Tensor structure on $kC$-mod and cohomology rings

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1. Basic concepts

- \( \mathcal{C} \) a small category (we will assume it is finite throughout this talk);

- \( k \) a field and \( Vect_k \) the category of \( k \)-vector spaces.

There are three well defined mathematical subjects:

- the category algebra \( k\mathcal{C} \);

- the functor category \( Vect^\mathcal{C}_k \);

- the classifying space \( BC \).
The \textit{k-category algebra} is defined as a \textit{k}-vector space \textit{k}\text{Mor} \mathcal{C}, in which the multiplication is given on base elements by

$$\alpha \ast \beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \text{ and } \beta \text{ are composable in } \mathcal{C}; \\ 0, & \text{otherwise}. \end{cases}$$

**An example** Let \textit{k} be a field and \mathcal{C} the following category

\[
\begin{array}{c}
1_x \xrightarrow{\alpha} y \\
\downarrow g \\
1_y
\end{array}
\]

with \(g^2 = 1_y\) and \(\alpha = g\alpha\). Then the category algebra is

\[
k\mathcal{C} = k1_x + k\alpha + k1_y + kg,
\]

with an identity \(1_{k\mathcal{C}} = 1_x + 1_y\).
**Definition** A $k$-representation of $\mathcal{C}$ is a covariant functor $\lambda : \mathcal{C} \to \text{Vect}_k$.

**Slogan** Representations = Functors !!!
• an identity $1_C = \sum_{x \in \text{Ob}C} 1_x$;

• an equivalence $kC\text{-mod} \simeq Vect^C_k$ (B. Mitchell 1972) given by: $M \mapsto F_M, F_M(x) = 1_x \cdot M$ and $F \mapsto \bigoplus_{x \in \text{Ob}C} F(x)$. Thus, **covariant functors** $C \to Vect_k = k$-representations of $C = \text{left } kC$-modules. Moreover this equivalence gives $kC\text{-mod}$ a monoidal category structure: there exists a tensor product $\otimes_k$ and a tensor identity $k$;

• The canonical diagonal functor

$$\Delta : C \to C \times C$$

induces, by linearization, a co-multiplication $\Delta : kC \to kC \otimes_k kC$,

$$\Delta(\sum_i \lambda_i \alpha_i) = \sum_i \lambda_i \alpha_i \otimes \alpha_i.$$
2. Tensor structure on $kC$-mod

Let $M, N \in kC$-mod. The (internal) tensor product $M \hat{\otimes}_k N \in kC$-mod is defined by

$$(M \hat{\otimes}_k N)(x) = M(x) \otimes_k N(x),$$

where $x \in \text{Ob } C$. Note that $M \hat{\otimes}_k N \subset M \otimes_k N$ as vector spaces, and the category algebra acts on it via $\Delta : kC \to kC \hat{\otimes}_k kC \subset kC \otimes_k kC$.

There is a \textbf{tensor identity} $k \in kC$-mod, which as an element in $\text{Vect}_k^C$ is the constant functor taking $k$ as its values. The tensor identity $k$ is also called the \textbf{trivial module} of $kC$.

\textbf{Lemma} The trivial module can be realized by $k \text{Ob } C$. In fact, there exists a surjection

$$\epsilon : kC \to k \text{Ob } C,$$

defined on base elements by $\epsilon(\alpha) = t(\alpha)$, the target of the morphism $\alpha$. This gives $k \text{Ob } C$ a $kC$-module structure and it is isomorphic to $k$.

The map $\epsilon : kC \to k$ plays the role of a co-unit.
The algebra \( kC \) is almost a **co-commutative bialgebra** in the sense that we can produce the following structure maps:

1. **co-associativity**

\[
\begin{align*}
\Delta: kC & \rightarrow kC \otimes kC \\
\Delta \downarrow & \downarrow \Delta \otimes 1 \\
kC \otimes kC & \rightarrow kC \otimes kC \otimes kC;
\end{align*}
\]

2. **co-unitary property**

\[
\begin{align*}
\cong & \quad \cong \\
\Delta & \quad \Delta \\
kC & \rightarrow kC \otimes kC \\
kC \otimes kC & \rightarrow kC \otimes kC \otimes kC ;
\end{align*}
\]

