# 1 Collision of viscoelastic particles with adhesion

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**Abstract:** The collision of convex bodies is considered for small impact velocity, when plastic deformation and fragmentation may be disregarded. In this regime the contact is governed by forces according to viscoelastic deformation and by adhesion. The viscoelastic interaction is described by a modified Hertz law while for the adhesive interactions the model by Johnson, Kendall and Roberts (JKR) is adopted. We solve the general contact problem of convex viscoelastic bodies in quasistatic approximation, which implies that the impact velocity is much smaller than the speed of sound in the material and that the viscosity relaxation time is much smaller than the duration of a collision. We estimate the threshold impact velocity which discriminates restitutive and sticking collisions. If the impact velocity is not large as compared with the thresold velocity, adhesive interaction becomes important, thus limiting the validity of the pure viscoelastic collision model.

## 1.1 Introduction

The large set of phenomena observed in granular systems, ranging from sand and powders on Earth to granular gases in planetary rings and protoplanetary discs, is caused by the specific particle interaction. Besides elastic forces, common for molecular or atomic materials (solids, liquids and gases), colliding granular particles exert also dissipative forces. These forces correspond to the dissipation of mechanical energy in the bulk of the grain material as well as on their surfaces. The dissipated energy transforms into energy of the internal degrees of freedom of the grains, that is, the particles are heated. In many applications, however, the increase of temperature of the particle material may be neglected (see, e.g. [5]).

The dynamical properties of granular materials depend sensitively on the details of the dissipative forces acting between contacting grains. Therefore, choosing the appropriate model of the dissipative interaction is crucial for the adequate description of these systems. In real granular systems the particles may have a complicated non-spherical shape, they may be nonuniform and even composed of smaller grains, kept together by adhesion. The particles may differ in size, mass and in their material properties. In what follows we consider the contact of granular particles under simplifying conditions: We assume that the particles are smooth, convex and of uniform material. The latter assumption allows to describe the particle deformation by continuum mechanics, disregarding their molecular structure.

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It is assumed that particles exert forces on each other exclusively via pairwise mechanical contact, i.e., electromagnetic interaction as well as gravitational attraction is not considered.

## 1.2 Forces between granular particles

### 1.2.1 Elastic forces

When particles deform each other due to a static (or quasistatic) contact they feel an elastic interaction force. Elastic deformation implies that after separation of the contacting particles, they recover their initial shape, i.e., there is no plastic deformation. The stress tensor $\sigma_{\rm el}^{ij}(\vec{r})$ describes the $i$-component of the force, acting on a unit surface which is normal to the direction $j$ $(i, j = \{x, y, z\})$. In the elastic regime the stress is related to the material deformation

$$u_{ij}(\vec{r}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \tag{1.1}$$

where $\vec{u}(\vec{r})$ is the displacement field at the point $\vec{r}$ in the deformed body, via the linear relation,

$$\sigma_{\rm el}^{ij}(\vec{r}) = E_1\left(u_{ij}(\vec{r}) - \frac{1}{3}\delta_{ij}u_{ll}(\vec{r})\right) + E_2\delta_{ij}u_{ll}(\vec{r}) , \tag{1.2}$$

Repeated indices are imlicitely summed over (Einstein convention). The coefficients $E_1$ and $E_2$ read

$$E_1 = \frac{Y}{(1+\nu)}, \qquad E_2 = \frac{Y}{3(1-2\nu)} , \tag{1.3}$$

where $Y$ is the Young modulus and $\nu$ is the Poisson ratio. Let the pressure $\vec{P}(x, y)$ act on the surface of an elastic semispace, $z > 0$, leading to a displacement field in the bulk of the semispace [17]:

$$u_i = \int\int G_{ik}\left(x - x', y - y', z\right) P_k\left(x', y'\right) dx' dy' , \tag{1.4}$$

where $G_{ik}\left(x, y, z\right)$ is the corresponding Green function. For the contact problem addressed here we need only the $z$-component of the displacement on the surface $z = 0$, that is, we need only the component

$$G_{zz}(x, y, z = 0) = \frac{(1-\nu^2)}{\pi Y}\frac{1}{\sqrt{x^2+y^2}} = \frac{(1-\nu^2)}{\pi Y}\frac{1}{r} \tag{1.5}$$

of the Green function [17].

Consider a contact of two convex smooth bodies labeled as 1 and 2. We assume that only normal forces, with respect to the contact area, act between the particles. In the contact region their surfaces are flat. For the coordinate system centered in the middle of the contact region, where $x = y = z = 0$, the following relation holds true:

$$B_1x^2 + B_2y^2 + u_{z1}(x, y) + u_{z2}(x, y) = \xi , \tag{1.6}$$

where $u_{z1}$ and $u_{z2}$ are respectively the $z$-components of the displacement in the material of the first and of the second bodies on the plane $z = 0$. The sum of the compressions of both bodies in the center of the contact area defines $\xi$. The constants $B_1$ and $B_2$ are related to the radii of curvature of the surfaces in contact [17]:

$$
\begin{aligned}
2\,(B_1 + B_2) &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1'} + \frac{1}{R_2'} \\
4\,(B_1 - B_2)^2 &= \left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 + \left(\frac{1}{R_1'} - \frac{1}{R_2'}\right)^2 + 2\cos 2\varphi \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{R_1'} - \frac{1}{R_2'}\right).
\end{aligned}
\tag{1.7}
$$

Here $R_1$, $R_2$ and $R_1'$, $R_2'$ are respectively the principle radii of curvature of the first and the second body at the point of the contact and $\varphi$ is the angle between the planes corresponding to the curvature radii $R_1$ and $R_1'$. The equations (1.6, 1.7) describe the general case of a contact of two smooth bodies (see [17] for detail). The physical meaning of (1.6) is easy to see from the case of a contact of a soft sphere of a radius $R$ ($R_1 = R_2 = R$) with a hard, undeformed plane ($R_1' = R_2' = \infty$). In this case $B_1 = B_2 = 1/R$, the compressions of the sphere and of the plane are respectively $u_{z1}(0,0) = \xi$ and $u_{z2} = 0$, and the surface of the sphere before the deformation is given by $z(x,y) = (x^2 + y^2)/R$. Then (1.6) reads in the flattened area, $u_{z1}(x,y) = \xi - z(x,y)$, that is, it gives the condition for a point $z(x,y)$ on the body's surface to touch the plane $z = 0$.

