Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$, revisited

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Consider $\mathbb{R}^4$ with canonical complex structure $J$ such that $\omega(.,.) = \langle J.,. \rangle$ where $\langle .,. \rangle$ is the scalar product on $\mathbb{R}^4$ and $\omega$ the standard symplectic form.
Lagrangian surfaces

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- An immersion $f : M \to \mathbb{R}^4$ of a Riemann surface $M$ into $\mathbb{R}^4 = \mathbb{C}^2$ is called Lagrangian if $f^*\omega = 0$.

- The Gauss map $\gamma$ of a Lagrangian immersion has values in the space of Lagrangian subspaces $\text{Lag}(\mathbb{R}^4)$:

$$\gamma : M \to \text{Lag}(\mathbb{R}^4).$$
Lagrangian angle and Maslov form

$U(2)$ operates on $\text{Lag}(\mathbb{R}^4)$

$$\text{Lag}(\mathbb{R}^4) = U(2)/S0(2)$$
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The Lagrangian angle $\beta$ is the lift of $s$ to the universal cover:

$$s = e^{i\beta}.$$ 

Moreover, when $M = T^2 = \mathbb{C}/\Gamma$ is a 2-torus,

$$\beta(z) = 2\pi < \beta_0, z>$$

where $\beta_0 \in \Gamma^* \subset \mathbb{C}$ is called the Maslov form.
Variational problems

Consider Hamiltonian stationary Lagrangians (HSL), that is immersions $f : M \rightarrow \mathbb{C}^2$ which are critical points of the area functional

$$A(f) = \int_M |df|^2$$

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\mathcal{A}(f) = \int_M |df|^2
\]

under variations by Hamiltonian vector fields.

**Fact:** \( f : M \rightarrow \mathbb{C}^2 \) is Hamiltonian stationary Lagrangian if and only if its Lagrangian angle map \( \beta \) is harmonic.
Results

- Oh: first and second variational formulae of the area functional
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- Oh’s conjecture: Clifford torus minimizes the area in its Hamiltonian isotopy class
- Ilmanen, Anciaux: if there exists a smooth minimizer, it has to be the Clifford torus.
- Castro, Chen, Urbano: non-trivial examples.
- Helein-Romon: complete description of HSL tori by Fourier polynomials; frequencies lie on a circle whose radius is governed by the Maslov class.
Left normal of a HSL surface

Let $f : M \rightarrow \mathbb{R}^4$ be a conformal immersion. Then the Gauss map of $f$ is given by

$$(N, R) : M \rightarrow S^2 \times S^2 = Gr_2(\mathbb{R}^4).$$
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In fact, identifying $\mathbb{R}^4 = \mathbb{H}$ we can write

$$df = e^{\frac{i\beta}{2}} dz g$$

and the left normal

$$N = e^{i\beta} i$$

satisfies $\ast df = N df.$
Spectral curves

- Helein-Romon: family of flat connections

\[ d^\lambda = \lambda^{-2}\alpha_{-2} + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2 \]

where \( \alpha_j \) lie in the eigenspaces of an order 4 automorphism of the Lie algebra of the group of symplectic isometries of \( \mathbb{R}^4 \).
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Gives all weakly conformal Hamiltonian stationary Lagrangian tori

Gives Hamiltonian stationary Lagrangian tori with branch points and "no" control on the branch locus.
Spectral curves

- HSL tori $f : T^2 \to \mathbb{R}^4$ are conformal: have multiplier spectral curve (Schmidt, Taimanov, Bohle-L-Pedit-Pinkall)
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The left normal of a HSL torus $N : T^2 \rightarrow S^1$ is harmonic: have spectral curve of the harmonic left normal (Hitchin).
The family of flat connections

A map $N : M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ is harmonic if and only if the family of complex connections

$$d^\mu = d + (\mu - 1)A^{1,0} + (\mu^{-1} - 1)A^{0,1}$$

on the trivial bundle $\mathbb{H}$ is flat, where $A = \frac{1}{4}(\ast dN + NdN)$ and $\mu \in \mathbb{C}_*$. 
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on the trivial bundle $\mathbb{H}$ is flat, where $A = \frac{1}{4}(\ast dN + NdN)$ and $\mu \in \mathbb{C}_\ast$.