3. **co-commutativity**

\[
\begin{align*}
\Delta & \quad \Delta \\
\Delta & \quad \Delta \\
kC & \rightarrow kC \otimes kC \\
kC \otimes kC & \rightarrow kC \otimes kC ;
\end{align*}
\]

4. **multiplication and co-multiplication**

\[
\begin{align*}
\Delta \otimes \Delta & \quad \mu \\
\mu & \quad \Delta \\
kC \otimes kC & \rightarrow kC \\
kC \otimes kC \otimes kC \otimes kC & \rightarrow kC \otimes kC ;
\end{align*}
\]

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5. unit and co-multiplication:

\[
\begin{array}{ccc}
\mu & k \otimes k & \pi \circ (\iota \otimes \iota) \\
\downarrow & kC \quad \Delta & kC \hat{\otimes} kC;
\end{array}
\]

where \( \iota \) is the inclusion \( k = k1_C \hookrightarrow kC \) and \( \pi \) is the truncation map \( kC \otimes kC \rightarrow kC \hat{\otimes} kC \).

6. unit and co-unit

\[
\begin{array}{ccc}
\iota & k & \eta \\
\downarrow & kC \quad \epsilon & k,
\end{array}
\]

where the \( k \)-linear map \( \eta : k = k1_C \rightarrow k \) is defined by \( 1_C \mapsto \sum_{x \in \text{Ob} \ C} x \).

**Remark** a) We do not have a bialgebra in the usual sense; b) There is **no antipode** map in general. This is a significant difference between a category algebra and a cocommutative Hopf algebra.
3. Cohomology rings

We call $\text{Ext}^*_{k\mathcal{C}}(k, k)$ the **ordinary cohomology ring** of $k\mathcal{C}$ (with Yoneda splice).

There is also a **Hochschild cohomology ring** $\text{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C})$. Here $k\mathcal{C}^e$ is the category algebra of $\mathcal{C}^e := \mathcal{C} \times \mathcal{C}^{op}$, which is isomorphic to the enveloping algebra $(k\mathcal{C})^e := k\mathcal{C} \otimes_k (k\mathcal{C})^{op}$.

Note that $k\mathcal{C} \in k\mathcal{C}^e\text{-mod} \simeq \mathcal{C}^{\text{Vect}}e$ as a functor is described by $k\mathcal{C}(x, y) = k\text{Hom}_\mathcal{C}(y, x)$.

**Main Theorem** There is a natural split surjective algebra homomorphism

$$\text{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C}) \to \text{Ext}^*_{k\mathcal{C}}(k, k).$$

**Remark** Note that $k\mathcal{C}^e\text{-mod}$ is a monoidal category w.r.t. the tensor product $\otimes_{k\mathcal{C}}$ and tensor identity $k\mathcal{C}$. It has another monoidal category structure w.r.t. $\hat{\otimes}_k$ and tensor identity $\bar{k}$. 
Let $M, M', N, N' \in kC\text{-mod}$. We will define the cup product to be

$$
\cup: \operatorname{Ext}^i_{kC}(M,N) \otimes \operatorname{Ext}^j_{kC}(M',N') \to \operatorname{Ext}^{i+j}_{kC}(M \hat{\otimes} M', N \hat{\otimes} N').
$$

This gives $\operatorname{Ext}^\ast_{kC}(k, k)$ a ring structure and its action on $\operatorname{Ext}^\ast_{kC}(M, N)$ for arbitrary $M, N \in kC\text{-mod}$. Note that $\operatorname{Ext}^\ast_{kC}(k, F) \cong \operatorname{H}^\ast(C; F) \cong \lim \leftarrow C F$ for any $F \in kC\text{-mod}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two complexes of $kC$-modules.

- The product $(\mathcal{C} \hat{\otimes} \mathcal{D})_n = \bigoplus_{i+j=n} \mathcal{C}_i \hat{\otimes} \mathcal{D}_j$, with the differential (a natural transformation)

$$
\partial_x (a \otimes b) = \partial_x^\mathcal{C} a \otimes b + (-1)^{\deg(a)} a \otimes \partial_x^\mathcal{D} b,
$$

where $a \in \mathcal{C}_i(x)$ and $b \in \mathcal{D}_j(x)$, for each $x \in \text{Ob} \mathcal{C}$.

