Transformations on Willmore surfaces

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Chapter 0

Introduction

In surface theory one frequently considers surface classes which come by variational principles: one tries to find all surfaces which are critical points of a certain “energy” associated with the surface. The most famous examples are the *minimal surfaces* which are critical points of the area functional. Given a boundary curve, one tries to find the surface with the least energy having this boundary. Further examples of surface classes arising from variational principles are *constant mean curvature surfaces* which are the critical points of the area functional under the constraint to preserve the volume, and the *Willmore surfaces*, the critical points of the bending energy $\mathcal{W}(f) = \int H^2$ where H is the mean curvature of $f : M \to \mathbb{R}^3$.

Over the past 25 years significant progress has been made in the classification and construction of these surfaces, e.g. [Cal67], [Bry82], [Bry84], [Hit90], [PS89], and [Bob94]. The theory of each of these surface classes is closely linked to the theory of harmonic maps into some associated symmetric space: in the case of a minimal surface, the harmonic map is given by a conformal parametrization, for constant mean curvature surfaces it is given by the Gauss map, and for Willmore surfaces by the conformal Gauss map. The harmonic map equations for the various cases turn out to be *completely integrable* partial differential equations.

In the simplest case these equations lead to solutions given in terms of holomorphic functions, for example the classical Weierstraß representation of minimal surfaces. In the case of tori, solutions to the harmonic map equations can be obtained from theta functions on an auxiliary Riemann surface, the so–called *spectral curve* [PS89], [Hit90], [Bob91], [McI01]. For some surface classes the spectral curves have finite genus (e.g. CMC tori and Willmore tori), which allows for explicit parametrizations in terms of theta functions of *all* the tori in this class [Bob91], [Sch02], [FPPS92].

Despite of these results, a number of basic questions remain unanswered. For instance, what are the minimal values of the variational problems, or on which tori are they attained. For higher genus the situation is even more unsatisfactory: there are only few explicit examples of higher genus constant mean curvature or Willmore surfaces, and little is known about an appropriate generalization of the "spectral curve".

The analytic difficulties come from the fact that the fundamental equations of a surface in 3-space, the Gauss-Codazzi equations, are a nonlinear, third order system. Moreover, the Gauss-Codazzi equations for various classes of surfaces give rise to qualitatively different systems of differential equations, each demanding its own theory.

In comparison, the theory of holomorphic curves into complex projective space is very rich: explicit examples for every genus are studied, global properties and methods applied. One of the reasons for the difference to surface theory is that the fundamental equation of algebraic curve theory is the linear, first order Cauchy-Riemann equation, and much more can be said about solutions. Moreover, the Kodaira correspondence gives the link between holomorphic curves and holomorphic line bundles, and the theory of algebraic curves can be formulated in the language of holomorphic line bundles. Cornerstones for the theory of algebraic curves are the Riemann-Roch Theorem, the Clifford estimate, the Plücker relations and the Abel map.

Over the past years, a completely new theory, the so–called Quaternionic Holomorphic Geometry, [PP98], [BFL+02], [FLPP01], has been built to combine the theory of algebraic curves with the theory of conformal maps of a Riemann surface into 3– or 4–space. In Quaternionic Holomorphic Geometry, the conformal geometry of the 4–sphere is modeled by the projective line $\mathbb{HP}^1$ on which the group of orientation preserving Möbius transformations acts by $\mathrm{Gl}(2, \mathbb{H})$. The holomorphic maps in this theory are exactly the conformal maps into the 4–sphere. The appeal of this model is that the geometric background gives insights in how to build a quaternionic extension of Complex Analysis, and, conversely, well–known results for complex holomorphic curves in $\mathbb{CP}^n$ can be translated to the quaternionic setting and give, when applied to surfaces in 3– and 4–space, new results in surface theory.

In *Chapter 1* we give a fairly coherent summary of Quaternionic Holomorphic Geometry: on one hand it will help the reader to get a flavor of the ideas and motivations of the theory, on the other hand it will setup the notations and tools needed for the results on transformations of conformal maps.

Conformal maps $f : M \to \mathbb{R}^3$ from a Riemann surface M with complex structure J into 3–space satisfy a Cauchy–Riemann–type equation $df(JX) = Ndf(X)$ with varying "i". Here, $N : M \to S^2$ is the normal to the surface and the multiplication is the multiplication in the quaternions where we identify $\mathbb{R}^3$ with the imaginary quaternions $\operatorname{Im}\mathbb{H}$. If f takes values in a 2–plane its unit normal is a constant map, say $N = i$, in which case we recover the usual Cauchy–Riemann equation for a holomorphic map $f : M \to \mathbb{C}$.

Using this observation as a starting point, the quaternionic holomorphic theory can be derived analogous to the complex case [PP98],[BFL+02], [FLPP01]: the holomorphic functions in the quaternionic setting are the conformal maps into the 4–sphere. A key observation is that a map $f : M \to S^4 = \mathbb{HP}^1$ from a Riemann surface M into the 4–sphere is the same as the quaternionic line bundle L over M whose fiber at $p \in M$ is given by $L_p = f(p)$. We explain how the quaternionic setup can be used to study conformal geometry of surfaces in 3– and 4–space. In particular, the basic notions for conformal immersions from a Riemann surface into the 4–sphere and for surfaces in Euclidean space are expressed in terms of quaternionic calculus, e.g., the mean curvature sphere, the Hopf fields, the Willmore functional, the mean curvature vector, and the Gauss and normal curvature.

The conformality of f gives rise to a complex structure J and a holomorphic structure on the dual line bundle L^{-1}. Here a quaternionic *holomorphic structure* on a quaternionic vector bundle V with complex structure J is a quaternionic linear map $D : \Gamma(V) \to \Gamma(\bar{K}V)$ which satisfies the Leibniz rule $D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)''$ for $\lambda : M \to \mathbb{H}$ where ω'' denotes the $(0,1)$-part of a 1–form ω.

The complex *Kodaira correspondence* translates to the quaternionic setup: a quaternionic holomorphic curve $f : M \to \mathrm{G}_k(\mathbb{H}^n)$ into the k–plane Grassmannian corresponds, up to Möbius equivalence, to a basepoint free n–dimensional subspace $H \subset H^0(V)$ of the space of holomorphic sections of a k–dimensional quaternionic holomorphic vector bundle V [FLPP01]. In particular, a holomorphic curve $[f_1 : f_2 : \ldots : f_{n+1}] : M \to \mathbb{HP}^n$ from a Riemann surface into $\mathbb{HP}^n$ gives rise to a family of conformal maps: the coordinate maps $f_i : M \to \mathbb{H}$ are conformal.

Moreover, the *Riemann–Roch theorem* holds verbatim for quaternionic holomorphic vector bundles [PP98], [FLPP01]. However, the quaternionic *Plücker formula* involves a new quaternionic invariant: the *Willmore energy* $\mathcal{W}(D)$ of the holomorphic structure D. A quaternionic holomorphic structure decomposes $D = \bar{\partial} + Q$ in J commuting and anti-commuting parts, and the Willmore energy is given by $\mathcal{W}(D) = 2\int_M < Q \wedge *Q >$. In particular, $\mathcal{W}(D)$ measures the deviation from the complex case: for $Q \equiv 0$ we recover the theory of complex vector bundles. If the holomorphic structure is induced via the Kodaira embedding of a conformal immersion $f : M \to S^4$ of a compact Riemann surface into the 4–sphere then the Willmore energy of D is exactly the classical Willmore functional $\mathcal{W}(f) = \int |\mathcal{H}|^2 - K - K^\perp$ where $\mathcal{H}$ is the mean curvature vector and K and $K^\perp$ are the Gauss and normal curvature of f.

As mentioned before, applying results of this theory to surfaces in 3– and 4–space gives substantial insight into classical problems of surface geometry: the quaternionic Plücker formula provides lower bounds for the energy of harmonic tori in the 2–sphere and the area of constant mean curvature tori in 3–space in terms of their spectral genus. In particular, for a constant mean curvature torus of spectral genus > 6 the Willmore and Lawson conjectures are both satisfied [FLPP01].

Transformations which preserve special surface classes in 3– and 4–space play an important role in surface geometry. One of the motivations for the study of these transformations comes from the fact that they allow to construct more complicated surfaces from given simple surfaces. Historical examples include the Bäcklund transformation on surfaces of constant Gaussian curvature [Bia80] and the Darboux transformation on isothermic surfaces [Dar99].

In *Chapter 2* we discuss a generalized Darboux and Bäcklund transformation on conformal maps $f : M \to S^4$ of a Riemann surface into the 4–sphere. To define the Darboux transformation we use a geometric construction [BLPP]. A sphere congruence S *envelopes* a conformal immersion $f : M \to S^4$ if for all $p \in M$ the sphere $S(p)$ passes through $f(p)$ and the oriented tangent spaces of f and $S(p)$ coincide. It is a classical result [Dar99] that if $f : M \to \mathbb{R}^3$ allows a sphere congruence enveloping f and a second surface $f^\sharp : M \to \mathbb{R}^3$ then both f and $f^\sharp$ are isothermic, and $f^\sharp$ is a Darboux transform of f. To generalize this transformation it is necessary to refine the enveloping property: we say that the sphere congruence S *left-envelopes* f if $S(p)$ goes through f at p and the oriented tangent spaces are left–parallel, that is if their associated oriented great circles on S^3 correspond via left translation in the group S^3. Given a conformal map $f : M \to S^4$ a conformal map $f^\sharp$ is called *Darboux transform* of f if there is a sphere congruence enveloping f and left–enveloping $f^\sharp$.

The *spectral curve* of a conformal torus $f : T^2 \to S^4$ with trivial normal bundle can be defined [BLPP], loosely speaking, as the set of all closed Darboux transforms of f. For each point $p \in T^2$, the images $f^\sharp(p)$ of the Darboux transforms $f^\sharp$ of f canonically embed the spectral curve into S^4 as a twistor projection of a holomorphic curve $\tilde{F}(p, .) : \Sigma \to \mathbb{CP}^3$. It can be shown that Σ is a Riemann surface of possibly infinite genus.

In the case of a constant mean curvature torus $f : T^2 \to \mathbb{R}^3$ this definition of the spectral curve coincides [CLP] with the "classical" one given by the eigenvalues of the holonomy of a family of flat connections [Hit90]. We show in Chapter 2 that the Darboux transforms corresponding to points on the spectral curve are isothermic even though the general Darboux transformation only coincides for very special points on the spectral curve with the classical Darboux transformation of a constant mean curvature torus in $\mathbb{R}^3$.

To define [LP03] a Bäcklund transformation on conformal maps $f : M \to S^4$, recall the enveloping and osculating construction for conformal maps into the 4–sphere, or more general, for holomorphic curves into $\mathbb{HP}^n$. The *first osculate* of a holomorphic curve $f : M \to \mathbb{HP}^n$ is obtained by intersecting the tangent of f with a fixed hyperplane. Conversely, an *envelope* $\tilde{f}$ of f is given by integrating prescribed tangents so that f is the first osculate of $\tilde{f}$. In [LP05] we extend the Bäcklund transformation to holomorphic curves in $\mathbb{HP}^n$, and show that the Bäcklund transform is given by an envelope of f. We use this geometric picture to show in Chapter 2 a Bianchi permutability theorem. In particular, Bäcklund transforms of Frenet curves are Frenet, and the $(n+1)$–step Bäcklund transform $\tilde{f}$ of a holomorphic curve $f : M \to \mathbb{HP}^n$ can be computed solely by differentiation and algebraic operations so that the $(n+1)$–step Bäcklund $\tilde{f}$ is globally defined.

The study of *Willmore surfaces*, the critical points of the bending energy $\int H^2$ of a surface, goes at least back to Blaschke's school in the 1920's [Bla29]. About 40 years later Willmore [Wil68] reintroduced the problem and focused on minimizers for the bending energy, nowadays called the *Willmore energy*, over compact surfaces of fixed genus. He showed that the round sphere is the minimum among genus 0 surfaces and formulated the conjecture that the minimum over tori is given by the Clifford torus with Willmore energy $2\pi^2$. In the 1980's Bryant [Bry84] classified all Willmore spheres in 3-space as inverted minimal spheres with planar ends in $\mathbb{R}^3$. Subsequently Ejiri [Eji88] and recently Montiel [Mon00] proved an analogous result for Willmore spheres in 4–space — in addition to inverted minimal spheres in $\mathbb{R}^4$ we also have twistor projections to S^4 of rational curves in $\mathbb{CP}^3$. The case of Willmore tori is more involved: To a Willmore torus $f : T^2 \to S^4$ with trivial normal bundle one can associate its spectral curve, namely the Riemann surface defined by possible monodromies of the associated S^1–family of flat connections [FPPS92]. The spectral curve of a Willmore torus has finite genus [Sch02], and the Willmore torus is then parametrized by theta functions on the spectral curve, [FPPS92], [Sch02]. In fact, the recent preprint [Sch02] by Schmidt seems to go some way toward proving the Willmore conjecture.

In *Chapter 3* we study Willmore curves in $\mathbb{HP}^n$: A holomorphic curve $f : M \to \mathbb{HP}^n$ from a compact Riemann surface into $\mathbb{HP}^n$ is Willmore [LP03] if f is a critical point of the Willmore energy under variations of f which have at least $n+1$ holomorphic sections in the associated holomorphic line bundle of f.

An important aspect of the theory of Willmore surfaces is its connection to harmonic maps: the *conformal Gauß map* or *mean curvature sphere congruence* S of a Willmore surface is harmonic. Similar to the $\bar\partial$ and ∂ sequence of harmonic maps $h : M \to \mathbb{CP}^n$ the $(0,1)$ and $(1,0)$–part Q and A of the derivative of the conformal Gauss field give new (possibly branched) conformal immersions $\tilde f, \hat f : M \to S^4$ provided $A \not\equiv 0$ and $Q \not\equiv 0$. We show that the conformal Gauss maps of $\tilde f$ and $\hat f$ extend smoothly into the branch points, and both surfaces are Willmore.

A degree computation shows that all mean curvature spheres of a Willmore sphere $f : S^2 \to S^4$ go through a constant point $q \in f(S^2)$ on the surface if $A, Q \not\equiv 0$. The stereographic projection across q gives a minimal surface in $\mathbb{R}^4$. In the case when $A \equiv 0$ (or $Q \equiv 0$) then f (or its dual curve) is a twistor projection of a holomorphic curve in $\mathbb{CP}^3$ [FLPP01], and we recover Montiel's result [BFL$^+$02]. In [Les] this result is generalized to Willmore spheres in $\mathbb{HP}^n$: a Willmore sphere in $\mathbb{HP}^n$ has integer Willmore energy and is given by complex holomorphic data [Les].

The case of Willmore tori is more involved: there are examples of Willmore tori in the 4–sphere constructed by integrable system methods which are neither inverted minimal surfaces nor twistor projections of elliptic curves [Pin85], [FP90], [BB93]. However, if the Willmore torus f has non–trivial normal bundle then we showed in [LPP05] that f comes from the twistor projection of a holomorphic curve in $\mathbb{CP}^3$ or from a minimal surface in $\mathbb{R}^4$. The method of proof in [LPP05] is to examine the possible monodromies of the associated S^1–family of flat connections. In Chapter 3 we present a different approach by using sequences of Willmore surfaces [LP]. This approach can be extended to the case of Willmore tori in $\mathbb{HP}^n$ with $\deg L \neq 0$, and we show that such a Willmore torus has integer

Willmore energy.

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Chapter 1

Introduction to Quaternionic Holomorphic Geometry

In this chapter, we review the main techniques and results developed in Quaternionic Holomorphic Geometry, [PP98], [BFL$^+$02], [FLPP01]. The objective is to use a "quaternionified complex analysis" to obtain results in surface theory. We generalize the theory of algebraic curves $h : M \to \mathbb{CP}^n$ of a Riemann surface M into complex projective space to a theory of holomorphic curves $f : M \to \mathbb{HP}^n$ into quaternionic projective space. This is geometrically motivated by the observation that the conformality condition for a map $f : M \to S^4$ is, when expressed in terms of quaternions, a Cauchy-Riemann equation with varying "i" on the target space. Here, we use the fact that the conformal geometry of the 4–sphere can be modeled by the quaternionic projective space $\mathbb{HP}^1 = S^4$ where the group of orientation preserving Möbius transformations is given by $\mathrm{GL}(2, \mathbb{H})$. Various results of algebraic curve theory can be extended to the quaternionic setting. The Kodaira correspondence links holomorphic curves in $\mathbb{HP}^n$ with quaternionic holomorphic line bundles. Analytically, the Cauchy–Riemann operator $\bar\partial$ on a complex vector bundle is replaced in the quaternionic theory by an elliptic operator $\bar\partial + Q$ on a quaternionic vector bundle L with complex structure J, where Q is a $(0, 1)$–form with values in the J–complex antilinear endomorphisms of L. The L^2–norm of Q provides an invariant of the quaternionic theory, the *Willmore energy* of the quaternionic holomorphic structure $\bar\partial + Q$. The properties of quaternionic holomorphic structures are similar to the properties of a $\bar\partial$ operator in the complex case: the vanishing orders of holomorphic sections are well–defined, the Riemann–Roch theorem holds verbatim, and the Plücker formula gives estimates on the Willmore energy.

1.1 Holomorphic curves in $\mathbb{HP}^n$

We use the example of a conformal map $f : M \to \mathbb{R}^3$ to motivate the definition of a holomorphic curve $f : M \to \mathbb{HP}^1$, and more general, of a holomorphic curve in $\mathbb{HP}^n$. The conformal Gauss map of a conformal map $f : M \to S^4$ will turn out to be an important ingredient to built a theory of quaternionic holomorphic curves and quaternionic holomorphic vector bundles. The conformal Gauss map gives a complex structures S on the trivial $\mathbb{H}^2$–bundle $V = \underline{\mathbb{H}}^2$. Generalizing the conformal Gauss map, we obtain the so–called *canonical complex structure* of a holomorphic curve $f : M \to \mathbb{HP}^n$. In general, the conformal Gauss map of a conformal map $f : M \to S^4$ does not extend into the branch points of f. A holomorphic curve for which the mean curvature sphere congruence, or more generally the canonical complex structure, extends to M is called a Frenet curve. The Willmore energy of a Frenet curve is the analog of the Willmore energy $W(f) = \int H^2$ of a conformal immersion $f : M \to \mathbb{R}^3$ where H is the mean curvature of f.

1.1.1 Conformal maps into the 4–sphere

We use the example of a conformal immersion $f : M \to \mathbb{R}^3$ of a Riemann surface M into 3–space to motivate how to define holomorphic curves in $S^4 = \mathbb{HP}^1$ or, more generally, in $\mathbb{HP}^n$. The Riemann surface M comes with a complex structure J_{TM} which gives the 90° degree rotation in the tangent space TM. In particular, an immersion $f : M \to \mathbb{R}^3$ is conformal if and only if for all $X \in TM$:

$$df(J_{TM}X) \perp df(X) \quad \text{and} \quad |df(J_{TM}X)| = |df(X)| \,. \tag{1.1}$$

Choosing the sign of the unit normal vector N appropriately, (1.1) reads as

$$df(J_{TM}X) = N \times df(X) \,. \tag{1.2}$$

Figure 1.1: Conformal maps into 3–space

We rewrite (1.2) in terms of quaternions: The Euclidean 3-space can be identified with the imaginary quaternions

$$\operatorname{im}\mathbb{H} = \operatorname{Span}_{\mathbb{R}}\{i, j, k\} = \mathbb{R}^3 \subset \mathbb{R}^4 = \operatorname{Span}_{\mathbb{R}}\{1, i, j, k\}$$

where $i^2 = j^2 = k^2 = -1$. The conjugation in the quaternions is defined by

$$\bar{x} = x_0 - x_1$$

for $x = x_0 + x_1$ with $x_0 \in \mathbb{R}, x_1 \in \operatorname{im}\mathbb{H}$, so that

$$x \in \operatorname{im}\mathbb{H} \quad \text{if and only if} \quad \bar{x} = -x \,.$$

The inner product on $\mathbb{R}^4$ is given in terms of quaternions as

$$< x, y >= \operatorname{Re}(\bar{x}y)$$

so that the cross product for orthogonal vectors in $\mathbb{R}^3$ becomes the quaternionic multiplication. In particular, (1.2) reads as

$$df(J_{TM}X) = N df(X)\,.$$

Denoting by $* : \Omega^1(TM) \to \Omega^1(TM)$ the negative Hodge star, i.e.,

$$*df(X) = df(J_{TM}X), \text{ for all } X \in TM\,,$$

we see that an immersion $f : M \to \mathbb{R}^3$ is conformal if and only if

$$* \, df = N df \tag{1.3}$$

with $N : M \to S^2$. Note that

$$S^2 = \{N \in \operatorname{im}\mathbb{H} \mid N^2 = -1\}$$

since $|N|^2 = N\bar{N} = -N^2$. If $f : M \to \mathbb{R}^3$ is a conformal immersion, then

$$- * df = \overline{*df} = \overline{Ndf} = \overline{df}\ \bar{N} = dfN$$

shows that

$$* \, df = N df = -dfN\,. \tag{1.4}$$

Furthermore, we observe that (1.3) is a Cauchy–Riemann equation with varying "i": if $f : M \to \mathbb{C} = \operatorname{Span}\{j, k\} \subset \operatorname{im}\mathbb{H}$, then f has constant normal $N = \pm i$ and f is conformal if and only if $*df = \pm i df$, i.e., f is holomorphic or antiholomorphic. An antiholomorphic map can be interpreted as a holomorphic map if we equip $\mathbb{C}$ with the complex structure $-i$ instead of i. This leads to the following definition:

Definition 1.1. *A map $f : M \to \mathbb{R}^3$ is called* holomorphic *if there exists $N : M \to S^2$ with $*df = -dfN$.*

Note that we dropped the assumption that f is immersed. However, a holomorphic map is conformal. We will see below that if df does not vanish identically, the holomorphicity of $f : M \to \mathbb{R}^3$ implies that df has only isolated zeros. In particular, f is branched conformal immersion. To generalize our notion of holomorphicity to maps $f : M \to \mathbb{R}^4$ into the 4–space, recall the Fundamental Lemma:

Lemma 1.2 (Fundamental Lemma, e.g. [BFL+02, Lemma 2])**.** *Every 2–dimensional subspace $E \subset \mathbb{R}^4 = \mathbb{H}$ is given by a pair $(N, R) \in S^2 \times S^2$ by*

$$E = \{x \in \mathbb{H} \mid Nx + xR = 0\}$$

N and R are called the left normal *and* right normal *of E. The pair (N, R) is unique up to sign. The orthogonal complement $E^\perp$ of E is given by*

$$E^\perp = \{x \in \mathbb{H} \mid Nx - xR = 0\}\,.$$

Note that the left and right normal vector of E are in general *not* orthogonal to E. However, since $N, R \in S^2$, we have $Nx, xR \in E$ and $Nx \perp x$ and $xR \perp x$ for all $x \in E$ as well as $|Nx| = |xR| = |x|$.

Applying the Fundamental Lemma in the case of an immersion $f : M \to \mathbb{R}^4 = \mathbb{H}$, we see that the tangent space and normal space of f are given at each point $p \in M$ by a pair $(N, R) \in S^2 \times S^2$ via

$$d_p f(T_p M) = \{x \in \mathbb{R}^4 \mid Nx + xR = 0\} \tag{1.5}$$

and

$$\perp_p M = \{x \in \mathbb{R}^4 \mid Nx - xR = 0\}\,. \tag{1.6}$$

Corollary 1.3. *Let $f : M \to \mathbb{R}^4 = \mathbb{H}$ be an immersion. Then f is conformal if and only if there exist $N, R : M \to S^2$ with*

$$* \, df = N df = -df R\,. \tag{1.7}$$

If $f : M \to \mathbb{R}^3 = \operatorname{im} \mathbb{H}$, then $R = N$ is a unit normal vector field to f.

Remark 1.4. Note that in the case of an immersion $f : M \to \mathbb{H}$ the two conditions

(1) there exists $N : M \to S^2$ with $*df = N df$

(2) there exists $R : M \to S^2$ with $*df = -df R$

are equivalent. In particular, in Corollary 1.3 we could replace (1.7) by either one of the conditions (1) or (2).

