Lecture 1: Higher categories: introduction and background

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Some sources of higher categorical ideas

- Algebraic topology: notion of homotopy coherence (eg loop spaces, operads), models of homotopy types.
- Mathematical physics: pursuit of models for TQFT and higher cobordism categories.
- Algebraic geometry: pursuit of the notion of higher and derived stacks as well as higher non-abelian cohomology.
- Logic: homotopy type theory.
- Computer science: quantum computing.
- Representation theory: low dimensional cases.

Categories

In a category we have objects, morphisms, compositions of morphisms and identity morphisms for each object, such that compositions are associative and unital.

- When each morphism is invertible we obtain a groupoid.
- A one-object groupoid is the familiar notion of a group, while a one-object category is a monoid.

The maps between categories are the functors: These associate objects to objects and arrows to arrows in a way that is compatible with the composition and with the identities.

- Given paths $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, the two paths

  $$x \xrightarrow{(h*g)*f} w \quad \text{and} \quad x \xrightarrow{h*(g*f)} w$$

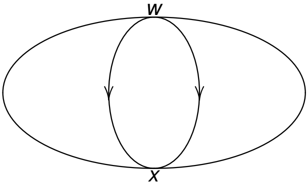
  are not the *same* path, but merely *homotopic*.

- We can think of this homotopy as a 2-morphism

$$x \underset{h*(g*f)}{\overset{(h*g)*f}{\underset{\Downarrow}{\rightrightarrows}}} w$$

2-tracks

- We have a notion of 'higher homotopy' between 2-morphisms:



- The equivalence classes of 2-morphisms are called *2-tracks*.

Two motivating examples

Two prototype examples in dimension 2:

a) The 2-dimensional structure with
   Objects = categories
   1-morphisms = functors
   2-morphisms = natural transformations.

b) The 2-dimensional structure with
   Objects = points of a space X
   1-morphisms = paths in X
   2-morphisms = 2-tracks (equivalence classes of homotopies between paths).

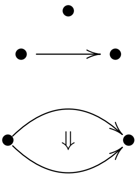
Two motivating examples, cont.

- Objects are also called 0-cells and k-morphisms are called k-cells.
- In both examples, we can use the pictorial representation

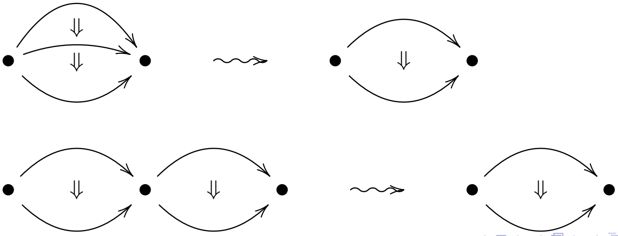
Objects

1-morphisms

2-morphisms



Vertical and horizontal compositions



Two motivating examples, cont.

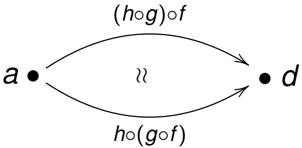
Main difference between examples a) and b):

a) All compositions are associative and unital. This is a strict 2-category.

b) Composition of paths is associative and unital only up to homotopy; given paths

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

there is a homotopy



The structure we obtain is a weak 2-category.

Strict versus weak n-categories

Idea of strict n-category: in a strict n-category there are cells in dimension $0, \dots, n$, identity cells and compositions which are associative and unital. Each k-cell has source and target which are $(k-1)$-cells, $1 \leq k \leq n$.

Idea of weak n-category: in a weak n-category there are cells in dimension $0, \dots, n$, identity cells and compositions which are associative and unital up to an invertible cell in the next dimension, in a coherent way.

Definition

Strict n-categories are defined by iterated enrichment

$$1\text{-}Cat = Cat, \qquad n\text{-}Cat = ((n-1)\text{-}Cat)\text{-}Cat$$

- In dimensions $n = 2, 3$ it is possible to give an explicit definition of weak n-category with the notions of bicategory and tricategory.
- For general n there are several different models of weak n-categories and weak n-groupoids.
- The comparison of different models of weak n-category is largely an open problem.

- We can study spaces by breaking them into smaller pieces, via the Postnikov decomposition.
- These pieces (Postnikov sections) are the n-types (that is spaces with trivial homotopy groups in dimension greater than n) and are patched together using k-invariants.
- The Postnikov sections and their k-invariants (Postnikov systems) determine the space up to homotopy.

Motivating question: find an algebraic-categorical description of Postnikov systems.