- A Künneth formula: for each integer $n$

$$
\operatorname{H}_n(\mathcal{C} \hat{\otimes} \mathcal{D}) \cong \bigoplus_{i+j=n} \operatorname{H}_i(\mathcal{C}) \hat{\otimes} \operatorname{H}_j(\mathcal{D}).
$$
Suppose $\zeta \in \text{Ext}^m_{kC}(M, N)$ is represented by
\[0 \to N \to L_{m-1} \to \cdots \to L_0 \to M \to 0,\]
and $\zeta' \in \text{Ext}^n_{kC}(M', N')$ is represented by
\[0 \to N' \to L'_{n-1} \to \cdots \to L'_0 \to M' \to 0.\]
We construct an exact sequence
\[0 \to N \hat{\otimes} N' \to (L_{m-1} \hat{\otimes} N) \oplus (N \hat{\otimes} L'_{n-1}) \to \cdots \to L_0 \hat{\otimes} L'_0 \to M \hat{\otimes} M' \to 0,\]
and it is defined to be the cup product of $\zeta$ and $\zeta'$, $\zeta \cup \zeta' \in \text{Ext}^{m+n}_{kC}(M \hat{\otimes} M', N \hat{\otimes} N').$

**Lemma** Let $\zeta, \zeta'$ be as above. The cup product $\zeta \cup \zeta'$ is the Yoneda splice of
\[\zeta \hat{\otimes} \text{Id}_{N'} \in \text{Ext}^i_{kC}(M \hat{\otimes} N', N \hat{\otimes} N')\]
with
\[\text{Id}_M \hat{\otimes} \zeta' \in \text{Ext}^j_{kC}(M \hat{\otimes} M', M \hat{\otimes} N').\]
It says that “cup product=Yoneda splice” on $\text{Ext}^*_C(k, k)$. We demonstrate they are the same as the “simplicial cup product”.

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**Proposition** With the above cup product,

\[ \text{Ext}^*_{k^*}(k, k) \cong H^*(BC, k) \]

as algebras.

The ring \( H^*(BC, k) \) is computed from the nerve \( N_*C \) of \( C \).

- \( N_0C = \text{Ob} \ C \);

- \( N_nC = \{ x_0 \xrightarrow{\alpha_1} x_1 \rightarrow \cdots \rightarrow x_n | \alpha_i \in \text{Mor} \ C \} \), if \( n > 0 \).

On the simplicial complex \( kN_*C \to 0 \), for each base element \( x_0 \xrightarrow{\alpha_1} x_1 \rightarrow \cdots \xrightarrow{\alpha_n} x_n \in kN_nC \),

\[
\delta(x_0 \xrightarrow{\alpha_1} x_1 \rightarrow \cdots \xrightarrow{\alpha_n} x_n) = \sum_{i=0}^n (-1)^i x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} x_i \rightarrow \cdots \xrightarrow{\alpha_n} x_n.
\]

The cohomology ring \( H^*(BC, k) \) is the homology of the cochain complex \( 0 \to \text{Hom}_k(kN_*C, k) \), and the cup product is given by the Alexander-Whitney map on \( kN_*C \to 0 \).
In order to compare \( \text{Ext}^*_k(k, k) \) with \( H^*(BC, k) \),
We note that each \( kN_nC \) has a \( kC \)-module structure, and especially \( kN_0C = k \text{Ob} \mathcal{C} \cong k \).

**The bar resolution** \( \mathcal{B}^C_* = \mathcal{B}_* \) of \( k \). For each \( x \in \text{Ob} \mathcal{C} \), \( \mathcal{B}_n(x) \) is the \( k \)-vector space with base elements of the form

\[
x_0 \xrightarrow{\alpha_1} x_1 \rightarrow \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x
\]

with \( x_i, x \in \text{Ob} \mathcal{C}, \alpha_i, \alpha \in \text{Mor}(\mathcal{C}), \) and a non-negative \( n \in \mathbb{Z} \). The differential, as a natural transformation, is defined subsequently as

\[
\delta_x(x_0 \xrightarrow{\alpha_1} x_1 \rightarrow \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x) = \sum_{i=0}^n (-1)^i x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} \bar{x_i} \rightarrow \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x.
\]