Since we consider conformal maps up to Möbius equivalence, we extend our interest to conformal maps $f : M \to S^4$ of a Riemann surface into the 4–sphere. We model the Möbius geometry of S^4 by the projective geometry of the (quaternionic) projective line $\mathbb{HP}^1$: A map $f : M \to S^4$ is given by the line subbundle $L \subset V$ of the trivial $\mathbb{H}^2$–bundle V over M, where the fiber L_p over $p \in M$ is the projective point $L_p = f(p)$. In other words, $L = f^*\mathcal{T}$ where $\mathcal{T}$ is the *tautological bundle* over $\mathbb{HP}^1$. In what follows, all vector spaces are, if not mentioned otherwise, quaternionic right vector spaces, and $\operatorname{Hom}(V, W)$ denotes the space of quaternionic linear maps from a (quaternionic) vector space V to a (quaternionic) vector space W. Since the tangent space of $\mathbb{HP}^1$ is given by $T_{[v]}\mathbb{HP}^1 = \operatorname{Hom}([v], \mathbb{H}^2/[v]))$, the *derivative* of $f : M \to \mathbb{HP}^1$ is given by

$$\delta = \pi d|_L \in \Omega^1(\operatorname{Hom}(L, V/L))\,, \tag{1.8}$$

where $\pi : V \to V/L$ and d is the trivial connection on $V = M \times \mathbb{H}^2$. In particular, if $f : M \to \mathbb{H}$ is a conformal immersion with $*df = -df R$, then the line bundle L of $g = [f, 1] : M \to \mathbb{HP}^1$ is given by $L = \psi\mathbb{H}$ where

$$\psi = \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

Moreover, the derivative of g is

$$\delta\psi = \pi \begin{pmatrix} df \\ 0 \end{pmatrix} .$$

Since f is conformal we can equip the line bundle L with the complex structure $J \in \Gamma(\mathrm{End}(L))$, $J^2 = -1$, defined by

$$J\psi = -\psi R .$$

For $\varphi \in \Gamma(L)$ we have $\varphi = \psi\lambda$ for some quaternionic valued function $\lambda : M \to \mathbb{H}$, and therefore, away from the zeros of λ,

$$J\varphi = -\varphi\lambda^{-1}R\lambda .$$

Moreover,

$$*\delta\psi = \pi \begin{pmatrix} *df \\ 0 \end{pmatrix} = -\pi \begin{pmatrix} df R \\ 0 \end{pmatrix} = \delta J\psi ,$$

shows that $*\delta = \delta J$ for a conformal immersion $g : M \to \mathbb{H}$. This observation motivates the following definition:

Definition 1.5. *A map $f : M \to \mathbb{HP}^1$ is called a* holomorphic curve *if there exists a complex structure $J \in \Gamma(\mathrm{End}(L))$ with*

$$*\delta = \delta J .$$

Corollary 1.6. *Let $f : M \to \mathbb{H}$ be an immersion and $g = [f, 1] : M \to \mathbb{HP}^1$. Then f is conformal if and only if g is a holomorphic curve.*

After choosing a point at infinity on $\tilde{S}$, an oriented round 2–sphere $\tilde{S}$ in $\mathbb{HP}^1$ is given in affine coordinates by an oriented affine 2–dimensional subspace $E \subset \mathbb{H}$. By the Fundamental Lemma there exist $(N, R) \in S^2 \times S^2$ and $H \in \mathbb{H}$ with

$$E = \{x \in \mathbb{H} \mid Nx + xR = H\}$$

with $NH = HR$. Thus, in affine coordinates,

$$\tilde{S} = \{[x, 1] \mid Nx + xR = H\} \cup [1, 0] .$$

If we consider

$$S = \begin{pmatrix} N & -H \\ 0 & -R \end{pmatrix}$$

then $S^2 = -1$ due to $NH = HR$ and $N^2 = R^2 = -1$, in other words $S \in \mathrm{End}(\mathbb{H}^2)$ is a complex structure. Moreover, $S[v] = [v]$ if and only if $[v] \in \tilde{S}$. Therefore, an oriented round 2–sphere $\tilde{S}$ in $\mathbb{HP}^1$ can be identified with a complex structure $S \in \mathrm{End}(\mathbb{H}^2)$, $S^2 = -1$, via

$$\tilde{S} = \{[v] \in \mathbb{HP}^1 \mid S[v] = [v], v \in \mathbb{H}^2\} .$$

The line bundle $\tilde{L}$ of the conformal map $\tilde{S}$ is S–stable, and the conformality equation of the embedded round sphere $\tilde{S}$ is $*\tilde{\delta} = \tilde{\delta} S$ where $\tilde{\delta}$ is the derivative of $\tilde{S}$. Moreover, S induces a complex structure on V/L and

$$*\tilde{\delta} = \tilde{\delta} S = S\tilde{\delta} .$$

A *sphere congruence* assigns to each point $p \in M$ an oriented round 2–sphere $S(p)$. In other words, a sphere congruence is a complex structure $S \in \Gamma(\mathrm{End}(V)), S^2 = -1$, on V. We say that a sphere congruence $S \in \Gamma(\mathrm{End}(V)), S^2 = -1$ *envelopes* a conformal map $f : M \to \mathbb{HP}^1$, if for all points $p \in M$ the sphere $S(p)$ passes through $f(p)$, and if the oriented tangent space of $f(p)$ coincides with the oriented tangent space of $S(p)$ at $f(p)$ over immersed points $p \in M$, i.e.:

$$SL = L \quad \text{and} \quad * \delta = S\delta = \delta S , \tag{1.9}$$

where $L = f^*\mathcal{T}$ is the line bundle of f.

Figure 1.2: S envelopes f

Definition 1.7. *Let $f : M \to \mathbb{HP}^1$ be a holomorphic curve. Then $S \in \Gamma(\mathrm{End}(V)), S^2 = -1$, is called an* adapted complex structure *if*

$$*\delta = S\delta = \delta S .$$

Adapted complex structures are not unique: if S and $\tilde{S}$ are adapted, then

$$R = \tilde{S} - S \in \Gamma(\mathcal{R}_-) ,$$

where

$$\mathcal{R} = \{R \in \mathrm{End}(\mathbb{H}^2) \mid R|_L = 0, \mathrm{im}\, R \subset L\} , \tag{1.10}$$

and, denoting by $\mathrm{End}_\pm(V)$ the S commuting and anticommuting endomorphisms,

$$\mathcal{R}_- = \mathrm{End}_-(V) \cap \mathcal{R} = \{R \in \mathcal{R} \mid SR = -RS\} .$$

To single out a particular adapted complex structure, we recall the *mean curvature sphere congruence* (also called the *conformal Gauss map*) of a conformal immersion $f : M \to \mathbb{H}$ which plays the role of the Gauss map in Möbius geometry: To each point $p \in M$, assign the sphere $S(p)$ so that $S(p)$ goes through $f(p)$ with coinciding oriented tangent spaces, and $S(p)$ has the same mean curvature vector $\mathcal{H}(p)$ as f at $f(p)$. To compute the conformal Gauss map of a conformal immersion $f : M \to S^4$, we first recall (1.5) that

$$Ndf(Y) + df(Y)R = 0 ,$$

for $X, Y \in \Gamma(TM)$ and thus

$$dN(X)df(Y) + N(X \cdot df(Y)) + (X \cdot df(Y))R - df(Y)dR(X) = 0\,.$$

Using the above equation we obtain the second fundamental form II of f as

$$\begin{aligned} 2II(X,Y) &= 2(X \cdot df(Y))^{\perp} = (X \cdot df(Y) - NX \cdot df(Y)R) \\ &= *df(Y)dR(X) - dN(X) * df(Y)\,. \end{aligned}$$

The *mean curvature vector* $\mathcal{H}$ of f is given by $\mathcal{H} = \frac{1}{2}\operatorname{tr} II$ so that

$$4|df(X)|^2\mathcal{H} = II(X,X) + II(JX,JX) = (-df \wedge dR - dN \wedge df)(X, JX)\,.$$

Since $dN \wedge df = d(Ndf) = -d(dfR) = df \wedge dR$ and

$$df \wedge dR(X, JX) = df(X)(*dR(X) + RdR(X))$$

we obtain $2\overline{df}\mathcal{H} = -(*dR + RdR)$ and

$$2\bar{\mathcal{H}}df = *dR + RdR\,. \tag{1.11}$$

Defining H by $\bar{\mathcal{H}} = HN$ we get

$$2Hdf = dR - R * dR\,. \tag{1.12}$$

Note that for a conformal immersion $f : M \to \mathbb{R}^3$ the *mean curvature* of f is given by $H = -\bar{\mathcal{H}}N$. A similar computation with N instead of R gives

$$-2df\bar{\mathcal{H}} = *dN + NdN\,,$$

and, if we assume $NH = HR$,

$$2dfH = dN - N * dN\,. \tag{1.13}$$

Let $L = f^*\mathcal{T}$ is the line bundle of the immersion $f : M \to \mathbb{H}$, and $e = \begin{pmatrix}1\\0\end{pmatrix}$ a *point at infinity* not intersecting f, i.e.,

$$V = L \oplus e\mathbb{H}\,.$$

An adapted complex structure S induces a holomorphic structure $\tilde{J}$ on $e\mathbb{H} = V/L$ via this splitting. In particular, S is given in the splitting $V = L \oplus e\mathbb{H}$ by

$$S = \begin{pmatrix} J & B \\ 0 & \tilde{J} \end{pmatrix}\,, \tag{1.14}$$

where $*\delta = \tilde{J}\delta = \delta J$ and $B \in \Gamma(\operatorname{Hom}(e\mathbb{H}, L))$. Since $S^2 = -1$, we have

$$JB + B\tilde{J} = 0\,.$$

Note that if $\psi = \begin{pmatrix} f \\ 1 \end{pmatrix} \in \Gamma(L)$ then

$$*\delta = \tilde{J}\delta = \delta J \iff *df = Ndf = -dfR\,,$$

where $N, R : M \to S^2$ are defined by $J\psi = -\psi R$ and $\tilde{J}e = eN$.

The trivial connection d on $V = L \oplus e\mathbb{H}$ is given by

$$d = \begin{pmatrix} \nabla^L & 0 \\ \delta & \tilde{\nabla} \end{pmatrix}.$$

Since d is flat, both connections ∇^L on L and $\tilde{\nabla}$ on $e\mathbb{H} = V/L$ are flat, and

$$d^{\hat{\nabla}}\delta = 0\,,$$

where $\hat{\nabla}$ is the induced connection on $\mathrm{Hom}(L, e\mathbb{H})$. Since the mean curvature sphere congruence has to satisfy a second order condition, we compute

$$dS = \begin{pmatrix} \nabla^L J - B\delta & \nabla B \\ 0 & \delta B + \tilde{\nabla}\tilde{J} \end{pmatrix}.$$

Here ∇ is the connection on $\mathrm{Hom}(e\mathbb{H}, L)$ induced by ∇^L and $\tilde{\nabla}$. The $(1,0)$ and $(0,1)$–part of dS with respect to S are given by

$$(dS)' = \frac{1}{2}(dS - S * dS) = \begin{pmatrix} (\nabla^L J)' & (\nabla B)' - \frac{1}{2}(B * \delta B + B * \tilde{\nabla}\tilde{J}) \\ 0 & \delta B + (\tilde{\nabla}\tilde{J})' \end{pmatrix}$$

and

$$(dS)'' = \frac{1}{2}(dS + S * dS) = \begin{pmatrix} (\nabla^L J)'' - B\delta & (\nabla B)'' + \frac{1}{2}(B * \delta B + B * \nabla\tilde{J}) \\ 0 & (\tilde{\nabla}\tilde{J})'' \end{pmatrix}$$

respectively. We will show that the mean curvature sphere congruence S of f satisfies

$$Be = -\psi H \tag{1.15}$$

where $H = -\bar{\mathcal{H}}N$ is given by the mean curvature vector of f. Writing $Be = -\psi\tilde{B}$ with $\tilde{B} : M \to \mathbb{H}$ we obtain

$$2(dS)''\psi = 2(\nabla^L J)'\psi - 2B\delta\psi = \psi(2\tilde{B}df - (dR - R * dR))\,. \tag{1.16}$$

Thus, $\tilde{B} = -\bar{\mathcal{H}}N$ if and only if $(dS)''|_L = 0$.

We apply this computation to an embedded round sphere $\tilde{f} : S^2 \to S^4$ in S^4. The mean curvature sphere congruence $\tilde{S}$ of $\tilde{f}$ is constant (given by $\tilde{S}(p) = \tilde{f}(S^2)$ for all $p \in S^2$). Thus $d\tilde{S} = 0$ and the above computation shows that $\tilde{S}$ is given by (1.14) with $B\psi = -\psi\tilde{H}$ where $\tilde{H} = \overline{\tilde{\mathcal{H}}}N$ and $\tilde{\mathcal{H}}$ is the mean curvature vector of $\tilde{f}$.

In particular, an adapted complex structure S is the mean curvature sphere congruence of f, i.e., the mean curvature vectors $\tilde{\mathcal{H}}$ of $S(p)$ at p and $\mathcal{H}$ of f at p coincide for all $p \in M$, if and only if $(dS)''|_L = 0$.

Since $S^2 = -1$ the derivative of S anticommutes with S. Decomposing the derivative of S into $(1,0)$ and $(0,1)$–parts

$$dS = 2(*Q - *A) \in \Omega^1(\mathrm{End}_-(V))\,, \tag{1.17}$$

where $A \in \Gamma(K\,\mathrm{End}_-(V))$ and $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V))$, we see that a complex structure $S \in \Gamma(\mathrm{End}(V)), S^2 = -1$, is the mean curvature sphere congruence of f if and only if

$$*\,\delta = S\delta = \delta S\,, \text{ and } Q|_L = 0\,. \tag{1.18}$$

Since $S^2 = -1$ we have $NH = HR$, so that a similar computation, using (1.13), shows that (1.18) is equivalent to

$$*\delta = S\delta = \delta S\,, \text{ and } AV \subset L\,.$$

For latter use, we give the *Hopf fields* A and Q of an adapted complex structure in the splitting $V = L \oplus e\mathbb{H}$

$$A = \frac{1}{2}(*dS)' = \begin{pmatrix} A^L & \frac{1}{2}(*\nabla B)' + \frac{1}{4}(B\delta B + B\tilde{\nabla}\tilde{J}) \\ 0 & \tilde{A} + \frac{1}{2} * \delta B \end{pmatrix} \tag{1.19}$$

and

$$Q = -\frac{1}{2}(*dS)'' = \begin{pmatrix} Q^L + \frac{1}{2}B * \delta & -\frac{1}{2}(*\nabla B)'' + \frac{1}{4}(B\delta B + B\tilde{\nabla}\tilde{J}) \\ 0 & \tilde{Q} \end{pmatrix} \tag{1.20}$$

where $(\nabla^L J) = 2(*Q^L - *A^L)$ and $\tilde{\nabla}\tilde{J} = 2(*\tilde{Q} - *\tilde{A})$ are the splittings of the derivatives of J and $\tilde{J}$ into $(0,1)$ and $(1,0)$ parts respectively. In particular, S is the mean curvature sphere congruence of f if and only if

$$\tilde{A} = -\frac{1}{2} * \delta B \tag{1.21}$$

or, equivalently,

$$Q^L = -\frac{1}{2}B * \delta\,. \tag{1.22}$$

The *Willmore functional* of a conformal immersion $f : M \to S^4$ from a compact Riemann surface into the 4–sphere is given [Wil93] by

$$\mathcal{W}(f) = \int_M (|\mathcal{H}|^2 - K - K^\perp)|df|^2\,. \tag{1.23}$$

Here $\mathcal{H}$ is the mean curvature vector of f, K the Gaussian curvature, and $K^\perp$ the curvature of the normal bundle. In our notation, the Willmore integrand computes [BFL$^+$02, Prop. 11] to

$$|\mathcal{H}|^2 - K - K^\perp = \frac{1}{4}|dR + R * dR|^2 \,.$$

Let $< B > := \text{Re tr}(B)$ then (1.19) gives

$$< A \wedge *A > = -2\text{Re tr}\, A^2 = \frac{1}{8}|dR + R * dR|^2$$

and the Willmore energy of a conformal immersion $f : M \to S^4$ can be expressed as

$$\mathcal{W}(f) = 2 \int < A \wedge *A > \ . \tag{1.24}$$

For convenience, we collect how properties of conformal maps are phrased in the quaternionic formalism.

Geometric property	*Quaternionic formulation*
$f : M \to S^4$ map	quaternionic line bundle $L \to M$ $L \subset V = \underline{\mathbb{H}}^2$
$df : TM \to TS^4$	$\delta \in \Omega^1(\mathrm{Hom}(L, V/L))$ derivative of L
$f : M \to S^4$ holomorphic	there exists $J \in \Gamma(\mathrm{End}(L))$, $J^2 = -1$, $*\delta = \delta J$
round 2–sphere S in S^4	$S \in \mathrm{End}(\mathbb{H}^2), S^2 = -1$
point $[x]$ lies on sphere S	$S[x] = [x]$
sphere congruence S	$S \in \Gamma(\mathrm{End}(\mathbb{H}^2)), S^2 = -1$
sphere congruence S envelopes $f : M \to S^4$	S is adapted: $*\delta = S\delta = \delta S$
$f : M \to S^4$ conformal immersion	δ is nowhere vanishing, and there exists an adapted complex structure $S \in \Gamma(\mathrm{End}(V))$
Hopf fields A, Q	$A = \frac{1}{2}(*dS)'$, $Q = -\frac{1}{2}(*dS)''$
S mean curvature sphere congruence of f (S conformal Gauss map of f)	S adapted, $Q''\|_L = 0$
Willmore functional $\mathcal{W}(f) = \int(\|\mathcal{H}\|^2 - K - K^\perp)\|df\|^2$	$\mathcal{W}(f) = 2\int < A \wedge *A >$, S conformal Gauss map of f

1.1.2 Holomorphic curves in $\mathbb{HP}^n$

In the previous section, we discussed holomorphic maps $f : M \to S^4 = \mathbb{HP}^1$ from a Riemann surface into the 4–sphere. In analogy to the theory of complex curves $f : M \to \mathbb{CP}^n$, we extend our definition of holomorphicity to maps $f : M \to \mathbb{HP}^n$ from a Riemann surface M into quaternionic projective space.

Consider maps $f : M \to G_{k+1}(V)$ from a Riemann surface M into the $(k+1)$–Grassmannian of the trivial $\mathbb{H}^{n+1}$ bundle V over M. Let $\mathcal{T} \to G_{k+1}(V)$ be the tautological $(k+1)$–plane bundle whose fiber over $V_k \in G_{k+1}(\mathbb{H}^{n+1})$ is $\mathcal{T}_{V_k} = V_k \subset V$. A map $f : M \to G_{k+1}(V)$ can be identified with a rank $k+1$ subbundle $V_k \subset V$ via $V_k = f^*\mathcal{T}$, i.e., $(V_k)_p = \mathcal{T}_{f(p)} = f(p)$ for $p \in M$. From now on, we will make no distinction between a map f into the Grassmannian $G_{k+1}(V)$ and the corresponding subbundle $V_k \subset V$.

The derivative of $V_k \subset V$ is given by the $\mathrm{Hom}(V_k, V/V_k)$ valued 1–form

$$\delta_k = \pi_{V_k} d|_{V_k} \,, \tag{1.25}$$

where $\pi_{V_k} : V \to V/V_k$ is the canonical projection and d is the trivial connection on V. Under the identification $TG_{k+1}(V) = \mathrm{Hom}(\mathcal{T}, V/\mathcal{T})$ the 1–form δ_k is the derivative df of $f : M \to G_{k+1}(V)$.

Analog to the case $n = 1$, i.e. conformal maps $f : M \to S^4 = \mathbb{HP}^1$, we define a holomorphic curve into a Grassmannian:

Definition 1.8. *Let $V = \underline{\mathbb{H}}^{n+1}$ be the trivial quaternionic $(n+1)$–plane bundle over a Riemann surface M and d the trivial connection on V.*

A map $f : M \to G_{k+1}(V)$ from a Riemann surface M into the Grassmannian $G_{k+1}(V)$ is called a holomorphic curve *in $G_{k+1}(V)$ if there exists a complex structure $J \in \Gamma(End(V_k))$, $J^2 = -1$, such that*

$$*\delta_k = \delta_k J \,,$$

where $V_k = f^*\mathcal{T}$ and $\delta_k = \pi d|_{V_k}$ is the derivative of V_k.

A holomorphic curve $f : M \to G_{k+1}(V)$ is called full *if V_k is not contained in a lower dimensional Grassmannian.*

We are particularly interested in the case when $f : M \to \mathbb{HP}^n$ is a holomorphic curve into quaternionic projective space. We usually denote by $L = f^*\mathcal{T}$ the associated line bundle. If $f = [f_1, \dots, f_n, 1] : M \to \mathbb{HP}^n$ is given in affine coordinates then

$$\delta\psi = \pi \begin{pmatrix} df_1 \\ \vdots \\ df_n \\ 0 \end{pmatrix} \qquad \text{for } \psi = \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 1 \end{pmatrix} \in \Gamma(L) \,.$$

If we define $R : M \to S^2$ by $J\psi = -\psi R$, then f is a holomorphic curve with $*\delta = \delta J$ if and only if $[f_i, 1] : M \to \mathbb{HP}^1$ are conformal maps to S^4, all of them having the same right normal R.

Example 1.9. A good example to keep in mind are the holomorphic curves in $\mathbb{HP}^n$ which arise as the twistor projections of smooth curves in $\mathbb{CP}^{2n+1}$. We view $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$ via the complex structure I given by right multiplication by i. The *twistor projection* maps a complex line in $\mathbb{H}^{n+1}$ into the corresponding quaternionic line via

$$\mathbb{C}^{2n+2} \supset E \mapsto \pi(E) = E \oplus Ej \subset \mathbb{H}^{n+1} .$$

If $h : M \to \mathbb{CP}^{2n+1}$ is a smooth curve with corresponding line bundle $E \subset \mathbb{C}^{2n+2}$ given by $E_p = h(p)$, then its twistor projection

$$L = \pi(E) = E \oplus Ej \subset V = M \times \mathbb{H}^{n+1}$$

is a (quaternionic) holomorphic curve if and only if the $(0,1)$–part of the derivative $\delta_E = \pi_E d|_E$ of E with respect to the complex structure I on $\mathbb{H}^{n+1}$ satisfies

$$\delta''_E \in \Gamma(\bar{K} \operatorname{Hom}_{\mathbb{C}}(E, L/E)) . \tag{1.26}$$

To see this, define a complex structure J on L by regarding E as the $+i$ eigenspace of J. For $\varphi \in \Gamma(E)$ we have $J\varphi = \varphi i$, and thus

$$\delta''_E \varphi = \frac{1}{2}\pi_E(d\varphi + (*d\varphi)i) = \frac{1}{2}\pi_E(d\varphi + *d(J\varphi)) .$$

In particular, (1.26) holds if and only

$$\pi_L(\hat{D}\psi) = 0$$

for $\psi \in \Gamma(L)$ where $\hat{D}$ is the *mixed structure*

$$\hat{D} = \frac{1}{2}(d + *dJ) \tag{1.27}$$

on L. Since the derivative of f is given by $\delta = \pi_L(d|_L)$, we obtain

$$\pi_L(\hat{D}\psi) = \delta\psi + *\delta J\psi$$

for $\psi \in \Gamma(L)$ so that (1.26) is equivalent to

$$*\delta\psi = \delta J\psi ,$$

which means that L is a holomorphic curve.

In particular, if $h : M \to \mathbb{CP}^n$ is a complex holomorphic curve in $\mathbb{CP}^n$, i.e.,

$$\delta''_E = 0 ,$$

Figure 1.3: Twistor projection of a holomorphic curve in $\mathbb{CP}^3$

then its twistor projection is a holomorphic curve in $\mathbb{HP}^n$.

Conversely, if $f : M \to \mathbb{HP}^n$ is a holomorphic curve, then the complex structure J on L defines a splitting

$$L = E \oplus Ej$$

where E and Ej are the $\pm i$ eigenspaces of J respectively. The complex line bundle E gives a smooth map $h : M \to \mathbb{CP}^{2n+1}$, the *twistor lift* of f.

Definition 1.10. *The* Frenet flag *of a full holomorphic curve $f : M \to \mathbb{HP}^n$ is a full flag $V_0 = L \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$ of quaternionic subbundles of rank $V_k = k+1$ together with complex structures J_k on the quotient bundles V_k/V_{k-1}, $J_0 = J$, such that*

1. $d\Gamma(V_k) \subset \Omega^1(V_{k+1})$.
2. *The derivatives $\delta_k = \pi_{V_k} d : V_k/V_{k-1} \to T^*M \otimes V_{k+1}/V_k$ satisfy*

$$*\delta_k = J_{k+1}\delta_k = \delta_k J_k \,.$$

Note that the Frenet flag exists smoothly away from isolated points on M, the so–called *Weierstraß points* of f, and extends continuously into the Weierstraß points [FLPP01, Lemma 4.10]. In case the Frenet flag extends smoothly into the Weierstraß points, these are exactly the zeros of the derivatives δ_k. In particular, a conformal immersion $f : M \to S^4$ has no Weierstraß points.

Example 1.11. Consider the line bundle $L = f^*\mathcal{T} \subset \underline{\mathbb{H}}^2$ induced by the holomorphic map $f : M \to S^4 = \mathbb{HP}^1$. In this case, the flag bundles exist globally and are given by $L \subset V_1 = \underline{\mathbb{H}}^2$. This flag is a Frenet flag if there exists J on V/L with $*\delta = J\delta$. In other words, $[f, 1] : M \to \mathbb{HP}^1$ is a holomorphic curve with Frenet flag if and only if there exist left and right normals $N, R : M \to S^2$ with $*df = Ndf = -dfR$.

Similar to the case of a holomorphic curve in S^4 we call a complex structure adapted if it induces the complex structures given by the Frenet flag:

Definition 1.12. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag $L \subset V_1 \subset \ldots \subset V$. A complex structure $S \in \Gamma(\mathrm{End}(V))$ is called* adapted *if*

$$*\delta_k = S\delta_k = \delta_k S \,,$$

where $\delta_k : V_k \to V/V_k$ are the derivatives of the flag bundles V_k.

Lemma 1.13. *A holomorphic curve $f : M \to \mathbb{HP}^n$ has a Frenet flag if and only if there exists a smooth adapted complex structure on V.*

Proof. The existence of S on M renders δ_0 into a complex holomorphic section, [Les]. In particular, the zeros fo δ_0 are isolated and the image of δ_0 defines a smooth bundle. By defining inductively the flag bundles as the smooth bundles given by the derivatives δ_k, one shows that the Frenet flag exists smoothly on M, see [LP03]. For the converse, choose a splitting $V = L \oplus L_1 \oplus \ldots \oplus L_n$ such that the flag spaces of f are given by

$$V_k = \bigoplus_{i=0}^{k} L_i \,,$$

and define a complex structure S by

$$S = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{pmatrix} ,$$

where J_i are the induced complex structures on L_i when identifying $L_i = V_i/V_{i-1}$ via the splitting. By definition, S is an adapted complex structure. □

Remark 1.14. The derivative δ_0 of a holomorphic curve $f : M \to \mathbb{HP}^1 = S^4$ which has a smooth adapted complex structure and $df \not\equiv 0$, has isolated zeros [Les]. Therefore, a map $f : M \to \mathbb{R}^4$ with smooth $(N, R) : M \to S^2 \times S^2$ satisfying $*df = N df = -df R$ is a branched conformal immersion.

In particular, a holomorphic map $f : M \to \mathbb{R}^3$ is a branched conformal immersion since the left normal equals (1.4) the right normal $N : M \to S^2$ given by the holomorphicity condition.

