Fibered aspects of Yoneda's regular span

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Abstract

In his pioneering 1960 paper [7], Nobuo Yoneda presents a formal categorical setting in order to formulate the classical theory of $\mathbf{Ext}^n$ functors avoiding the request of having enough projectives. The starting point is to study, in an additive category, the functor sending any exact sequence of length n

$$0 \longrightarrow b \xrightarrow{j} e_n \longrightarrow e_{n-1} \longrightarrow \cdots \qquad \longrightarrow e_1 \xrightarrow{p} a \longrightarrow 0$$

to the pair (a, b).

The basic observation is that, thanks to the dual properties of pushouts and pullbacks in Yoneda's additive setting, it is possible to define translations and cotranslations of exact sequences. This analysis leads Yoneda to identify a set of formal properties of a functor

$$S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$$

in order to get the axioms of what he calls a *regular span*.

In fact, his Classification Theorem in [7, §3.2] follows in a purely formal way from the axioms of regular span, once one considers connected components of the fibers of S (called *similarity classes* in [*loc. cit.*]). We are able to interpret his result by saying that, with any regular span S, it is possible to associate what is nowadays called (see [4] and also [6]) a *two-sided discrete fibration* $\bar{S}\colon \bar{\mathcal{X}} \to \mathcal{A} \times \mathcal{B}$, together with a factorization of S through a functor $Q\colon \mathcal{X} \to \bar{\mathcal{X}}$.

In this talk, I will start by the key observation that Yoneda's notion of a regular span $S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$ can be interpreted as a special kind of morphism, that we call *fiberwise opfibration*, in the 2-category $\mathbf{Fib}(\mathcal{A})$. We study the relationship between these notions and those of internal opfibration and two-sided fibration. This fibrational point of view makes it possible to interpret Yoneda's Classification Theorem as the result of a canonical factorization, and to extend it to a non-symmetric situation, where the fibration given by the product projection $Pr_0\colon \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ is replaced by any split fibration over $\mathcal{A}$. This new setting allows us to transfer Yoneda's theory of extensions to the non-additive analog given by crossed extensions for the cases of groups (see [3]) and other algebraic structures.

This is a join work with A. S. Cigoli, G. Metere and E. M. Vitale.

References

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