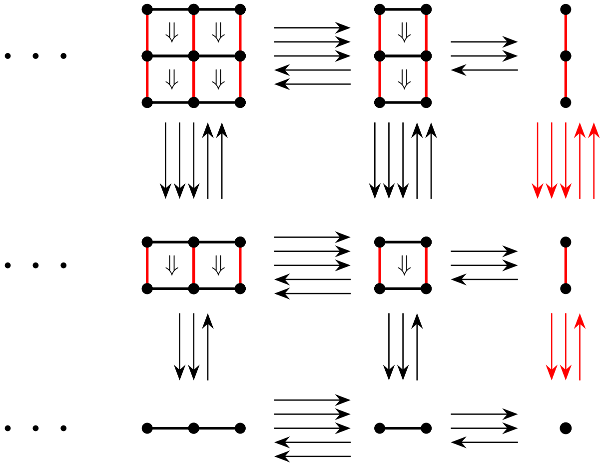
$X \in \mathsf{Cat}^2_{\mathsf{wg}}$ consists of a simplicial object $X \in [\Delta^{op}, \mathsf{Cat}]$ such that

- $X_0 \in \mathsf{Cat}_{\mathsf{hd}}$.
- The Segal maps $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ for each $k \geq 2$ are isomorphisms.
- The induced Segal maps $X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$ for each $k \geq 2$ are equivalences of categories.

Corner of the double nerve of a weakly globular double category X



Geometric picture of the corner of the double nerve of a weakly globular double category



Main results.

Theorem A. There is a functor *rigidification*

$$Q_n : \mathrm{Ta}^n_{\mathrm{wg}} \to \mathrm{Cat}^n_{\mathrm{wg}}$$

and for each $X \in \mathrm{Ta}^n_{\mathrm{wg}}$ an n-equivalence natural in X

$$s_n(X) : Q_n X \to X.$$

Theorem B. There is a functor *discretization*

$$Disc_n : \mathrm{Cat}^n_{\mathrm{wg}} \to \mathrm{Ta}^n$$

and, for each $X \in \mathrm{Cat}^n_{\mathrm{wg}}$, a zig-zag of n-equivalences in $\mathrm{Ta}^n_{\mathrm{wg}}$ between X and $Disc_n X$.

Theorem C. The functors

$$Q_n : \mathrm{Ta}^{\mathrm{n}} \leftrightarrows \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} : Disc_n$$

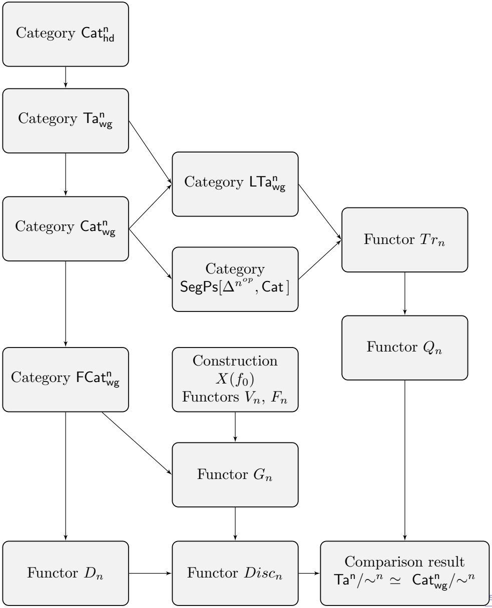
induce an equivalence of categories after localization with respect to the n-equivalences

$$\mathrm{Ta}^{\mathrm{n}}/\sim^n \simeq \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}/\sim^n \quad .$$

Theorem D. There is an equivalence of categories

$$\mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}}/\sim^n \simeq \mathrm{Ho}\,(n\text{-types})\ .$$

Diagram of connections between the topics



Overall Summary

The three Segal-type models and Segalic pseudo-functors

Definition: $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}}$

Definition: $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$

Definition: $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

Definition: $\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

Theorem

$St \; : \; \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

Rigidification of weakly globular Tamsamani *n*-categories

Definition: $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$

Theorem

$Tr_n \; : \; \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}]$

Theorem: Rigidification functor

$Q_n \; : \; \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n}$

$\to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$

Weakly globular *n*-fold categories as a model of weak *n*-categories

Definition: $\mathrm{FCat}^{\mathrm{n}}_{\mathrm{wg}}$

Definitions: $\mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}}$, $\mathrm{GTa}^{\mathrm{n}}_{\mathrm{wg}}$, $\mathrm{GTa}^{\mathrm{n}}$