Remark 1.15. As in the case $n = 1$, an adapted complex structure can be seen as a congruence of osculating $\mathbb{CP}^n$'s: Let $E \subset \mathbb{H}^{n+1}$ be a complex line subbundle of $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$ such that $E \oplus Ej = \mathbb{H}^{n+1}$ where we consider $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$ via the complex structure I given by right multiplication by i. The map

$$E \supset w\mathbb{C} \mapsto w\mathbb{H} \subset \mathbb{H}^{n+1}$$

is injective and defines a $\mathbb{CP}^n \subset \mathbb{HP}^n$. On the other hand, splittings $E \oplus Ej = \mathbb{H}^{n+1}$ are the same as complex structures $S \in \operatorname{End}(\mathbb{H}^{n+1})$, $S^2 = -1$, by declaring E to be the $+i$ eigenspace of S. The condition that the congruence of $\mathbb{CP}^n$ passes through a holomorphic curve f is given by

$$SL = L \,,$$

where $L = f^*\mathcal{T}$ is the line bundle of f. Since the tangent space to $\mathbb{CP}^n$ at L_p is given by

$$T_{L_p}\mathbb{CP}^n = \operatorname{Hom}_+(L, \mathbb{H}^{n+1}/L)_p \,,$$

the congruence of $\mathbb{CP}^n$'s envelopes f if and only if

$$\delta \in \Gamma(K \operatorname{Hom}_+(L, \mathbb{H}^{n+1}/L)) \,.$$

We decompose the derivative of a complex structure $S \in \Gamma(\mathrm{End}(V)), S^2 = -1$, into $(1,0)$ and $(0,1)$–parts with respect to S:

$$dS = 2(*Q - *A)\,,$$

where $A \in \Gamma(K\,\mathrm{End}_-(V))$ and $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V))$. Given a holomorphic curve $f : M \to \mathbb{HP}^n$ and a *hyperplane at infinity* $\alpha \in (\mathbb{H}^{n+1})^*$ such that $V = L \oplus \ker\alpha$, an adapted complex structure is given in this splitting as

$$S = \begin{pmatrix} J & B \\ 0 & \tilde{S} \end{pmatrix}$$

where $B \in \Gamma(\mathrm{Hom}(\ker\alpha, L))$, $\tilde{S} \in \Gamma(\mathrm{End}(\ker\alpha))$ is a complex structure on $\ker\alpha$, and

$$JB + B\tilde{S} = 0\,.$$

Since S is adapted we also have

$$*\delta = \tilde{S}\delta = \delta J\,.$$

The same computation as in the 1–dimensional case (1.19), (1.20) gives the formulas for the Hopf fields A and Q of an adapted complex structure S in the splitting $V = L \oplus \ker\alpha$:

$$A = \frac{1}{2}(*dS)' = \begin{pmatrix} A^L & \frac{1}{2}(*\nabla B)' + \frac{1}{4}(B\delta B + B\tilde{\nabla}\tilde{J}) \\ 0 & \tilde{A} + \frac{1}{2} * \delta B \end{pmatrix} \tag{1.28}$$

and

$$Q = -\frac{1}{2}(*dS)'' = \begin{pmatrix} Q^L + \frac{1}{2}B * \delta & -\frac{1}{2}(*\nabla B)'' + \frac{1}{4}(B\delta B + B\tilde{\nabla}\tilde{J}) \\ 0 & \tilde{Q} \end{pmatrix}\,. \tag{1.29}$$

Example 1.16. Let $h : M \to \mathbb{CP}^{2n+1}$ be a smooth curve such that its twistor projection $f : M \to \mathbb{HP}^n$ is a holomorphic curve (1.26), i.e.,

$$\delta_E'' \in \Gamma(\bar{K}\,\mathrm{Hom}_{\mathbb{C}}(E, L/E))\,.$$

If S is a complex structure on V which induces J on L, i.e.,

$$S|_L = J\,,$$

then

$$\delta_E'' = A|_E\,.$$

This follows from the fact that $L/E = Ej$ so that the projection

$$\pi_E : L \to L/E$$

is the projection onto the $-i$ eigenspace of $J = S|_L$. Therefore, we have for $\varphi \in \Gamma(E)$:

$$\delta''_E \varphi = \frac{1}{4}((d\varphi + *d(S\varphi) + Sd(S\varphi) - S * d\varphi)|_L = A\varphi\,.$$

In particular, $f : M \to \mathbb{HP}^n$ is the twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^n$ in $\mathbb{CP}^n$, i.e., $\delta''_E = 0$, if and only if

$$A|_L = 0 \tag{1.30}$$

for any complex structure S with $S|_L = J$.

Moreover, if the n^{th} flag space E_n of the holomorphic curve $h : M \to \mathbb{CP}^n$ does not contain a quaternionic subspace, i.e.,

$$E_n \oplus E_n j = \mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$$

then E_n defines a complex structure S on $\mathbb{H}^{n+1}$. The flag spaces $V_k = E_k \oplus E_k j$ of L are S–stable and $\delta_k|_{E_k} = \delta_{E_k}$ shows that S is adapted.

Example 1.17. In what follows, the example of the dual curve of a holomorphic curve $f : M \to \mathbb{HP}^n$ with Frenet flag will play an important role. For any subbundle V_k of V let

$$V_k^\perp := \{\alpha \in V^* \mid <\alpha, \psi>= 0 \text{ for all } \psi \in V_k\}\,,$$

where V^* is the dual bundle of V.

The *dual curve* $L^\dagger$ of a holomorphic curve $L \subset V$ with Frenet flag is the holomorphic curve in V^* defined by

$$L^\dagger := V_{n-1}^\perp\,.$$

In the case of a holomorphic curve $f : M \to S^4$ with adapted complex structure, the dual curve is given by the antipodal map since $L^\dagger = V_{n-1}^\perp = L^\perp$. The Frenet flag of the dual curve $L^\dagger$ of L is given by

$$V_k^\dagger = V_{n-1-k}^\perp\,. \tag{1.31}$$

with derivatives $\delta_k^\dagger = -\delta_{n-1-k}^*$ and complex structures $J_k^\dagger = J_{n-k}^*$. In particular, if S is an adapted complex structure of L then S^* is an adapted complex structure of the dual curve $L^\dagger$. Since

$$(dS)^* = d^* S^*$$

where d^* is the dual connection of d, one easily verifies that the Hopf fields of $f^\dagger$ are given by

$$A^\dagger = -Q^* \in \Gamma(K\,\text{End}_-(V^*)) \quad \text{and} \quad Q^\dagger = -A^* \in \Gamma(\bar{K}\,\text{End}_-(V^*))\,. \tag{1.32}$$

Example 1.18. We use the above concept of the dual curve to define a holomorphic curve in $\mathbb{HP}^n$ by prescribing an n–dimensional subbundle of V. Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve in $\mathbb{HP}^n$ and ω be a nowhere vanishing closed 1–form with $\operatorname{im}\omega = L$ where $L = f^*\mathcal{T}$ so that $\ker\omega$ is a n–dimensional subbundle of V. Moreover, assume that $\omega \in \Gamma(K\operatorname{Hom}(V, L))$ where the complex structure J on L is given by the holomorphic curve f, i.e.,

$$*\omega = J\omega\,.$$

For $\psi \in \Gamma(\ker\omega)$ we see

$$0 = d(\omega\psi) = (d\omega)\psi - \omega \wedge d\psi = -\omega \wedge \pi_{\ker\omega} d\psi\,,$$

where $\pi_{\ker\omega} : V \to V/\ker\omega$ is the canonical projection. Therefore, the derivative $\delta_\omega = \pi_{\ker\omega} d|_{\ker\omega}$ of $\ker\omega$ satisfies

$$*\delta_\omega = J\delta_\omega\,,$$

where J is the complex structure on $V/\ker\omega$ induced by the bundle isomorphism $\omega : V/\ker\omega \to L$. Since $\hat{L} = (\ker\omega)^\perp$ has derivative $\hat{\delta} = -\delta_\omega^*$, we see that

$$*\hat{\delta} = \hat{\delta}^*\,,$$

in other words, $\hat{L} = (\ker\omega)^\perp$ is a holomorphic curve. Note that $\hat{L}$ is contained in a constant subbundle $\hat{V} \subset V^*$ if and only if $\hat{V}^\perp \subset \ker\omega$. In particular, $\hat{f}$ is a full curve in $\mathbb{HP}^n$ if and only if $\ker\omega$ does not contain a constant subbundle.

As a consequence we can give a holomorphic curve by prescribing the n^{th} flag space:

Corollary 1.19. *Let $\omega \in \Gamma(K\operatorname{Hom}(V, L))$ be a nowhere vanishing 1–form with $d\omega = 0$ such that $\ker\omega$ does not contain a constant subbundle. Then a holomorphic curve $\tilde{f} : M \to \mathbb{HP}^n$ with flag bundles $\tilde{V}_k$ is the dual curve of $\hat{L} = (\ker\omega)^\perp$ if and only if*

$$\tilde{V}_{n-1} = \ker\omega\,.$$

We discuss the existence of $\omega \in \Gamma(K\operatorname{Hom}(V, L))$ wit $d\omega = 0$ in Chapter 2.

1.1.3 Frenet curves in $\mathbb{HP}^n$

The analog of the conformal Gauss map of a conformal immersion $f : M \to S^4$ is the canonical complex structure of a holomorphic curve in $\mathbb{HP}^n$. Recall that in the case $n = 1$, the conformal Gauss map corresponds to an adapted complex structure S with $Q|_L = 0$. Similarly, we define:

Definition 1.20. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag $L \subset V_1 \subset \ldots \subset V$.*

An adapted complex structure $S \in \Gamma(\mathrm{End}(V))$ *with*

$$Q|_{V_{n-1}} = 0 \tag{1.33}$$

is called the canonical complex structure *of* f. *Here*

$$dS = 2(*Q - *A) \tag{1.34}$$

is the decomposition of the derivative of S *into* $(1,0)$ *and* $(0,1)$*–parts* $A \in \Gamma(K\,\mathrm{End}_-(V))$ *and* $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V))$ *respectively.*

Lemma 1.21 ([Les])**.** *Let* $V_k \subset V$ *be stable with respect to a complex structure* $S \in \Gamma(\mathrm{End}(V))$ *on* V, *and* A *and* Q *the Hopf fields of* S. *Then* $*\delta_k = S\delta_k = \delta_k S$ *implies that* V_k *is* A *and* Q *stable. Here* $\delta_k = \pi_{V_k} d|_{V_k}$ *is the derivative of* V_k.

Using the above result we show that $Q|_{V_{n-1}} = 0$ is equivalent to $AV \subset \Omega^1(L)$:

Corollary 1.22. *An adapted complex structure* $S \in \Gamma(\mathrm{End}(V))$ *is the canonical complex structure of* f *if and only if*

$$AV \subset \Omega^1(L)\,. \tag{1.35}$$

Proof. If $Q|_{V_{n-1}} = 0$ then (1.34) yields

$$\begin{aligned} \frac{1}{2}\pi_L d(Q + A)|_{V_k} &= \frac{1}{2}\pi_L d(S(*Q - *A))|_{V_k} = \pi_L(*Q - *A) \wedge (*Q - *A)|_{V_k} \\ &= \pi_L(Q \wedge Q + A \wedge A)|_{V_k} = \pi_L A \wedge A|_{V_k}\,, \end{aligned} \tag{1.36}$$

where we used $Q|_{V_{n-1}} = 0$ and $A \wedge Q = Q \wedge A = 0$ by type considerations. Since $dQ|_{V_k} = Q \wedge \delta_k = 0$ by type (1.36) becomes

$$\pi_L dA|_{V_k} = \pi_L A \wedge A|_{V_k}\,.$$

Because A stabilizes the flag spaces V_k for all k, we obtain $\pi_L A \wedge A|_L = 0$ and

$$0 = \pi_L dA|_L = \delta_0 \wedge A|_L + \pi_L A \wedge \delta_0 = \pi_L A \wedge \delta_0$$

shows that $AV_1 \subset \Omega^1(L)$. Proceeding inductively, we conclude $AV \subset \Omega^1(L)$.

Conversely, if $AV \subset \Omega^1(L)$ we use the dual curve $L^\dagger$ to show that $Q|_{V_{n-1}} = 0$: Since $Q^\dagger = -A^*$, we see that

$$L \supset \operatorname{im} A = (\ker A^*)^\perp = (\ker Q^\dagger)^\perp$$

which shows that $V^\dagger_{n-1} = L^\perp \subset \ker Q^\dagger$. By the previous argument this implies

$$V^\perp_{n-1} = L^\dagger \supset \operatorname{im} A^\dagger = (\ker A^*)^\perp = (\ker Q)^\perp$$

so that

$$V_{n-1} \subset \ker Q\,.$$

∎

The proof of the previous Corollary is exemplary for latter arguments: using the dual curve we transfer properties of the Hopf field Q to properties of A. In particular, a complex structure S is the canonical complex structure of f if and only if S^* is the canonical complex structure of its dual curve $f^\dagger$.

Example 1.23. Let $f : M \to \mathbb{HP}^n$ be the twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^{2n+1}$, and assume that the n^{th}–flag space of h does not contain a quaternionic subspace, i.e,

$$E_n \oplus E_n j = \mathbb{H}^{n+1} .$$

As discussed before, E_n induces an adapted complex structure S on V. By the same arguments as in Example 1.16, we see that

$$0 = \delta''_{E_k} = A|_{E_k}$$

since E_k is the $+i$ eigenspace of the complex structure S restricted to the k^{th} flag space $V_k = E_k \oplus E_k j$ of f. Thus,

$$A = 0 ,$$

and S is the canonical complex structure of f.

In general, the Frenet flag and the canonical complex structure of a holomorphic curve $f : M \to \mathbb{HP}^n$ only exist [FLPP01, Lemma 4.1] away from the Weierstraß points of f. Whereas the Frenet flag of a holomorphic curve extends continuously into the Weierstraß points [FLPP01, Lemma 4.10], the canonical complex structure may become singular: If $f : M \to \mathbb{HP}^1$ is the twistor projection of a complex holomorphic curve $h : M \to \mathbb{CP}^3$ then the tangent $W_1 \subset V$ of h can become quaternionic, i.e., $W_1 = W_1 j$ at some $p \in M$, see [Pet04]. In this case the canonical complex structure S degenerates to a point at $p \in M$ and thus S cannot be extended into $p \in M$. To avoid these difficulties, we consider holomorphic curves $f : M \to \mathbb{HP}^n$ which have a smooth canonical complex structure. For conformal maps $f : M \to \mathbb{HP}^1$ this means that the mean curvature sphere congruence extends smoothly across the branch points.

Definition 1.24. *A* Frenet curve $f : M \to \mathbb{HP}^n$ *is a holomorphic curve which has a smooth canonical complex structure on M.*

Recalling Lemma 1.13 we have

Corollary 1.25 ([LP03])**.** *The Frenet flag of a Frenet curve $f : M \to \mathbb{HP}^n$ is smooth on M.*

A trivial example of a Frenet curve is an *unramified* curve $f : M \to \mathbb{HP}^n$, that is a holomorphic curve without Weierstraß points. In the case $n = 1$ an unramified curve is a conformal immersion $f : M \to S^4$ so that the conformality gives an adapted complex structure on V, and we can solve (1.21) smoothly for B. Moreover, the dual curve $f^\dagger$ of a Frenet curve $f : M \to \mathbb{HP}^n$ is Frenet.

Definition 1.26. *The* Willmore energy *of a Frenet curve $f : M \to \mathbb{HP}^n$ over a compact Riemann surface M is given by*

$$\mathcal{W}(f) = 2\int_M < A \wedge *A > , \tag{1.37}$$

where A is the Hopf field of the canonical complex structure S of f.

Remark 1.27. As we will see below, the definition of the Willmore energy is a Möbius invariant. In the case $n = 1$, we obtain (1.24) the usual Willmore functional (1.23)

$$\mathcal{W}(f) = \int_M (|\mathcal{H}|^2 - K - K^{\perp})|df|^2 .$$

Again, we summarize:

Geometric property	*Quaternionic formulation*
$f : M \to \mathbb{HP}^n$ map	line bundle $L \to M$, $L \subset V = \underline{\mathbb{H}}^{n+1}$
$df : TM \to T\mathbb{HP}^n$	$\delta \in \Omega^1(\mathrm{Hom}(L, V/L))$
$f : M \to \mathbb{HP}^n$ holomorphic	there exists $J \in \Gamma(\mathrm{End}(L)), J^2 = -1$, $*\delta = \delta J$
Frenet flag of f	$L = V_0 \subset V_1 \subset \ldots \subset V_n = V$ with $J_k \in \Gamma(\mathrm{End}(V_k/V_{k-1})), J_k^2 = -1$, such that $d\Gamma(V_k) \subset \Omega^1(V_{k+1})$ and $*\delta_k = \delta_k J_k = J_{k+1}\delta_k$
S adapted complex structure	$S \in \Gamma(\mathrm{End}(V)), S^2 = -1$, and $*\delta_k = S\delta_k = \delta_k S$
$f^\dagger$ dual curve of f	$L^\dagger = V_{n-1}^\perp$
Hopf fields of S	$A = \frac{1}{2}(*dS)', \quad Q = -\frac{1}{2}(*dS)''$
S canonical complex structure	S adapted, $Q\|_{V_{n-1}} = 0$
f Frenet curve	f has smooth canonical complex structure
Willmore energy of a Frenet curve	$\mathcal{W}(f) = 2\int < A \wedge *A >$

1.2 Holomorphic line bundles

In this section we briefly explain the notion of a (quaternionic) holomorphic structure on a quaternionic vector bundle, and how holomorphic curves $f : M \to \mathbb{HP}^n$ are in one-to-one correspondence to holomorphic line bundles [FLPP01]. Analytically, the Cauchy–Riemann operator $\bar{\partial}$ on a complex vector bundle is replaced in the quaternionic theory by an elliptic operator $D = \bar{\partial} + Q$ on a complex quaternionic vector bundle (W, J), where Q is a $(0, 1)$–form with values in the complex antilinear endomorphisms of W and $\bar{\partial}$ is a complex holomorphic structure on the complex bundle (W, J). The quaternionic holomorphic line bundle (W, D) inherits a new invariant, the *Willmore energy* $\mathcal{W}(W, D) = 2\int_M < Q \wedge * Q >$ of the holomorphic structure D. The properties of quaternionic holomorphic structures are in many ways similar to the properties of complex holomorphic structures, for example the vanishing orders of holomorphic sections are well–defined, and the Riemann–Roch theorem holds verbatim. The quaternionic Plücker formula involves the new invariant, and gives eventually estimates on the Willmore energy.

1.2.1 Holomorphic vector bundles

In what follows, let (W, J) be a quaternionic vector bundle with complex structure J.

Definition 1.28. *A quaternionic* holomorphic structure *on* (W, J) *is given by a quaternionic linear operator*

$$D : \Gamma(W) \to \Gamma(\bar{K}W) \tag{1.38}$$

satisfying the Leibniz rule $D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)''$*, where* $\psi \in \Gamma(W)$ *and* $\lambda : M \to \mathbb{H}$*. The quaternionic vector space of* holomorphic sections *of* W *is denoted by*

$$H^0(W) = \ker(D) \tag{1.39}$$

and has finite dimension $h^0(W) := \dim H^0(W)$ *for compact* M*. A linear subspace* $H \subset H^0(W)$ *is called a* linear system*.*

The zeros of a holomorphic section $\psi \in H^0(W)$ are isolated, and the vanishing order $\operatorname{ord}_p \psi$ of ψ at a zero $p \in M$ is well–defined [FLPP01, Def. 3.5]. The *Weierstraß gap sequence* of H is given by $n_0(p) < \ldots < n_n(p)$ where $n_0(p)$ is the minimal vanishing order of a holomorphic section in H at p, and each $n_k(p)$ is defined inductively by letting $n_k(p)$ be the minimal vanishing order of holomorphic sections in H at p strictly greater than $n_{k-1}(p)$ for $k \geq 1$.

The *order* of a linear system H is defined [FLPP01, Def. 4.2] by

$$\operatorname{ord}(H) = \sum_{p \in M} \operatorname{ord}_p(H)\,, \tag{1.40}$$

where $\operatorname{ord}_p(H) = \sum_{k=0}^{n}(n_k(p) - k)$ is the order of H at p and $n_0(p) < \ldots < n_n(p)$ is the Weierstraß gap sequence of H. Away from isolated points the Weierstraß gap sequence is $n_k(p) = k$ and $\operatorname{ord}_p H = \sum_{k=0}^{n-1} n_k - k$ measures the deviation from the generic sequence.

A quaternionic vector bundle W with complex structure J over a Riemann surface M decomposes into $W = W_+ \oplus W_-$, where $W_\pm$ are the $\pm i$–eigenspaces of J. By restriction J induces complex structures on $W_\pm$ and $W_- = W_+ j$ gives a complex linear isomorphism between W_+ and W_-. The decomposition of D into J commuting and anticommuting parts gives

$$D = \bar\partial + Q\,. \tag{1.41}$$

Here $\bar\partial = \bar\partial \oplus \bar\partial$ is the double of a complex holomorphic structure on W_+ and $Q \in \Gamma(\bar K \operatorname{End}_-(W))$ is a $(0,1)$–form with values in J–complex J–antilinear endomorphisms of W. The endomorphism $Q \in \Gamma(\bar K \operatorname{End}(V))$ "measures" the difference to the complex theory of complex holomorphic structures $\bar\partial \oplus \bar\partial$ on $E \oplus E$.

Definition 1.29. *The L^2–norm*

$$\mathcal{W}(W) = \mathcal{W}(W, D) = 2\int_M < Q \wedge *Q > \tag{1.42}$$

of Q is called the Willmore energy *of the holomorphic bundle (W, D) where $<,>$ denotes the trace pairing on* $\operatorname{End}(W)$*. The special case $Q = 0$, for which $\mathcal{W}(W) = 0$, describes (doubles of) complex holomorphic bundles $W = W_+ \oplus W_+$.*

A connection ∇ on a complex quaternionic vector bundle (W, J) decomposes

$$\nabla = \nabla_+ + \nabla_- \tag{1.43}$$

into J–commuting and J–anticommuting parts, in particular $\hat\nabla = \nabla_+$ is a complex connection on W. Furthermore, we denote the $(0,1)$ and $(1,0)$–parts of ∇_+ and ∇_- by

$$\hat\nabla = \bar\partial + \partial \qquad \text{and} \qquad \nabla_- = Q + A\,,$$

where

$$*\bar\partial = -S\bar\partial = -\bar\partial S\,, \quad *\partial = S\partial = \partial S\,,$$

and $Q \in \Gamma(\bar K \operatorname{End}_-(W))$ and $A \in \Gamma(K \operatorname{End}_-(W))$. Note that

$$\nabla J = 2(*Q - *A)$$

since $\hat\nabla J = 0$. The $(0,1)$–part of ∇ defines a holomorphic structure

$$\nabla'' = \bar\partial + Q$$

on (W, J). Moreover, if ∇ is a flat connection then

$$0 = \hat R + d^{\hat\nabla}(Q + A) + Q \wedge Q + A \wedge A\,,$$

where we used that $A \wedge Q = Q \wedge A = 0$ by type considerations. Splitting this equation into J commuting and J–anticommmuting parts, we get

$$\hat{R} = -(Q \wedge Q + A \wedge A)\,, \tag{1.44}$$

and

$$d^{\hat{\nabla}}(Q + A) = 0\,.$$

We denote by $\bar{W}$ the complex vector bundle $(W, -J)$ of the complex quaternionic vector bundle (W, J). Then ∇' is the $(0,1)$–part of the connection ∇ with respect to $-J$ and $(\bar{W}, \nabla')$ is a holomorphic vector bundle. We call ∇' an *anti–holomorphic structure* on (W, J).

The *degree* of the quaternionic bundle W with complex structure J is defined as the degree of the underlying complex vector bundle

$$\deg W := \deg W_+\,, \tag{1.45}$$

which is half of the usual degree of W when viewed as a complex bundle with J. The degree of a complex bundle is given by the curvature of a complex connection on W. Therefore, if $\nabla = \hat{\nabla} + A + Q$ is a flat connection on (W, J) then (1.44) gives

$$\deg W = \frac{1}{2\pi}\int_M < J\hat{R} >= \frac{1}{2\pi}\int_M < A \wedge *A > - < Q \wedge *Q > \,. \tag{1.46}$$

Given two quaternionic bundles W and $\tilde{W}$ with complex structures J and $\tilde{J}$ the complex linear homomorphisms $\mathrm{Hom}_+(W, \tilde{W})$ are complex linearly isomorphic to $\mathrm{Hom}_{\mathbb{C}}(W_+, \tilde{W}_+)$, in particular

$$\deg \mathrm{Hom}_+(W, \tilde{W}) = \deg \tilde{W} - \deg W\,. \tag{1.47}$$

On the other hand, the complex antilinear homomorphisms $\mathrm{Hom}_-(W, \tilde{W})$ are complex linearly isomorphic to $\mathrm{Hom}_+(\bar{W}, \tilde{W})$, where the complex structure on a homomorphism bundle is induced by the target complex structure.

Finally, if V_1 and V_2 are two complex holomorphic vector bundles with complex holomorphic structures $\bar{\partial}_k$, then $\mathrm{Hom}_+(V_1, V_2)$ inherits a complex holomorphic structure $\bar{\partial}$ via

$$(\bar{\partial}A)\psi := \bar{\partial}_2(A\psi) - A(\bar{\partial}_1\psi)\,. \tag{1.48}$$

The usual tensor product construction for complex holomorphic structures induces a complex holomorphic structure on $K\,\mathrm{Hom}_+(V_1, V_2)$.

We summarize:

Geometric property	*Quaternionic formulation*
complex quaternionic vector bundle	quaternionic vector bundle W with complex structure $J \in \Gamma(\mathrm{End}(W))$
degree of a complex quaternionic vector bundle (W, J)	$\deg W = \deg W_+$ where W_+ is the $+i$ eigenspace of J
holomorphic structure D on (W, J)	$D = \bar{\partial} + Q$, $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(W))$
complex holomorphic structure	$D = \bar{\partial}$
Willmore energy of (W, D)	$\mathcal{W}(W, D) = 2\int < Q \wedge *Q >$, $D = \bar{\partial} + Q$
space of holomorphic sections of (W, D)	$H^0(W) = \ker D$
linear system of holomorphic sections	$H \subset H^0(W)$ linear subspace
order of a linear system H at $p \in M$	$\mathrm{ord}_p H = \sum(n_k(p) - k)$ where $n_k(p)$ is the Weierstraß gap sequence

1.2.2 The Kodaira correspondence

We now turn to the question, how holomorphic curves $f : M \to \mathbb{HP}^n$ correspond to holomorphic line bundles (W, D). In the case of complex holomorphic curves $f : M \to \mathbb{CP}^n$ the Kodaira correspondence gives a 1:1 correspondence between holomorphic curves and holomorphic line bundles with basepoint free linear systems H. In fact, given a holomorphic curve $f : M \to \mathbb{CP}^n$ the corresponding holomorphic line bundle is given by

E^{-1} where E is the line bundle associated to the holomorphic curve via $E_p = f(p)$. As in the case of algebraic curves in $\mathbb{CP}^n$, the canonically associated holomorphic line bundle of a holomorphic curve $f : M \to \mathbb{HP}^n$ is given by the dual bundle L^{-1} of $L = f^*\mathcal{T}$ via the *Kodaira correspondence*:

A holomorphic curve $f : M \to \mathbb{HP}^n$ induces, [FLPP01, Thm. 2.3], a unique holomorphic structure D on the dual bundle L^{-1} of $L = f^*\mathcal{T}$ such that linear forms $\alpha \in V^*$ on V restricted to L give holomorphic sections

$$\alpha|_L \in H^0(L^{-1}) .$$

Thus $H = V^* \subset H^0(L^{-1})$ is a $(n + 1)$–dimensional linear system. By transversality H is *basepoint free*, that is, there exists a nowhere vanishing holomorphic section of the holomorphic line bundle L^{-1} in H.

Conversely, a holomorphic line bundle L^{-1} and a basepoint free $(n+1)$–dimensional linear system $H \subset H^0(L^{-1})$ give a holomorphic curve $f : M \to \mathbb{HP}^n = \mathbb{P}(H^*)$ by the *Kodaira embedding* of L with respect to the linear system H which is defined as

$$\mathrm{ev}^*(L) \subset V = (M \times H)^* .$$

Here the bundle map

$$\mathrm{ev} : M \times H \to L^{-1}, (p, \psi) \mapsto \psi_p$$

evaluates the holomorphic section ψ at the point p.