Modelling *n*-types

Categorical models of *n*-types of spaces amount to a category of higher groupoids $\mathcal{G}_n$ generalizing the fundamental groupoid of a space: that is, there is a functor

$$Q_n : Top \to \mathcal{G}_n$$

inducing an equivalence of categories

$$Ho(n\text{-types}) \;\simeq\; \mathcal{G}_n/\sim$$

- Fact: Strict n-groupoids do not model n-types when $n > 2$.
- Thus a more general higher structure is needed to model algebraically the Postnikov systems of spaces.

This was one of the motivations for the development of weak n-categories: in the weak n-groupoid case it gives an algebraic model of n-types (homotopy hypothesis).

Types of higher structures

a) *n*-Truncated globular structures

strict *n*-categories $\subset$ weak *n*-categories (several models)

b) *n*-Fold structures

Cat^n (iterated internal categories)

c) Infinity structures

strict ω-categories $\subset$ weak ω-categories (complicial sets)

$\cup$

(∞, n)-categories (several models)

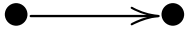
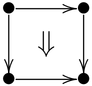
Example: double categories

- Let $X \in \mathsf{Cat}(\mathsf{Cat})$

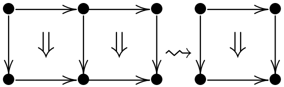
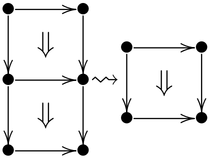
  $X_0 \in \mathsf{Cat}$ has

  objects •   morphisms 

  $X_1 \in \mathsf{Cat}$ has

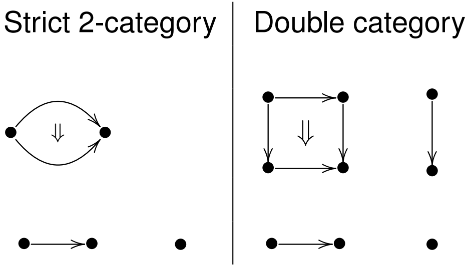
  objects    morphisms 

  Thus squares can be composed horizontally and vertically

  All compositions are associative and unital; interchange law.

Strict 2-categories versus double categories



Note: the picture on the right becomes the one on the left when all vertical morphisms are identities.

Strict n-categories versus n-fold categories

There is an embedding

$$n\text{-Cat} \hookrightarrow \text{Cat}^n.$$

A strict n-category $X \in n$-Cat is a n-fold category in which certain substructures are discrete (that is just sets).

- This discreteness condition is called the globularity condition.
- The sets underlying these discrete substructures are the sets of cells in the strict n-category.

A motivating question

The category n-Cat is too small to model weak n-category while Cat^n is too large. Is there an intermediate category

$$n\text{-Cat} \hookrightarrow ? \hookrightarrow \mathrm{Cat}^n$$

which is a model of weak n-categories?

The answer is provided by the category $\mathrm{Cat}^n_{\mathrm{wg}}$ of weakly globular n-fold categories.

Some historical development

- A pioneering work on the use of n-fold structures in connection with homotopy theory is Loday's Cat^n-groups as a model of path-connected $(n+1)$-types.

- This was also investigated by Bullejos-Cegarra-Duskin with a different approach, and led Brown to a proof a higher order Van-Kampen theorem with interesting computational applications.

- A combinatorially different model was also given by Porter and by Ellis-Steiner in terms of crossed n-cubes.

The notion of weak globularity first arose in a special case in relating Loday's model to the Tamsamani-Simpson model in the path-connected case ([P., Adv. Math. 2009]).

An environment for higher categories

To build a model of weak n-category we need a combinatorial machinery that allows to encode:

i) The sets of cells in dimension 0 up to n.
ii) The behavior of the compositions (including their coherence laws).
iii) The higher categorical equivalences.

Multi-simplicial objects are a good environment for the definition of higher categorical structures because there are natural candidates for the compositions given by the Segal maps.

Simplicial combinatorics

- Let Δ be the simplicial category. Its objects are finite ordered sets

$$[n] = \{0 < 1 < \cdots < n\}$$

  for integers $n \geq 0$ and its morphisms are non decreasing monotone functions.

- The functor category $[\Delta^{op}, \mathcal{C}]$ is the category of simplicial objects and simplicial maps in $\mathcal{C}$.

Simplicial combinatorics, cont.