Theorem : Discretization functor

$Disc_n \; : \; \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n}}$

Theorem: $\mathrm{Ta}^{\mathrm{n}} / \sim^n \; \simeq \; \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} / \sim^n$

Theorem: $\mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}} / \sim^n \; \simeq \; \mathrm{Ho}(n\text{-types})$

Overall Summary

The three Segal-type models and Segalic pseudo-functors

Definition: $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$

Definition: $\mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$

Definition: $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$

Definition: $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

Theorem

$St \ : \ \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \rightarrow \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$

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Rigidification of weakly globular Tamsamani *n*-categories

Definition: $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$

Theorem

$Tr_n \ : \ \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}} \rightarrow \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

Theorem: Rigidification functor

$Q_n \ : \ \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}} \xrightarrow{P_n} \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}} \xrightarrow{Tr_n}$

$\rightarrow \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$

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Weakly globular *n*-fold categories as a model of weak *n*-categories

Definition: $\mathsf{FCat}^{\mathsf{n}}_{\mathsf{wg}}$

Definitions: $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$, $\mathsf{GTa}^{\mathsf{n}}_{\mathsf{wg}}$, $\mathsf{GTa}^{\mathsf{n}}$

Theorem : Discretization functor

$Disc_n \ : \ \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \ \rightarrow \ \mathsf{Ta}^{\mathsf{n}}$

Theorem: $\mathsf{Ta}^{\mathsf{n}}/\sim^n \ \simeq \ \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n$

Theorem: $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n \ \simeq \ \mathsf{Ho}(n\text{-types})$

The three Segal-type models and Segalic pseudo-functors

Definition: Category $\mathsf{Cat}^n_{\mathsf{hd}}$ of homotopically discrete n-fold categories

Definition: Category $\mathsf{Ta}^n_{\mathsf{wg}}$ of weakly globular Tamsamani n-categories

Definition: Category $\mathsf{Cat}^n_{\mathsf{wg}}$ of weakly globular n-fold categories

Proposition: Criterion for an n-fold category to be weakly globular

Definition: Category $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$ of Segalic pseudo-functors

Proposition: n-Fold categories levelwise equivalent via pseudo-natural transformation to Segalic pseudo-functors are weakly globular

Theorem: Functor St : $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \to \mathsf{Cat}^n_{\mathsf{wg}}$

The idea of homotopically discrete *n*-fold category

- A homotopically discrete category is an equivalence relation.

A homotopically discrete *n*-fold category is an *n*-fold category suitably equivalent to a discrete one both 'globally' and in each simplicial dimension.

- We further require the existence of truncation functors

$$\mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}} \xrightarrow{p^{(n-1)}} \mathrm{Cat}^{\mathrm{n}-1}_{\mathrm{hd}} \xrightarrow{p^{(n-2)}} \cdots \mathrm{Cat}_{\mathrm{hd}} \xrightarrow{p^{(0)}} \mathrm{Set}\ .$$

The formal definition of the category Cat^n_{hd}

Definition

Let $\mathrm{Cat}^0_{hd} = \mathrm{Set}$. Suppose, inductively, we defined the subcategory

$\mathrm{Cat}^{n-1}_{hd} \subset \mathrm{Cat}^{n-1}$ of homotopically discrete $(n-1)$-fold categories. We say that the n-fold category $X \in \mathrm{Cat}^n$ is homotopically discrete if:

- X is a levelwise equivalence relation.
- $p^{(n-1)}X \in \mathrm{Cat}^{n-1}_{hd}$.

We denote $\mathrm{Cat}^1_{hd} = \mathrm{Cat}_{hd}$.

Definition

$X \in \mathsf{Cat}^2_{\mathsf{hd}}$ consists of a simplicial object $X \in [\Delta^{op}, \mathsf{Cat}]$ such that

- $X_k \in \mathsf{Cat}_{\mathsf{hd}}$ for all $k \geq 0$.
- The Segal maps $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ for each $k \geq 2$ are isomorphisms.
- For each $X \in \mathsf{Cat}^2_{\mathsf{hd}}$, $p^{(1)} X \in \mathsf{Cat}_{\mathsf{hd}}$ where $(p^{(1)} X)_k = p(X_k)$.

Definition

- Given $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}}$ let $\gamma^{(n-1)}_X : X \to p^{(n-1)} X$ be the morphism given by

$$(\gamma^{(n-1)}_X)_{s_1 \dots s_{n-1}} : X_{s_1 \dots s_{n-1}} \to qX_{s_1 \dots s_{n-1}} = pX_{s_1 \dots s_{n-1}}$$

for each $(s_1, ..., s_{n-1}) \in {\Delta^{n-1}}^{op}$.