Here $\mathbb{C} = \text{span}\{1, I\}$ where the complex structure $I$ on $\mathbb{H}$ is defined by right multiplication by $i$.

Moreover, for $\omega \in \Omega^1$

$$\omega^{1,0} = \frac{1}{2}(\omega - I \ast \omega), \quad \omega^{0,1} = \frac{1}{2}(\omega + I \ast \omega)$$

denote the $(1, 0)$ and $(0, 1)$ parts with respect to the complex structure $I$. 
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- If \( M = \mathbb{C}/\Gamma \) is a 2-torus, the parallel sections \( \alpha \in \Gamma(\mathbb{H}) \) of \( d^\mu \) with \textit{multiplier}, that is

\[
\gamma^* \alpha = \alpha h_\gamma, \quad \gamma \in \Gamma, \ h_\gamma \in \mathbb{C}_*, \ C = \text{span}\{1, i\},
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  are the eigenvectors of the monodromy of $d^\mu$.

- The spectral curve $\Sigma_e$ of $N : T^2 \to S^2$ is the normalization of
  \[ \text{Eig} := \{ (\mu, h) \mid \exists \alpha : d^\mu \alpha = 0, \gamma^* \alpha = \alpha h_\gamma, \gamma \in \Gamma \} \]
The eigenline bundle [Hitchin]

Let \( N : M \to S^2 \) and \( d^\mu \) the associated family of flat connections.

- Generically, the space of parallel sections of \( d^\mu \) with a given multiplier is 1-dimensional, and one obtains the eigenline bundle \( \mathcal{E} \to \Sigma_e \).
Let $N : M \to S^2$ and $d^\mu$ the associated family of flat connections.

- Generically, the space of parallel sections of $d^\mu$ with a given multiplier is 1-dimensional, and one obtains the eigenline bundle $\mathcal{E} \to \Sigma_e$.
- The harmonic map can be reconstructed by linear flow in the Jacobian of $\Sigma_e$. 
The spectral curve of the left normal

Let $f : T^2 \to \mathbb{C}^2$ be a Hamiltonian stationary Lagrangian torus with harmonic left normal $N$ and family of flat connections $d\mu$. 

**Theorem (L-Romon, Moriya)**

All parallel sections with multiplier can be computed explicitly:

$$\alpha_{\mu}^\pm = e^j\beta 2 \left( 1 \mp k\sqrt{\mu - 1} \right)e^{\pm 2\pi \left( <A\mu, .> + i <C\mu, .> \right)}$$

with $A_{\mu} = i\beta_0 4 \left( \sqrt{\mu - 1} - \sqrt{\mu} \right)$, $C_{\mu} = \beta_0 4 \left( \sqrt{\mu - 1} + \sqrt{\mu} \right)$. 

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HSL tori

HSL in $\mathbb{C}^2$

HSL tori

The Hitchin spectral curve

$\mu$-Darboux transforms

The multiplier spectral curve

Darboux transforms
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$$

*with $A^\mu = \frac{i \beta_0}{4} (\sqrt{\mu^{-1}} - \sqrt{\mu})$, $C^\mu = \frac{\beta_0}{4} (\sqrt{\mu^{-1}} + \sqrt{\mu})$.***
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- \( \Sigma_e \) compactifies with \( \tilde{\Sigma}_e = \mathbb{CP}^1 \).
- \( \mu : \tilde{\Sigma}_e \rightarrow \mathbb{CP}^1, (\mu, h) \mapsto \mu \) is a 2-fold covering, branched over \( 0, \infty \).
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- The eigenline bundle \( \mathcal{E} \) extends holomorphically to \( \bar{\Sigma}_e \).
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- Let \( J \in \Gamma(\text{End}(\mathbb{H})) \), \( J^2 = -1 \), be the complex structure given by the quaternionic extension of

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J|_{\mathcal{E}_{x_\infty}} = I|_{\mathcal{E}_{x_\infty}}, \quad \mu(x_\infty) = \infty.
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Then \( J \) is in fact the complex structure given by left multiplication by \( N \).
\( \mu \)-Darboux transforms

Let \( f : M \to \mathbb{C}^2 \), be a Hamiltonian stationary Lagrangian torus with harmonic left normal \( N \).

