Graded Blocks of Group Algebras

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University of Leicester, 21 June 2010
Motivation

Theorem (Rouquier, 2001)

Let $D^b(A\text{-mod}) \cong D^b(B\text{-mod})$. If $A$ is graded, then $B$ is graded.
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Let $D^b(A\text{-mod}) \cong D^b(B\text{-mod})$. If $A$ is graded, then $B$ is graded.

**Problem**

How do we effectively transfer gradings between derived equivalent algebras $A$ and $B$?
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**Problem**

How do we effectively transfer gradings between derived equivalent algebras $A$ and $B$?

If the grading on $A$ has some interesting properties, does the resulting grading on $B$ have the same properties?
Overview

1 Preliminaries
   ▶ Brauer tree algebras
   ▶ Graded algebras

2 The tilting complex

3 Example

4 The general case

5 Properties of the resulting grading

6 Classification of gradings
Brauer tree algebras

- $\Gamma \rightsquigarrow$ finite connected tree with $e$ edges.

- $\Gamma$ is a Brauer tree of type $(m, e)$ if given:
  - Cyclic ordering of the edges adjacent to a given vertex,
  - The exceptional vertex $v$, to whom we assign the multiplicity $m$. 

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\begin{array}{c}
\circ \\
V_1 \quad U_r \\
S \\
V_t \quad U_1 \\
\circ
\end{array}
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- $V \sim V_1, V_2, \ldots, V_t,$
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V & \quad & S \\
\end{array}
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- $V \sim V_1, V_2, \ldots, V_t$,
- $U \sim U_1, U_2, \ldots, U_r; S, U_1, U_2, \ldots, U_r; \ldots; S, U_1, U_2, \ldots, U_r.$
Brauer tree algebras

- $\Gamma$ is of type $(2, 4)$
Brauer tree algebras

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\[ P_{S_1} = \begin{array}{ccc} S_1 & S_2 & S_3 \\ S_2 & S_1 & S_4 \end{array}, \quad P_{S_2} = S_1, \quad P_{S_3} = \begin{array}{ccc} S_3 & S_4 & S_2 \\ S_2 & S_3 & S_1 \end{array} \]
Brauer tree algebras

- The Brauer star of type \((m, e)\)
Brauer tree algebras

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- The projective indecomposable module \(P_j\) is uniserial.
Brauer tree algebras

- Any two Brauer tree algebras associated with the same Brauer tree $\Gamma$ and defined over the same field $k$ are Morita equivalent.

- A basic Brauer tree algebra corresponding to a Brauer tree $\Gamma$ is isomorphic to the algebra $kQ/I$. 
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A basic Brauer tree algebra corresponding to a Brauer tree $\Gamma$ is isomorphic to the algebra $kQ/I$.

$Q$ and $I$ are easily constructed from $\Gamma$.

$A_\Gamma := kQ/I$
**Graded algebras**

**Definition**

A is a **graded algebra** if $A$ is the direct sum of subspaces $A = \bigoplus_{i \in \mathbb{Z}} A_i$, such that $A_i A_j \subset A_{i+j}$, $i, j \in \mathbb{Z}$.
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- $a_i \in A_i \rightarrow \deg(a_i) = i$
- If $A_i = 0$ for $i < 0$, then $A$ is positively graded.
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**Definition**

An $A$-module $M$ is graded if $M = \bigoplus_{i \in \mathbb{Z}} M_i$, and $A_i M_j \subset M_{i+j}$. 

N$_\langle n \rangle$ denotes the graded module given by $N_j = M_{n+j}$, $j \in \mathbb{Z}$. 

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- $N = M\langle n \rangle$ denotes the graded module given by $N_j = M_{n+j}$, $j \in \mathbb{Z}$.

**Definition**

An $A$-module homomorphism $f$ between two graded modules $M$ and $N$ is a homomorphism of graded modules if $f(M_i) \subset N_i$, for all $i \in \mathbb{Z}$.
Graded algebras

- $X = (X^i, d^i)_{i \in \mathbb{Z}}$ is a graded complex if $X^i$ is a graded module, $d^i$ is a homomorphism of graded $A$-modules, for all $i$.

- $f = \{f^i\}_{i \in \mathbb{Z}} \in \text{Hom}_A(X, Y)$ is a homomorphism of graded complexes if $f^i$ is a homomorphism of graded modules.