The displacements $u_{z1}$ and $u_{z2}$ may be expressed in terms of the normal pressure $P_z(x,y)$ which acts between the compressed bodies in the plane $z = 0$. Using (1.4) and (1.5) we rewrite (1.6) as

$$
\frac{1}{\pi}\left(\frac{1-\nu_1^2}{Y_1} + \frac{1-\nu_2^2}{Y_2}\right)\iint \frac{P_z(x',y')}{r}dx'dy' = \xi - B_1 x^2 - B_2 y^2\,, \tag{1.8}
$$

where $r = \sqrt{(x-x')^2 + (y-y')^2}$ and integration is performed over the contact area. Equation (1.8) is an integral equation for the unknown function $P_z(x,y)$. We compare this equation with the mathematical identity [17],

$$
\iint \frac{dx'dy'}{r}\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}} = \frac{\pi ab}{2}\int_0^\infty \left[1 - \frac{x^2}{a^2+t} - \frac{y^2}{b^2+t}\right]\frac{dt}{\sqrt{(a^2+t)(b^2+t)t}} \tag{1.9}
$$

where integration is performed over the elliptical area $x'^2/a^2 + y'^2/b^2 = 1$. The left-hand sides of both equations contain integrals of the same type, while the right-hand sides contain quadratic forms of the same type. Therefore, the contact area is an ellipse with the semi-axes $a$ and $b$ and the pressure is of the form $P_z(x,y) = \text{const}\sqrt{1 - x^2/a^2 - y^2/b^2}$. The constant may be found from the total elastic force $F_{\text{el}}$ acting between the bodies. Integrating $P_z(x,y)$ over the contact area we obtain

$$
P_z(x,y) = \frac{3F_{\text{el}}}{2\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,. \tag{1.10}
$$

We substitute (1.10) into (1.8) and replace the double integration over the contact area by integration over the variable $t$, according to the identity (1.9). Thus, we obtain an equation containing terms proportional to $x^2$, $y^2$ and a constant. Equating the corresponding coefficients we obtain

$$\xi = \frac{F_{\rm el} D}{\pi} \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)t}} = \frac{F_{\rm el} D}{\pi} \frac{N(x)}{b} \tag{1.11}$$

$$B_1 = \frac{F_{\rm el} D}{\pi} \int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)t}} = \frac{F_{\rm el} D}{\pi} \frac{M(x)}{a^2 b} \tag{1.12}$$

$$B_2 = \frac{F_{\rm el} D}{\pi} \int_0^\infty \frac{dt}{(b^2+t)\sqrt{(a^2+t)(b^2+t)t}} = \frac{F_{\rm el} D}{\pi} \frac{M(1/x)}{a b^2} \tag{1.13}$$

where

$$D \equiv \frac{3}{4} \left( \frac{1-\nu_1^2}{Y_1} + \frac{1-\nu_2^2}{Y_2} \right) , \tag{1.14}$$

$x \equiv a^2/b^2$ is the ratio of the contact ellipse semiaxes, and where we introduce the short-hand notations[2]:

$$N(x) = \int_0^\infty \frac{dt}{\sqrt{(1+xt)(1+t)t}} \tag{1.15}$$

$$M(x) = \int_0^\infty \frac{dt}{(1+t)\sqrt{(1+t)(1+xt)t}} . \tag{1.16}$$

From these relations follow the size of the contact area, $a$, $b$, and the compression $\xi$ as functions of the elastic force $F_{\rm el}$ and the geometrical coefficients $B_1$ and $B_2$.

The dependence of the force $F_{\rm el}$ on the compression $\xi$ may be obtained from scaling arguments: If we rescale $a^2 \to \alpha a^2$, $b^2 \to \alpha b^2$, $\xi \to \alpha \xi$ and $F_{\rm el} \to \alpha^{3/2} F_{\rm el}$, with $\alpha$ constant, equations (1.11-1.13) remain unchanged. That is, when $\xi$ changes by the factor $\alpha$, the semiaxis $a$ and $b$ change by the factor $\alpha^{1/2}$ and the force by the factor $\alpha^{3/2}$, i.e., $a \sim \xi^{1/2}$, $b \sim \xi^{1/2}$ and

$$F_{\rm el} = {\rm const}\, \xi^{3/2} . \tag{1.17}$$

The dependence (1.17) holds true for all smooth convex bodies in contact. To find the constant in (1.17) we divide (1.13) by (1.12) and obtain the transcendental equation

$$\frac{B_2}{B_1} = \frac{\sqrt{x} M\,(1/x)}{M(x)} \tag{1.18}$$

for the ratio of semiaxes $x$. Let $x_0$ be the root of Eq. (1.18), then $a^2 = x_0 b^2$ and we obtain,

$$\xi = \frac{F_{\rm el} D}{\pi} \frac{N(x_0)}{b} \tag{1.19}$$

$$B_1 = \frac{F_{\rm el} D}{\pi} \frac{M(x_0)}{x_0 b^3} , \tag{1.20}$$

[2]The function $N(x)$ and $M(x)$ may be expressed as a combination of the Jacobian elliptic functions $E(k)$ and $K(k)$ [1]

where $N(x_0)$ and $M(x_0)$ are pure numbers. Equations (1.19,1.20) allow to find the semiaxes $b$ and the elastic force $F_{\rm el}$ as a function of the compression $\xi$. Hence we obtain for the force, i.e. we get the constant in (1.17):

$$F_{\rm el} = \frac{\pi}{D}\left(\frac{M(x_0)}{B_1 x_0 N(x_0)}\right)^{1/2} \xi^{3/2} = C_0 \xi^{3/2} \tag{1.21}$$

For the special case of contacting spheres ($a = b$),

$$B_1 = B_2 = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\frac{1}{R^{\rm eff}} . \tag{1.22}$$

In this case $x_0 = 1$, and $N(1) = \pi$, $M(1) = \pi/2$ and the solution of Eqs. (1.19,1.20) yields,

$$a^2 = R^{\rm eff}\, \xi \tag{1.23}$$

$$F_{\rm el} = \rho \xi^{3/2} ; \qquad \rho \equiv \frac{2Y}{3(1-\nu^2)}\sqrt{R^{\rm eff}} , \tag{1.24}$$

where we use the definition of the constant $D$, (1.14). This contact problem has been solved by Heinrich Hertz in 1882 [14]. It describes the force between *elastic* particles. For inelastically deforming particles it describes the repulsive force in the static case.

### 1.2.2 Viscous forces

When the contacting particles move with respect to each other, i.e. the deformation changes with time, an additional dissipative force arises, which acts in opposit direction to the relative particle motion. The dissipative processes occurring in the bulk of the body cause a viscous contribution to the stress tensor. For small deformation the respective component of the stress tensor is proportional to the deformation rate $\dot{u}_{ij}(\vec{r}\,)$, according to the general relation [7]:

$$\sigma_{\rm dis}^{ij}(\vec{r},t) = E_1 \int_0^t d\tau \psi_1(t-\tau)\left[\dot{u}_{ij}(\vec{r},\tau) - \frac{1}{3}\delta_{ij}\dot{u}_{ll}(\vec{r},\tau)\right] + E_2 \int_0^t d\tau \psi_2(t-\tau)\delta_{ij}\dot{u}_{ll}(\vec{r},\tau) , \tag{1.25}$$

where the (dimensionless) functions $\psi_1(t)$ is the relaxation function for the distortion deformation and $\psi_2(t)$ for the dilatation deformation.