**Theorem (L-Romon)**

Let \( \alpha \in \Gamma(\mathbb{H}) \) be a parallel section of \( d^\mu \) and put

\[
T^{-1} = \frac{1}{2} (N\alpha (a - 1)\alpha^{-1} + \alpha b \alpha^{-1})
\]

\[
a = \frac{\mu + \mu^{-1}}{2}, \quad b = i \frac{\mu^{-1} - \mu}{2}.
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Let $f : M \to \mathbb{C}^2$, be a Hamiltonian stationary Lagrangian torus with harmonic left normal $N$.

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Then

$$\hat{N} = -TNT^{-1}$$

is a harmonic map $\hat{N} : M \to S^2$ of $M$ into the 2-sphere.
\(\mu\)-Darboux transforms

Let \(f : M \rightarrow \mathbb{C}^2\), be a Hamiltonian stationary Lagrangian torus with harmonic left normal \(N\).

**Theorem (L-Romon)**

Let \(\alpha\) be a parallel section with multiplier and put

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$\mu$-Darboux transforms

Let $f : M \to \mathbb{C}^2$, be a Hamiltonian stationary Lagrangian torus with harmonic left normal $N$.

**Theorem (L-Romon)**

Let $\alpha$ be a parallel section with multiplier and put

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$$a = \frac{\mu + \mu^{-1}}{2}, \quad b = i\frac{\mu^{-1} - \mu}{2}.$$  

Then $T^{-1}$ is again globally defined and

$$\hat{N} = -TNT^{-1}$$

is a harmonic map $\hat{N} : M \to S^2$ from $M$ into the 2-sphere.
$\mu$-Darboux transforms

Let $f : T^2 \to \mathbb{C}^2$, be a Hamiltonian stationary Lagrangian torus with harmonic left normal $N$ and $df = e^{\frac{j}{2} \beta} dz g$.

Theorem (L-Romon)

If $\alpha \in \Gamma(\mathbb{H})$ is a parallel section with multiplier, than $\hat{N}$ is the left normal of a HSL torus

$$\hat{f} = f + TH^{-1},$$

where $H = \pi g^{-1} \bar{\beta}_0 e^{\frac{j}{2} \beta} k$. 
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where $H = \pi g^{-1} \beta_0 e^{\frac{i\beta}{2}} k$.

We call $\hat{f}$ a **$\mu$-Darboux transform** of $f$. 
Remark

- *Locally, a $\mu$-Darboux transform is always at least constrained Willmore*.

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- A similar theorem holds both for $\mu$-Darboux transforms of CMC tori (Carberry-L-Pedit), and (constrained) Willmore tori (Bohle).
Remark

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- $f^\#$ is called a classical Darboux transformation of $f$ if there exists a sphere congruence enveloping $f$ and $f^\#$. 
\( \mu \)-Darboux transforms

Remark

- The \( \mu \)-Darboux transformation is a generalization of the classical Darboux transformation.
- \( f^\# \) is called a classical Darboux transformation of \( f \) if there exists a sphere congruence enveloping \( f \) and \( f^\# \).
- The \( \mu \)-Darboux transformation satisfies a weaker enveloping condition.
Special cases

For special parameter $\mu$ the transform on the left normal is trivial
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Question: Is $\hat{f}$ for $\mu > 0$ the original HSL torus?
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More generally, does the Lagrangian angle $\beta$ determine $f$?

What is the condition for the existence of a HSL torus with Lagrangian angle $\beta$?
Recall: A Hamiltonian stationary Lagrangian immersion $f$ has Lagrangian angle $\beta \iff *df = N df$ with $N = e^{i\beta}i$. 
Recall: A Hamiltonian stationary Lagrangian immersion $f$ has Lagrangian angle $\beta \iff *df = Ndf$ with $N = e^{i\beta}i$.