Theorem 1.30 (Kodaira correspondence, see [FLPP01, Thm. 2.8])**.** *There is a 1:1 correspondence between holomorphic curves $f : M \to \mathbb{HP}^n$ (up to Möbius equivalence) and holomorphic line bundles L^{-1} together with basepoint free linear systems $H \subset H^0(L^{-1})$.*

In particular, every holomorphic curve $f : M \to \mathbb{HP}^n$ gives rise to a family of conformal maps into S^4: choosing a basepoint free 2–dimensional linear system $\check{H} \subset H$ yields via the Kodaira embedding of $L \subset \check{V} = (\check{H})^*$ a conformal map $\check{f} : M \to S^4$.

If we choose $\alpha \in H^0(L^{-1})$, $\alpha|_L \neq 0$, then there is a unique $\psi \in \Gamma(L)$ with $< \alpha, \psi >= 1$. We define $R : M \to S^2$ by $J\psi = -\psi R$ so that for all $\beta \in H^0(L^{-1})$ the map

$$f_\beta =< \beta, \psi >: M \to \mathbb{H}$$

is conformal with right normal vector R since the Leibniz rule for the holomorphic structure D gives

$$0 = D\beta = \alpha(d\bar{f}_\beta + R * d\bar{f}_\beta) .$$

In other words, if we fix a point at infinity α, then every 2–dimensional linear system $H \subset H^0(L^{-1})$ with $\alpha \in H$ gives rise to a conformal map $f : M \to \mathbb{H}$ with right normal R.

Conversely, every conformal map $f : M \to \mathbb{H}$ with right normal R defines a holomorphic section $\beta_f = \alpha\bar{f} \in H^0(L^{-1})$. This way, we can think of a holomorphic curve $f : M \to \mathbb{HP}^n$ with $\dim H^0(L^{-1}) = n + 1$ as the family of *all* conformal maps $f : M \to \mathbb{H}$ with the same right normal vector.

Example 1.31. Consider the dual curve $f^\dagger : M \to \mathbb{HP}^n$ of a holomorphic $f : M \to \mathbb{HP}^n$ with Frenet flag. Then $L^\dagger = V_{n-1}^\perp$ and $(L^\dagger)^{-1} = V/V_{n-1}$. Since

$$(\pi_{n-1} d|_{V_{n-1}})' = \delta'_{n-1} = 0\,,$$

where $\pi_{n-1} : V \to V/V_{n-1}$ is the canonical projection, the operator $D : \Gamma((L^\dagger)^{-1}) \to \Gamma(\bar{K}(L^\dagger)^{-1})$ given by

$$D[\alpha] = (\pi_{n-1} d\alpha)'' \tag{1.49}$$

for $\alpha \in \Gamma(V)$ is well–defined. One easily verifies that D is a holomorphic structure on V/V_{n-1}. In fact, we obtain the holomorphic structure given by the Kodaira correspondence on $V/V_{n-1} = (L^\dagger)^{-1}$.

Remark 1.32. Note that the notion of the dual curve depends on the linear system of $f : M \to \mathbb{HP}^n = P(H^*)$: if $L \subset H^*$ has dual curve $L^\dagger \subset H$ and $\hat{H} \subsetneqq H$ is basepoint free, then in general

$$\pi(L)^\dagger \neq \pi(L^\dagger)\,,$$

where $\pi : H^* \to \hat{H}^*$ denotes the canonical projection. This is due to the fact that if $\dim H = n+1$ then the $n-1$ first derivative of $L \subset H$ occurs in the dual curve $L^\dagger$ whereas in $\pi(L)^\dagger$ only lower derivatives appear.

Since a holomorphic curve $f : M \to \mathbb{HP}^n$ corresponds to a holomorphic line bundle L^{-1}, we can associate the *Willmore energy* to a holomorphic curve:

Definition 1.33. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve. The* Willmore energy *of f is defined by*

$$\mathcal{W}(f) := \mathcal{W}(L^{-1}, D) = \int_M < Q \wedge *Q >$$

where $D = \bar\partial + Q$ is the induced holomorphic structure of L^{-1}.

Note that the Willmore energy is by definition a Möbius invariant since it only depends on the holomorphic structure D on L^{-1}.

Example 1.34. A holomorphic curve $f : M \to \mathbb{HP}^n$ is the twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^n$ if and only if (1.30)

$$A^L = 0$$

where A^L is the Hopf field of the complex structure J on L given by the holomorphicity of f. Since in this case the holomorphic structure on L^{-1} is given by $\bar\partial + Q$ where $Q^* = -A^L$, we see that twistor projections of holomorphic curves in $\mathbb{CP}^n$ have zero Willmore energy.

Conversely, if $f : M \to \mathbb{HP}^n$ has zero Willmore energy then $0 = Q^* = -A^L$, that is, the complex bundle E given by the $+i$ eigenspace of J on L has $\delta''_E = A^L|_E = 0$. This shows that f is the twistor projection of the holomorphic curve $h : M \to \mathbb{CP}^{2n+1}$ given by the complex bundle E.

For a Frenet curve $f : M \to \mathbb{HP}^n$ with canonical complex structure S, the complex structure on L^{-1} is given by $(S|_L)^* = J$, and the Hopf fields are related by

$$(A|_L)^* = -Q^{L^{-1}} ,$$

where A is the Hopf field with respect to S whereas $Q^{L^{-1}}$ is given by the $(0,1)$–part of J. In particular, the Willmore energy is given by (1.37) in case of a Frenet curve

$$\mathcal{W}(f) = 2\int < Q \wedge *Q >= 2\int < (A|_L)^* \wedge (*A|_L)^* >= 2\int < A \wedge *A > .$$

The holomorphic structure $\bar\partial$ of a complex holomorphic vector bundle E^{-1} induces a holomorphic structure $\bar\partial^*$ on E by dualization. For quaternionic line bundles this dualization process fails: the dual D^* on V^* of a quaternionic holomorphic structure D on V is a *mixed structure* [FLPP01, Lemma 2.1] but *not* a holomorphic structure.

To equip the line bundle L of a holomorphic curve $f : M \to \mathbb{HP}^n$ with a holomorphic structure, one can choose a hyperplane at infinity not intersecting f: the basepoint free linear system H has a nowhere vanishing holomorphic section and thus we can choose $\alpha \in V^*$ so that $V = L \oplus \ker\alpha$. In this splitting we decompose the trivial connection d on V as

$$d|_L = \nabla^L + \delta . \tag{1.50}$$

Here δ is the derivative of f expressed via the identification $V/L = \ker\alpha$ and the induced connection ∇^L on L is flat since $\ker\alpha \subset V$ is constant. We now equip the line bundle L with the holomorphic structure

$$D = (\nabla^L)'' \tag{1.51}$$

coming from the connection ∇^L on L. Note that L always has the canonical nowhere vanishing holomorphic section $\psi \in H^0(L)$ for which $< \alpha, \psi >= 1$: since $\nabla^L\psi = 0$ the section ψ also has $D\psi = 0$ and thus is holomorphic. In particular, the complete linear system $H^0(L)$ is basepoint free. However, the holomorphic structure D on L depends on the choice of α. We will return in Chapter 2 to the holomorphic curves given by the Kodaira embedding $L^{-1} \subset (H^0(L))^*$.

1.2.3 The Riemann–Roch Theorem

In what follows, we frequently will make use of the bundle KL of L–valued $(1,0)$–forms. By [PP98] there is a unique holomorphic structure D^{KL} on KL compatible with the pairing

$$L^{-1} \times KL \to \Lambda^2 TM^* \otimes \mathbb{H} :$$

given $\psi \in \Gamma(L^{-1})$ and $\omega \in \Gamma(KL)$ the holomorphic structure is determined by the Leibniz rule

$$d < \psi, \omega >=< D\psi \wedge \omega > + < \psi, D^{KL}\omega > . \tag{1.52}$$

The Riemann–Roch theorem relates the dimensions of the spaces of holomorphic sections of L^{-1} and KL. It is verbatim the same statement as in the complex case. In fact, since the index of the elliptic operator $D = \bar\partial + Q$ does not change under continuous deformations, we have $\operatorname{index}(\bar\partial + tQ) = \operatorname{index}(\bar\partial)$ for $t \in [0, 1]$. The holomorphic structure D^{KL} on KL is the adjoint elliptic operator to D [FLPP01, Thm. 2.2] so that

$$\operatorname{index}(D) = \dim H^0(L^{-1}) - \dim H^0(KL)\,.$$

Using the Riemann–Roch theorem for complex holomorphic vector bundles we see:

Theorem 1.35 (Riemann–Roch Theorem, see [FLPP01, Thm. 2.2])**.** *Let L^{-1} be a holomorphic line bundle over a Riemann surface M and KL be equipped with the holomorphic structure so that L^{-1} and KL are paired holomorphic bundles. Then*

$$\dim H^0(L^{-1}) - \dim H^0(KL) = \deg L^{-1} - g + 1\,, \tag{1.53}$$

where g is the genus of M.

Two descriptions of the holomorphic structure D^{KL} on KL will be useful for our purposes: first, since $L \subset V$ is a subbundle, we can take exterior derivatives of $\omega \in \Gamma(KL)$ with respect to the trivial connection d on V. But $\pi d\omega = \delta \wedge \omega = 0$ by type considerations which implies that $d\omega \in \Omega^2(L)$ is again L–valued. One immediately checks (1.52) for d and thus

$$D^{KL} = d\,. \tag{1.54}$$

Second, if $\nabla^{L^{-1}}$ is any connection on L^{-1} adapted to the holomorphic structure D on L^{-1}, i.e., $D = (\nabla^{L^{-1}})''$, then the exterior derivative of $\omega \in \Gamma(KL)$ with respect to the dual connection ∇^L on L also gives

$$D^{KL} = d^{\nabla^L}\,. \tag{1.55}$$

This follows again from the fact that d^{∇^L} satisfies (1.52).

Lemma 1.36. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with $f = [f_1, \ldots, f_n, 1]$ in affine coordinates, and denote by*

$$\psi = \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 1 \end{pmatrix} \in \Gamma(L)\,.$$

Then $\omega \in H^0(KL)$ if and only if there exists locally a map $g : U \subset M \to \mathbb{H}$ such that

$$\omega = \psi dg \quad \text{and} \quad df_k \wedge dg = 0$$

for some $1 \le k \le n$.

Proof. Since f is holomorphic, we know that $*\delta = \delta J$ for some complex structure J on L. Let $R : M \to S^2$ be defined by $J\psi = -\psi R$, then $*\delta\psi = \delta J\psi$ is equivalent to $*df_i = -df_i R$ for all i. In particular, $df_k \wedge dg = 0$ for some $1 \le k \le n$ implies $df_k \wedge dg = 0$ for all $1 \le i \le n$.

Moreover, a 1–form $\omega \in \Gamma(KL)$ is d^{∇_L}–closed if and only if there exists locally a map g with $\omega = \psi dg$. In this case $*\omega = J\omega$ is equivalent to

$$0 = \delta \wedge \omega = \delta\psi \wedge dg = \pi \begin{pmatrix} df_1 \wedge dg \\ \vdots \\ df_n \wedge dg \\ 0 \end{pmatrix}.$$

□

1.2.4 The Plücker formula

The Plücker formula [FLPP01, Thm. 4.7] gives a lower bound on the Willmore energy $\mathcal{W}(D)$ of a holomorphic bundle (L, D) over a compact Riemann surface M in terms of the genus of M, the degree of L, and the vanishing orders of holomorphic sections of L. We give a proof of the Plücker formula in the case when L is the holomorphic line bundle given by a Frenet curve $f : M \to \mathbb{HP}^n$.

In the first step, we compute the degrees of the flag spaces of f.

Lemma 1.37. *Let $f : M \to \mathbb{HP}^n$ be a Frenet curve, S its canonical complex structure and $L \subset V_1 \subset \ldots \subset V_n = V$ its Frenet flag with corresponding derivatives δ_k. Then the degree of the bundle V_k/V_{k-1} with respect to S is given by*

$$\deg V_k/V_{k-1} = \sum_{i=0}^{k-1} \operatorname{ord} \delta_i - k \deg K + \deg L, \;\; 0 \le k \le n\,. \tag{1.56}$$

where we put $V_{-1} := \{0\}$.

Proof. The degree of a complex holomorphic line bundle E is given by the vanishing order of any holomorphic section of E. In [Les] it is shown that the derivatives of the flag spaces are holomorphic sections

$$\delta_i \in H^0(K \operatorname{Hom}_+(V_i/V_{i-1}, V_{i+1}/V_i))$$

provided there exists an adapted complex structure S on V. Here the complex holomorphic structure on $K \operatorname{Hom}_+(V_i/V_{i-1}, V_{i+1}/V_i)$ is given by the tensor construction (1.48) and the complex holomorphic structures $\bar\partial = d''_+$ on V_i/V_{i-1} and V_{i+1}/V_i. Therefore, (1.47) shows that

$$\operatorname{ord}\delta_i = \deg(K\operatorname{Hom}_+(V_i/V_{i-1}, V_{i+1}/V_i)) = \deg K + \deg V_{i+1}/V_i - \deg V_i/V_{i-1}\,.$$

Telescoping this identity, we get

$$\sum_{i=0}^{k-1}\operatorname{ord}\delta_i = k\deg K + \deg V_k/V_{k-1} - \deg L\,. \tag{1.57}$$

□

Remark 1.38. The degree of the dual curve $f^\dagger$ of a Frenet curve $f: M \to \mathbb{HP}^n$ is given by

$$\deg L^\dagger = n\deg K - \deg L - \sum_{i=0}^{n-1}\operatorname{ord}\delta_i\,, \tag{1.58}$$

since $(V/V_{n-1})^{-1} = V_{n-1}^\perp = L^\dagger$.

We now prove the quaternionic Plücker relation [FLPP01, Thm. 4.7] in the case of a Frenet curve:

Theorem 1.39 (Plücker formula)**.** *Let $f: M \to \mathbb{HP}^n$ be a Frenet curve and S its canonical complex structure. Let $L \subset V_1 \subset \ldots \subset V_n = V$ be the Frenet flag of f and δ_k the derivatives of V_k. If M is a compact Riemann surface of genus g, then*

$$\deg(V,S) = \frac{1}{4\pi}(\mathcal{W}(f) - \mathcal{W}(f^\dagger)) = (n+1)(n(1-g) + \deg L) + \operatorname{ord} H\,, \tag{1.59}$$

where $\operatorname{ord} H$ *is the order of the linear system $H = V^* \subset H^0(L^{-1})$.*

Remark 1.40. The expression (1.40) for the order of H simplifies for a Frenet curve $f: M \to \mathbb{HP}^n$ to

$$\operatorname{ord} H = \sum_{i=0}^{n-1}(n-i)\operatorname{ord}\delta_i\,. \tag{1.60}$$

Proof. Since $V = \bigoplus_{k=0}^n V_k/V_{k-1}$ as complex vector bundles, we obtain the right hand sight of the equation by using (1.57)

$$\begin{aligned}\deg(V,S) &= \sum_{k=0}^n \deg V_k/V_{k-1} = \sum_{k=0}^n\Big(\sum_{i=0}^{k-1}\operatorname{ord}\delta_i - k\deg K + \deg L\Big)\\ &= \Big(\sum_{k=0}^n\sum_{i=0}^{k-1}\operatorname{ord}\delta_i\Big) - \frac{n(n+1)}{2}\deg K + (n+1)\deg L\,.\end{aligned}$$

On the other hand, recalling (1.46)

$$2\pi \deg(V,S) = \int_M < A \wedge *A > - < Q \wedge *Q >$$

as well as (1.37) and (1.32)

$$\mathcal{W}(f) = 2\int_M < A \wedge *A >, \quad \mathcal{W}(f^\dagger) = 2\int_M < Q \wedge *Q >$$

we get

$$4\pi \deg(V,S) = \mathcal{W}(f) - \mathcal{W}(f^\dagger).$$

□

Remark 1.41. Let $h : M \to \mathbb{CP}^{2n+1}$ be a holomorphic curve in $\mathbb{CP}^n$ such that the twistor projection $f : M \to \mathbb{HP}^n$ of h is a Frenet curve, see Example 1.23. Since $\mathcal{W}(f) = 0$ the Plücker relation shows that the Willmore energy of the dual curve $f^\dagger$ of f is given by

$$\mathcal{W}(f^\dagger) = 4\pi \deg(V,S) \in 4\pi\mathbb{N}.$$

The general case of the Plücker formula is proved [FLPP01, Thm. 4.7] by using the holomorphic jet complex of the holomorphic line bundle L^{-1}. Away from the Weierstraß points $\{p_1, \dots, p_m\}$, the k^{th} jet bundle is given by the dual $\mathcal{L}_k = V_k^*$ of the flag space V_k. Moroever there exists an isomorphism $P : H \to \mathcal{L}_n$ of the basepoint free linear system $H \subset H^0(L^{-1})$ and the n^{th} jet space of L^{-1} on $M_0 = M \setminus \{p_1, \dots, p_m\}$. The trivial connection on H can be pushed forward to a trivial connection ∇ on $\mathcal{L}_n$ over M_0. Moreover, the holomorphic jet complex $\mathcal{L}_n$ has a canonical complex structure, which is given on M_0 by the dual of the canonical complex structure $\mathcal{S} = S^*$ of f. Decomposing ∇ with respect to the complex structure $\mathcal{S}$ we get

$$\nabla = \hat{\nabla} + \mathcal{A} + \mathcal{Q},$$

where $\hat{\nabla}$ is a $\mathcal{S}$–complex connection with only logarithmic singularities at the Weierstraß points [FLPP01, Sec. 4.3], and

$$\int_M < \mathcal{S}\hat{R} >= 2\pi(\deg \mathcal{L}_n - \operatorname{ord} H).$$

Recall (1.44) that

$$\hat{R} = -(\mathcal{A} \wedge \mathcal{A} + \mathcal{Q} \wedge \mathcal{Q})$$

and $\mathcal{W}(L) = 2\int < \mathcal{Q} \wedge *\mathcal{Q} >$, which shows that

$$\mathcal{W}(L^\dagger) := \int < \mathcal{A} \wedge *\mathcal{A} >$$

is finite. Computing the degree of the n^{th}–jet bundle, one derives the Plücker formula in the general case. For the complete argument see [FLPP01, Sec. 4.3].

Discussing the orders of zeros of holomorphic sections carefully [FLPP01, Sec. 4.5] the Plücker formula gives estimates on the Willmore energy of a conformal map $f : M \to S^4$ in dependence of the dimension of the space of holomorphic sections, e.g., in the case of a conformal torus with $\dim H^0(L^{-1}) = n + 1$ one has [FLPP01, Rem. 6]

$$\mathcal{W}(f) \geq \begin{cases} \pi(n+1)^2 \,, & n \text{ odd} \\ \pi((n+1)^2 - 1) \,, & n \text{ even} \end{cases} .$$

Using these kinds of estimates, one derives a new estimate for the eigenvalues of the Dirac operator of a Riemannian spin bundle [FLPP01, Thm. 5.5], area estimates for a constant mean curvature torus in terms of its spectral genus g [FLPP01, Thm. 6.5]

$$\text{area}(f) \geq \begin{cases} \frac{\pi}{4}(g+2)^2 \,, & g \text{ even} \\ \frac{\pi}{4}((g+)^2 - 1) \,, & g \text{ odd} \end{cases} ,$$

and estimates on the Willmore energy of a constant mean curvature torus in $\mathbb{R}^3$ or S^3 [FLPP01, Thm. 6.7]

$$\mathcal{W}(f) \geq \begin{cases} \frac{\pi}{4}(g+2)^2 \,, & g \text{ even} \\ \frac{\pi}{4}((g+)^2 - 1) \,, & g \text{ odd} \end{cases} .$$

In particular, to verify the Willmore conjecture on minimal tori in the 3–sphere it suffices to consider minimal tori of spectral genus at most 3.

The case of equality in the Plücker formula are the so–called soliton surfaces, which are discussed for example in [Pet04], [BP].

Chapter 2

Transformations on conformal maps

Transformations which preserve special surface classes in 3– and 4–space play an important role in surface geometry. One of the motivations to study those transformations comes from the fact that they allow to construct more complicated surfaces from given simple ones. Historical examples include the Bäcklund transformation on surfaces of constant Gaussian curvature [Bia80] and the Darboux transformation on isothermic surfaces [Dar99]. More recently, also a Bäcklund transformation and a Darboux transformation on Willmore surfaces in S^4 have been studied [BFL$^+$02].

In this chapter we discuss a Darboux and a Bäcklund transformation on conformal maps $f : M \to S^4$ of a Riemann surface M into the 4–sphere [BLPP] [LP05], [Pet04], [BP]. Both transformations satisfy Bianchi permutability: The Darboux transformation arises from integrating a Ricatti–type equation whereas the Bäcklund transformation comes from solving Abelian integrals (which explains to some extend our choice of notation).

2.1 The Darboux transformation

The classical Darboux transformation of isothermic surfaces goes back to [Dar99]. Geometrically, the Darboux transform of an isothermic surface $f : M \to S^4$ is given by a sphere congruence enveloping both f and its Darboux transform $f^\sharp$. To define a Darboux transformation on arbitrary conformal maps, we relax the enveloping condition [BLPP]. We show that the transformation defined this way satisfies Bianchi permutability. In particular, we can define the spectral curve of a conformal torus $f : T^2 \to S^4$ in S^4 as the set of all Darboux transforms: For each point $p \in T^2$, the images $f^\sharp(p)$ of the Darboux transforms $f^\sharp$ of f canonically embed the spectral curve into S^4 as a twistor projection of a holomorphic curve $\tilde{F}(p, .) : \Sigma \to \mathbb{CP}^3$.

In the special case of a constant mean curvature torus $f : T^2 \to \mathbb{R}^3$ this definition of the spectral curve coincides [CLP] with the "classical" one given by the eigenvalues of the holonomy of a family of flat connections [Hit90]. We show that the Darboux transforms corresponding to points on the spectral curve are isothermic even though the general Darboux transformation only coincides for very special points on the spectral curve with the classical Darboux transformation of a constant mean curvature torus in $\mathbb{R}^3$.

2.1.1 The classical Darboux transformation on isothermic surfaces

The classical Darboux transformation of isothermic surfaces in 3–space goes back to [Dar99]. We recall the notion of an isothermic surface $f : M \to S^4$, see for example [HJ02, Sec. 7.4], [Bur]:

Definition 2.1. *A conformal map $f : M \to S^4$ is called* isothermic *if there exists a closed 1–form non–trivial $\omega \in \Omega^1(\mathcal{R})$, i.e.,*

$$d\omega = 0 \qquad \textit{and} \qquad \operatorname{im}\omega \subset L \subset \ker\omega\,. \tag{2.1}$$

In this case, any local parallel section $\hat\psi$ of $d + \omega$ gives rise to a quaternionic line bundle $\hat L = \hat\psi\mathbb{H}$ over an open set U of M. The corresponding map $\hat f : U \subset M \to S^4$ is called a classical Darboux transform *of f.*

In [HJ02, Sec. 7.4] it is shown that the 1–form $\omega \in \Omega^1(\mathcal{R})$ is unique up to multiplication by a real constant $\rho \in \mathbb{R}$, at least away from the umbilics of f. The condition (2.1) implies that $d + \omega\rho$ is a flat connection on V for all $\rho \in \mathbb{R}$. We fix $\omega \in \Omega^1(\mathcal{R})$ for reference and choose a point at infinity not intersecting f then ω is written in the splitting $L \oplus e\mathbb{H} = V$ as

$$\omega = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix}.$$

Note that ω is closed if and only

$$0 = d\omega = \begin{pmatrix} -\eta\wedge\delta & d^\nabla\eta \\ 0 & \delta\wedge\eta \end{pmatrix}. \tag{2.2}$$

If $\hat\psi \in \Gamma(\operatorname{pr}^* V)$ is a parallel section of $d + \omega\rho$, $\rho \in \mathbb{R}$, on the universal cover $\operatorname{pr} : \tilde M \to M$ of M then

$$\hat\psi = \begin{pmatrix} \alpha + f\beta \\ \beta \end{pmatrix}$$

with $d\alpha = -df\beta$ since

$$d\hat\psi = -\omega\hat\psi\rho \in \Omega^1(\operatorname{pr}^* L)\,. \tag{2.3}$$

Note that

$$d\alpha = -df\beta$$

shows that $\gamma^*\hat{\psi} = \hat{\psi} h_\gamma$ for all $\gamma \in \pi_1(M)$ with $h_\gamma : M \to \mathbb{H}_*$, so that the classical Darboux transform $\hat{L} = \hat{\psi}\mathbb{H}$ is globally defined on M. Moreover,

$$T = \alpha\beta^{-1}$$

is a global solution of the *Ricatti equation*

$$dT = -df + T\hat{\eta}\rho T\,, \tag{2.4}$$

where $\hat{\eta}\rho = -d\beta\alpha^{-1}$. Note that (2.3) shows

$$\omega e = \eta e = \begin{pmatrix} f \\ 1 \end{pmatrix} \hat{\eta}\,,$$

and ω is closed (2.2) if and only if

$$d\hat{\eta} = 0\,, \quad df \wedge \hat{\eta} = 0\,, \text{ and } \hat{\eta} \wedge df = 0 \tag{2.5}$$

Using $d\alpha = -df\beta$ one easily verifies

$$(d\hat{\eta})T = -\hat{\eta} \wedge df \quad \text{ and } \quad df \wedge \hat{\eta} = 0\,,$$

so that we proved:

Corollary 2.2. *If $d + \omega$ is a connection with $\omega \in \Omega^1(\mathcal{R})$ and if there exists*

$$\hat{\psi} = \begin{pmatrix} \alpha + f\beta \\ \beta \end{pmatrix} \in \Gamma(\mathrm{pr}^* V)$$

with $d\hat{\psi} = -\omega\hat{\psi}$, then ω is closed if and only if $d\beta\alpha^{-1}$ is closed.