In many important applications the viscous stress tensor may be simplified significantly: If the relative velocity of the colliding bodies is much smaller than the speed of sound in the particle material and if the characteristic relaxation times of the dissipative processes $\tau_{\rm vis,\,1/2}$ are much smaller than the duration of the collision $t_c$,

$$\tau_{\rm vis,\,1/2} \equiv \int_0^\infty \psi_{1/2}(\tau) d\tau \ll t_c , \tag{1.26}$$

the viscous constants $\eta_1$ and $\eta_2$ may be used instead of the functions $\psi_1(t)$ and $\psi_2(t)$. Thus

$$\eta_{1/2} = E_{1/2}\tau_{\rm vis,\,1/2} = E_{1/2}\int_0^\infty \psi_{1/2}(\tau) d\tau \tag{1.27}$$

and the dissipative stress tensor reads (see [7] for details)

$$\sigma_{\rm dis}^{ij}(\vec{r},t) = \eta_1 \left[ \dot{u}_{ij}(\vec{r},\tau) - \frac{1}{3}\delta_{ij}\dot{u}_{ll}(\vec{r},\tau) \right] + \eta_2 \delta_{ij}\dot{u}_{ll}(\vec{r},\tau) \,. \tag{1.28}$$

It may be also shown that the above conditions are equivalent to the assumption of quasistatic deformation [7, 6]. When the material is deformed quasistatically, the displacement field $\vec{u}(\vec{r})$ in the particles coincides with that for the static case $\vec{u}_{\rm el}(\vec{r})$, which is the solution of the elastic contact problem. The field $\vec{u}_{\rm el}(\vec{r})$, in its turn, is completely determined by the compression $\xi$, which varies with time during the collision, i.e. $\vec{u}_{\rm el} = \vec{u}_{\rm el}(\vec{r},\xi)$. Therefore, the corresponding displacement rate may be approximated as

$$\dot{\vec{u}}(\vec{r},t) \simeq \dot{\xi}\frac{\partial}{\partial\xi}\vec{u}_{\rm el}(\vec{r},\xi) \tag{1.29}$$

and the dissipative stress tensor reads, respectively

$$\sigma_{\rm dis}^{ij} = \dot{\xi}\frac{\partial}{\partial\xi}\left[ \eta_1 \left( u_{ij}^{\rm el} - \frac{1}{3}\delta_{ij}u_{ll}^{\rm el} \right) + \eta_2\delta_{ij}u_{ll}^{\rm el} \right] \,. \tag{1.30}$$

From (1.30) and (1.2) follows the relation between the elastic and dissipative stress tensors within the quasistatic approximation,

$$\sigma_{\rm dis}^{ij} = \dot{\xi}\frac{\partial}{\partial\xi}\sigma_{\rm el}^{ij}\left( E_1 \leftrightarrow \eta_1, E_2 \leftrightarrow \eta_2 \right) \,, \tag{1.31}$$

where we emphasize that the expression for the dissipative tensor may be obtained from the corresponding expression for the elastic tensor after substituting the elastic constants by the relative viscous constants and application of the operator $\dot{\xi}\partial/\partial\xi$.

The component $\sigma_{\rm el}^{zz}$ of the elastic stress is equal to the normal pressure $P_z$ at the plane $z = 0$ of the elastic problem, Eq. (1.10)

$$\begin{aligned} \sigma_{\rm el}^{zz}(x,y,0) &= E_1\frac{\partial u_z}{\partial z} + \left( E_2 - \frac{E_1}{3} \right)\left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \\ &= \frac{3F_{\rm el}}{2\,\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \,. \end{aligned} \tag{1.32}$$

Now we compute the total dissipative force acting between the bodies. Instead of a direct computation of the dissipative stress tensor, we employ the method proposed in [7, 6]: We transform the coordinate axes as

$$x = \alpha x' \,, \qquad y = \alpha y' \,, \qquad z = z' \tag{1.33}$$

with

$$\alpha = \left( \frac{\eta_2 - \frac{1}{3}\eta_1}{\eta_2 + \frac{2}{3}\eta_1} \right)\left( \frac{E_2 + \frac{2}{3}E_1}{E_2 - \frac{1}{3}E_1} \right) \qquad \beta = \frac{(\eta_2 - \frac{1}{3}\eta_1)}{\alpha(E_2 - \frac{1}{3}E_1)} \tag{1.34}$$

$$a = \alpha a' \qquad b = \alpha b' \,. \tag{1.35}$$

and perform the transformations:

$$
\begin{aligned}
&\eta_1 \frac{\partial u_z}{\partial z} + \left(\eta_2 - \frac{\eta_1}{3}\right)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
&= \beta \left[E_1 \frac{\partial u_z}{\partial z'} + \left(E_2 - \frac{E_1}{3}\right)\left(\frac{\partial u_x}{\partial x'} + \frac{\partial u_y}{\partial y'} + \frac{\partial u_z}{\partial z'}\right)\right] \\
&= \beta \frac{3F_{\rm el}}{2\,\pi a' b'}\sqrt{1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2}} = \beta\alpha^2 \frac{3F_{\rm el}}{2\,\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,.
\end{aligned} \tag{1.36}
$$

Applying the operator $\dot{\xi}\partial/\partial\xi$ to the last expression in the right-hand side we obtain the dissipative stress tensor. Subsequent integration over the contact area yields finally the total dissipative force acting between the bodies:

$$
F_{\rm dis} = A\dot{\xi}\frac{\partial}{\partial \xi}F_{\rm el}(\xi)\,, \tag{1.37}
$$

where

$$
A \equiv \alpha^2\beta = \frac{1}{3}\frac{(3\eta_2 - \eta_1)^2}{(3\eta_2 + 2\eta_1)}\left[\frac{(1-\nu^2)(1-2\nu)}{Y\nu^2}\right]\,. \tag{1.38}
$$

Using the scaling relations for the elastic force, Eq. (1.17), and for the semiaxes of the contact ellipse, we obtain,

$$
\frac{\partial F_{\rm el}}{\partial \xi} = \frac{3}{2}\frac{F_{\rm el}}{\xi}\,, \qquad \frac{\partial a}{\partial \xi} = \frac{1}{2}\frac{a}{\xi}\,, \qquad \frac{\partial b}{\partial \xi} = \frac{1}{2}\frac{b}{\xi}\,. \tag{1.39}
$$

Then from (1.36) and (1.21), the distribution of the dissipative pressure in the contact area may be found:

$$
P_z^{\rm dis}(x,y) = \frac{3A}{4\pi}\,\frac{AC_0}{ab}\dot{\xi}\sqrt{\xi}\,\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2}\,, \tag{1.40}
$$

where the constant $C_0$ is defined in (1.21).

We wish to stress, that to derive the above expressions, we only assumed that the surfaces of the two bodies in the vicinity of the contact point before the deformation, are described by quadratic the forms, $z_1 = \kappa_{ij}^{(1)}x_i x_j$ and $z_2 = \kappa^{(2)}x_i x_j$ $(i,j = x,y,z)$, where $\kappa_{ij}^{(1/2)}$ are symmetric tensors [17]). Therefore the obtained relations are valid for a contact of arbitrarily shaped convex bodies. For spherical particles of the identical material, (1.37) and (1.24) yield [7, 6]

$$
F_{\rm dis} = \frac{3}{2}A\rho\dot{\xi}\sqrt{\xi}\,, \tag{1.41}
$$

with $\rho$ as defined in (1.24). Hence, the total force acting between viscoelastic spheres takes the simple form [7, 6]

$$
F = \rho\left(\xi^{3/2} + \frac{3}{2}A\sqrt{\xi}\dot{\xi}\right)\,. \tag{1.42}
$$

The range of validity of (1.42) for the viscoelastic force is determined by the quasistatic approximation. The impact velocity must be significantly smaller than the speed of sound. On the other hand, the impact velocity must not be too small in order to neglect adhesion. We also neglect plastic deformation in the material.