- $X_{\langle j \rangle}$ denotes the complex $\left( X_{\langle j \rangle}^i \right)_{i \in \mathbb{Z}}$:
  
  - $X_{\langle j \rangle}^i = X^i$ for all $i$.
  - $d^i_{X_{\langle j \rangle}} = d^i$.

- Theorem: If $X$ and $Y$ are graded $A$-modules (or complexes), then
  
  $$\text{Hom}_A(X, Y) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_{\text{gr}}}(X_{\langle i \rangle}, Y_{\langle i \rangle}).$$

- $\text{Hom}_{A_{\text{gr}}}(X, Y) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_{\text{gr}}}(X_{\langle i \rangle}, Y_{\langle i \rangle})$. The subspace $H_{\text{t}}(X, Y)$ of zero homotopic maps is homogeneous.

- From this we get a grading on $\text{Hom}_{K_{\text{b}}}(P_{A_{\text{gr}}}(X, Y))$:
  
  $$\text{Hom}_{K_{\text{b}}}(P_{A_{\text{gr}}}(X, Y)) := \text{Hom}_{A_{\text{gr}}}(X, Y) / H_{\text{t}}(X, Y).$$

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- The subspace $Ht(X, Y)$ of zero homotopic maps is homogeneous.
- From this we get a grading on $\text{Hom}_{K^b(P_A)}(X, Y)$

$$\text{Homgr}_{K^b(P_A)}(X, Y) := \text{Homgr}_A(X, Y)/Ht(X, Y)$$
The tilting complex

**Theorem (Rickard 1989)**

*Up to derived equivalence, a Brauer tree algebra is determined by the number of edges and the multiplicity of the exceptional vertex.*
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- $\Gamma \rightsquigarrow$ an arbitrary Brauer tree of type $(m, e)$
- $S \rightsquigarrow$ the Brauer star of type $(m, e)$
- $D^b(A_S\text{-mod}) \cong D^b(A_{\Gamma}\text{-mod})$

$A_S$ is tightly graded, i.e. $A_S \cong \bigoplus_{i=0}^{\infty} (\text{rad} A_S)^i / (\text{rad} A_S)^{i+1}$.

$A_{\Gamma}$ is graded. Question: What is the corresponding grading on $A_{\Gamma}$?
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- $A_\Gamma$ is graded.

**Question**

What is the corresponding grading on $A_\Gamma$?
The tilting complex

- There exist a tilting complex $T$ such that $\text{End}_{K^b(P_{A_S})}(T)^{op} \cong A_\Gamma$. 

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$T$ can be constructed by taking Green's walk around $\Gamma$.

There are many such tilting complexes (studied by Rickard, Schaps & Zakay-Illouz).
The tilting complex

- There exist a tilting complex $T$ such that $\text{End}_{K^b(P_{A_S})}(T)^{op} \cong A_\Gamma$.
- $T$ can be constructed by taking Green’s walk around $\Gamma$.
- There are many such tilting complexes (studied by Rickard, Schaps & Zakay-Illouz).
- $T$ is a direct sum of $e$ indecomposable complexes with at most two non-zero terms.
- The summand $T_i$ corresponds to the edge $S_i$ of $\Gamma$. 
Calculating \( \text{End}_{K^b(P_A S)}(T)^{op} \)

- If \( T \) is a graded complex, then \( \text{Endgr}_{K^b(P_A S)}(T)^{op} \) is a graded algebra.
Calculating $\text{End}_{K^b(P_{AS})}(T)^{op}$

- If $T$ is a graded complex, then $\text{Endgr}_{K^b(P_{AS})}(T)^{op}$ is a graded algebra.

- When identifying $\text{End}_{K^b(P_{AS})}(T)^{op}$ with $A_\Gamma$, $\text{Hom}_{K^b(P_{AS})}(T_i, T_j)$ corresponds to the space spanned by all paths starting at $i$ and ending at $j$ in the quiver of $A_\Gamma$. 

"Theorem (B. 2007)"
Calculating $\text{End}_{K^b(P_{A_S})}(T)^{op}$

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- When identifying $\text{End}_{K^b(P_{A_S})}(T)^{op}$ with $A_\Gamma$, $\text{Hom}_{K^b(P_{A_S})}(T_i, T_j)$ corresponds to the space spanned by all paths starting at $i$ and ending at $j$ in the quiver of $A_\Gamma$.