Figure 2.1: Darboux pair

Definition 2.3. *Let f and $\hat{f}$ be conformal maps into the 4–sphere such that $L \oplus \hat{L} = V$. The pair $(f, \hat{f})$ is called a* (classical) Darboux pair *if both surfaces f and $\hat{f}$ are simultaneously enveloped by a sphere congruence S.*

Recall (1.9) that S envelopes f if and only if

$$*\delta = S\delta = \delta S\,.$$

We write the trivial connection d and the sphere congruence S in the splitting $V = L \oplus \hat{L}$:

$$d = \begin{pmatrix} \nabla^L & \hat{\delta} \\ \delta & \hat{\nabla} \end{pmatrix} \tag{2.6}$$

and

$$S = \begin{pmatrix} J & 0 \\ 0 & \hat{J} \end{pmatrix}, \tag{2.7}$$

where we used $SL = L$ and $S\hat{L} = \hat{L}$ since S envelopes f and $\hat{f}$. Moroever, the enveloping condition

$$*\delta = \hat{J}\delta = \delta J \quad \text{and} \quad *\hat{\delta} = J\hat{\delta} = \hat{\delta}\hat{J}$$

is equivalent to

$$\hat{\delta} \wedge \delta = 0 \quad \text{and} \quad \delta \wedge \hat{\delta} = 0\,. \tag{2.8}$$

An isothermic immersion $f : M \to S^4$ and each of its classical Darboux transforms $\hat{f} : M \to S^4$ form a Darboux pair $(f, \hat{f})$. Since the argument is of local nature we may assume that $\hat{\psi} \in \Gamma(\hat{L})$ is a parallel section of $d + \omega$ where $\omega \in \Omega^1(\mathcal{R})$ is closed. Since $d\hat{\psi} = -\omega\hat{\psi} \in \Omega^1(L)$ we get, using the splitting $L \oplus \hat{L} = V$,

$$\omega\hat{\psi} = -\hat{\delta}\hat{\psi}\,.$$

For $\hat{\psi} \in \Gamma(\hat{L})$ we have

$$0 = (d\omega)\hat{\psi} = d(\omega\hat{\psi}) + \omega(d\hat{\psi})$$

so that

$$0 = \pi_L d(\omega\hat{\psi}) = \delta \wedge \omega\hat{\psi} = -\delta \wedge \hat{\delta}\psi$$

and

$$*\hat{\delta} = J\hat{\delta}\,, \tag{2.9}$$

where J is the complex structure on $V/\hat{L} = L$ given by the conformality of f. Similarly, for $\psi \in \Gamma(L)$ we have

$$0 = (d\omega)\psi - \omega \wedge \pi_L d\psi = -\omega \wedge \delta\psi\,,$$

i.e., using $\hat{\delta} = -\omega$,

$$*\hat{\delta} = \hat{\delta}\hat{J}\,, \tag{2.10}$$

where $\hat{J}$ is the complex structure on $V/L = \hat{L}$ given by f. In particular, the classical Darboux transform $\hat{f} : M \to S^4$ of an isothermic surface f is conformal. Moreover, we can define a sphere congruence S on $V = L \oplus \hat{L}$ by

$$S|_L = J \quad \text{and} \quad S|_{\hat{L}} = \hat{J}$$

and S envelopes f and $\hat{f}$ by (2.10) and (2.9).

Conversely, the surfaces f and $\hat{f}$ of a Darboux pair are isothermic: If we define

$$\omega = \begin{pmatrix} 0 & \hat{\delta} \\ 0 & 0 \end{pmatrix} \in \Omega^1(\mathcal{R})$$

in the splitting $V = L \oplus \hat{L}$ then ω is closed since

$$d\omega = \begin{pmatrix} -\hat{\delta} \wedge \delta & d^\nabla \hat{\delta} \\ 0 & \delta \wedge \hat{\delta} \end{pmatrix} = 0\,,$$

where we used (2.8) and $d^\nabla \hat\delta = 0$ by the flatness of d. Interchanging the roles of L and $\hat L$ we see that $\hat f$ is isothermic, too.

One of the important features of the Darboux transformation on isothermic surfaces is the *Bianchi permutability*, see for example [HJ02, Sec. 7.6]:

Theorem 2.4 (Bianchi permutability)**.** *Given two Darboux transforms $f_1, f_2 : M \to S^4$ of an isothermic surface $f : M \to S^4$ then there exists an isothermic surface $\hat f : M \to S^4$ such that $\hat f$ is simultaneously Darboux transform of f_1 and f_2.*

Proof. Let $\rho_1, \rho_2 \in \mathbb{R}$ such that L_i is $d + \omega\rho_i$ parallel, and let $\psi_i \in \Gamma(\mathrm{pr}^* L_i)$ be parallel sections with respect to $d+\omega\rho_i$. Since $d\psi_i = -\omega\rho_i\psi_i \in \Gamma(\mathrm{pr}^* KL)$, there exists $\rho : \tilde M \to \mathbb{H}$ such that

$$d\psi_2 = d\psi_1\rho\,.$$

Define $\hat f : M \to S^4$ by $\hat L = \hat\psi\mathbb{H}$ where

$$\hat\psi = \psi_2 - \psi_1\rho\,.$$

To see that $\hat f$ is a Darboux transform of f_1, we write

$$\psi_i = \begin{pmatrix} \alpha_i + f\beta_i \\ \beta_i \end{pmatrix}$$

in the splitting $L \oplus e\mathbb{H} = V$ so that

$$\omega\rho_i e = \begin{pmatrix} f \\ 1 \end{pmatrix} d\beta_i\alpha_i^{-1}$$

and

$$\rho = (\rho_1\alpha_1)^{-1}(\rho_2\alpha_2)\,.$$

The Darboux transforms are given in affine coordinates by

$$[f_i, 1] : M \to S^4$$

where $f_i = f + T_i$ and $T_i = \alpha_i\beta_i^{-1}$ satisfy the Ricatti equation (2.4). Therefore, $\hat\psi$ is given in the splitting $L_1 \oplus e\mathbb{H} = V$ as

$$\hat\psi = \begin{pmatrix} \alpha + f_1\beta \\ \beta \end{pmatrix}$$

where

$$\beta = \beta_2 - \beta_1\rho = ((\rho_2 T_2)^{-1} - (\rho_1 T_1)^{-1})\rho_2\alpha_2$$

and

$$\alpha = (T_2 - T_1)\beta_2\,.$$

Using the Ricatti equation (2.4) and $d\alpha_2 = -df\beta_2$ we obtain

$$d\beta = -(\rho_1 T_1)^{-1} df T_1^{-1}\rho_2\alpha\,,$$

and

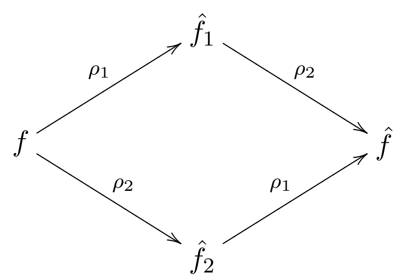
$$(d\beta)\alpha^{-1} = -(\rho_1 T_1)^{-1} df T_1^{-1} \rho_2 \,.$$

Applying again the Ricatti equation this yields

$$\begin{aligned} d((d\beta)\alpha^{-1}) &= -((\rho_1 T_1)^{-1} df T_1^{-1} + \hat\eta) \wedge df T_1^{-1}\rho_2 + (\rho_1 T_1)^{-1} df \wedge (T_1^{-1} df T_1^{-1} - \hat\eta \rho_1)\rho_2 \\ &= 0\,, \end{aligned}$$

where we also used that $\hat\eta \wedge df = df \wedge \hat\eta = 0$ by (2.5). Corollary 2.2 shows that $\hat f$ is a classical Darboux transform of f_1. Interchanging the roles of f_1 and f_2 we see that $\hat f$ and f_2 form a Darboux pair, too. □

If $(f, \hat f)$ form a Darboux pair then $\hat f$ is a Darboux transform of f and vice versa. If we choose the reference $\hat\omega \in \Omega^1(\hat{\mathcal{R}})$ according to this symmetry, the parameters in the Bianchi permutability are given by the following diagram, see [HJ02, Sec. 7.6]:



Note also that $\hat f$ can be computed in terms of f_1 and f_2 by only using differentiation and algebraic operation. For more results on isothermic surfaces see for example [Bur], [HJP97], [HJ02], [HJ97], [KPP98].

2.1.2 The spectral curve of constant mean curvature tori

To motivate how we generalize the classical Darboux transformation on isothermic surfaces to a Darboux transformation on conformal maps $f : M \to S^4$, we first explain how points on the spectral curve of a constant mean curvature torus $f : T^2 \to \mathbb{R}^3$ can be seen as conformal maps which satisfy a generalized enveloping condition.

Let $f : M \to \mathbb{R}^3 = \operatorname{im}\mathbb{H}$ be a constant mean curvature surface, and without loss of generality assume that f has constant mean curvature $H = -1$. Thus, the mean curvature sphere congruence of f is given in the splitting $V = L \oplus e\mathbb{H}$ by (1.14) and (1.15)

$$S = \begin{pmatrix} J & R \\ 0 & \tilde J \end{pmatrix}$$

with $Re = \psi$. Using the mean curvature sphere condition (1.21)

$$*\tilde A = \frac{1}{2}\delta R$$

and

$$(\nabla R)e = \nabla^L(Re) - R(\tilde{\nabla}e) = \nabla^L\psi = 0\,,$$

we get

$$d^{\tilde{\nabla}} * \tilde{A} = (\frac{1}{2}d^{\nabla}\delta)R - \frac{1}{2}\delta\nabla R = 0\,, \tag{2.11}$$

where we used that $d^{\nabla}\delta = 0$ by the flatness of d. Thus $*\tilde{A}$ is a closed 1–form which implies by standard arguments, see for example [BFL$^+$02, Prop. 5], that $\tilde{J}$ is harmonic. Let $\lambda = a + b\tilde{J}$, $a, b \in \mathbb{R}$, $|\lambda| = 1$, then

$$\tilde{\nabla}^{\lambda} = \tilde{\nabla} + (\lambda - 1)\tilde{A} \tag{2.12}$$

defines a family of connections on $e\mathbb{H}$. Note that

$$\tilde{\nabla}\lambda = b(\tilde{\nabla}\tilde{J}) = 2b(*\tilde{Q} - *\tilde{A})$$

and

$$0 = d^{\tilde{\nabla}} * \tilde{A} = d^{\tilde{\nabla}}(\tilde{J}\tilde{A}) = \tilde{J}(d^{\tilde{\nabla}}A - 2\tilde{A} \wedge \tilde{A})$$

since $\tilde{Q} \wedge \tilde{A} = 0$ by type considerations.

This gives for $\omega^{\lambda} = (\lambda - 1)\tilde{A}$

$$d^{\tilde{\nabla}}\omega^{\lambda} + \omega^{\lambda} \wedge \omega^{\lambda} = 2b(*\tilde{Q} - *\tilde{A}) \wedge \tilde{A} + 2(\lambda - 1)\tilde{A} \wedge \tilde{A} + |\lambda - 1|^2\tilde{A} \wedge \tilde{A} = 0\,,$$

where we used that $|\lambda| = 1$. Thus, the connection $\tilde{\nabla}^{\lambda} = \tilde{\nabla} + \omega^{\lambda}$ is flat. The family $\tilde{\nabla}^{\lambda}$ of flat quaternionic connections extends to a family of flat *complex* connections [FLPP01, Lemma 6.3] if we introduce the complex structure I on $V/L = e\mathbb{H}$ given by right multiplication by i: For $\mu = \alpha + \beta I \in \mathbb{C}_*$ the connections

$$\tilde{\nabla}^{\mu} = \tilde{\nabla} + \frac{\mu + \mu^{-1} - 2}{2}\tilde{A} + \frac{\mu^{-1} - \mu}{2}I * \tilde{A}$$

are flat by the same computation as before since $a = \frac{\mu + \mu^{-1}}{2}$ and $b = \frac{\mu^{-1} - \mu}{2}$ satisfy $a^2 - b^2 = 1$.

Fixing a base point $p \in M$, we get a family of holonomy representations $H^{\mu} : \pi_1(M) \to SL(2, \mathbb{C})$ for $\mu \in \mathbb{C}_*$. We now restrict to the case of a torus $M = T^2$ so that the fundamental group $\pi_1(M)$ is abelian and we have common eigenvectors for the holonomy of ∇^{μ}. The set of eigenvalues of H^{μ} compactifies to a hyperelliptic Riemann surface $\Sigma \to \mathbb{CP}^1$, the so–called *spectral curve* of f, of finite genus [Hit90]. The eigenlines of H^{μ} define a holomorphic line bundle $\mathcal{L} \to \Sigma$ over the spectral curve. Note that these considerations were done with respect to a fixed base point $p \in M$. When changing p the spectral curve Σ remains unchanged however the point in the Jacobian corresponding to $\mathcal{L}$ changes linearly with the base point on (the abelian group) T^2. For details how to reconstruct the harmonic map $N : T^2 \to S^2$ and thus the constant mean curvature torus from algebraic curve data compare [Hit90], [McI01].

In order to deal with a quaternionic connection instead of a complex one, introduce [CLP] the complex structure S^x on $V/L = e\mathbb{H}$ as the quaternionic extension of the complex structure I on the complex E_μ bundle to the quaternionic bundle $E_\mu \oplus E_\mu j = e\mathbb{H}$ and define

$$\tilde{\nabla}^x = \tilde{\nabla} + \tilde{A}\frac{\mu + \mu^{-1} - 2}{2} + *\tilde{A}S^x\frac{\mu^{-1} - \mu}{2}\,, \tag{2.13}$$

where we denote in abuse of notation $\mu = \alpha + \beta S^x$. Then $\tilde{\nabla}^x$ is a flat quaternionic connection on $e\mathbb{H}$ with

$$\tilde{\nabla}^x S^x = 0$$

and

$$\nabla^x|_{\Gamma(E_\mu)} = \tilde{\nabla}^\mu|_{\Gamma(E_\mu)}\,.$$

Moreover, for $\mu(x) \in S^1$ the connection ∇^x is the quaternionic connection defined in (2.12).

Lemma 2.5 ([CLP]). *A point $x \in \Sigma$ on the spectral curve gives a unique pair $(\tilde{\nabla}^x, S^x)$ on $e\mathbb{H}$ with*

$$\tilde{\nabla}^x = \tilde{\nabla} - \tilde{A} + \tilde{A}a + *\tilde{A}S^x b\,, \quad a^2 - b^2 = 1\,,$$

and $\tilde{\nabla}^x S^x = 0$.

We want to understand points on the spectral curve geometrically. Let $\varphi \in \Gamma(\mathrm{pr}^* V/L)$ be a parallel section with respect to

$$\tilde{\nabla}^x = \tilde{\nabla} + \omega$$

on the universal cover $\mathrm{pr} : \tilde{M} \to M$ of M. Here we again identify $V/L = e\mathbb{H}$ and abbreviate

$$\omega = \tilde{A}\frac{\mu + \mu^{-1} - 2}{2} + *\tilde{A}S^x\frac{\mu^{-1} - \mu}{2}\,.$$

For a lift $\psi \in \Gamma(\mathrm{pr}^* V)$ of φ, i.e.,

$$\pi_L(\psi) = \varphi\,,$$

where $\pi_L : V \to V/L$ is the canonical projection, we write

$$\psi = \varphi_L + \varphi$$

in the splitting $V = L \oplus e\mathbb{H}$. There is a well–defined map $B \in \mathrm{Hom}(e\mathbb{H}, L)$ given by

$$\delta B = \omega \tag{2.14}$$

since $\delta_X : L \to e\mathbb{H}$ is for $X \neq 0$ a bundle isomorphism and $*\omega = \tilde{J}\omega$ and $*\delta = \tilde{J}\delta$. Because $d\psi = \nabla^L \varphi_l + \delta\varphi_L - \omega\varphi$, we see that

$$\pi_L(d\psi) = 0 \quad \text{if and only if} \quad \varphi_L = B\varphi\,.$$

We summarize:

Lemma 2.6. *Let $\varphi \in \Gamma(\mathrm{pr}^* V/L)$ be a parallel section of a connection $\tilde\nabla + \omega$ on V/L where $\tilde\nabla$ is the trivial connection on $V/L = e\mathbb{H}$ and ω a 1–form with values in V/L with $*\omega = \tilde J\omega$. Then there is a unique lift $\psi \in \Gamma(\mathrm{pr}^* V)$ of φ so that*

$$\pi_L(d\psi) = 0\,.$$

Note, that ψ has the same (multiplicative) monodromy as φ, and ψ is nowhere vanishing since $\pi_L(\psi) = \varphi$ is a parallel section of $\tilde\nabla^\mu$. Therefore, ψ defines a line subbundle of V by

$$L^\mu = \psi\mathbb{H} \subset V\,,$$

and V splits into the direct sum

$$V = L \oplus L^\mu\,.$$

Writing the trivial connection d in this splitting, i.e., identifying $V/L = L^\mu$ and $V/L^\mu = L$, we have

$$d = \begin{pmatrix} \nabla^L & \delta^\mu \\ \delta & \nabla^\mu \end{pmatrix}$$

Note that ∇^μ is indeed the flat connection $\tilde\nabla^x$ if we identify $e\mathbb{H} = V/L = L^\mu$ by $\varphi \mapsto \psi$ since $\nabla^\mu\psi = \pi_L d\psi = 0$. Together with the flatness of d we obtain

$$0 = R^\mu = -\delta \wedge \delta^\mu\,.$$

Therefore, the map $f^\mu : T^2 \to S^4$ given by L^μ is conformal since

$$*\delta^\mu = J\delta^\mu\,,$$

where J is the complex structure on V/L^μ given by the identification $V/L^\mu = L$ and the conformality of f.

We call f^μ a *(generalized) Darboux transform* of f: the sphere congruence given by $S|_L = J$ and $S|_{L^\mu} = \tilde J$ envelopes f and *left–envelopes* f^μ, that is $SL^\mu = L^\mu$ and

$$*\delta^\mu = J\delta^\mu\,.$$

We have shown:

Lemma 2.7. *If $f : M \to \mathbb{R}^3$ is a constant mean curvature torus with spectral curve Σ then points $x \in \Sigma$ give (generalized) Darboux transforms f^μ of f.*

2.1.3 The Darboux transformation on conformal maps into the 4–sphere

As we have seen in the previous section, points on the spectral curve of a constant mean curvature torus $f : T^2 \to \mathbb{R}^3$ give (generalized) Darboux transforms $\hat f : T^2 \to S^4$. The Darboux transformation can be defined for conformal maps $f : M \to S^4$, and gives eventually rise to the spectral curve of a conformal torus $f : T^2 \to S^4$ in the 4–sphere, [BLPP].

The Darboux transformation is defined by relaxing the enveloping condition: a sphere congruence *left envelopes* a conformal map $f^\sharp : M \to S^4$ if $S(p)$ goes through $f^\sharp(p)$ and the tangent planes of f and $S(p)$ are left–parallel for all $p \in M$. Two planes in $\mathbb{R}^4$ through the origin are called *left–parallel*, if their oriented intersection great circles in $S^3 \subset \mathbb{R}^4$ correspond via right translation in the group S^3.

Definition 2.8 ([BLPP]). *Let $f : M \to S^4$ be a conformal map. If there exists a sphere congruence S and a conformal map $f^\sharp$ such that $L \oplus L^\sharp = V$ and S envelopes f and left–envelopes $f^\sharp$ then $f^\sharp$ is called a* Darboux transform *of f.*

Figure 2.2: Homogeneous torus with various Darboux transforms

Note that the sphere congruence S is in this case given in the splitting $V = L \oplus L^\sharp$ as

$$S = \begin{pmatrix} J & 0 \\ 0 & J^\sharp \end{pmatrix} .$$

Moreover, writing the trivial connection $d = \begin{pmatrix} \nabla^L & \delta^\sharp \\ \delta & \nabla^\sharp \end{pmatrix}$ as before in the splitting, the touching and left touching condition read as

$$*\delta = J^\sharp \delta = \delta J \qquad \text{and} \qquad * \delta^\sharp = J\delta^\sharp .$$

In other words, $f^\sharp$ is a Darboux transform of f if and only if

$$\delta \wedge \delta^\sharp = 0 . \tag{2.15}$$

The flatness of d implies that (2.15) is equivalent to the flatness of the connection $\nabla^\sharp$ on $L^\sharp$. Since

$$d|_{\Gamma(L^\sharp)} = \nabla^\sharp + \delta^\sharp$$

a section $\psi^\sharp \in \Gamma(\mathrm{pr}^* L^\sharp)$ satisfies

$$d\psi^\sharp \in \Omega^1(\mathrm{pr}^* L)$$

if and only if

$$\nabla^\sharp \psi^\sharp = 0 .$$

We summarize:

Corollary 2.9 ([BLPP]). *Let $f : M \to S^4$ be a conformal immersion and $f^\sharp : M \to S^4$ such that $V = L \oplus L^\sharp$. The following statements are equivalent:*

1. *$f^\sharp$ is a Darboux transform of f.*
2. *$\nabla^\sharp$ is a flat connection on $L^\sharp$.*
3. *There exists a non–trivial section $\psi^\sharp \in \Gamma(\mathrm{pr}^* L^\sharp)$ with monodromy such that $d\psi^\sharp \in \Omega^1(\mathrm{pr}^* L)$.*

Note that
$$(\nabla^\sharp)'' = D = (\pi d)''$$
is the holomorphic structure (1.49) on V/L induced by the dual surface $f^\dagger$ of f, and is thus independent of $f^\sharp$. Moreover, parallel sections of $\nabla^\sharp$ give holomorphic sections of D.

Conversely, a holomorphic section $\varphi^\sharp \in H^0(\mathrm{pr}^* V/L)$ with monodromy has a unique lift $\psi^\sharp \in \mathrm{pr}^* V$ with the same monodromy and $\pi_L d\psi^\sharp = 0$: note that in the proof of Lemma 2.6 we only used the fact that
$$(\tilde\nabla\varphi^\sharp)'' = 0\,,$$
where $\tilde\nabla$ is the trivial connection on V/L induced by d and the splitting $V = L \oplus e\mathbb{H}$. But $(\tilde\varphi^\sharp)'' = 0$is satisfied in our situation since for any lift $\psi = \varphi^\sharp + \varphi_L^\sharp$ of a holomorphic section $\varphi^\sharp \in H^0(\mathrm{pr}^* V/L)$ we have
$$(\pi_L(d\psi))'' = D\varphi^\sharp = 0\,.$$
If this unique lift $\psi^\sharp$ of a holomorphic section $\varphi \in H^0(\mathrm{pr}^* V/L)$ is nowhere vanishing then $\psi^\sharp\mathbb{H} = L^\sharp$ defines a conformal map $f^\sharp$, and $f^\sharp$ is a Darboux transform of f. More general, we call the map $f^\sharp$ which is defined away from the zeros of $\psi^\sharp$ a *singular Darboux transform*.

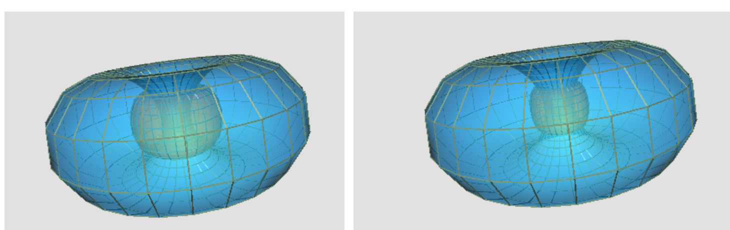


Figure 2.3: Darboux transforms of a homogeneous torus

Note that the holomorphicity conditions is equivalent to a Ricatti type equation: if $\varphi \in H^0(\mathrm{pr}^* V/L)$ is a holomorphic section with monodromy, and $f^\sharp$ a Darboux transform of f, then the Ricatti–type equation
$$\omega = -(\nabla B) + B\delta B \tag{2.16}$$
for $B \in \mathrm{Hom}(e\mathbb{H}, L)$ where $*\omega = J\omega$ has a solution since there exists a unique lift $\psi^\sharp \in \Gamma(\mathrm{pr}^* V)$ of φ with $\pi_L d\psi^\sharp = 0$. Conversely, if B is a solution of this equation for some ω

with $*\omega = J\omega$ then $\nabla + \delta B$ is a flat connection on V/L since

$$d^{\nabla}(\delta B) + \delta B \wedge \delta B = \delta \wedge \omega = 0\,.$$

Any parallel section $\varphi \in \Gamma(\mathrm{pr}^* V/L)$ of $\nabla + \delta B$ is holomorphic in V/L since $(\nabla + \delta B)'' = \nabla'' = D$, in other words,

$$(\varphi + B\varphi)\mathbb{H}$$

is a Darboux transform of f.

Remark 2.10. Note that a connection $d+\omega$ such that $L^\sharp$ is $d+\omega$ parallel and $\omega \in \Omega^1(\mathcal{R})$ is uniquely given by (2.16). Moreover, as we have seen in Section 2.1, f and $f^\sharp$ form a classical Darboux pair if and only if $d\omega = 0$.

As in the case of the Darboux transformation of isothermic surfaces, we have Bianchi permutability:

Theorem 2.11 ([BLPP]). *Given two Darboux transforms $f^\sharp$ and $f^\flat$ of a conformal map $f : M \to S^4$ then there is a common Darboux transform $\hat{f}$ of $f^\sharp$ and $f^\flat$.*

Proof. Let $\varphi^\sharp, \varphi^\flat \in H^0(\mathrm{pr}^* V/L)$ be the holomorphic sections with monodromy, such that $f^\sharp$ and $f^\flat$ are given by the unique lifts $\psi^\sharp, \psi^\flat \in \Gamma(\mathrm{pr}^* V)$ of $\varphi^\sharp$ and $\varphi^\flat$ respectively with $\pi_L(d\psi^\sharp) = \pi_L(d\psi^\flat) = 0$. In other words,

$$d\psi^\sharp = d\psi^\flat \rho\,,$$

where $\rho : \tilde{M} \to \mathbb{H}$. Define, as in the case of isothermic surfaces,

$$\hat{\psi} = \psi^\flat - \varphi^\sharp \rho\,,$$

then $d\hat{\psi} \in \Omega^1(\mathrm{pr}^* L^\sharp)$ so that

$$\hat{L} = \hat{\psi}\mathbb{H}$$

is a Darboux transform of $f^\sharp$. Similarly, $\hat{\psi}\rho^{-1}$ exhibits $\hat{L}$ as a Darboux transform of $f^\flat$. □

2.1.4 The Darboux transformation on constant mean curvature surfaces

It is a well–known fact that a constant mean curvature surface $f : M \to \mathbb{R}^3$ (without loss of generality with $H = -1$) is isothermic: Recalling (1.13) we see

$$(dN)' = -df \quad \text{and} \quad (dN)'' = dg\,,$$

where $g = f + N : M \to \mathrm{im}\,\mathbb{H}$. In particular,

$$df \wedge dg = dg \wedge df = 0$$

so that

$$\omega = \begin{pmatrix} 0 & dg \\ 0 & 0 \end{pmatrix} \in \Omega^1(\mathcal{R})$$

is a closed 1–form. We fix ω as the reference 1–form used in Section 2.1.

The Darboux transformation is even in this case a genuine generalization of the classical Darboux transformation: only for special parameters $x \in \Sigma$ of the spectral curve Σ of a constant mean curvature torus we obtain a sphere congruence enveloping both surfaces f and f^μ where $\mu = \mu(x)$.

Lemma 2.12. *Let $f : T^2 \to \mathbb{R}^3$ be a constant mean curvature torus in $\mathbb{R}^3$. Let*

$$\tilde{\nabla}^x = \tilde{\nabla} + \tilde{A}(a - 1 + \tilde{J}S^x b)$$

be the flat connection (2.13) given by a point $x \in \Sigma$ of the spectral curve Σ of f where $\mu = \mu(x)$ and $a = \frac{\mu+\mu^{-1}}{2}$ and $b = \frac{\mu^{-1}-\mu}{2}$. The Darboux transform f^μ given by ∇^x is a classical Darboux transform of f if and only $\mu \in \mathbb{R}$ or $\mu \in S^1$.