### 1.2.3 Adhesion of particles in a contact

**Models of adhesive interaction**

The Hertz theory has been derived for the contact of non-adhesive particles. Adhesion becomes important when the distance of the particle surfaces approaches the range of molecular forces. Johnson, Kendall and Roberts (JKR) [16] extended the Hertz theory by taking into account adhesion in the flat contact region. They show that the contact area is enlarged by the action of the adhesive force. Therefore, they introduced an apparent Hertz load $F_H$ which would cause this enlarged contact area. To simplify the notations we consider the contact of identical spheres. The contact area is then a circle of radius $a$, which corresponds to the compression $\xi_H$ for the Hertz load $F_H$. In reality, however, this contact radius occurs at the compression $\xi$ which is smaller than $\xi_H$. In the JKR theory it is assumed that the difference between the Hertz compression $\xi_H$ and the actual one, $\xi$, may be attributed to the additional stress

$$P_B(x,y) = \frac{F_B}{2\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^{-1/2}, \tag{1.43}$$

which refers to the solution of the classical Boussinesq problem [27]: This distribution of the normal surface traction gives rise to a constant displacement over a circular region of an elastic body. The displacement $\xi_B$ corresponding to the contact radius $a$ and the total load $F_B$ are related by

$$\xi_B = \frac{2}{3} D \frac{F_B}{a}, \tag{1.44}$$

where the constant $D$ is defined in (1.14).

The value of $F_B < 0$ mimics the additional surface forces, such that the pressure is positive (compressive) in the center of the contact area, while it is negative (tensile) near the boundary [16]. Hence, the shape of the body is determined by the action of two effective forces $F_H$ and $F_B$. The total force between the particles is their difference, $F = F_H - F_B$. Johnson et al. assumed that the elastic energy stored in the deformed spheres may be found as a difference of the elastic energy coresponding to the Hertz force $F_H$ and that due to the force $F_B$ [16]. Using

$$U_s = -\pi\gamma a^2 \tag{1.45}$$

for the surface energy, where $\gamma > 0$ is twice the surface free energy per unit area of the solid in vacuum or gas, and minimizing the total energy, we obtain

$$F_B = -2\pi a^2 \sqrt{\frac{3\gamma}{2\pi D a}}, \tag{1.46}$$

and, thus, the contact radius corresponding to the total force $F$:

$$a^3 = \frac{1}{2} D\, R \left( F + \frac{3}{2}\pi\gamma R + \sqrt{3\pi\gamma R F + \left(\frac{3}{2}\pi\gamma R\right)^2} \right) \tag{1.47}$$

and respectively the compression

$$\xi = \frac{2a^2}{R} - \sqrt{\frac{8\pi\gamma D a}{3}}\,. \tag{1.48}$$

The first term in (1.48) is the Hertz compression $\xi_H$, which coincides with (1.23) for $R^{\text{eff}} \to R/2$. Equation (1.47) may be solved to express the total force as a function of the contact radius:

$$F(a) = \frac{2a^3}{DR} - \sqrt{\frac{6\pi\gamma}{D}} a^{3/2}\,. \tag{1.49}$$

For vanishing applied load the contact radius $a_0$ is finite:

$$a_0^3 = \frac{3}{2} D\pi\gamma R^2\,. \tag{1.50}$$

For negative applied load the contact radius decreases and the condition for a real solution of 1.47 yields the maximal negative force which the adhesion forces can resist,

$$F_{\text{sep}} = -\frac{3}{4}\pi\gamma R\,, \tag{1.51}$$

corresponding to the contact radius

$$a_{\text{sep}}^3 = \frac{3}{8} D\pi\gamma R^2 = \frac{1}{4} a_0^3\,. \tag{1.52}$$

For a larger (in the absolute value) negative force the spheres separate. For spheres of dissimilar radii, in (1.47-1.52) $R$ should be substituted by $2R^{\text{eff}}$.

Another approach to the problem of the adhesive contact was developed by Derjaguin, Muller and Toporov (DMT). They assumed that the Hertz profile of the pressure distribution on the surface stays unaffected by adhesion and obtain the pull-off force $F_{\text{sep}} = -2\pi\gamma R^{\text{eff}}$ [9]. The assumption of the Hertz profile allows to avoid the singularities of the pressure distribution (1.43) on the boundary of the contact zone. Since the experimental measurement of $\gamma$ is problematic, it is not possible to check the validity of the JKR and DMT theories, i.e., to resolve their disagreement.

In later studies [19, 20] a more accurate theoretical analysis has been performed: The elastic equations have been solved numerically for a simplified microscopic model of adhesive surfaces with Lennard-Jones interaction. Within this microscopic approach the relative accuracy of different theories has been estimated for a wide range of model parameters. It was found that the DMT theory is valid for small adhesion and for small, hard particles. JKR theory is more reliable for large, soft particles with large adhesion forces, which, however, should be short-ranged.

In [2] the Lennard-Jones continuum model of solids was studied. The adhesive forces between the surfaces read then

$$P_s(h) = \frac{H}{6\pi h^3}\left[\frac{z_0^6}{h^6} - 1\right] . \tag{1.53}$$

Here $P_s(h)$ describes the forces acting per unit area between the surfaces, $h = h(r)$ is the actual microscopic distance betweem them. $H$ is the Hamaker constant, characterizing the van der Waals attraction of the particles in a gas or vacuum and $z_0$ is the equilibrium separations of the surfaces. The surface energy in this model is defined by

$$\gamma = \frac{H}{16\pi z_0^2} . \tag{1.54}$$

It was observed in [2] that the accuracies of different theories vary depending of the value of the Tabor parameter $\mu$, [25]

$$\mu^{3/2} \equiv \frac{3}{2}\gamma D\sqrt{R^{\rm eff}/z_0^3} . \tag{1.55}$$

In agreement with [19, 20] it has been shown [2] that small values of $\mu$ (small hard particles with low surface energies) favor for the DMT theory ($\mu < 10^{-2}$) while for $\mu \sim 1 - 10$ the JKR theory occurs to be rather accurate. Both JKR and DMT fail for large $\mu$ when the strong adhesion is combined with a soft material of the contacting bodies. In this limit, the surfaces jump into contact which corresponds to a spontaneous non-equilibrium transition (see e.g. [24]). Similar analysis has been performed later [11], where the author conclude that the DTM theory generally fails both in original and corrected forms. One of the main conclusions of [2, 11] is that the JKR theory, albeit simple, gives relatively accurate predictions for basic quantities in the range of its validity ($\mu \sim 1 - 10$).

Among the theories developed to cover the DMT-JKR transition regimes [19, 20, 11, 25, 18] the theory by Maugis [18] is the most frequently used. It uses a simplified model of adhesive forces: The adhesive force of a constant intensity $P_D$ is extended over a fixed distance $h_D$ above the surface, yielding the surface tension $\gamma = P_D h_D$. The description of a contact in this model is based on two coupled analytical equation which are to be solved numerically. The recently developed double-Hertz model [12, 13] constructs the solution for the adhesive contact as a sum of two Hertzian solutions, which make the theory analytically more tractable than the Maugis model. Combining in an adopted manner the successful assumptions of the JKR and the modified DMT model, a generalized analytical theory for the adhesive contact has been proposed [23].