- The discretization map is the composite

$$\gamma_{(n)} : X \xrightarrow{\gamma^{(n-1)}} p^{(n-1)} X \xrightarrow{\gamma^{(n-2)}} p^{(n-2)} p^{(n-1)} X \to \cdots \xrightarrow{\gamma^{(0)}} X^d$$

where $X^d = p^{(0)} p^{(1)} ... p^{(n-1)} X$.

Internal equivalence relations

$\mathcal{C}$ category with finite limits, $f : A \to B$ morphism in $\mathcal{C}$

$$A[f] \in \mathrm{Gpd}\,\mathcal{C} \qquad (A \times_B A) \times_A (A \times_B A) \xrightarrow{m} A \times_B A \underset{\underset{s}{\longleftarrow}}{\overset{p_1}{\underset{p_2}{\rightrightarrows}}} A$$

where p_1, p_2 are the two projections and s is the diagonal map.

- When $\mathcal{C} = \mathrm{Set}$ and f is a surjection, $A[f]$ is an equivalence relation.

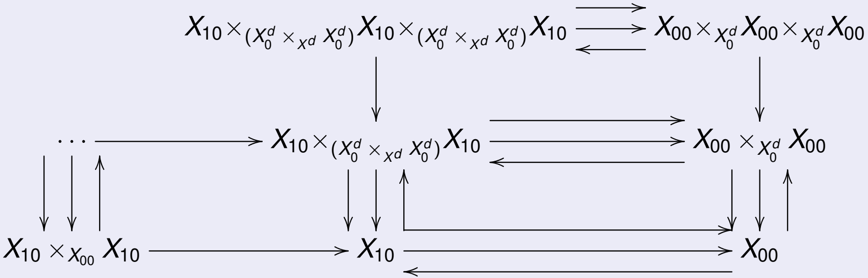
The following is a description of Cat^n_{hd} as (iterated) internal equivalence relations.

Proposition

Let $X \in \mathsf{Cat}^n$*. Then* $X \in \mathsf{Cat}^n_{hd}$ *if and only if* $X = A[f]$ *for a morphism* $f : A \to B$ *in* Cat^{n-1} *with* $B \in \mathsf{Cat}^{n-1}_{hd}$ *and* f *a levelwise surjection in* Set*.*

Example: $n = 2$

Let $X \in \mathsf{Cat}^2_{\mathsf{hd}}$; then $p^{(1)}X$ is the equivalence relation associated to the surjective map of sets $\gamma : X_0^d = (p^{(1)}X)_0 \to p^{(0)}(p^{(1)}X) = X^d$ and X has the form



The idea of n-equivalences in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$

- The n-equivalences in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$ should generalize a functor which is fully faithful and essentially surjective on objects.
- We need a notion of 'hom $(n-1)$-category' $X(a,b) \in \mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{hd}}$ for each $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$.
- We use the functor $p^{(n-1)}$ for the higher version of fully faithfulness.

Definition

Given $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$, for each $a, b \in X_0^d$ denote by $X(a, b)$ the fiber at (a, b) of the map

$$X_1 \xrightarrow{(d_0, d_1)} X_0 \times X_0 \xrightarrow{\gamma(n) \times \gamma(n)} X_0^d \times X_0^d \ .$$

Definition

For $n = 1$, a 1-equivalence in $\mathsf{Cat}_{\mathsf{hd}}$ is an equivalence of categories.

Suppose, inductively, that we defined $(n-1)$-equivalences in $\mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{hd}}$. Then a map $f : X \to Y$ in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$ is an n-equivalence if

a) For all $a, b \in X_0^d$, $f(a, b) : X(a, b) \to Y(fa, fb)$ is a $(n-1)$-equivalence.

b) $p^{(n-1)} f$ is a $(n-1)$-equivalence.

Properties of n-Equivalences in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$

Lemma

A map $f : X \to Y$ in $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$ is an n-equivalence if and only if it induces an isomorphism $X^d \cong Y^d$.

Corollary

Let $X \in \mathsf{Cat}^{\mathsf{n}}_{\mathsf{hd}}$. Then

a) *The discretization map $\gamma_{(n)} : X \to X^d$ is an n-equivalence.*

b) *For each $s \geq 2$ the induced Segal map*

$$\hat{\mu}_s : X_1 \times_{X_0} \overset{s}{\cdots} \times_{X_0} X_1 \to X_1 \times_{X_0^d} \overset{s}{\cdots} \times_{X_0^d} X_1$$

is a $(n-1)$-equivalence.