Proof. Let φ be a parallel section of $\tilde{\nabla}^x$, and ψ its unique lift to V such that $\pi_L(d\psi) = 0$. The Darboux transform of f given by ∇^x is given by $L^\mu = \psi\mathbb{H}$.

Writing $\psi = \varphi + B\varphi$ with $B \in \text{Hom}(e\mathbb{H}, L)$ and

$$\omega = \begin{pmatrix} 0 & -\nabla B + B\delta B \\ 0 & 0 \end{pmatrix}$$

then $d + \omega$ is the unique connection with $(d + \omega)\hat{\psi} = 0$ and $\omega \in \Omega^1(L)$. Recall (2.14) that

$$\delta B = \omega^x = \tilde{A}(a - 1 + \tilde{J}S^x b)\,.$$

Since the connection $\tilde{\nabla}^x = \tilde{\nabla} + \omega^x$ is flat, we have

$$0 = dw^x + \omega^x \wedge \omega^x = \delta \wedge (-\nabla B + B\delta B)\,,$$

so that

$$d^\nabla(-\nabla B + B\delta B) = (\nabla B - B\delta B) \wedge \delta B\,.$$

Because

$$d\omega = \begin{pmatrix} (-\nabla B + B\delta B) \wedge \delta & d^\nabla(-\nabla B + B\delta B) \\ 0 & \delta \wedge (-\nabla B + B\delta B) \end{pmatrix}$$

it remains with Remark 2.10 to show that

$$0 = (\nabla B - B\delta B) \wedge \delta$$

if and only if $\mu \in \mathbb{R}$ or $\mu \in S^1$.

We compute δ in the splitting $L \oplus e\mathbb{H} = V$. Let $R : e\mathbb{H} \to L$ as before be defined by $Re = \psi$. Then R^{-1} is parallel with respect to the induced connections ∇^L on L and $\tilde{\nabla}$

on V/L, and R^{-1} is antilinear with respect to the complex structures on L and V/L, i.e., $R^{-1}J = -\tilde{J}R^{-1}$. Since $\tilde{J}e = eN$ we have

$$-4 * \tilde{A}e = (\tilde{\nabla}\tilde{J} - \tilde{J} * \tilde{\nabla}\tilde{J})e = 2e(dN)' = -2edf = -2\delta\psi$$

so that

$$\tilde{A}(a - 1 + \tilde{J}S^x b) = \delta B = -2\tilde{A}\tilde{J}R^{-1}B\,,$$

and

$$R^{-1}B = \tilde{J}\frac{a-1}{2} - S^x\frac{b}{2}\,. \tag{2.17}$$

Since μ is ∇^x–parallel and $a^2 - b^2 = 1$, we get

$$\begin{aligned}
R^{-1}(\nabla B - B\delta B) &= \nabla(R^{-1}B) - R^{-1}B\delta B \\
&= (\tilde{\nabla}^x\tilde{J})\frac{a-1}{2} - [\delta B, R^{-1}B] - R^{-1}B\delta B \\
&= *\tilde{Q}(a-1) + *\tilde{A}\left(1 - a - \frac{1}{2}(a-1)^2 + \frac{1}{2}b^2\right) \\
&= *\tilde{Q}(a-1)\,.
\end{aligned}$$

But $*\tilde{Q}(a-1)\wedge\delta = 0$ if and only if a commutes with $\tilde{J}$. Since $a = \frac{\mu+\mu^{-1}}{2}$ this only holds for $a \in \mathbb{R}$ (since $\tilde{J} = \pm S^x$ is impossible as $\nabla^x\tilde{J} \neq 0$), so that $\mu \in \mathbb{R}$ or $\mu \in S^1$. □

Remark 2.13. Implicitly, the above computation also gives the Ricatti equation

$$\tilde{\nabla}\tilde{T} = \tilde{T} * \tilde{A}\tilde{T}(1-a) - 2 * \tilde{Q} \tag{2.18}$$

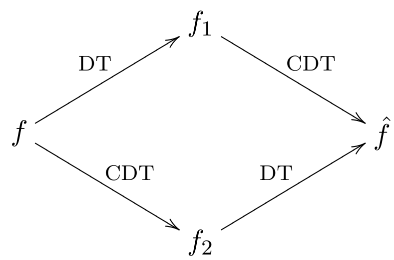
for $\tilde{T} = 2R^{-1}B(1-a)^{-1}$.

Though the Darboux transformation is in general not the classical Darboux transformation on constant mean curvature tori, the Darboux transforms of a constant mean curvature torus $f : T^2 \to \operatorname{im}\mathbb{H}$ are still isothermic:

Theorem 2.14. *Let $f : T^2 \to \mathbb{R}^3$ be a constant mean curvature torus with spectral curve Σ. For every $x \in \Sigma$ the corresponding Darboux transform $f^\mu : M \to S^4$ is isothermic.*

Proof. To show that the Darboux transform f^μ of f is isothermic it is enough to find a conformal map $\hat{f}$ so that $\hat{f}$ and f^μ form a classical Darboux pair. Choose $x_2 \in \Sigma$ such that $\mu(x_2) \in \mathbb{R}$ (or S^1), then the corresponding Darboux transform $f^{\mu(x_2)}$ is isothermic. Using Bianchi permutability there is a common Darboux transform $\hat{f}$ of f^μ and $f^{\mu(x_2)}$. We show that $\hat{f}$ and f^μ form a classical Darboux pair (whereas in general $\hat{f}$ and $f^{\mu(x_2)}$

do not). For simplicity of notation we abbreviate $f_1 = f^\mu$ and $f_2 = f^{\mu(x_2)}$.



Let $\varphi_i \in H^0(\mathrm{pr}^* V/L)$ be the parallel section of

$$\nabla^i = \tilde{\nabla} + \tilde{A}(a_i - 1 + \tilde{J}S^i b_o)$$

and $\psi_i = \varphi_i + B_i\varphi_i \in \Omega^1(L_i)$ the unique lift of φ_i with $\pi_L d\psi_i = 0$. We write ψ_i in coordinates

$$\psi_i = \begin{pmatrix} \alpha_i + f\beta_i \\ \beta_i \end{pmatrix},$$

where $d\alpha_i = -df\beta_i$. Moreover, we denote as before by $f_i = f + T_i$ with $T_i = \alpha_i\beta_i^{-1}$.

Since

$$B_i e = \begin{pmatrix} f \\ 1 \end{pmatrix} T_i^{-1},$$

and

$$2 * \tilde{Q}(1 - a_i)e = edg\lambda_i\,,$$

where $\lambda_i = \frac{1-a_i}{2}$ in the trivialization $V/L = e\mathbb{H}$ and $g = f + N$, the Ricatti type equation (2.16) reads in coordinates as

$$d(T^{-1}) = T^{-1}dfT^{-1} - dg\lambda \tag{2.19}$$

On the other hand, the Ricatti equation (2.18) gives

$$d((\lambda T)^{-1}) = (\lambda T)^{-1}dfT^{-1} - dg\,. \tag{2.20}$$

Note that the above two equations are equivalent if $\lambda \in \mathbb{R}$ but in general, i.e., for a conformal map $f : M \to \mathbb{HP}^1$, the Ricatti type equation (2.16) does not imply (2.18).

The common Darboux transform $\hat{f}$ is given by

$$\hat{\psi} = \begin{pmatrix} \alpha + \hat{f}_1\beta \\ \beta \end{pmatrix}$$

where

$$\beta = ((\lambda_2 T_2)^{-1} - (\lambda_1 T_1)^{-1})\lambda_2\alpha_2$$

and

$$\alpha = (T_2 - T_1)\beta_2 .$$

Examining the computations in the proof of the Bianchi permutability Theorem 2.4 for isothermic surfaces one verifies that we only used $\lambda_2 \in \mathbb{R}$, $df \wedge dg = dg \wedge df = 0$, and the equations (2.19) and (2.20). Therefore, the same computation as before gives $d(d\beta\alpha^{-1}) = 0$, so that $\hat{f}$ and f_1 form a Darboux pair by Corollary 2.2. □

2.1.5 The spectral curve of a conformal torus

Figure 2.4: Spectral curve of a homogeneous torus

Motivated by the definition of the spectral curve in case of a constant mean curvature torus, we expect the spectral curve of a conformal torus $f : T^2 \to S^4$ to basically be the set of all (singular) Darboux transforms of f. To turn this set into a Riemann surface [BLPP], we have to assume that f has trivial normal bundle. As we have seen, Darboux transforms of f are given by holomorphic setions $\varphi \in H^o(\mathrm{pr}^* V/L)$ with monodromy. If

$$\gamma^*\varphi = \varphi h_\gamma$$

where $h : \Gamma \to \mathbb{H}_*$ is a representation of the fundamental group Γ of $T^2 = \mathbb{C}/\Gamma$ then the section $\tilde{\varphi} = \varphi\lambda, \lambda \in \mathbb{H}_*$, has monodromy

$$\gamma^*(\tilde{\varphi}) = \tilde{\varphi}\lambda^{-1}h_\gamma\lambda .$$

In partiuclar, to represent a Darboux transform of f, we can choose $\varphi \in H^0(\mathrm{pr}^* V/L)$ such that $h_\gamma \in \mathbb{C}_*$ where $\mathbb{C} = \mathrm{Span}\{1, i\}$. We define

$$\mathrm{Spec}(D) = \{h : \Gamma \to \mathbb{C}_* \mid \text{ there exists } \varphi \in H^0(\mathrm{pr}^* V/L) \text{ with } \gamma^*\varphi = \varphi h_\gamma \text{ for } \gamma \in \Gamma\}/\mathbb{C}_* .$$

Note that $\gamma^*(\varphi j) = \varphi j \bar{h}_\gamma$ so that we have an involution $\tau(h) = \bar{h}$ on $\mathrm{Spec}(D)$. It can be shown [BLPP] that $\mathrm{Spec}(D)$ defines a Riemann surface, the so–called *spectral curve* Σ of f, of possibly infinite genus. In case of a constant mean curvature torus $f : M \to \mathbb{R}^3$ it coincides [CLP] with the spectral curve of [Hit90].

Figure 2.5: Spectral curve of a homogeneous torus, with original and Darboux transformed torus, with Darboux transform

The space of holomorphic sections of $\mathrm{pr}^* V/L$ with a given monodromy is generically 1–dimensional, and defines a complex holomorphic line bundle $\mathcal{L} \to \Sigma$ over the spectral curve which moves linearly in the Jacobian of Σ tangent to its Abel image due to Bianchi permutability [BLPP]. On the other hand, holomorphic sections $\varphi^\sharp$ give singular Darboux transforms $f^\sharp$ of f. Thus, if we fix a point $p \in T^2$ on the torus and evaluate $f^\sharp(p)$ for all points on the spectral curve we obtain a realization of the abstract Riemann surface Σ in S^4.

If Σ is of *finite type* then the original surface gives a marked point $\infty \in \Sigma$. Tracing the marked point on the spectral curve under the flow of the holomorphic line bundle $\mathcal{L}$ parametrizes the conformal torus f. In particular, we get a recipe to construct finite type conformal immersions of 2–tori via theta functions on the spectral curve [BLPP].

2.1.6 The Darboux transformation on holomorphic curves

Similarly to the case of conformal maps $f : M \to S^4$ we can define a Darboux transformation on holomorphic curves by giving holomorphic sections with monodromy.

Let $f : M \to \mathbb{HP}^n$ be an unramified Frenet curve. Its dual curve $f^\dagger$ defines by Kodaira correspondence a holomorphic structure D on $V/V_{n-1} = (L^\dagger)^{-1}$. The holomorphic jet complex [FLPP01, Thm. 4.3] of $(L^\dagger)^{-1}$ is given by

$$\mathcal{L}_k = V/V_{n-1-k}\,,$$

with projections $\pi_k : V/V_{n-1-k} \to V/V_{n-k}$ and $\mathcal{N}_k = \ker \pi_k = V_{n-1-k}/V_{n-2-k}$. In particular, for a holomorphic section $\varphi \in H^0(\mathrm{pr}^* V/V_{n-1})$ with monodromy there is a unique section $\hat\psi \in \Gamma(V)$, the so–called n^{th} *prolongation* of φ, such that for the successively defined sections $\varphi_{i+1} = \pi_i \varphi_i$, $\varphi_0 = \hat\psi$, the following conditions hold [FLPP01, Corollary 3.2]:

$$\pi_i d\varphi_i = 0\,, \quad \text{and} \quad \varphi_n = \varphi\,.$$

In particular, $\hat\psi$ has the same (multiplicative) monodromy as φ and we can define a map $\hat f : M \to \mathbb{HP}^n$ by

$$\hat L = \hat\psi \mathbb{H}\,.$$

We call $\hat f$ a *Darboux transform* of f.

If $f : M \to S^4$ is a conformal map such that $\dim H^0(L^{-1}) = n+1 > 2$ then the Kodaira embedding of $L \subset (H^0(L^{-1}))^*$ gives a holomorphic curve $\check f : M \to \mathbb{HP}^n$. The following diagram

$$\begin{array}{ccc} \check{f}: M \to \mathbb{HP}^n & \xrightarrow{\text{DT}} & \widehat{\check{f}}: M \to \mathbb{HP}^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ f: M \to \mathbb{HP}^1 & \xrightarrow{\text{DT}} & \hat{f}: M \to \mathbb{HP}^1 \end{array}$$

is commutative that is the projection of a Darboux transform $\widehat{\check{f}}$ of $\check{f}$ is a Darboux transform of the projection f of $\check{f}$. Little more is known about the Darboux transformation on holomorphic curves in $\mathbb{HP}^n$, in particular it is not yet understood how the existence of many holomorphic sections of a conformal torus $f : T^2 \to S^4$ affect the spectral curve of f.

2.2 The Bäcklund transformation

A Bäcklund transformation is defined by solving Abelian Integrals [LP05], [BP]: The holomorphic structure on KL is defined such that holomorphic sections in KL are exactly the closed 1–forms of L. Integrating such a closed 1–forms $\omega \in H^0(KL)$ on the universal cover $\mathrm{pr} : \tilde{M} \to M$ of M, we get sections $\varphi \in \Gamma(\mathrm{pr}^* L)$ with $d\varphi = \omega$. The line bundle $\mathrm{pr}^* L$ can be equipped with a holomorphic structure D such that the φ's are holomorphic, and span a linear system $H \subset H^0(L)$. The Kodaira embedding of the line bundle $L^{-1} \subset H^*$ defines a holomorphic curve in $\mathbb{HP}^k$ when $\dim H = k+1 \geq 2$. The Riemann Roch theorem gives control over the dimension of the space of holomorphic sections of KL, and thus of $\dim H^0(L)$.

More general, the choice of a hyperplane at infinity equips the line bundle L with a holomorphic structure (1.51). Any linear system $\hat{H} \subset H^0(L)$ of dimension at least 2 gives by the Kodaira embedding of L^{-1} in $\hat{H}^*$ a holomorphic curve, a *Bäcklund transform* of f.

Geometrically, the Bäcklund transformation is the quaternionic analog of a well–known transformation in algebraic curve theory: given a holomorphic curve in $h : M \to \mathbb{CP}^n$ the intersection of the tangent of h with a fixed hyperplane gives a holomorphic curve in $\tilde{h} : M \to \mathbb{CP}^{n-1}$, the *osculate* of h. Conversely, prescribing the tangents of $\tilde{h} : M \to \mathbb{CP}^{n-1}$ one can define an *envelope* $h : M \to \mathbb{CP}^n$ such that $\tilde{h}$ is an osculate of h [LP03]. This geometric construction is closely related to the Bäcklund transformation: a Bäcklund transform of f turns out to be a projection of an envelope of f.

Though a 1-step Bäcklund transform is in general only defined on the universal cover $\tilde{M}$ of M, we show that the $(n+1)$–step Bäcklund transform given by holomorphic $\omega_i \in H^0(KL)$ is given globally by differentiation and algebraic operations.

2.2.1 The Bäcklund transformation on holomorphic curves

As we have seen in (1.51) the choice of a hyperplane at infinity $\alpha \in (\mathbb{H}^{n+1})^*$ not intersecting the holomorphic curve $f : M \to \mathbb{HP}^n$ induces a holomorphic structure on L: if $V = L \oplus \ker \alpha$ then the trivial connection d on L induces (1.50) a trivial connection ∇^L on L and (1.51)

$$D = (\nabla^L)''$$

is the induced holomorphic structure on L. If $\hat{H} \subset H^0(L)$ is a basepoint free linear system of dimension at least 2, then the Kodaira embedding of $L^{-1} \subset \hat{H}^{-1}$ gives a holomorphic curve $\hat{f} : M \to \mathbb{P}(\hat{H}^*)$. To obtain a Bianchi permutability theorem and a geometric interpretation of the Bäcklund transform we define, in contrast to [LP05], the Bäcklund transform as the dual curve of $\hat{f}$.

Definition 2.15. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve and $\alpha \in V^*$ such that $V = L \oplus \ker \alpha$. If $\hat{H} \subset H^0(L)$ is a basepoint free linear system of dimension at least 2 such that $\hat{f}$ has a Frenet flag, then the dual curve*

$$B_{\alpha,\hat{H}}(f) = \hat{f}^\dagger : M \to \mathbb{P}(\hat{H}),$$

of $\hat{f}$ is called the forward Bäcklund transform of f with parameters α and $\hat{H}$*. If $\hat{H} = H^0(L)$ is the complete linear system, we call $B_\alpha(f)$ the* forward Bäcklund transform of f with parameter α*.*

We will discuss the inverse transformation, the backward Bäcklund transformation, in Section 2.2.5. In the following, the adjective *forward* will be dropped until we return to the topic of the backward transformation.

Remark 2.16. The assumption that $\hat{f}$ has a Frenet flag is of rather technical nature. For example, if

$$\hat{H} = <\psi, -\psi g> \subset H^0(L)$$

is a 2–dimensional linear system then $\hat{f} : M \to \mathbb{HP}^1$ is given by

$$\hat{L} = \begin{pmatrix} -\bar{g} \\ 1 \end{pmatrix} \mathbb{H}$$

and its dual curve is

$$\tilde{L} = \hat{L}^\dagger = \begin{pmatrix} g \\ 1 \end{pmatrix} \mathbb{H}.$$

Since $\hat{L} = L^{-1}$ is a holomorphic curve in $\hat{H}^*$, we see that

$$*\tilde{\delta} = \tilde{J}\tilde{\delta},$$

where $\tilde{J}$ is the induced complex structure on $V^*/\tilde{L} = \hat{L}^{-1}$. Thus, $\tilde{f} : M \to \mathbb{HP}^1$ is a conformal map. We call $\tilde{f}$ a (generalized) Bäcklund transform. The requirement that $\hat{L}$

has a Frenet flag turns $\tilde{f}$ into a holomorphic curve, i.e., there exists a complex structure J on $\tilde{L}$ such that

$$*\tilde{\delta} = \tilde{\delta} J \, .$$

More generally, if $\dim H \geq 2$ then we can define the dual curve of $\hat{f}$ away from the Weierstraß points. Since the Frenet flag extends at least continuously across the Weierstraß points, we obtain a continuous map $\tilde{f} : M \to P(H^*)$ which is a holomorphic curve away from the Weierstraß points.

To estimate the dimension $h^0(L)$ of the space of holomorphic sections of L, in the case when M is compact of genus g, recall the Riemman-Roch Theorem (1.53)

$$h^0(L) - h^0(KL^{-1}) = \deg L - g + 1 \, . \tag{2.21}$$

From (1.55) we know that the holomorphic structure on KL^{-1} is given by the exterior derivative $d^{\nabla^{L^{-1}}}$ with respect to the dual connection $\nabla^{L^{-1}}$ on L^{-1}. Since $\alpha|_L \in H^0(L^{-1})$ is parallel with respect to $\nabla^{L^{-1}}$ the linear map

$$H^0(L^{-1}) \to H^0(KL^{-1}) : \beta \mapsto \nabla^{L^{-1}} \beta$$

has kernel spanned by $\alpha|_L$ and we obtain

$$h^0(KL^{-1}) = h^0(L^{-1}) - 1 \geq n \, .$$

Applying the Riemann–Roch relation (1.53) we get [LP05] that the dimension of the space of holomorphic sections of L is given by

$$h^0(L) = h^0(L^{-1}) + \deg L - g \, .$$

In particular, if the degree of the line bundle $L = f^*\mathcal{T}$ satisfies

$$\deg L \geq 1 + g - n \, ,$$

then the complete linear system $H^0(L)$ is at least 2–dimensional.

2.2.2 Construction of Bäcklund transforms from Abelian integrals

We now will explain how one can use Abelian integrals to construct linear systems $\hat{H} \subset H^0(L)$ see [LP05], [BP]. The holomorphic structure on L is induced by the splitting $V = L \oplus \ker \alpha$ and is given (1.50), (1.51) by the $(0,1)$–part of the flat connection ∇^L on L. On the other hand, the bundle KL has a canonical holomorphic structure entirely determined by the holomorphic curve $f : M \to \mathbb{HP}^n$ expressed by (1.55). Therefore, the integrals of holomorphic sections $\omega \in H^0(KL)$ give holomorphic sections $\varphi \in H^0(L)$ — at least on the universal cover $\mathrm{pr} : \tilde{M} \to M$. Since the section $\psi \in H^0(L)$ with $< \alpha, \psi >= 1$ has no zeros any linear system $\tilde{H} \subset H^0(\mathrm{pr}^* L)$ containing ψ is basepoint free.

Given a linear system $H_{KL} \subset H^0(KL)$ the linear system $\hat{H} \subset H^0(\mathrm{pr}^* L)$ obtained by integrating sections in H_{KL} and including the section ψ with $< \alpha, \psi >= 1$ is called the *linear system obtained by integration* of H_{KL}. Since we have included $\psi \in H^0(L)$, which appears as the constant of integration, in the linear system $\hat{H}$ this procedure is well–defined. Moreover, because $\psi \in \hat{H}$ has no zeros, the linear system $\hat{H}$ is basepoint free. To calculate the dimension of $\hat{H}$ in case M is compact and $H_{KL} = H^0(KL)$ is the complete linear system, we use the Riemann–Roch theorem (1.53) applied to L^{-1}:

$$h^0(KL) = h^0(L^{-1}) + \deg L + g - 1$$

and therefore

$$\dim \hat{H} = 1 + h^0(KL) = h^0(L^{-1}) + \deg L + g\,.$$

Lemma 2.17 ([LP05])**.** *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve. Then, for any choice of hyperplane $\alpha \in V^*$ not intersecting f, the linear system $\hat{H} \subset H^0(\mathrm{pr}^* L)$ obtained by integration of $H^0(KL)$ is basepoint free and has dimension*

$$\dim \hat{H} = h^0(L^{-1}) + \deg L + g\,.$$

Assuming that $m + 1 = \dim \hat{H} \geq 2$, we obtain a Bäcklund transform on the universal cover $\tilde{M}$ of M.

If we only are concerned about local surface theory then spaces of holomorphic sections are infinite dimensional, Abelian integrals have no periods, and we always obtain Bäcklund transforms by integrating sections in $H^0(KL)$.

In the case of compact surfaces, genus 0 is exceptional since there are no periods to close. Moreover, for a holomorphic sphere $f : S^2 \to \mathbb{HP}^n$ of

$$\deg L \geq -n + 1$$

the previous Lemma implies that $m + 1 = \dim \hat{H} \geq 2$ since $h^0(L^{-1}) \geq n + 1$. Thus we always have Bäcklund transforms $\tilde{f} : S^2 \to \mathbb{HP}^m$. In [Pet04], [BP] the Bäcklund transformation is used to construct soliton spheres.

For surfaces $f : M \to \mathbb{HP}^n$ with genus of M $g \geq 1$, we have to close the periods of the Bäcklund transform. In case, of a torus this can be done if we allow discrete points on M where the conformality of $\tilde{f}$ fails, see [LP05].

Figure 2.6: 1–step Bäcklund transform of a Willmore sphere in S^4, [Pet04],[Hel02]

For simplicity of notation we denote by $B_{\alpha,\omega} : \tilde{M} \to \mathbb{HP}^1$ the (generalized) Bäcklund transform of $f : M \to \mathbb{HP}^n$ with respect to α and the linear system $H_L \subset H^0(L)$ obtained by integration of $H_{KL} = \mathrm{Span}\{\omega\}$.

Definition 2.18. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve and let*

$$\tilde{f} = B_{\alpha,\omega}(f) : \tilde{M} \to \mathbb{HP}^1$$

be the Bäcklund transform with respect to the nowhere vanishing $\omega \in H^0(KL)$. Let $\tilde{L} = \tilde{f}^*\mathcal{T}$ and $\tilde{H} = H_L^* \subset H^0(\tilde{L}^{-1})$ the corresponding basepoint free linear system given by the Kodaira correspondence.

The Kodaira embedding of $\tilde{L}$ into any $(k+1)$–dimensional linear system

$$\check{H} \subset H^0(\tilde{L}^{-1}) \quad \text{with} \quad \tilde{H} \subset \check{H}\,, k \geq 1\,,$$

is called a 1–step Bäcklund transform *of f. In particular, $B_{\alpha,\omega}(f)$ is a 1–step Bäcklund transform of f.*

2.2.3 Envelopes and Osculates

In this section we will give a geometric interpretation of the Bäcklund transformations. Let us first recall the construction of the tangent curves of a holomorphic curve in $\mathbb{CP}^n$: The successive higher derivatives of a holomorphic curve in $\mathbb{CP}^n$ form a holomorphic flag, the Frenet flag. The intersection of the k^{th} osculating flag with a complementary $\mathbb{CP}^{n-k}$ gives [GH94, Ch. 2.4] a new holomorphic curve in $\mathbb{CP}^{n-k}$. The analogous construction for a holomorphic curve in $\mathbb{HP}^n$ also requires the existence of a smooth osculating flag.

Definition 2.19. *Given a holomorphic curve $f : M \to \mathbb{HP}^n$ with Frenet flag, we get a holomorphic curve $f_+ : M \to \mathbb{HP}^{n-1}$, the* tangent curve *or* (first) osculate *of f, by intersecting the first flag space V_1 of the Frenet flag of f with a $\mathbb{HP}^{n-1} \subset \mathbb{HP}^n$.*

Conversely, $f_- : M \to \mathbb{HP}^{n+1}$ is called an envelope *of $f : M \to \mathbb{HP}^n$ if f is a tangent curve of f_-, i.e.,*

$$(f_-)_+ = f\,.$$

Remark 2.20. The tangent construction preserves holomorphic curves with Frenet flag, and Frenet curves [LP03].