In what follows we assume that the parameters of our system belong to the range of validity of the JKR model, $\mu \sim 1 - 10$, and will use this simple analytical theory to describe the adhesive contacts between spheres. Moreover, we assume small adhesive force. Then from (1.47) and (1.48) we obtain the total force as a function of the compression. Keeping only the leading-order terms with respect to the parameter $\gamma$ it reads

$$F \approx \rho\xi^{3/2} - q_0\sqrt{\pi\gamma/D}\left(R^{\rm eff}\right)^{3/4}\xi^{3/4} , \tag{1.56}$$

where $q_0 = (\sqrt{6} - \sqrt{8/3})$.

**Viscoelasticity in adhesive interactions**

The adhesive forces between particles cause the additional deformation in the contacting bodies as compared to a pure Hertzian deformation, hence in the corresponding dynamical problem an additional deformation rate arises. Hence the dissipative forces must have an additional component attributed to the adhesive interactions. The adhesive contact of viscoelastic spheres has been studied numerically [13], where the double Hertz model has been used. In [8] the quasistatic condition for the colliding viscoelastic adhesive spheres was used and an analytical expression for the interaction force has been derived for the JKR model. Similar as for the case of non-adhesive particles, it was assumed that in the quasistatic approximation, the deformation field may be parameterized by the value of the compression $\xi$. (Note, that this assumption neglects the possible hysteresis which can happen for the negative total force [2]). Performing the same transformation which lead to the expression (1.37) for the case of non-adhesive contact, and using the approximation (1.56) we obtain for the dissipative forces [8]

$$F_{\rm dis} = \frac{3}{2} A \rho \dot{\xi} \sqrt{\xi} + \frac{3}{4} B \sqrt{\pi \gamma / D} \left( R^{\rm eff} \right)^{3/4} \dot{\xi} \xi^{-1/4} \tag{1.57}$$

$$B \equiv \alpha \beta q_0 = \frac{(3\eta_2 - \eta_1) Y \nu}{3(1+\nu)(1-2\nu)} q_0 \,. \tag{1.58}$$

Note the singularity in the second term of (1.57) at $\xi = 0$[3]. It is attributed to the quasistatic approximation for JKR theory and physically reflects the fact that the adhesive particles can jump into a contact [24] with the discontinuous change of the compression $\xi$.

Consider now how the above forces determine the particle dynamics.

## 1.3 Collision of granular particles

We turn now to the description of the collisions of granular particles. We assume that they are smooth and do not exchange the tangential forces over the contact surface (see [5] where a consistency of such assumption is discussed). Hence only the normal motion is considered. Let the particles be spheres of the same material, which begin to collide at time $t = 0$ with the relative normal velocity $g$. Then the time dependent compression reads

$$\xi(t) = R_i + R_j - \left| \vec{r}_i(t) - \vec{r}_j(t) \right| , \tag{1.59}$$

where $\vec{r}_i(t)$ and $\vec{r}_j(t)$ define the positions of the colliders at time $t$ (see Fig. 1.1). The problem of relative normal motion of particles at a collision is equivalent to the problem of motion of a point particle with the effective mass

$$m^{\rm eff} = \frac{m_i m_j}{m_i + m_j} \,. \tag{1.60}$$

We assume that only elastic and dissipative forces act between the colliding particles and neglect for a moment the adhesion forces, i.e. we assume the interparticle force, Eq. (1.42)

[3]This is a weak, or integrable singularity, that is $\int_0^\epsilon \xi^{-1/4} \xi \sim \epsilon^{3/4} \to 0$ for $\epsilon \to 0$. Hence for practical application of (1.57) one can use $\xi > \epsilon$, where $\epsilon$ may be very small but a finite number.

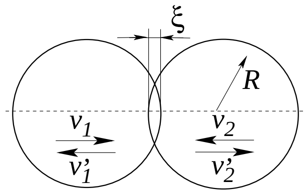


**Figure 1.1:** Two colliding spheres. The fi gure illustrates the compression $\xi \equiv 2R - |\vec{r}_1 - \vec{r}_2|$ and the compression rate $\dot{\xi} = v_1 - v_2$. The head on collision of identical spheres is shown.

for a viscoelastic collision. Hence the equation of motion and the initial conditions read [4]:

$$\begin{aligned} &\ddot{\xi} + \frac{\rho}{m^{\mathrm{eff}}}\left(\xi^{3/2} + \frac{3}{2}A\sqrt{\xi}\dot{\xi}\right) = 0 \\ &\dot{\xi}(0) = g \\ &\xi(0) = 0\,. \end{aligned} \tag{1.61}$$

This equation describes generally the collision of the viscoelastic particles. In many cases the details of the collision are not important and only the result of a collision is of interest. This is given by the coefficient of restitution.

This quantity characterizes the most important property of the adhesive collision: It gives the relative interparticle velocity after a collision in terms of the precollisional relative velocity. In the case of normal motion, the pre-collisional relative velocity is $\dot{\xi}(0) = g$, while the after-collisional quantity is $\dot{\xi}(t_c)$, where the collision starts at $t = 0$ and ends at $t = t_c$. Hence the normal restitution coefficient reads,

$$\varepsilon = -\dot{\xi}\left(t_c\right)/\dot{\xi}(0) = -\dot{\xi}(t_c)/g\,, \tag{1.62}$$

$\varepsilon$ can be found by numerical integration of Eq. (1.61) [15, 7, 6] and analytically [22, 21] but requires considerable efforts [22]. Here we give a derivation which is based on the dimension analysis of the equation of motion (1.61) [21]. The method of dimensional analysis was first developed by Tanaka *et al.* [26] to show that a constant coefficient of restitution is not consistent with physical reality.

### 1.3.1 Coefficient of restitution: The dimensional analysis

To perform the general analysis we adopt the following form of the elastic and dissipative forces:

$$\begin{aligned} F_{\mathrm{el}} &= m^{\mathrm{eff}} D_1\, \xi^{\alpha} \\ F_{\mathrm{diss}} &= m^{\mathrm{eff}} D_2\, \xi^{\gamma} \dot{\xi}^{\beta}\,, \end{aligned} \tag{1.63}$$

with $D_{1/2}$ being material parameters. With these notations the equation of motion for colliding particles reads

$$\ddot{\xi} + D_1\, \xi^{\alpha} + D_2\, \xi^{\gamma}\, \dot{\xi}^{\beta} = 0\,, \qquad \xi(0) = 0\,, \qquad \dot{\xi}(0) = g\,. \tag{1.64}$$

[4]Note that the elastic force acts against compression, while the dissipative force against the compression rate

As the characteristic length $\xi_0$ of the problem we choose the maximal compression for the elastic case. It may be found from the condition that the initial kinetic energy, $m^{\rm eff} g^2/2$, equals the maximal elastic energy $m^{\rm eff} D_1 \xi_0^{\alpha+1}/(\alpha+1)$, which yields

$$\xi_0 \equiv \left(\frac{\alpha+1}{2\,D_1}\right)^{1/(1+\alpha)} g^{2/(1+\alpha)} \,. \qquad (1.65)$$

For the characteristic time of the problem $\tau_0$ we choose the value $\tau_0 \equiv \xi_0/g$ and arrive at the following dimensionless variables

$$\hat{\xi} \equiv \xi/\xi_0 \,, \qquad \dot{\hat{\xi}} \equiv \dot{\xi}/g \,, \qquad \ddot{\hat{\xi}} = \left(\xi_0/g^2\right) \ddot{\xi} \,. \qquad (1.66)$$