The operator $D : \Gamma(\mathbb{H}) \to \Gamma(\bar{K}\mathbb{H})$

$$D := \frac{1}{2}(d + J * d)$$

is a (quaternionic) holomorphic structure where the complex structure $J$ on $\mathbb{H}$ is given by left multiplication by $N$. 
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Goal: given $N = e^{i\beta}i$ find all holomorphic sections $\alpha \in \ker D$. 
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Note: $d^{\mu} \alpha = 0 \implies D\alpha = 0.$
Holomorphic sections with multiplier

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Goal: given $N = e^{i\beta}i$ find all holomorphic sections $\alpha$ with multiplier, that is $\alpha \in \ker D$ with $\gamma^*\alpha = \alpha h_\gamma$, $\gamma \in \Gamma$.

Note: $d^\mu \alpha = 0 \implies D\alpha = 0$. 
Holomorphic sections with multiplier

Let $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ be a Hamiltonian stationary torus. For $(A, B) \in \mathbb{C}^2$ consider

$$|\delta - B|^2 - |A|^2 = \frac{|\beta_0|^2}{4}, \quad <A, \delta - B> = 0 \quad (1)$$

with $\delta \in \Gamma^* + \frac{\beta_0}{2}$. 
Holomorphic sections with multiplier

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$$|\delta - B|^2 - |A|^2 = \frac{|\beta_0|^2}{4}, \quad <A, \delta - B> = 0$$

with $\delta \in \Gamma^* + \frac{\beta_0}{2}$. Denote by

$$\Gamma^*_{A,B} = \{ \delta \in \Gamma^* + \frac{\beta_0}{2} \mid \delta \text{ satisfies (1)} \}$$

the set of admissible frequencies.
Holomomorphic sections with multiplier

Let $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ be a Hamiltonian stationary torus, and $D$ the quaternionic holomorphic structure given by the complex structure $J$.

**Theorem (L-Romon)**

- **Multipliers of holomorphic sections are exactly given by**

  $$h^{A,B} = e^{2\pi(\langle A, \cdot \rangle - i\langle B, \cdot \rangle)}$$

  with $\Gamma^*_{A,B} \neq \emptyset$.
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- **Multipliers of holomorphic sections are exactly given by**
  
  $$h^{A,B} = e^{2\pi(\langle A, \cdot \rangle - i\langle B, \cdot \rangle)}$$

  with $\Gamma_{A,B}^* \neq \emptyset$.

- **For** $\delta \in \Gamma_{A,B}^*$
  
  $$\alpha_\delta = e^{i\beta/2} (1 - k\lambda_\delta) e^{2\pi(\langle A, \cdot \rangle + \langle \delta - B, \cdot \rangle)}$$

  with $\lambda_\delta \in \mathbb{C}_*$ is a *(monochromatic)* holomorphic section.
HSL tori with prescribed Lagrangian angle [Helein-Romon, L-Romon]

Let $\Gamma$ be a lattice in $\mathbb{C}$, and let $\beta_0 \in \Gamma^*$. Then $\beta = 2\pi < \beta_0, \cdot >$ is a Lagrangian angle of a Hamiltonian stationary torus $f$ if and only if

$$\Gamma_{0,0}^* \supset \{ \pm \frac{\beta_0}{2} \}$$
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In this case, all HSL tori with Lagrangian angle \( \beta \) are (up to translation) of the form

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f = \sum_{\delta \in \Gamma^*_{0,0} \setminus \{ \pm \frac{\beta_0}{2} \}} \alpha_\delta m_\delta, \quad m_\delta \in \mathbb{C}.
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In this case, all HSL tori with Lagrangian angle $\beta$ are (up to translation) of the form

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HSL tori with prescribed Lagrangian angle
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Holomorphic sections with multiplier

**Theorem (L-Romon)**

*Every holomorphic section with multiplier $h^{A,B}$ is given by

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\alpha = \sum_{\delta \in \Gamma_{A,B}^*} \alpha_\delta m_\delta
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where $m_\delta \in \mathbb{C}$.\)
Holomorphic sections with multiplier