Figure 2.7: Osculate of a Willmore torus in $\mathbb{HP}^2$, [Hel02]

For a Frenet curve $f : M \to \mathbb{HP}^n$ with corresponding line bundle $L \subset V$ any choice of a nowhere vanishing holomorphic section $\omega \in H^0(KL)$ gives an envelope: if $\varphi \in \Gamma(V)$ satisfies $\nabla\varphi = \omega$ then $\psi_- = \varphi \oplus 1$ is a nowhere vanishing section of the trivial $\mathbb{H}^{n+1}$–bundle $V_- = \text{pr}^*(V) \oplus \underline{\mathbb{H}}$ over the universal cover $\tilde{M}$ of M, and defines the quaternionic line bundle $L_- = \psi_-\mathbb{H} \subset V_-$. For $\alpha_- \in V_-^*$ nowhere vanishing and $\ker\alpha_- = V$ we see that

$$L = \ker\alpha_- \cap (L_- \oplus \operatorname{im}\delta_-) = \ker\alpha_- \cap V_1 \subset V\,, \tag{2.22}$$

since $\delta_-\psi_- = \omega \in H^0(KL)$. Moreover, ω defines $N \in \operatorname{im}\mathbb{H}$ by $*\omega = \omega N$ and gives a complex structure J on L_- by quaternionic linear extension of

$$J\psi_- = \psi_- N\,.$$

In particular, L_- is a holomorphic curve

$$f_- : \tilde{M} \to \mathbb{HP}^{n+1}\,,$$

and due to (2.22) an envelope of f. Moreover, f_- has a smooth Frenet flag with $(V_-)_0 = L_-$ and

$$(V_-)_k = L_- \oplus V_{k-1} \quad \text{for} \quad k > 0\,.$$

Lemma 2.21 ([LP03]). *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag and let $\omega \in H^0(KL)$ be nowhere vanishing. Then the envelope f_- given by ω is a holomorphic curve with Frenet flag. If f is a Frenet curve so is f_-.*

Proof. It remains to show that f_- is Frenet if f is Frenet. Since $\omega = \delta_-\psi_-$ is nowhere vanishing we can smoothly define $B \in \operatorname{Hom}(V, L_-)$ by

$$\delta_- B = 2 * A$$

where A is the Hopf field of the canonical complex structure S of f. Define the complex structure S_- on $V_- = L_- \oplus V$ by

$$S_- = \begin{pmatrix} J & B \\ 0 & S \end{pmatrix}$$

then S_- is the canonical complex structure of f_- by (1.35). □

Definition 2.22. *The k^{th}* osculate *$f_k : M \to \mathbb{HP}^{n-k}$ is inductively defined to be the first osculate*

$$f_k = (f_{k-1})_+$$

of the $k-1^{st}$ osculate $f_{k-1} : M \to \mathbb{HP}^{n-k+1}$. Similarly, the k^{th} envelope *$f_{-k} : \tilde{M} \to \mathbb{HP}^{n+k}$ is defined inductively.*

In other words, if the linear system of f is given by $H = \{\alpha, \alpha_1, \dots, \alpha_{n-1}, \beta\}$ with $\alpha_k|_{L_k} \neq 0$ then the k^{th} osculate is given by

$$L_k = V_k \cap \ker\alpha_{k-1}\,,$$

where $L \subset V_1 \subset \ldots \subset V_n$ is the Frenet flag of f.

For

$$\check{H} = \{\alpha, \dots, \alpha_{n-1}\} \overset{\text{incl}}{\hookrightarrow} H \subset H^0(L^{-1})$$

denote by $\pi = \text{incl}^* : V \to \check{V}$ the induced projection, where $V = H^*$ and $\check{V} = \check{H}^*$. We can identify $\check{V} = V_{n-1}$ via the splitting $V_{n-1} \oplus L_n = V$. The trivial connection $\check{d}$ on $\check{V}$ is given by the trivial connection d on V via the splitting, i.e.,

$$d = \begin{pmatrix} \check{d} & 0 \\ \delta_{n-1} & \tilde{\nabla} \end{pmatrix}.$$

Since $\alpha \in \check{H}$, the linear system $\check{H}$ is basepoint free. If we denote by $\check{L}$ the Kodaira embedding of $L \subset \check{V}$, then the flag spaces of $\check{f}$ are given by

$$\check{V}_j = V_j, \quad j \leq n-1,$$

where we again use the splitting $V_{n-1} \oplus L_n = V$ to identify $\pi(V_j)$ with V_j. In particular, the osculates of $\check{f}$ are given by $\check{L}_j = \pi(L_j) = L_j$, so that the diagram

$$\begin{array}{ccccccccc} f & \longrightarrow & f_1 & \longrightarrow & \cdots & \longrightarrow & f_{n-1} & \longrightarrow & f_n \\ \downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & \\ \check{f} & \longrightarrow & \check{f}_1 & \longrightarrow & \cdots & \longrightarrow & \check{f}_{n-1} & & \end{array}$$

commutes. Moreover, if $f : M \to \mathbb{HP}^n$ is a Frenet curve with canonical complex structure given in the splitting $V = V_{n-1} \oplus L_n$ by

$$S = \begin{pmatrix} \check{S} & B \\ 0 & \tilde{J} \end{pmatrix}$$

then $\check{f}$ is a Frenet curve with canonical complex structure $\check{S}$: we compute the Hopf field A of S in this splitting

$$A = \begin{pmatrix} \check{A} & \frac{1}{2}(*\nabla B)' + \frac{1}{4}(B\delta B + B\tilde{\nabla}\tilde{J}) \\ 0 & 0 \end{pmatrix}.$$

Since $\operatorname{im} A \subset L$, we see that $\operatorname{im} \check{A} \subset \check{L} = L$, and $\check{S}$ is the canonical complex structure of $\check{f}$. We summarize:

Lemma 2.23. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag. Then the linear system $\check{H}$ is basepoint free. Moreover, the Kodaira embedding of L into $\check{V} = (\check{H})^*$ gives a holomorphic curve $\check{f} : M \to \mathbb{HP}^{n-1}$ with Frenet flag*

$$\check{V}_j = \pi V_j, \quad j \leq n-1,$$

and j^{th}–osculate

$$\check{L}_j = \pi(L_j).$$

Moreover, if $f : M \to \mathbb{HP}^n$ is a Frenet curve so is the projection $\check{f} : M \to \mathbb{HP}^{n-1}$.

The Bäcklund transformation is given in by the two transformations above: a Bäcklund transform is a projection of an envelope of f.

Theorem 2.24 ([LP05]). *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag and let $\omega \in H^0(KL)$ be a holomorphic 1–form without zeros. Moreover, let $\alpha \in (\mathbb{H}^{n+1})^*$ be a hyperplane at infinity not intersecting f.*

Then the envelope $f_- : \tilde{M} \to \mathbb{HP}^{n+1}$ of f given by

$$L_- = \psi_- \mathbb{H} \subset \underline{\mathbb{H}}^{n+2}$$

with $\nabla \psi_- = \omega$ projects onto the 1–step Bäcklund transform $B_{\alpha,\omega}(f)$. In particular, the 1–step Bäcklund transform $B_{\alpha,\omega}(f)$ is a holomorphic curve with Frenet flag. If f is a Frenet curve, so is $B_{\alpha,\omega}(f)$.

Proof. Let $\psi \in H^0(L)$ be the nowhere vanishing holomorphic section with $< \alpha, \psi >= 1$. By Lemma 1.36 we can write

$$\omega = \psi dg$$

with $g : \tilde{M} \to \mathbb{H}$ and

$$H = \text{Span}\{\psi, -\psi g\} \subset H^0(\text{pr}^* L)$$

is the linear system obtained by integration of $H_{KL} = \text{Span}\{\omega\}$. The Bäcklund transform $B_{\alpha,\omega}(f)$ is therefore given by

$$B_{\alpha,\omega}(f) = [g, 1] \, .$$

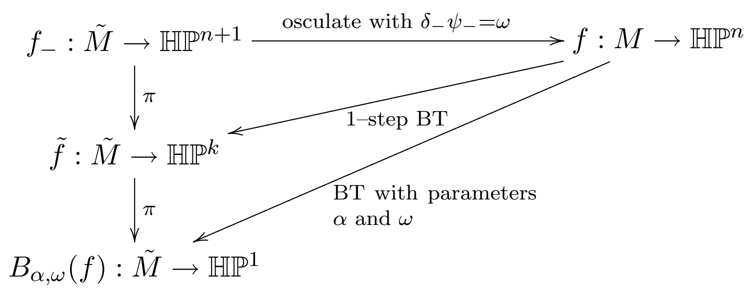
On the other hand, the Kodaira embedding of $L_- \subset \check{H}^*$ with respect to the linear system $\check{H} = \text{Span}\{\alpha_-, \alpha\} \subset H_- \subset H^0(L_-^{-1})$, is also given by $[g, 1]$ since

$$d < \alpha, \psi_- >=< \alpha, \nabla\psi_- >=< \alpha, \omega >= dg$$

and $< \alpha_-, \psi_- >= 1$. The remaining statements follow from Lemma 2.21 and Lemma 2.23. □

Corollary 2.25. *Let $f : M \to \mathbb{HP}^n$ a holomorphic curve with Frenet flag and $\omega \in H^0(KL)$ without zeros. Let $f_- : \tilde{M} \to \mathbb{HP}^{n+1}$ be the envelope of f with respect to ω, and $\hat{f} = B_{\alpha,\omega}(f)$ the Bäcklund transform of f with respect to α and ω.*

Any 1–step Bäcklund transform $\tilde{f}$ of f with linear system $\tilde{H} \subset H^0(L_-)$ with $\hat{H} \subset \tilde{H} \subset H_-$ is a projection of the envelope f_- and projects onto the Bäcklund transform $B_{\alpha,\omega}(f)$. In particular, such a 1–step Bäcklund transform is a holomorphic curve with Frenet flag. If f is a Frenet curve, then the 1–step Bäcklund transform is Frenet, too.



2.2.4 The (n+1)–step Bäcklund transformation

The 1–step Bäcklund transform of a holomorphic curve $f : M \to \mathbb{HP}^n$ is defined by Abelian integrals, in particular it is only given on the universal cover $\tilde{M}$ of M. However, we will show that after $n+1$ successive 1–step Bäcklund transforms, the resulting holomorphic curve, the so–called $(n+1)$–step Bäcklund transform, is globally defined.

Given a holomorphic curve $f_i^j : M \to \mathbb{HP}^n$, we denote an osculate of f_i^j by

$$f_{i+1}^j : M \to \mathbb{HP}^{n-1}\,,$$

a 1–step Bäcklund transform of f_i^j by

$$f_i^{j+1} : M \to \mathbb{HP}^n\,,$$

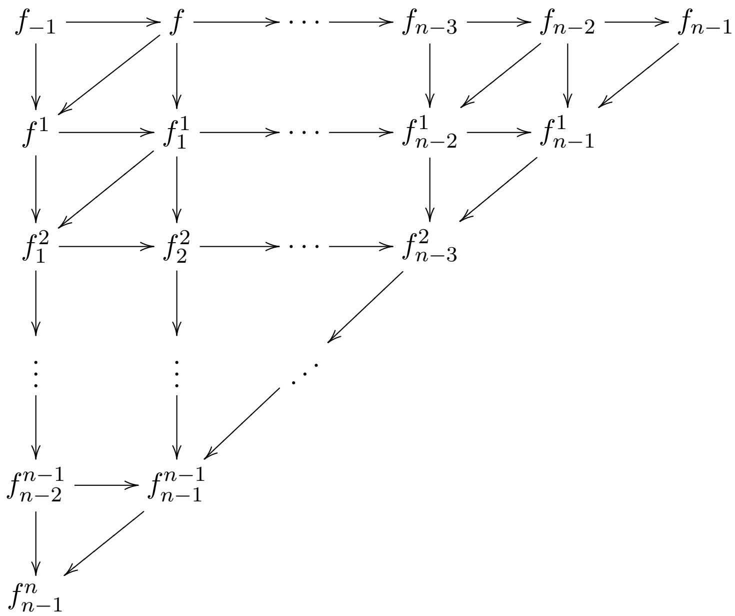
and a projection of f_i^j by

$$f_{i+1}^{j+1} : M \to \mathbb{HP}^{n-1}\,.$$

We omit the index 0, i.e.,

$$f^j := f_0^j\,, \quad f_j := f_j^0\,.$$

We first prove the commutativity of the following diagram



(where $\to$ denotes the osculating construction, $\downarrow$ a projection, and $\swarrow$ a 1–step Bäcklund transformation).

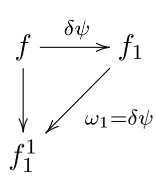
Lemma 2.26. *The 1–step Bäcklund transform $B_{\alpha,\omega}(f) : \tilde{M} \to \mathbb{HP}^1$ of a holomorphic curve $f : M \to \mathbb{HP}^n$, $n \geq 2$, is given by a n–step Bäcklund transform of the $(n-1)^{st}$–osculate f_{n-1} of f.*

Proof. In the case $n = 2$, the first osculate with respect to $\alpha \in (\mathbb{H}^3)^*$ with $L \oplus \ker\alpha = V$, is a map $f_1 : M \to \mathbb{HP}^1$. Denote by $\psi \in \Gamma(L)$ the section with $< \alpha, \psi >= 1$.

Choose $\alpha_1 \in (\mathbb{H}^2)^*$ such that $L_1 \oplus \ker\alpha_1 = \underline{\mathbb{H}}^2$, and let $\psi_1 \in \Gamma(L_1)$ be the section with $< \alpha_1, \psi_1 >= 1$. The derivative of L satisfies

$$\delta\psi = \psi_1 d < \alpha_1, \psi >,$$

and therefore $\omega_1 = \delta\psi \in H^0(KL_1)$, see Lemma 1.36.



By Theorem 2.24 the 1–step Bäcklund transform of f_1 with respect to ω_1 and α_1 is given by the projection f_1^1 of f since f is an envelope of f_1 with respect to ω_1.

Let now $n > 2$, and let L_j be the j^{th}–osculate of f and f_{-1} the envelope of f with respect to ω. Let

$$\alpha_{-1}, \alpha, \alpha_1, \dots, \alpha_{n-1}, \beta \in (\mathbb{H}^{n+2})$$

such that

$$H_- = \operatorname{Span}\{\alpha_{-1}, \alpha, \alpha_1, \dots, \alpha_{n-1}, \beta\}$$

is the basepoint free linear system of f_{-1} and

$$\alpha_j|_{L_j} \neq 0 \,.$$

The linear system

$$H^1 = \operatorname{Span}\{\alpha_{-1}, \alpha, \alpha_1, \dots, \alpha_{n-1}\}$$

induces a projection $\pi : V_{-1} \to V^1$, where $V_{-1} = H_{-1}^*$ and $V^1 = (H^1)^*$, so that

$$f^1 = \pi(f_{-1})$$

is a 1–step Bäcklund transform of f by Corollary 2.25. By Lemma 2.23 f^1 has j^{th}–osculate f^1_j where each f^1_j is the projection of f_{j-1}.

Applying the above argument for $n = 2$, the map f^1_{n-1} is the 1–step Bäcklund transform of f_{n-1} with respect to the holomorphic 1–form

$$\omega_{n-1} = \delta_{n-2}\psi_{n-2} \,.$$

Proceeding inductively, f^k_{n-1} is a Bäcklund transform of f^{k-1}_{n-1} and, by construction, also the 1–step Bäcklund transform of f_{n-k} with parameters α_k and $\omega_k = \delta_k\psi_k$. □

Theorem 2.27. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag, and let $\omega \in \Omega^1(\operatorname{End}(V))$ be a nowhere vanishing 1-form with $d\omega = 0, \operatorname{im}\omega = L$, and $*\omega = J\omega$. Moreover, assume that $\hat{L} = (\ker\omega)^\perp$ is a holomorphic curve with Frenet flag in $\mathbb{HP}^n$. Then the dual curve $\tilde{f} : M \to \mathbb{HP}^n$ of $\hat{f}$ is a $(n+1)$–step Bäcklund transform of f. In particular, the $(n+1)$–step Bäcklund transform is globally defined.*

Remark 2.28. The assumptions of the theorem can be relaxed. If $\ker\omega$ does not contain a constant subbundle then $\hat{L}$ is a holomorphic cure in $\mathbb{HP}^n$ see Example 1.18, and similar arguments as below can be used away from the Weierstraß points of $\hat{f}$ to show that the $(n+1)$–step Bäcklund transform is defined on M without isolated points. Moreover, the assumption that $\hat{f}$ is a full curve guarantees that the successive 1–step Bäcklund transforms are not constant maps, and can be dropped if we allow more general Bäcklund transforms.

Proof. Choose a basis $a_1, \dots, a_{n+1}$ of $\mathbb{H}^{n+1}$ and let $\alpha_1, \dots, \alpha_n$ denote the dual basis. Since

$$d\omega_i = (d\omega)a_i = 0 \,,$$

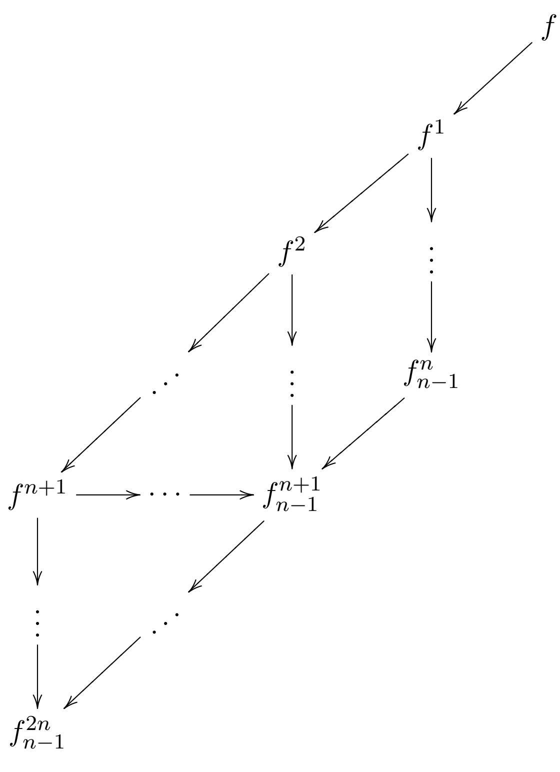
we can define holomorphic 1–forms $\omega_i^0 \in H^0(KL)$ by

$$\omega_i^0 := \omega a_i\,.$$

Away from the zeros M_0 of ω_1^0 we can define the 1–step Bäcklund transform $f_n^{n+1} : \tilde{M} \setminus M_0 \to \mathbb{HP}^1$ of f with respect to ω_1^0 and α_1, i.e.,

$$f_n^{n+1} = B_{\alpha_1,\omega_1^0}(f)\,.$$

We will show that the dual curve $\tilde{L}$ of $\hat{L} = (\ker \omega)^\perp$ projects onto an n–step Bäcklund transform of f_n^{n+1} away from the Weierstraß points of $\hat{f}$. Moreover, we will see that $\tilde{f}$ is a $n-1^{\text{st}}$ envelope of a 2–step Bäcklund transform of f. On the other hand, the following diagram is commutative as we have seen before where $f_{n-1}^{n+1} : M \to \mathbb{HP}^1$ is a 2–step Bäcklund transform of f, and $f^{n+1} : M \to \mathbb{HP}^n$ and $f_{n-1}^{2n} : M \to \mathbb{HP}^1$ are $(n+1)$–step Bäcklund transforms.



In other words, we will show that $\tilde{f} = f^{n+1}$.

Define $g_j^1 : M \to \mathbb{H}$ away from the isolated zeros M_0 of ω_1^0 by

$$\omega_{j+1}^0 = \omega_1^0 g_j^1, \qquad j = 1, \dots, n\,. \tag{2.23}$$

This way, we obtain on $M^0 = M \setminus M_0$

$$\ker \omega = \operatorname{Span}\{a_1 g_j^1 - a_{j+1}\}$$

and the nowhere vanishing section

$$\hat{\psi} = \sum_{k=1}^{n-1} \alpha_{k+1} \bar{g}_k^1 + \alpha_1$$

spans $\hat{L} = (\ker \omega)^{\perp}$ over M^0. In affine coordinates, this reads as

$$\hat{L} = \begin{pmatrix} \bar{g}_n^1 \\ \vdots \\ \bar{g}_1^1 \\ 1 \end{pmatrix} \mathbb{H} .$$

Since $\hat{f}$ is a holomorphic curve with Frenet flag, we can compute the k^{th}–osculates $\hat{f}_k$ of $\hat{f}$ given by $\hat{V}_k \cap \ker a_k$. Note that $dg_j^1 \not\equiv 0$ since $\hat{f}$ is a full curve in $\mathbb{HP}^n$. In particular, dg_1^1 has isolated zeros M_1, and away from M_1 we can define g_j^2, $j = 1, \dots, n-1$ by

$$dg_{j+1}^1 = dg_1^1 g_j^2 .$$

Proceeding inductively we define on $\hat{M} = M \setminus \bigcup_{j=0}^{n-1} M_j$

$$dg_{j+1}^k = dg_1^k g_{j+}^{k+1} \tag{2.24}$$

so that the k^{th} osculate is given in affine coordinates by

$$\hat{L}_k = \begin{pmatrix} \bar{g}_{n-k}^{k+1} \\ \vdots \\ \bar{g}_1^{k+1} \\ 1 \end{pmatrix} \mathbb{H} .$$

Note that (2.24) implies that

$$dg_1^{k-1} \wedge dg_j^k = 0 ,$$

so that

$$[g_1^k, 1] : M^0 \to \mathbb{HP}^1$$

is a 1–step Bäcklund transform of $[g_1^{k-1}, 1]$.

Moreover, since ω_{j+1}^0 is closed, (2.23) shows that g_1^1 is a 1–step Bäcklund transform of f_n^{n+1}, and the map

$$f_{n-1}^{n+k} = [g_1^k, 1]$$

is a $(k+1)$–step Bäcklund transform of f.

Since $(\hat{V}_{n-1})^{\perp} \subset \hat{L}_{n-1}^{\perp}$ the dual curve $\tilde{f}$ of $\hat{f}$ is given in affine coordinates by

$$\begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \\ 1 \end{pmatrix}$$

with $f_n = -g_1^n$. In other words, $\tilde{f}$ projects onto a $(n+1)$–step Bäcklund transform of f. A lengthy but straightforward computation shows that the coordinates of $\tilde{f}$ are recursively given by

$$\tilde{f}_k = -g_{n-k+1}^k - \sum_{l=1}^{n-k} g_l^k \tilde{f}_{l+k}$$

and that the $n-1^{\text{st}}$ osculate of $\tilde{f}$ satisfies

$$\tilde{L}_{n-1} = \begin{pmatrix} -g_1^1 \\ 1 \end{pmatrix} \mathbb{H}\,.$$

Corollary 1.18 now shows that $\tilde{f}$ is a $(n+1)$–step Bäcklund transform of f since $\tilde{f}$ projects onto a $(n+1)$–step Bäcklund transform of f and envelopes a 2–step Bäcklund transform. □

Remark 2.29. The previous theorem can be interpreted as Bianchi permutability: the proof shows that if we prescribe $n+1$ holomorphic $\omega_i^0 \in H^0(KL)$ then there is a common n–step Bäcklund transform of the 1–step Bäcklund transforms $B_{\alpha,\omega_i^0}(f)$ of f. Moreover, this common n–step Bäcklund transform is, up to Möbius equivalence, algebraically given by $\ker \omega$ where $\omega \in \Omega^1(\text{End}(V))$ is defined by $\omega a_i = \omega_i$ after a choice of a basis a_i of $\mathbb{H}^{n+1}$.

2.2.5 The backward Bäcklund transformation

The *backward Bäcklund transformation* is the inverse transformation of the forward Bäcklund transformation.

Theorem 2.30 (and Definition). *If $f: M \to \mathbb{HP}^n$ is a holomorphic curve with Frenet flag and $\hat{\omega} \in \Omega^1(\text{End}(V))$ a nowhere vanishing closed 1–form with*

$$\ker \hat{\omega} = V_{n-1}\,,$$

and $*\hat{\omega} = \hat{\omega} S$ for some adapted complex structure S of f such that $\operatorname{im} \hat{\omega}$ is not contained in a proper constant subbundle of V, then

$$\hat{L} = \operatorname{im} \hat{\omega}$$

defines a holomorphic curve $\hat{f}: M \to \mathbb{HP}^n$, the so–called (n+1)–step backward Bäcklund transform *of f.*

Proof. Let $f^\dagger : M \to \mathbb{HP}^n$ be the dual curve of f. Example 1.17 shows that $f^\dagger$ is a holomorphic curve with Frenet flag $L_k^\dagger = V_{n-k-1}^\perp$ and complex structure $J_k^\dagger = J_{n-k}^*$ on V_k/V_{k-1}. If we define

$$\omega = \hat{\omega}^* \in \Omega^1(V^*)\,,$$

then ω is a nowhere vanishing closed 1–form with

$$\operatorname{im}\omega = (\ker\omega)^\dagger = L^\dagger\,.$$

Moreover, $\omega \in \Gamma(KL^\dagger)$ since $S^*|_{L^\dagger} = J_n$ is the complex structure of the dual curve on $L^\dagger$, and

$$*\omega = S^*\omega\,.$$

Thus, the $(n+1)$–step forward Bäcklund transformation of $f^\dagger$ is the holomorphic curve given by

$$\widetilde{(L^\dagger)} = ((\ker\omega)^\perp)^\dagger\,.$$

In particular,

$$\hat L = (\operatorname{im}\hat\omega) = (\ker\omega)^\perp = \widetilde{(L^\dagger)}^\dagger$$

is a holomorphic curve with Frenet flag. □

Corollary 2.31. *Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve with Frenet flag, and $\tilde f : M \to \mathbb{HP}^n$ be the $(n+1)$–step forward Bäcklund transform of f given by $\omega \in \Gamma(K\operatorname{Hom}(V, L))$.*

Then f is a $(n+1)$–step backward Bäcklund transform of $\tilde f$, i.e.,

$$\hat{\tilde f} = f\,.$$

Conversely, given a backward Bäcklund transform $\hat f : M \to \mathbb{HP}^n$ of f with respect to $\hat\omega \in \Omega^1(\operatorname{End}(V))$, then f is a $(n+1)$–step forward Bäcklund transform of $\hat f$, i.e.,

$$\tilde{\hat f} = f\,.$$

Proof. The 1–form ω has

$$\ker\omega = \tilde V_{n-1}$$

where $\tilde V_k$ are the flag spaces of the forward Bäcklund transform $\tilde f$. Since ω is closed and nowhere vanishing and satisfies $*\omega = \omega\tilde S$ for some adapted complex structure $\tilde S$ of $\tilde f$, this yields that

$$L = \operatorname{im}\omega$$

is a backward Bäcklund transform of $\tilde f$. □

The 1–step backward Bäcklund transform of a holomorphic curve $f : M \to \mathbb{HP}^n$ with Frenet flag is given by a closed 1–form $\hat\omega \in \Omega^1(L)$ with $*\hat\omega = \hat\omega J$: the dual form

$$\omega = \hat\omega^* \in H^0(KL^{-1})$$

is a holomorphic section in KL^{-1}. If $\beta \in \Gamma(\operatorname{pr}^* L^{-1})$ satisfies $d\varphi = \omega$ then β is a holomorphic section in $\operatorname{pr}^*(L^{-1})$. Assume that $\beta \notin H$ then H and β span a $(n+2)$–dimensional basepoint free linear system $\tilde H$ and the Kodaira embedding of L into $\tilde H^*$ is a holomorphic curve $\tilde{f}$. Its derivative $\tilde{\delta}$, gives after the choice of a hyperplane $\alpha \in \tilde{H}$ at infinity not intersecting $\tilde{f}$, a line bundle

$$\hat{L} = \tilde{V}_1 \cap \ker \alpha \,,$$

the *1–step backward Bäcklund transform* of f. Here $\tilde{V}_1$ is the first flag space of $\tilde{f}$.