Now we recast the equation of motion and its initial conditions, Eq. (1.61), into dimensionless form:

$$\ddot{\hat{\xi}} + \varkappa \hat{\xi}^{\gamma}\, \dot{\hat{\xi}}^{\beta} + \frac{1+\alpha}{2}\,\hat{\xi}^{\alpha} = 0 \,, \qquad \hat{\xi}(0) = 0 \,, \qquad \dot{\hat{\xi}}(0) = 1 \qquad (1.67)$$

with

$$\varkappa = \varkappa(g) = D_2 \left(\frac{1+\alpha}{2\,D_1}\right)^{(1+\gamma)/(1+\alpha)} g^{2(\gamma-\alpha)/(1+\alpha)+\beta} \,. \qquad (1.68)$$

None of the terms in (1.67) depends neither on material properties nor impact velocity except for $\varkappa$. Therefore, if the motion of the particles depends on material properties and on impact velocity, it may depend only via $\varkappa$, i.e., in the combination of the parameters as given by (1.68). Hence, any function of the the impact velocity, including the restitution coefficient must be of the form $\varepsilon(g) = \varepsilon[\varkappa(g)]$. A similar result for $\varepsilon \to 0$, $\beta = 1$ and $\alpha = 3/2$ has been obtained in [10].

If we assume that the restitution coefficient does not depend on the impact velocity $g$, then it follows that

$$2(\gamma-\alpha) + \beta\,(1+\alpha) = 0 \qquad (1.69)$$

For a linear dependence of the dissipative force on the velocity, i.e. for $\beta = 1$ (this seems to be the most realistic for small $\dot{\xi}$), one obtains the constant restitution coefficient for the following cases,

- For linear elastic force $F_{\rm el} \propto \xi$, (i.e. $\alpha = 1$), the condition (1.69) implies $\gamma = 0$, and thus the linear dissipative force, $F_{\rm dis} \propto \dot{\xi}$. This is the linear one-dimensional dashpot force model.
- For the Hertz law for 3d-spheres (1.24), (i.e. $\alpha = 3/2$), the condition (1.69) requires $\gamma = \frac{1}{4}$, i.e., $F_{\rm dis} \propto \dot{\xi}\,\xi^{1/4}$. As far as we can see there is no physical argument by which this functional form of the dissipative force could be supported.

Therefore, we conclude that a constant coefficient of restitution is in agreement with physical mechanics only in case of (quasi-) one-dimensional systems. For three-dimensional spheres the assumption $\varepsilon = \text{const}$ disagrees with basic mechanical laws.

We now ask the question: What kind of $\varepsilon(g)$ dependence corresponds to the collision of viscoelastic spheres? As it follows from (1.42) for this case $\alpha = 3/2$, $\beta = 1$, and $\gamma = 1/2$. From (1.68) can be obtained

$$\varkappa = \frac{3}{2}\left(\frac{5}{4}\right)^{3/5} A \left(\frac{\rho^{\text{ eff}}}{m}\right)^{2/5} g^{1/5} \tag{1.70}$$

and, therefore,

$$\varepsilon = \varepsilon\left[A\left(\frac{\rho^{\text{ eff}}}{m}\right)^{2/5} g^{1/5}\right] . \tag{1.71}$$

If we assume that the function $\varepsilon(g)$ is a sufficiently smooth function which can be expanded into a Taylor series, and if we realize that $\varepsilon(0) = 1$, we conclude that the restitution coefficient should have the following functional form for small impact velocities,

$$\varepsilon = 1 - C_1 A\kappa^{2/5} g^{1/5} + C_2 A^2 \kappa^{4/5} g^{2/5} \mp \dots \tag{1.72}$$

where

$$\kappa \equiv \left(\frac{3}{2}\right)^{5/2}\left(\frac{\rho}{m^{\text{eff}}}\right) = \left(\frac{3}{2}\right)^{3/2} \frac{Y\sqrt{R^{\text{eff}}}}{m^{\text{eff}}\left(1-\nu^2\right)} , \tag{1.73}$$

and the coefficients $C_1, C_2, \dots$ are pure numbers which are to be found. The coefficient $C_1$, $C_2$ were first obtained in [22] and $C_3$, $C_4$ in the later study [21]. Here give a simple derivation of these coefficients according to the method proposed in [21].

### 1.3.2 Restitution coefficient for spheres

**Small inelasticity expansion**

Using $d/d\xi = \dot{\hat{\xi}} d/d\hat{\xi}$ we write the equation of motion for a collision in the form

$$\frac{d}{d\hat{\xi}}\left(\frac{1}{2}\dot{\hat{\xi}}^2 + \frac{1}{2}\hat{\xi}^{5/2}\right) = -\varkappa\dot{\hat{\xi}}\sqrt{\hat{\xi}} = \frac{dE(\hat{\xi})}{d\hat{\xi}} , \qquad \hat{\xi}(0) = 0 , \qquad \dot{\hat{\xi}}(0) = 1 , \tag{1.74}$$

where we introduce the mechanical energy

$$E \equiv \frac{1}{2}\dot{\hat{\xi}}^2 + \frac{1}{2}\hat{\xi}^{5/2} . \tag{1.75}$$

The collision of particle has two stages: The first stage starts with zero compression and ends in the turning point with the maximal compression $\hat{\xi}_0$. At the second stage, the particles return to the undeformed state. The energy losses at the first stage read,

$$\int_0^{\hat{\xi}_0} \frac{dE}{d\hat{\xi}} d\hat{\xi} = -\varkappa \int_0^{\hat{\xi}_0} \dot{\hat{\xi}}\sqrt{\hat{\xi}} d\hat{\xi} . \tag{1.76}$$

In order to evaluate the right hand side of (1.76) one needs to know the dependence of the compression rate on the compression $\dot{\hat{\xi}} = \dot{\hat{\xi}}\left(\hat{\xi}\right)$.

For the case of elastic collisions, the maximal compression is $\hat{\xi}_0 = 1$, according to the definition of our dimensionless variables. Hence, it follows from the conservation of energy

$$\dot{\hat{\xi}}(\hat{\xi}) = \sqrt{1 - \hat{\xi}^{5/2}}\,, \tag{1.77}$$

i.e. the velocity $\dot{\hat{\xi}}$ vanishes at the turning point $\hat{\xi} = 1$. For inelastic collisions the maximal compression $\hat{\xi}_0$ is smaller than 1, therefore, an approximative relation for the inelastic case is

$$\dot{\hat{\xi}}(\hat{\xi}) \approx \sqrt{1 - (\hat{\xi}/\hat{\xi}_0)^{5/2}} \tag{1.78}$$

which also yields vanishing velocity $\dot{\hat{\xi}}$ at the turning point $\hat{\xi}_0$. Using (1.78) the integration in (1.76) may be performed yielding

$$\frac{1}{2}\hat{\xi}_0^{5/2} - \frac{1}{2} = -\varkappa\, b\, \hat{\xi}_0^{3/2} \tag{1.79}$$

where we take into account

$$\begin{aligned} E(\hat{\xi}_0) &= \frac{1}{2}\hat{\xi}_0^{5/2} \\ E(0) &= \frac{1}{2}\dot{\hat{\xi}}^2(0) = \frac{1}{2} \end{aligned} \tag{1.80}$$

and introduce a constant

$$b \equiv \int_0^1 \sqrt{x}\sqrt{1 - x^{5/2}}\, dx = \frac{\sqrt{\pi}\,\Gamma\,(3/5)}{5\,\Gamma\,(21/10)}\,. \tag{1.81}$$