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- Every holomorphic section with multiplier $h^{A,B}$ is given by
  \[ \alpha = \sum_{\delta \in \Gamma^*_{A,B}} \alpha_\delta m_\delta \]

- $m_\delta \in \mathbb{C}$.
- $|\Gamma^*_{A,B}| = 1$ away from a discrete set of pairs $(A, B)$. 
The multiplier spectral curve

Let \( \text{Spec} := \{ h | \exists \alpha \in \ker D : \gamma^* \alpha = \alpha h_\gamma, \gamma \in \Gamma \} \), and \( \Sigma \) its normalization.
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Then there exists a line bundle $\mathcal{L}$ such that
$$\mathcal{L}_\sigma = H^0_\sigma$$
for generic points $\sigma \in \Sigma$.
(see Bohle-L-Pedit-Pinkall for general conformal tori).
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**Theorem (L-Romon)**

- The spectral curves $\Sigma_e$ and $\Sigma$ of a Hamiltonian stationary torus are biholomorphic, and the eigenline bundle $\mathcal{E}$ and $\mathcal{L}$ coincide.
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**Theorem (L-Romon)**

- The spectral curves \( \Sigma_e \) and \( \Sigma \) of a Hamiltonian stationary torus are biholomorphic, and the eigenline bundle \( \mathcal{E} \) and \( \mathcal{L} \) coincide.
- In particular, the multiplier spectral curve of a Hamiltonian stationary torus can be compactified and has geometric genus 0.
Darboux transforms

Again, we can use holomorphic sections with multiplier to define a Darboux transform

\[ \hat{f} = f + TH^{-1} \]

of a HSL torus \( f \) where \( T = \alpha \beta^{-1} \), \( d\alpha = -dfH\beta \) and \( H = \pi g^{-1} \bar{\beta}_0 e^{\frac{i\beta}{2} k} \).
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- If $\alpha = \alpha_\delta$ is a monochromatic holomorphic section then $\hat{f}$ is HSL with (after reparametrization) Lagrangian angle $\beta$. 
Again, we can use holomorphic sections with multiplier to define a Darboux transform

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**Theorem (L-Romon)**

- If $\alpha = \alpha_\delta$ is a monochromatic holomorphic section then $\hat{f}$ is HSL with (after reparametrization) Lagrangian angle $\beta$.
- $f$ is obtained as limit of Darboux transforms with multiplier $\sigma \to \sigma_\infty \in \bar{\Sigma}$. 
The Darboux transformation is a further generalization of the $\mu$-Darboux transformation:

**Theorem (L-Romon)**

The monochromatic holomorphic sections are exactly the $d^\mu$-parallel sections for some $\mu \in \mathbb{C}_*$. 
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**Theorem (L-Romon)**

The monochromatic holomorphic sections are exactly the $d^\mu$-parallel sections for some $\mu \in \mathbb{C}_*$.

In other words, the monochromatic Darboux transforms are exactly the $\mu$-Darboux transforms.
Theorem (L-Romon)

If $|\Gamma_{0,0}^*| = 4$ then all monochromatic Darboux transforms are after reparametrization of $f$.
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- If $|\Gamma_{0,0}^*| = 4$ then all monochromatic Darboux transforms are after reparametrization $f$.
- However, there exist examples where $|\Gamma_{0,0}^*| > 4$, and the resulting monochromatic Darboux transforms are not Möbius transformations of $f$. 
Darboux transforms

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- Moreover, there exists HSL tori with polychromatic holomorphic sections $\alpha$. 
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- If $|\Gamma_{0,0}^*| = 4$ then all monochromatic Darboux transforms are after reparametrization $f$.
- However, there exist examples where $|\Gamma_{0,0}^*| > 4$, and the resulting monochromatic Darboux transforms are not Möbius transformations of $f$.
- Moreover, there exists HSL tori with polychromatic holomorphic sections $\alpha$ so that the corresponding Darboux transforms are not Lagrangian in $\mathbb{C}^2$. 
Homogeneous torus
Clifford torus
Thanks!