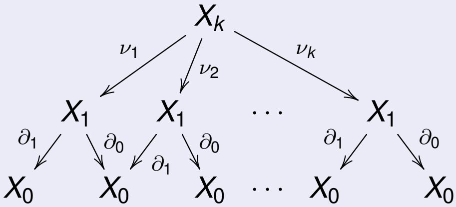
- To give a simplicial object X in $\mathcal{C}$ is the same as to give a sequence of objects $X_0, X_1, X_2, \ldots$ together with face operators $\delta_i : X_n \to X_{n-1}$ and degeneracy operators $\sigma_i : X_n \to X_{n+1}$ $(i = 0, \ldots, n)$ satisfying the simplicial identities; we denote $X([n]) = X_n$.

$$X \in [\Delta^{op}, \mathcal{C}] \qquad \cdots X_3 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} X_2 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow} X_1 \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow} X_0$$

Segal maps.

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks.
Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j-1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

Definition

- An *internal category* in a category $\mathcal{C}$ with pullbacks consists of a diagram in $\mathcal{C}$

$$C_1 \times_{C_0} C_1 \xrightarrow{c} C_1 \underset{s}{\overset{d_0, d_1}{\rightrightarrows}} C_0$$

  where these maps satisfies the axiom of a category.

- An *internal groupoid* in $\mathcal{C}$ is an internal category with all morphisms invertible. The inverses are given by the map $i : C_1 \to C_1$ such that $c(1_{c_1}, i) = sd_0$, $c(i, 1_{c_1}) = sd_1$.

- Denote by Cat $\mathcal{C}$ the category of internal categories and internal functors.

Segal maps and internal categories

- There is a nerve functor

$$N : \mathrm{Cat}\,\mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\,\mathcal{C}$

$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow}\; X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longleftarrow}\; X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps are isomorphisms.

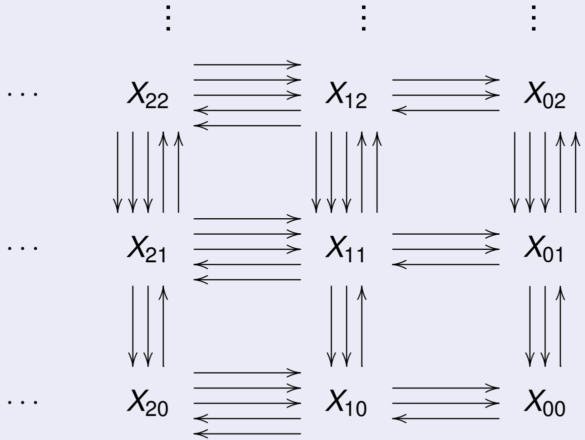
Multi-simplicial objects.

- Let $\Delta^{n^{op}} = \Delta^{op} \times \overset{n}{\cdots} \times \Delta^{op}$.
- Multi-simplicial objects in $\mathcal{C}$ are functors $[\Delta^{n^{op}}, \mathcal{C}]$.
- They have *n* different simplicial directions and every *n*-fold simplicial object in $\mathcal{C}$ is a simplicial object in $(n-1)$-fold simplicial objects in $\mathcal{C}$ in *n* possible ways:

$$[\Delta^{n^{op}}, \mathcal{C}] \underset{\xi_k}{\cong} [\Delta^{op}, [\Delta^{{n-1}^{op}}, \mathcal{C}]] \qquad 1 \leq k \leq n$$

Thus for each $X \in [\Delta^{n^{op}}, \mathcal{C}]$ we have Segal maps in each of the *n* simplicial directions.

Example: Bisimplicial object



Notational convention

- We identify $(n-1)$-fold simplicial sets X with those n-fold simplicial sets X for which the simplicial set $X_{k_1 \dots k_{n-1}}$ is discrete for all $(k_1, \dots, k_{n-1}) \in \Delta^{{n-1}^{op}}$.
- Under this convention, there are embeddings

$$\begin{gathered}\mathrm{Set} \hookrightarrow [\Delta^{op}, \mathrm{Set}] \hookrightarrow [\Delta^{2^{op}}, \mathrm{Set}] \hookrightarrow \cdots \\ \cdots \hookrightarrow [\Delta^{{n-1}^{op}}, \mathrm{Set}] \hookrightarrow [\Delta^{n^{op}}, \mathrm{Set}] \hookrightarrow \cdots\end{gathered}$$

Definition

n-fold categories are defined inductively as

$$\mathrm{Cat}^1 = \mathrm{Cat}$$

$$\mathrm{Cat}^{\mathrm{n}} = \mathrm{Cat}(\mathrm{Cat}^{\mathrm{n}-1})$$

- By iterating the nerve construction, we obtain fully faithful multinerve functors

$$N_{(n)} : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n^{op}}, \mathrm{Set}]$$

$$J_n : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n-1^{op}}, \mathrm{Cat}]$$

Notational conventions

- We identify $\mathrm{Cat}^{\mathrm{n}}$ with the essential image of the functor $N_{(n)}$.
- If $X \in \mathrm{Cat}^{\mathrm{n}}$ and $N^{(1)}$ is the nerve functor in direction 1, we denote for each $k \geq 0$, $(N^{(1)}X)_k = X_k$.