Let us define the *inverse collision*, the collision that starts with velocity $\varepsilon\, g$ and ends with velocity $g$. During the inverse collision the system gains energy, i.e., it is characterized by negative damping. The maximal compression $\hat{\xi}_0$ is, naturally the same for both collisions, since the inverse collision equals the direct collision except for the fact that time runs backwards. Hence one can write for the inverse collision

$$\frac{dE(\hat{\xi})}{d\hat{\xi}} = +\varkappa\dot{\hat{\xi}}\sqrt{\hat{\xi}}\,, \qquad \hat{\xi}(0) = 0\,, \qquad \dot{\hat{\xi}}(0) = \varepsilon\,. \tag{1.82}$$

This suggests an approximative relation for the inverse collision,

$$\dot{\hat{\xi}}(\hat{\xi}) \approx \varepsilon\sqrt{1 - (\hat{\xi}/\hat{\xi}_0)^{5/2}}\,, \tag{1.83}$$

with the additional pre-factor $\varepsilon$, which is the initial velocity for the inverse collision.

Integration of the energy *gain* for the first phase of the inverse collision (which equals up to its sign the energy loss in the second phase of the direct collision [22]) may be performed just in the same way as has been done for the direct collision, yielding the result

$$\frac{1}{2}\hat{\xi}_0^{5/2} - \frac{\varepsilon^2}{2} = +\varepsilon\,\varkappa\, b\, \hat{\xi}_0^{3/2}\,, \tag{1.84}$$

where we again use $E(\hat{\xi}_0) = \frac{1}{2}\hat{\xi}_0^{5/2}$ and $E(0) = \frac{1}{2}\varepsilon^2$.

Multiplying Eq. (1.79) with $\varepsilon$ and summing it up with Eq. (1.84) we find for the maximal compression, $\varepsilon = \hat{\xi}_0^{5/2}$. Substituting this into (1.79) we arrive at an equation for the coefficient of restitution

$$\varepsilon + 2\varkappa\, b\, \varepsilon^{3/5} = 1\,. \tag{1.85}$$

The formal solution to this equation may be written as a continuous fraction (which does not diverge in the limit $g \to \infty$):

$$\varepsilon^{-1} = 1 + 2\varkappa\, b(1 + 2\varkappa\, b(1 + \cdots)^{2/5} \cdots)^{2/5} \tag{1.86}$$

Another and, perhaps more appropriate way of representation of the coefficient of restitution $\varepsilon$ is a series expansion in terms of $\varkappa$. For practical applications it is convenient to return to dimensional units. We define the characteristic velocity $g^*$ such that

$$\varkappa \equiv \frac{1}{2b}\left(\frac{g}{g^*}\right)^{1/5}\,, \tag{1.87}$$

with $b$ being defined in (1.81). Using the definition (1.68) together with (1.42), which provides the values of $D_1$ and $D_2$ and (1.81) for $b$, we find for the characteristic velocity

$$\left(g^*\right)^{-1/5} = \frac{\sqrt{\pi}}{2^{1/5}5^{2/5}}\frac{\Gamma\left(3/5\right)}{\Gamma\left(21/10\right)}\left(\frac{3}{2}A\right)\left(\frac{\rho}{m^{\mathrm{eff}}}\right)^{2/5}\,. \tag{1.88}$$

With this new notation the coefficient of restitution reads

$$\varepsilon = 1 - a_1\left(\frac{g}{g^*}\right)^{1/5} + a_2\left(\frac{g}{g^*}\right)^{2/5} - a_3\left(\frac{g}{g^*}\right)^{3/5} + a_4\left(\frac{g}{g^*}\right)^{4/5} \mp \cdots\,, \tag{1.89}$$

with $a_1 = 1$, $a_2 = 3/5$, $a_3 = 6/25 = 0.24$, $a_4 = 7/125 = 0.056$. Rigorous but elaborated calculations [22, 21] shows that while the coefficients $a_1$ and $a_2$ are exact, the correct coefficients $a_3$ and $a_4$ are: $a_3 \approx 0.315$ and $a_4 \approx 0.161$. The coefficients $C_i$ of the expansion (1.72) can be obtained via

$$C_i = a_i C_1^i = a_i\left(g^*\right)^{-i/5}\,. \tag{1.90}$$

In particular,

$$\begin{aligned} C_1 &= \frac{\sqrt{\pi}}{2^{1/5}5^{2/5}}\frac{\Gamma\left(3/5\right)}{\Gamma\left(21/10\right)} \\ C_2 &= \frac{3}{5}C_1^2 \end{aligned} \tag{1.91}$$

and respectively, $C_3 \approx -0.483$, $C_4 \approx 0.285$. The convergence of the series is rather slow, and accurate results can be expected only for small enough $g/g^*$.

**Padé approximation**

For practical applications, such as molecular dynamics simulations, the expansion (1.89) is of limited value, since it diverges for large impact velocities, $g \to \infty$. However it is possible to construct a Padé approximant for $\varepsilon$ based on the above coefficients, which reveals the correct limits, $\varepsilon(0) = 1$ and $\varepsilon(\infty) = 0$. The dependence of $\varepsilon(g)$ is expected to be a smooth monotonically decreasing function, which suggests that the order of the numerator must be smaller than the order of the denominator. The 1-4 Padé-approximant,

$$\varepsilon = \frac{1 + d_1 \left(g/g^*\right)^{1/5}}{1 + d_2 \left(g/g^*\right)^{1/5} + d_3 \left(g/g^*\right)^{2/5} + d_4 \left(g/g^*\right)^{3/5} + d_5 \left(g/g^*\right)^{4/5}} \,. \tag{1.92}$$

satisfy these conditions. Standard analysis (e.g. [3]) yields the coefficients $d_k$ in terms of the coefficients $a_k$

$$\begin{aligned} d_0 &= a_4 - 2a_3 - a_2^2 + 3a_2 - 1 \\ d_1 &= \left[1 - a_2 + a_3 - 2a_4 + (a_2 - 1)(3a_2 - 2a_3)\right]/d_0 \approx 2.583 \\ d_2 &= \left[(a_3 - a_2)(1 - 2a_2) - a_4\right]/d_0 \approx 3.583 \\ d_3 &= \left[a_3 + a_2^2(a_2 - 1) - a_4(a_2 + 1)\right]/d_0 \approx 2.983 \\ d_4 &= \left[a_4(a_3 - 1) + (a_3 - a_2)(a_2^2 - 2a_3)\right]/d_0 \approx 1.148 \\ d_5 &= \left[2(a_3 - a_2)(a_4 - a_2a_3) - (a_4 - a_2^2)^2 - a_3(a_3 - a_2^2)\right]/d_0 \approx 0.326 \end{aligned} \tag{1.93}$$

Using the characteristic velocity $g^* = 0.32\,\mathrm{cm/s}$ for ice at very low temperature as a fitting parameter, we compare the theoretical prediction of $\varepsilon(g)$, given by Eq. (1.92), with the experimental results [4], see Fig. 1.2. The discrepancy with the experimental data at small $g$ follows from the fact that the extrapolation expression, $\varepsilon = 0.32/g^{0.234}$ used by [4] to fit the experimental data has an unphysical divergence at $g \to 0$ and does not imply the failing of the theory for this region. The scattering of the experimental data presented by [4] is large for small impact velocity according to experimental complications, therefore the fit formula of [4] cannot be expected to be accurate enough for velocities that are too small. For very high velocities the effects, such as brittle failure, fracture and others, may contribute to the dissipation, so that the mechanism of the viscoelastic losses could not be the primary one. In the region of very small velocity other than viscoelastic interactions e.g. adhesive interactions may be important.