The complex of \( kC \)-modules \( \mathcal{B}_* \to k \to 0 \) is a projective resolution, and for any \( M \in kC \)-mod

\[
\text{Hom}_{kC}(\mathcal{B}_n, M) \cong \prod_{x_0 \to x_1 \to \cdots \to x_n \in N_nC} M(x_n).
\]

Note that \( \mathcal{B}_0 = kC \).
Using the bar resolution $B_* \to k \to 0$, we can describe the cup product on $\text{Ext}^*_k(k, k)$.

A diagonal approximation map $D : B_* \to B_* \hat{\otimes} B_*$, as a natural transformation, is given by

$$D_x(x_0 \xrightarrow{\alpha_1} x_1 \to \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x) = \sum_{i=0}^n (x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} x_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha} x),$$

for any $x \in \text{Ob} C$ and integer $n$.

- $\text{Hom}_{kC}(B_*, k) \cong \text{Hom}_k(kN_*, k)$ as complexes;

- $D$ corresponds to the Alexander-Whitney map.

These imply that $\text{Ext}^*_k(k, k) \cong H^*(BC, k)$ as algebras.
4. Main theorem

There exists a natural split surjective algebra homomorphism

$$\phi_C : \text{Ext}_{kC}^*(kC, kC) \to \text{Ext}_{kC}^*(k, k).$$

- This algebra homomorphism is given by $- \otimes_{kC} k$;

- known to be true for groups (Cartan-Eilenberg 1956 etc) and posets (Gerstenhaber-Shack 1986), with distinct proofs;

- regarding groups and posets as small categories, our result generalizes both well known theorems.
Key features of the proof:

- Use $F(C)$, the category of factorizations in $C$, introduced by D. Quillen. It generalizes and replaces the diagonal subgroup $\Delta G \subset G \times G$ when $C = G$ is a group. Indeed there exists a commutative diagram

$$
\begin{array}{ccc}
F(C) & \xrightarrow{(t,s)} & C^e \\
\downarrow{t} & & \downarrow{pr} \\
C & & C \times C^{op}
\end{array}
$$

It was shown by Quillen that $t : F(C) \to C$ induces a homotopy equivalence.

- Use the left Kan extensions $LK_\tau$, $LK_{pr}$ and $LK_t$, which generalize various inductions (e.g. $\uparrow_{\Delta G}^{G \times G}$).
The preceding diagram gives rise to new commutative diagrams of module categories and functors among them:

\[
\begin{array}{ccc}
  kF(C)\text{-}mod & \xrightarrow{\text{Res}_\tau} & kC^e\text{-}mod \\
  \text{Res}_t & & \text{Res}_\text{pr} \\
  & kC\text{-}mod & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
  kF(C)\text{-}mod & \xrightarrow{LK_\tau} & kC^e\text{-}mod \\
  \text{LK}_t & & \text{LK}_\text{pr} \\
  & kC\text{-}mod & \\
\end{array}
\]

Here \(\text{Res}_\tau\) is the functor induced by \(\tau\) by pre-composition, and is called the \textbf{restriction} along \(\tau\). The functor \(LK_\tau\) is the well known left adjoint of it, called the \textbf{left Kan extension} of \(\tau\). The other two pairs of functors are constructed in the same way over \(t\) and \(pr\), respectively.

- \(\text{Res}\) is always exact and preserves \(k\);
- \(LK\) always preserves projectives.
Moving to cohomology:

1. $L K_t$ maps the bar resolution

   $$\mathcal{B}_*^{F(C)} \rightarrow k \rightarrow 0$$

   to a projective resolution of $k\mathcal{C}$-modules

   $$L K_t \mathcal{B}_*^{F(C)} \rightarrow L K_t k \cong k \rightarrow 0;$$

2. $L K_{\tau}$ takes the above bar resolution to a projective resolution of $k\mathcal{C}^e$-modules

   $$L K_{\tau} \mathcal{B}_*^{F(C)} \rightarrow L K_{\tau} k \cong k \mathcal{C} \rightarrow 0;$$

3. $L K_{pr}$ furthermore maps the projective resolution of $k\mathcal{C}$ in 2) to the projective resolution of the $k\mathcal{C}$-module $\underline{k}$ in 1).