Multi-simplicial descriptions

- We next describe strict n-categories and n-fold categories using multi-simplicial objects and their Segal maps.

These descriptions facilitate the geometric intuition of how to modify the structure to build weak models.

n-Fold categories and strict n-categories multi-simplicially.

An n-fold category is $X \in [\Delta^{n-1^{op}}, \mathrm{Cat}]$ such that the Segal maps in all directions are isomorphisms.

A strict n-category is $X \in [\Delta^{n-1^{op}}, \mathrm{Cat}]$ such that

i) The Segal maps in all directions are isomorphisms.

ii) $X_0, X(\overbrace{1, 1, \dots, 1}^{r}, 0, \text{-}) \qquad 1 \leq r < n-1$

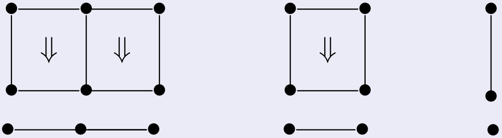
are constant functors taking values in a discrete category.

The globularity condition

- For a strict n-category X, condition ii) is equivalent to: $X_0 \in [\Delta^{n-2^{op}}, \mathrm{Cat}]$ and $X_{k_1 \dots k_r 0} \in [\Delta^{n-r-2^{op}}, \mathrm{Cat}]$ are constant functors taking value in a discrete category for all $1 \leq r \leq n-2$ and all $(k_1, \dots, k_r) \in \Delta^{r^{op}}$.
- This is called the globularity condition, as it gives rise to the globular shapes of the higher cells.

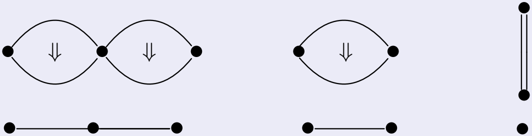
Example: $n = 2$

- Double category $X \in \mathrm{Cat(Cat)} \in [\Delta^{op}, \mathrm{Cat}]$



$$\cdots \quad X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

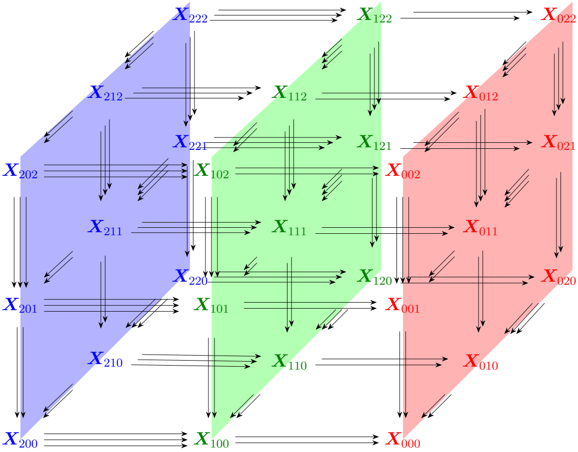
- Strict 2-category $X \in$ 2-Cat $\in [\Delta^{op}, \mathrm{Cat}]$



$$\cdots \quad X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

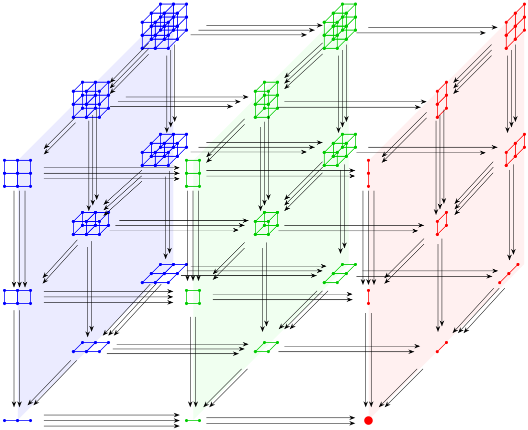
In the following picture, for all $i, j, k \in \Delta^{op}$

$$X_{2jk} \cong X_{1jk} \times_{X_{0jk}} X_{1jk}, \;\; X_{i2k} \cong X_{i1k} \times_{X_{i0k}} X_{i1k}, \;\; X_{ij2} \cong X_{ij1} \times_{X_{ij0}} X_{ij1} \,.$$



Geometric picture of the 3-fold nerve of a 3-fold category X

$$X \in \text{Cat}^3 \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \text{Set}]$$



Geometric picture of the 3-fold nerve of a strict 3-category X

$$\text{3-Cat} \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \text{Set}]$$