### 1.3.3 Coefficient of restitution for adhesive collisions

For very small velocities, when the kinetic energy of the relative motion of colliding particles becomes comparable with the surface interaction energy at a contact, the adhesive forces play an important role. They may even change the coefficient of restitution. Indeed, as it has been pointed out above, the adhesive particles in a contact are compressed even at zero external load; moreover one has to apply a tensile force to separate the particles. Therefore, at the second stage of the collision the moving back particles have to overcome the barrier due to the tensile interactions, which try to keep them together. The work against this tensile

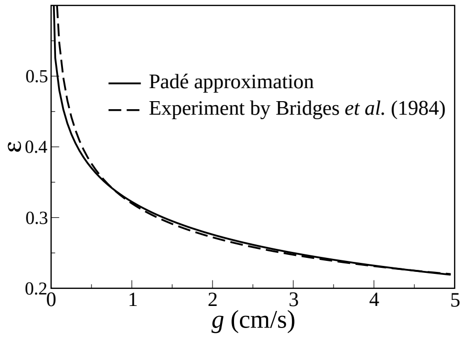


**Figure 1.2:** Dependence of the coeffi cient of normal restitution on the impact velocity for ice particles. The dashed line – experimental [30], solid line – the Pad´e-approximation (1.92) with the constants given by (1.93) and with the characteristic velocity for ice $g^* = 0.32$ cm/s.

force reduces the kinetic energy of the relative motion after the collision, that is it reduces the restitution coefficient. It may even happen that after a collision the kinetic energy of the relative motion is too small to overcome the attractive barrier. In this case the restitution coefficient occurs to be zero and the collision becomes a sticking.

A simplified analysis of an adhesive collisions has been performed in [8] to estimate the impact of the adhesive forces on the coefficient of restitution. This allows to estimate the range of validity of the viscoelastic collision model.

We assume that the JKR theory is adequate for the given system parameters. We also assume that the adhesion is small and that the adhesive interactions may be neglected when the force between the particles is purely repulsive. Hence we take into account the adhesive interactions only when the total force is attractive, that is when the force is determined by the adhesion. This happens at the very end of the collision. We also neglect the additional dissipative forces, which arise due to the adhesive interaction and assume that all dissipation during the collision may be attributed to the viscoelastic interactions.

At the second stage of a collision, when the particles move away from each other they pass the point where the contact area is $a_0$ and the total force vanishes. As the particles move away further, the force becomes negative, until it reaches at $a = a_{\rm sep}$ the maximum negative value $F = F_{\rm sep}$, Eq. (1.51). At this point the contact of the particles is terminated and they separate. According to our assumption, the work of the tensile force which acts against the particles separation reads,

$$W_0 = \int_{\xi(a_0)}^{\xi(a_{\rm sep})} F(\xi) d\xi = \int_{a_0}^{a_{\rm sep}} F(a) \frac{d\xi}{da} da \,. \tag{1.94}$$

Using Eq. (1.49) for the total force $F(a)$, Eq. (1.48) for the compression, which allows to obtain $d\xi/da$, and Eqs. (1.50,1.52) for $a_0$ and $a_{\rm sep}$, we arrive at the following expression for the work of the tensile forces:

$$W_0 = -q_1 \left( \pi^5 \gamma^5 D^2 R^4 \right)^{1/3} \,, \tag{1.95}$$

with the constant

$$q_1 = \frac{8}{5} I_{5/3} - \frac{2}{7} \left( \sqrt{\frac{8}{3}} + 4\sqrt{6} \right) I_{7/6} + I_{2/3}$$
$$I_\alpha \equiv (3/2)^\alpha - (3/8)^\alpha \,, \tag{1.96}$$

Let the impact velocity be $g$; at the end of the collision, before the tensile forces start to act, the relative velocity is $g' = \varepsilon g$. The final velocity $g''$, when the particles completely separate from each other, may be found from the conservation of energy:

$$\frac{1}{2} m^{\text{eff}} \left(g''\right)^2 - \frac{1}{2} m^{\text{eff}} \left(g'\right)^2 = W_0 \,. \tag{1.97}$$

From the latter equation we obtain the restitution coefficient for the adhesive collision, $\varepsilon_{\text{ad}}$,

$$\varepsilon_{\text{ad}}(g) = \frac{g''}{g} = \frac{\sqrt{\frac{1}{2} m^{\text{eff}} \varepsilon^2(g) g^2 - W_0}}{g} \,, \tag{1.98}$$

where $\varepsilon(g)$ is the restitution coefficient without the adhesive interaction. Hence we obtain the condition of the validity of the viscoelastic collision model,

$$\varepsilon(g) g \gg \sqrt{\frac{2W_0}{m^{\text{eff}}}} \,. \tag{1.99}$$

The threshold impact velocity $g_{\text{st}}$ which demarcates the restitutive, $g > g_{\text{st}}$ and sticking $g < g_{\text{st}}$ collisions may be obtained from the solution of the equation,

$$\frac{1}{2} m^{\text{eff}} \varepsilon^2(g) g^2 = W_0 \,. \tag{1.100}$$

Using Eq. (1.72) we obtain for viscoelastic spheres, in the leading order approximation with respect to the small dissipative parameter $A$:

$$g_{\text{st}} = \frac{2W_0}{m^{\text{eff}}} \left( 1 + C_1 A \kappa^{2/5} \left( \frac{2W_0}{m^{\text{eff}}} \right)^{-4/5} \right) \,. \tag{1.101}$$

If the relative normal velocity of the colliders is smaller than $g_{\text{st}}$ they move after the collision as a joint particle with the total mass $m_1 + m_2$.

## 1.4 Conclusion

We consider the main forces which act between particles in granular matter. We exclude very large particles, for which gravitational interparticle interactions could play a role, and very small particles for which the electrostatic interactions due to the residual surface charge of a few elementary units may be significant, and analyze the elastic, dissipative and adhesive forces. We use the classical Hertz theory for the elastic interactions at a contact. For the dissipative forces we adopt the simplest model of the viscoelastic interactions, where the viscous stress is linearly related to the strain rate. Moreover, we employ the quasistatic approximation, which implies that the impact velocity is much smaller then the speed of sound in the material and that the viscosity relaxation time is much smaller than the duration of a collision. Within this approximation we obtain the general solution for the contact problem for convex viscoelastic bodies. We discuss the available models of the adhesive forces and choose the simplest analytical model of Johnson, Kendall and Roberts (JKR), which is proved to be accurate in a range of parameters of a practical interest. Using the JKR model we discuss the additional dissipation which arises owing to the adhesive forces and estimate the range of validity of a pure viscoelastic contact. We also estimate the threshold value of the normal component of the impact velocity, which demarcates the restitutive and sticking collisions.

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