4. $\text{Res}_{\tau} k\mathcal{C} \cong \underline{k} \oplus N_{\mathcal{C}}$, as a $kF(C)$-module.
Along with the adjunctions with corresponding restrictions, the previous observations on the three left Kan extensions lead to a commutative diagram of cohomology rings

\[
\begin{array}{ccc}
\text{Ext}^*_{kF(C)}(k, k) & \xrightarrow{\tau^*} & \text{Ext}^*_{kC}(kC, kC) \\
\downarrow & & \downarrow \phi_C \\
\text{Ext}^*_k(k, k) & \xrightarrow{pr^*} & \end{array}
\]

The map \( pr^* \) is the same as the one induced by \(- \otimes_{kC} k\) and is often written as \( \phi_k \) or \( \phi_C \). The map \( t^* \) is an isomorphism. Then \( \tau^*(t^*)^{-1} \) becomes a right inverse to \( pr^* = \phi_C \).

**Remark** The map \( t^* \) can be identified with the homomorphism \( H^*(BF(C), k) \to H^*(BC, k) \) induced by the topological map \( Bt : BF(C) \to BC \) which is a homotopy equivalence,
Let $f, g$ be two cocycles representing two cohomology classes. Then we construct the following diagram

$$
\begin{align*}
& f \cup g : \\
& B^F(C) \xrightarrow{D^F(C)} B^F(C) \otimes B^F(C) \xrightarrow{f \otimes g} k \otimes k \cong k \\
& \downarrow LK_\tau \\
& LK_\tau(f \cup g) : \\
& LK_\tau B^F_*(C) \xrightarrow{LK_\tau(D^F(C))} LK_\tau(B^F_*(C) \otimes B^F_*(C)) \xrightarrow{LK_\tau(f \otimes g)} LK_\tau(k \otimes k) \cong kC \\
& \Theta_\tau \\
& LK_\tau(f) \cup LK_\tau(g) : \\
& LK_\tau B^F_*(C) \xrightarrow{D^c} LK_\tau B^F_*(C) \otimes kC \xrightarrow{LK_\tau(f) \otimes LK_\tau(g)} LK_\tau(k) \otimes kC \xrightarrow{\Theta_0} LK_\tau(k) \cong kC.
\end{align*}
$$

The commutativity of the lower two rows shows that $\tau^*$ preserves cup products.
More generally, for any $M \in kC^e$-mod,

$$\text{Ext}^*_{kC^e}(kC, M) \cong \text{Ext}^*_{kF(C)}(k, \text{Res}_\tau M).$$

The splitting of $\text{Res}_\tau kC \cong k \oplus N_C$ induces a surjective homomorphism $\rho : \text{Res}_\tau kC \hat{\otimes} \text{Res}_\tau M \to \text{Res}_\tau M$, and

$$\rho^* : \text{Ext}^*_{kF(C)}(k, \text{Res}_\tau kC \hat{\otimes} \text{Res}_\tau M) \to \text{Ext}^*_{kF(C)}(k, \text{Res}_\tau M).$$

The latter fits into the following commutative diagram

$$
\begin{align*}
\text{Ext}^*_{kF(C)}(k, \text{Res}_\tau kC) \otimes \text{Ext}^*_{kF(C)}(k, \text{Res}_\tau M) \ar[r] \ar[d] & \text{Ext}^*_{kF(C)}(k, \text{Res}_\tau kC \hat{\otimes} \text{Res}_\tau M) \ar[d] \ar[r]^{\rho^*} & \text{Ext}^*_{kF(C)}(k, \text{Res}_\tau M) \ar[d] \\
\text{Ext}^*_{kC^e}(kC, kC) \otimes \text{Ext}^*_{kC^e}(kC, M) \ar[r]_\cup & \text{Ext}^*_{kC^e}(kC, kC \otimes_{kC} M) \ar[r]_\cong & \text{Ext}^*_{kC^e}(kC, M).
\end{align*}
$$

Since $\text{Ext}^*_{kF(C)}(k, k)$ is a direct summand of $\text{Ext}^*_{kF(C)}(k, \text{Res}_\tau kC)$, it exhibits the action of $\text{Ext}^*_{kF(C)}(k, k)$ (and $\text{Ext}^*_{kC}(k, k)$) on $\text{Ext}^*_{kC^e}(kC, M)$. 
5. Examples

**Example 1** Let $k$ be a field of characteristic 2 and $\mathcal{C}$ the following category

![Diagram](image)

with $g^2 = 1_y$ and $\alpha = g\alpha$.