**Theorem (B. 2007)**

Let $A_\Gamma$ be a basic Brauer tree algebra corresponding to an arbitrary Brauer tree $\Gamma$. Then there exists a positive grading on $A_\Gamma$. 
Example
Example

\[ S_1 \]
Example
Example
Example
Example
Example

$T$ is the direct sum of:

\[
\begin{align*}
T_1 & : 0 \to P_1 \to 0 \\
T_2 & : 0 \to P_2 \to 0 \\
T_3 & : 0 \to P_3 \to 0 \\
T_4 & : 0 \to P_3 \to P_4\langle 5\rangle \to 0 \\
T_5 & : 0 \to P_3 \to P_5\langle 4\rangle \to 0 \\
T_6 & : 0 \to P_5\langle 4\rangle \to P_6\langle 9\rangle \to 0
\end{align*}
\]

- The differentials are maps of maximal ranks.
- This complex tilts from $A_S$ to $A_\Gamma$: $\text{End}_{K^b(P_{AS})}(T) \cong A_\Gamma^{op}$.
- $\text{Homgr}_{K^b(P_{AS})}(T_i, T_j) \rightsquigarrow \bullet \xrightarrow{i, \alpha} \bullet$
Example
The general case

- Edges \( i \) and \( j \) of a Brauer tree \( \Gamma \) are adjacent to the same vertex.
- \( i \) and \( j \) are at the same distance from the exceptional vertex:
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- Edges $i$ and $j$ of a Brauer tree $\Gamma$ are adjacent to the same vertex.
- $i$ and $j$ are at the same distance from the exceptional vertex:

\[
\begin{array}{c}
\bullet \\
S_i & \downarrow \alpha \\
S_j & \\
\end{array}
\quad \begin{array}{c}
\bullet \\
S_i & \downarrow \beta \\
S_j & \\
\end{array}
\]

- If $i > j$, then $\text{deg}(\alpha) = i - j$. If $i < j$, then $\text{deg}(\alpha) = e - (j - i)$.
- $\text{deg}(\beta) = 0$. 
The general case

- The distance of one the edges, say $i$, is one less than the distance of $j$: 

\[ S_l \leadsto S_i \leadsto S_f \]

\[ S_j \leadsto S_f \leadsto S_j \]

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The general case

- The distance of one of the edges, say $i$, is one less than the distance of $j$: 

  \[ \deg(\alpha) = m; \]
  \[ \deg(\beta) = 0. \]
The general case

\[ S_{11} \longrightarrow S_{10} \longrightarrow S_9 \bullet S_1 \longrightarrow S_6 \longrightarrow S_7 \]

\[ S_8 \]

\[ S_2 \]

\[ S_3 \quad S_5 \]

\[ S_4 \]
The general case
The subalgebra $A_0$
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- The quiver of $A_0$ is a union of directed rooted trees.
- $A_0$ is a quasi-hereditary algebra.
The subalgebra $A_0$

- The quiver of $A_0$ is a union of directed rooted trees.
- $A_0$ is a quasi-hereditary algebra.
- The global dimension of $A_0$ is bounded by the length (as a rooted tree) of the quiver of $A_0$.
- From $A_0$ we can recover $A_{\Gamma}$.
The group $\text{Out}^K(A_\Gamma)$

- Gradings are controlled by $\text{Out}(A)$.
The group $\text{Out}^K(A_\Gamma)$

- Gradings are controlled by $\text{Out}(A)$.
- $A = \bigoplus_{i \in \mathbb{Z}} A_i \leftrightarrow \pi : \mathbb{G}_m \to \text{Out}(A)$. 

A "good" case is when the maximal tori of $\text{Out}(A)$ are isomorphic to $\mathbb{G}_m$. In this case there is essentially unique grading on $A$. 

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The group $\text{Out}^K(A_\Gamma)$

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- $A = \bigoplus_{i \in \mathbb{Z}} A_i \iff \pi : G_m \rightarrow \text{Out}(A)$.
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- In this case there is essentially unique grading on $A$. 

The group $\text{Out}^K(A_\Gamma)$

**Theorem (B. 2008)**

Let $\Gamma$ be an arbitrary Brauer tree of type $(m, e)$. Then

$$\text{Out}^K(A_\Gamma) \cong \text{Aut}(k[x]/(x^{m+1})).$$
The group $\text{Out}^K(A_\Gamma)$

**Theorem (B. 2008)**

Let $\Gamma$ be an arbitrary Brauer tree of type $(m, e)$. Then

$$\text{Out}^K(A_\Gamma) \cong \text{Aut}(k[x]/(x^{m+1})).$$

**Corollary (B. 2008)**

There is a unique grading on $A_\Gamma$ up to graded Morita equivalence and rescaling.