1. Two indecomposable projective: $P_{x,k} = k 1_x + k\alpha$ and $P_{y,k} = k 1_y + kg$;

2. Two simples of dimension one: $S_{x,k}, S_{y,k}$.

The trivial module is projective, $k \cong P_{x,k}$. Hence $\text{Ext}^*_{k\mathcal{C}}(k, M) \cong M(x)$ for any $M \in k\mathcal{C}$-mod. Especially $\text{Ext}^*_{k\mathcal{C}}(k, k) \cong k$. However, both modules $\text{Ext}^*_{k\mathcal{C}}(S_{x,k}, S_{y,k})$ and $\text{Ext}^*_{k\mathcal{C}}(S_{y,k}, S_{y,k})$ are infinite dimensional.
Example 2 A category by Aurélien Djament: \( \mathcal{E} \) as follows

\[
\begin{array}{c}
\text{h} & \xrightarrow{1_x} & \text{x} \\
\text{g} & \xrightarrow{\alpha} & \text{y} \xrightarrow{\{1_y\}} \\
\end{array}
\]

where \( g^2 = h^2 = 1_x, gh = hg, \alpha h = \beta g = \alpha, \) and \( \alpha g = \beta h = \beta. \) We can compute the mod-2 ordinary and Hochschild cohomology rings.

The ordinary cohomology ring is computed as \( H^*(B\mathcal{E}, k) \) using the fact that \( B\mathcal{E} \simeq B(\mathbb{Z}_2 \times \mathbb{Z}_2)/B\mathbb{Z}_2. \) Using a long exact sequence we find that \( \text{Ext}^*_{k\mathcal{E}}(k, k) \) is a subring of the polynomial ring \( H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, k) \cong k[u, v] \) with \( \deg u = \deg v = 1: \)

\[
\text{Ext}^*_k(k, k) \cong \begin{cases} 
  k, & \text{at degree zero;} \\
  uk[u, v], & \text{at positive degree.}
\end{cases}
\]

The ordinary cohomology ring has no nilpotents and is not finitely generated.
By our Main Theorem, there exists a split surjection

\[ \phi_E : \text{Ext}^*_{kE}(kE, kE) \to \text{Ext}^*_{kE}(k, k). \]

Thus the Hochschild cohomology ring of \( kE \) modulo nilpotents is not finitely generated either. In fact, in this case, the kernel of \( \phi_E \) is explicitly calculated and consists of nilpotents only. As a reminder, \( kE \) is not Hopf. It is not even self-injective.

Further examples have been computed by Nicole Snashall:

There exists a 7-dimensional Koszul algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated, independent of the choice of the base field. This algebra is isomorphic to \( kE \) when \( \text{char } k = 2 \).
The category of factorizations in $\mathcal{E}$, $F(\mathcal{E})$, has the following shape

\[
\begin{array}{c}
\alpha \\
\downarrow \\
1x \\
\downarrow \\
h \\
\downarrow \\
g \\\n\downarrow \\
gh \\
\downarrow \\
g \\
\downarrow \\
\beta \\
\downarrow \\
1y
\end{array}
\]

in which $[1_x] \cong [h] \cong [g] \cong [gh]$ and $[\alpha] \cong [\beta]$. For the purpose of computation, we use the skeleton $F'(\mathcal{E})$ of $F(\mathcal{E})$. Note that $F(\mathcal{E}) \simeq F'(\mathcal{E})$. Hence $BF(\mathcal{E}) \simeq BF'(\mathcal{E})$ and $kF(\mathcal{E})\text{-}\text{mod} \simeq kF'(\mathcal{E})\text{-}\text{mod}$.

In the above category, next to each arrow is the set of homomorphisms in $F'(\mathcal{E})$ from one object to another.