Note that the 1–step forward Bäcklund transform of $\hat{f}$ given by $\omega = \tilde{\delta}\psi$ is f, where $\psi \in \Gamma(L)$ is given by $< \alpha, \psi >= 1$.

Remark 2.32. In the following, we will only prove statements for the forward (or backward) transform: as the proof of Theorem 2.30 shows, we can switch from one transformation to the other by considering the dual curve. The translation of properties of the forward Bäcklund transform to the corresponding properties of the backward transform is always given by straight forward arguments.

Chapter 3

Applications to Willmore curves

An immersion $f : M \to \mathbb{R}^4$ of a compact Riemann surface into the 4–space is called *Willmore surface* if f is a critical point under compactly supported variations of f of the Willmore functional

$$W(f) = \int_M (|\mathcal{H}|^2 - K - K^\perp)|df|^2 \,,$$

where $\mathcal{H}$ is the mean curvature vector of f, K the Gaussian curvature and $K^\perp$ the curvature of the normal bundle of f all computed with respect to the induced metric on M. Willmore surfaces have a long history attached to them: [Bla29], [Wil68], [Wei78], [LY82], [Bry84], [Eji88], [Sim93], [Mon00]. For an introduction to the subject see also [Wil93].

More general, if $f : M \to S^4 = \mathbb{HP}^1$ is a holomorphic curve with $\dim H^0(L^{-1}) = n+1 \geq 2$, we can ask under which conditions f is a critical point of the Willmore energy $\mathcal{W}(f) = 2\int_M < A \wedge *A >$ under variations $f_t : M \to \mathbb{HP}^1$ of f which preserve the dimension $\dim H^0(L_t^{-1}) \geq n+1$ of the space of holomorphic sections. For $n = 1$ we clearly obtain the classical Willmore surfaces $f : M \to S^4$. However, in general these critical points are not necessary Willmore surfaces in S^4 if $n > 1$: since we have a constraint on the variations, namely to preserve the dimension of the space of holomorphic sections, we allow fewer variations and thus a larger class of examples. Examples for such conformal maps are some soliton spheres, [Pet04].

On the other hand, using the Kodaira correspondence we can consider $L = f^*\mathcal{T}$ embedded in $(H^0(L^{-1}))^*$ and obtain a holomorphic curve in $\mathbb{HP}^n$. The constraint on the variation gives a variation of $f : M \to \mathbb{HP}^n$ by holomorphic curves $f_t : M \to \mathbb{HP}^n$. Since the Willmore energy is given by the holomorphic structure on the holomorphic line bundle L^{-1}, and thus independent of the linear system, f is a critical point under the variation f_t. In other words, we obtain a Willmore curve in $\mathbb{HP}^n$, that is a critical point of the Willmore energy under variations by holomorphic curves in $\mathbb{HP}^n$. Willmore curves in $\mathbb{HP}^n$ behave very much like Willmore surfaces in S^4. In particular, if $f : M \to \mathbb{HP}^n$ is Frenet, then the canonical complex structure is harmonic. Moreover, Willmore spheres in $\mathbb{HP}^n$

have integer Willmore energy [Les], and are given by complex holomorphic data — thus, also any projection $f : M \to S^4$ of a Willmore curve in $\mathbb{HP}^n$ has integer Willmore energy and is given by complex holomorphic data.

We will study the transformations discussed in Chapter 2 in the special case of a Willmore curve. The Darboux transformation gives a spectral curve of a Willmore torus $f : T^2 \to S^4$ with trivial normal bundle, and the Bäcklund transformation will allow to give a classification of Willmore spheres $f : S^2 \to S^4$ in terms of holomorphic data. This classification extends to Willmore tori $f : T^2 \to S^4$ with non–trivial normal bundle. The results for Willmore spheres and Willmore tori with non–trivial normal bundle have generalization to the case of Willmore spheres $f : S^2 \to \mathbb{HP}^n$ and $f : T^2 \to \mathbb{HP}^n$ in $\mathbb{HP}^n$.

3.1 Willmore curves in $\mathbb{HP}^n$

From now on, M will always denote a compact Riemann surface.

Definition 3.1 (see [LP03])**.** *A holomorphic curve $f : M \to \mathbb{HP}^n$ is called* Willmore *if f is a critical point of the Willmore energy under compactly supported variations of f by holomorphic curves where we allow the conformal structure on M to vary.*

Figure 3.1: Willmore sphere, [Hel02]

In case of a Frenet curve $f : M \to \mathbb{HP}^n$ the Willmore condition can be expressed in terms of harmonicity.

Definition 3.2. *The* energy functional *of $S : M \to \mathcal{Z} := \{S \in \mathrm{End}(V) \mid S^2 = -I\}$ is given by*

$$E(S) = \frac{1}{2}\int_M < dS \wedge *dS >= 2\int_M < Q \wedge *Q > + < A \wedge *A > . \qquad (3.1)$$

A map $S : M \to \mathcal{Z}$ is called harmonic *if it is a critical point of the energy functional.*

Let $d = \bar\partial + \partial + Q + A$ the decomposition of the trivial connection d on V with respect to a complex structure S. By changing the complex structure to $-S$ on the domain we get $K\,\mathrm{End}_-(V) = K\,\mathrm{Hom}_+(\bar V, V)$ and ∂ and $\bar\partial$ on V induce by (1.48) a complex holomorphic structure on $K\,\mathrm{End}_-(V)$. If we change the complex structure on $\bar K\,\mathrm{End}_-(V)$ to $-S$ then ∂ and $\bar\partial$ give a complex holomorphic structure $\bar\partial$ on $K\,\mathrm{End}_-(\bar V) = \overline{\bar K\,\mathrm{End}_-(V)}$, i.e., an antiholomorphic structure ∂ on $\bar K\,\mathrm{End}_-(V)$.

As in [BFL$^+$02, Prop. 5] one shows

Theorem 3.3. *Let $S : M \to \mathcal{Z}$. Then following are equivalent*

1. *S is harmonic.*
2. *$*Q$ is closed which due to (1.34) is the same as $*A$ is closed.*
3. *Q is antiholomorphic, i.e., $\partial Q = 0$.*
4. *A is holomorphic, i.e., $\bar{\partial} A = 0$.*

Moreover, if $f : M \to \mathbb{HP}^n$ is a Frenet curve and $S : M \to \mathcal{Z}$ its canonical complex structure, then S is conformal, i.e.,

$$< *dS, *dS >=< dS, dS > \ .$$

Since $\bar{\partial} + \partial$ is a complex connection on V with respect to the complex structure S, the degree of V is given by (1.46)

$$2\pi \deg(V, S) = \int_M < A \wedge *A > - < Q \wedge *Q > \ . \tag{3.2}$$

If $f : M \to \mathbb{HP}^n$ is a Frenet curve with canonical complex structure S then the Willmore energy is given in terms of the Hopf field A of S by $\mathcal{W}(f) = 2\int_M < A \wedge *A >$ so that we have

Corollary 3.4. *Let $f : M \to \mathbb{HP}^n$ be a Frenet curve with canonical complex structure S. Then*

$$E(S) + 4\pi \deg(V, S) = 2\mathcal{W}(f)\,.$$

Similar techniques as used in the S^4–case [BFL$^+$02, Thm. 3] give the classical relation between the Willmore condition and harmonicity.

Theorem 3.5 (see [LP03])**.** *A Frenet curve $f : M \to \mathbb{HP}^n$ is Willmore if and only if the canonical complex structure of f is harmonic, i.e.,*

$$d * A = 0\,.$$

Corollary 3.6. *Let $f : M \to \mathbb{HP}^n$ be a Frenet curve. Then*

$$(d * A)|_{V_{n-1}} = 0\,.$$

In particular, if $e \in \mathbb{H}^{n+1}$ such that $V = V_{n-1} \oplus e\mathbb{H}$ then f is Willmore if and only if

$$(d * A)e = 0\,.$$

Proof. For $\psi \in \Gamma(V_{n-1})$ we have

$$(d * A)\psi_{n-1} = (d * Q)\psi_{n-1} = d(*Q\psi_{n-1}) - Q \wedge d\psi_{n-1} = - * Q \wedge \delta_{n-1}\psi_{n-1}\,,$$

where we used (1.34) and that $Q|_{V_{n-1}} = 0$ for the Hopf field of the canonical complex structure (1.33). But $Q \wedge \delta_{n-1} = 0$ by type. □

Example 3.7. As we have seen in Example 1.34, holomorphic curves $f : M \to \mathbb{HP}^n$ with zero Willmore energy are exactly the twistor projections of holomorphic curves in $\mathbb{CP}^{2n+1}$. These provide the simplest examples of Willmore curves. Note that even in this case, the Willmore condition does not guarantee the smooth existence of the canonical complex structure S of the holomorphic curve in the Weierstraß points. However, a regularity result of Helein [Hél04] on harmonic maps, can be used to show that a Willmore curve f is Frenet if S can be extended continuously across the Weierstraß points [LP03].

Example 3.8. The notion of a Willmore curve carries over to the dual curve of a Frenet curve $f : M \to \mathbb{HP}^n$: the Hopf field $Q^\dagger$ of $f^\dagger$ satisfies (1.32)

$$d^* * Q^\dagger = -(d * A)^*$$

so that f is Willmore if and only if $f^\dagger$ is Willmore.

We have the Kodaira correspondence between holomorphic curves $f : M \to \mathbb{HP}^n$ and base point free linear systems $H \subset H^0(L^{-1})$. For a Willmore curve $f : M \to \mathbb{HP}^n$, it is natural to ask for which choices of basepoint free linear systems $\check{H} \subset H^0(L^{-1})$ the induced holomorphic curve $\check{L} \subset \check{H}^{-1}$ is again Willmore.

Proposition 3.9. *Let $f : M \to \mathbb{HP}^n$ be a Willmore curve. Let $L \subset V$ and $H \subset H^0(L^{-1})$ be the corresponding line bundle and basepoint free linear system. Let $\check{H} \subset H^0(L^{-1})$ be a linear system with $H = V^{-1} \subset \check{H}$ so that the map $\check{f} : M \to \mathbb{HP}^m$ given by the Kodaira correspondence has a canonical complex structure which extends continuously across the Weierstraß points. Then $\check{f}$ a Willmore curve in $\mathbb{HP}^m$ where $m = \dim \check{H}$.*

Proof. Let $\check{f}_t : M \to \mathbb{HP}^m$ be a variation of $\check{f}$ so that the compact support K does not contain Weierstraß points. Without loss of generality, we can assume that $\check{f}_t$ is unramified on K. Then $\pi : \check{V} = \check{H}^{-1} \to V$ defines a variation of f by Frenet curves $f_t : M \to \mathbb{HP}^n$ by $\pi(\check{L}_t) = L_t$. Since the Willmore energy only depends on the holomorphic structure on L^{-1} and not on the linear system, we see that

$$\frac{\partial}{\partial t}\mathcal{W}(\check{f}_t) = \frac{\partial}{\partial t}\mathcal{W}(f_t) = 0\,.$$

The usual arguments, see [LP03], show that the canonical complex structure of $\check{f}$ is harmonic on K, i.e.,

$$\check{d} * \check{A} = 0$$

away from the Weierstraß points. With [Hél04] the canonical complex structure extends smoothly into the Weierstraß points, and therefore $\check{f}$ is a Frenet curve. □

In general, projections of Willmore curves $L \subset V$ into flat subbundles $\check{V} \subset V$ fail to be Willmore, see [Pet04].

Proposition 3.10. *Let $f : M \to \mathbb{HP}^n$ be a Willmore curve with canonical complex structure S and let $H \subset H^0(L^{-1})$ be the corresponding linear system. Let $\check{H}$ be an S stable basepoint free linear system $\check{H} \subset H \subset H^0(L^{-1})$ with $m = \dim \check{H} \geq 2$. Then*

$$\check{L} = \pi(L) \subset \check{V} = \check{H}^{-1}$$

defines a Willmore curve $\check{f} : M \to \mathbb{HP}^m$. Here $\pi : V = H^{-1} \to \check{V}$ is the canonical projection.

Proof. Since f is a holomorphic curve, the line bundle L is full, i.e., L is not contained in a lower dimensional flat subbundle of V. The kernel $\check{H}^\perp = \ker \pi$ of π is d stable which shows that $\pi|_L \neq 0$. Since $\check{H}$ is a linear system the induced connection $\check{d}$ on $\check{V}$ satisfies $\pi d = \check{d}\pi$. Moreover,

$$\pi S =: \check{S}\pi$$

defines a complex structure on $\check{V}$ since $\ker \pi = \check{H}^\perp$ is S stable. The complex holomorphic structures $\check{d}''_+$ and d''_+ on $\check{V}$ and V given by the complex structures $\check{S}$ and S are related by

$$\check{d}''_+\pi = \pi d''_+ \,.$$

Since d''_+ and $\check{d}''_+$ stabilize L and πL respectively, the map $\pi|_L$ is a complex holomorphic map. In particular, the zeros of $\pi|_L$ are isolated and the complex bundle $\operatorname{im} \pi|_L$ can be extended smoothly across the zeros. In other words, $\operatorname{im} \pi|_L$ defines a complex quaternionic line bundle $\check{L}$. Note that $\check{L}_p = \pi L_p$ away from the isolated zeros of $\pi|_L$.

Let $L \subset V_1 \subset \ldots \subset V$ be the Frenet flag of f. Since $\pi\pi_L = \pi_{\check{L}}\pi$ we see

$$\check{\delta}_0\pi|_L = \pi\delta_0 \,.$$

If $\check{\delta}_0 = 0$ then V_1 is contained in the flat bundle $L + \ker \pi$ which has rank $\leq n$ since $\dim \ker \pi = \operatorname{rank} V - \operatorname{rank} \check{V} \leq n - 1$. This contradicts the assumption that L is a full curve in V, i.e., the assumption that $\delta_k \neq 0$ for $k = 0, \ldots, n-1$. Thus the map $\check{\delta}_0 \neq 0$ is complex holomorphic since

$$*\check{\delta}_0 = \check{S}\check{\delta}_0 = \check{\delta}_0\check{S} \,,$$

and defines a vector bundle $\check{V}_1$. Clearly, $\check{V}_1$ extends πV_1.

Proceeding inductively, we see that $\check{\delta}_k\pi|_{V_k} = \pi\delta_k$ and $\check{\delta}_k \neq 0$ for all $0 \leq k \leq \operatorname{rank} \check{V} - 2$. In particular, $\check{L}$ is a full curve in $\check{V}$ with Frenet flag $\check{V}_k = \pi V_k$. Moreover, $*\check{\delta}_k = \check{S}\check{\delta}_k = \check{\delta}_k\check{S}$ yields that $\check{S}$ is an adapted complex structure.

By construction $\check{A} = \frac{1}{2} * (\check{d}\check{S})'$ and $A = \frac{1}{2} * (dS)'$ satisfy $\check{A}\pi = \pi A$, hence $\check{S}$ is the canonical complex structure of $\check{f}$. In particular $\check{f}$ is a Frenet curve, and

$$\check{d} * \check{A}\pi = \pi d * A = 0 \,.$$

shows that $\check{f}$ is Willmore. □

Remark 3.11. If $\dim \check{H} = 1$ the same arguments as in the proof above show that $(\pi(L), \pi S, \pi d)$ defines a flat complex quaternionic line bundle.

Since the Hopf fields A and Q of a Willmore curve are holomorphic, the zeros of A and Q are isolated. Since $A|_L = 0$ at $p \in M$ implies [Les] that $A \equiv 0$ in a neighborhood of p, we obtain:

Corollary 3.12. *Let S be the canonical complex structure of a Willmore curve $f : M \to \mathbb{HP}^n$.*

1. *If $A \neq 0$ then the set*
$$\tilde{M} := \{p \in M \mid L_p \subset \ker A_p\}$$
has no inner points.

2. *If $Q \neq 0$ then the set*
$$\hat{M} := \{p \in M \mid \operatorname{im} Q_p \subset (V_{n-1})_p\}$$
has no inner points.

Remark 3.13. A flat connection ∇ on a complex quaternionic vector bundle (W, S) is called *Willmore connection* [FLPP01, Sec. 6.1] if S is harmonic, i.e., $d^\nabla * A = 0$.

Examples include the trivial connection on (V, S) where S is the canonical complex structure of a Willmore curve, and the family of flat connections (2.12) associated with a constant mean curvature surface where $W = V/L$ has rank 1, and $S = J$ is the complex structure given by the harmonic Gauss map of f.

In general, even if $W = V$ has rank 2, the harmonic complex structure S will not be the canonical complex structure of a Frenet curve. But if rank $A = 1$ then $\bar\partial A = 0$ implies that the image of A defines a $\bar\partial$ holomorphic line bundle. Thus, $L = \operatorname{im} A \subset V$ is a Willmore curve if S is the canonical complex structure of L, i.e., if the Hopf field $Q|_{V_{n-1}} = 0$ vanishes on the flag space V_{n-1} of L.

Moreover, if L is Willmore then the connections $\nabla^\lambda = \nabla + (\lambda - 1)A$ are flat for all $\lambda = \alpha + \beta S$, $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$. Denote by L^λ the line bundle L considered as subbundle in (V, ∇^λ). If we decompose ∇^λ with respect to the complex structure S of L then $Q^\lambda = Q$. Thus S is the canonical complex structure of L^λ and we obtain the associated family of Willmore curves L^λ. Its Willmore energy is given by $\mathcal{W}(L^\lambda) = \mathcal{W}(L)$.

Notice, that though ∇ is trivial, the Willmore curves of this family may have holonomy.

3.2 The Darboux transformation on Willmore surfaces in S^4

There are two ways to define a Darboux transformation on Willmore surfaces $f : M \to S^4$. One of them comes from the observation that Willmore surfaces behave in many ways like the rank 2 analog of constant mean curvature surfaces. Solving a Ricatti type equation on a rank 2 bundle we obtain a new Willmore surface. On the other hand, we defined a general Darboux transformation on conformal maps into the 4-sphere. We will see how these two transformations are related.

Let $f : M \to S^4$ be a Willmore immersion and let T be a solution of the Ricatti equation

$$dT = 2\rho T * QT - 2 * A \tag{3.3}$$

for $\rho \in \mathbb{R}$ with initial condition

$$(T - S)^2 = \rho^{-1} - 1 \tag{3.4}$$

at some point $p_0 \in M$. Here A and Q are the Hopf fields of the canonical complex structure S of f. In [BFL+02, Thm. 12] it is shown that the line bundle

$$\hat{L} = T^{-1}L$$

is a Willmore surface in S^4, and (3.4) holds on M. Note that (3.4) is equivalent to

$$T^{-2} = \rho(1 - ST^{-1} - T^{-1}S)$$

so that

$$(2b^{-1}(T + S\rho))^2 = 4b^{-2}\rho(1 - \rho) = -1$$

where $a^2 - b^2 = 1$, and $\rho = \frac{1-a}{2}$. Thus, if we define

$$S^\sharp = 2(T^{-1} + S\rho)b^{-1}$$

then $S^\sharp$ is a complex structure on V. Moreover, let $\lambda = a + bS^\sharp$ and define the connection $d^\sharp$ by

$$d^\sharp = d + 2 * AT^{-1} = d + A(\lambda - 1)\,.$$

Using the Ricatti equation (3.3), we get

$$\begin{aligned} d^\sharp(S^\sharp \frac{b}{2\rho}) &= d^\sharp(T^{-1}\rho^{-1} + S) \\ &= (dT^{-1})\rho^{-1} + 2 * Q - 2 * A + 2[*AT^{-1}, T^{-1}\rho^{-1} + S] \\ &= 0\,. \end{aligned}$$

If we define the family of I–complex connections

$$d^\mu = d + A(\mu - 1),$$

where $\mu = a + bI$, then the Willmore condition $d * A = 0$ gives that d^μ is a family of flat connections. Moreover,

$$d^\mu|_{E_\mu} = d^\sharp|_{E_\mu}$$

where E_μ is the $+i$ eigenspace of $S^\sharp$. In particular, $d^\sharp$ is a flat connection since $V = E_\mu \oplus E_\mu j$. Note that, since $d^\sharp S^\sharp = 0$, the $\pm i$ eigenspaces E_μ and $E_\mu j$ of $S^\sharp$ are also the eigenspaces of the monodromy of $d^\sharp$.

For any parallel section $\psi^\sharp \in \Gamma(\mathrm{pr}^* V)$ of $d^\sharp$ the projection $\varphi^\sharp = \pi_L(\psi^\sharp) \in H^0(\mathrm{pr}^* V/L)$ is a holomorphic section since

$$d\psi^\sharp = -(2 * AT^{-1})\psi^\sharp \in \Omega^1(\mathrm{pr}^* L)\,,$$

and thus

$$D\varphi^\sharp = (\pi d\psi^\sharp)'' = 0\,.$$

Moreover, $\pi_L(d\psi^\sharp) = 0$ shows that $\psi^\sharp$ is the canonical lift of the holomorphic section $\varphi^\sharp$ and $\psi^\sharp\mathbb{H}$ is a Darboux transform of f.

Lemma 3.14. *Every solution of the Ricatti equation (3.3) with initial condition (3.4) gives a Willmore surface $\hat f : M \to S^4$ and a Darboux transform $f^\sharp : M \to S^4$ of f.*

Remark 3.15. In [Sch02], see also[Boh], it is shown that the spectral curve of a Willmore curve has finite genus. It is still an open problem if the points of the spectral curve of a Willmore surface are again Willmore, and more specifically if the two transforms $\hat f$ and $f^\sharp$ of a Willmore surface in S^4 coincide. The examples of [Ber01], where Darboux transforms of the Clifford torus are constructed which are not constraint Willmore, shows that in the case of a Willmore torus a general Darboux transform will not even be constraint Willmore. We expect this kind of behavior only at exceptional points of the spectral curve.

3.3 The Bäcklund transformation on Willmore curves

Bryant [Bry84] gave a classification result for Willmore spheres $f : S^2 \to \mathbb{R}^3$. This result can be generalized to the case of Willmore spheres $f : S^2 \to S^4$, see [Eji88], [Mon00]: A Willmore sphere in S^4 is either the twistor projection of a holomorphic curve in $\mathbb{CP}^3$, the dual curve of such a twistor projection, or a minimal sphere in $\mathbb{R}^4$ with planar ends after choosing a suitable point at infinity. A similar result can be obtained [LPP05] for Willlmore tori $f : T^2 \to S^4$ with trivial normal bundle by discussing the monodromy of the associated family of flat connections. We present a different approach [LP] using the Bäcklund transformation to construct sequences of Willmore surfaces. This approach extends to the case of Willmore spheres in $\mathbb{HP}^n$ and shows that every Willmore sphere in $\mathbb{HP}^n$ has integer Willmore energy and is given by complex holomorphic data [Les]. Here we also give a generalization of [LP] to Willmore tori in $\mathbb{HP}^n$ with non–zero degree of the corresponding bundle L.

3.3.1 Sequences of Willmore curves

Recall that the Willmore condition of a Frenet curve can be expressed in terms of the harmonicity of the canonical complex structure. Recall the $\partial, \bar\partial$ transformation for harmonic maps $h : M \to \mathbb{CP}^n$: the $(0,1)$–part δ''_E of the derivative of the line bundle $E = h^*\mathcal{T}$ is holomorphic. The image of δ''_E defines a complex bundle of rank 2, and after orthogonal projection onto the orthogonal complement $E^\perp$ of E, one gets a line subbunde of $\underline{\mathbb{C}}^{n+1}$. The corresponding $\hat h : M \to \mathbb{CP}^n$ is a new harmonic map. This way, one constructs sequences of harmonic maps which can be used for classification of harmonic maps $h : M \to \mathbb{CP}^n$ with large (in terms of the genus of M) degree of h, [EW83], [Wol88].

In case of a Willmore curve in $\mathbb{HP}^n$ with harmonic canonical complex structure S the $(1,0)$ and $(0,1)$–parts of the derivative of S are essentially (1.34) the Hopf fields A and Q. The kernels and images of A and Q are smooth vector bundles and define holomorphic curves in $\mathbb{HP}^n$ since A and Q are holomorphic sections by the harmonicity of S see Theorem 3.3. Of course, the holomorphic curves $\operatorname{im} A = L$ and $(\ker Q)^\perp = L^\dagger$ give no new surfaces. However,

$$\tilde f = ((\ker A)^\perp)^\dagger$$

and

$$\hat f = \operatorname{im} Q$$

give the $(n+1)$–step forward and backward Bäcklund transform of f as defined in Chapter 2 for the special choices $\omega = *A$ and $\hat\omega = *Q$. In [Les] it is shown that if $f : M \to \mathbb{HP}^n$ is Willmore and Frenet such that $\tilde L = ((\ker A)^\perp)^\dagger$ is a Frenet curve too, then $\tilde f$ is Willmore and the Hopf field $\tilde Q$ of the canonical complex structure $\tilde S$ of $\tilde f$ satisfies

$$\tilde Q = A\,.$$

It remains to show that the $(n+1)$–step Bäcklund transformation on Willmore curves with $\omega = *A$ is Frenet. We prove the statement first for the 1–step Bäcklund transformation.

Lemma 3.16. *Let $f : M \to \mathbb{HP}^n$ be a Willmore curve with smooth canonical complex structure S such that $A \not\equiv 0$. Choose $b \in \mathbb{H}^{n+1}$ such that $V_{n-1} \oplus b\mathbb{H} = V$ and $\tilde V_{n-1} \oplus b\mathbb{H} = V$ where V_{n-1} and $\tilde V_{n-1}$ are the flag spaces of L and $\tilde L$ respectively. Then the 1–step Bäcklund transform $f^1 : M \to \mathbb{HP}^n$ given by $\omega = *Ab$ has a smooth canonical complex structure S^1 on the universal cover $\tilde M$ of M.*

Proof. Let $f_- : \tilde M \to \mathbb{HP}^{n+1}$ be the first osculate of f given by $d_-\psi_- = \omega$ where $\omega = *Ab \in H^0(KL)$, and let $\beta \in \Gamma(V^*)$ with $\beta(b) = 1$ and $\beta|_{\tilde V_{n-1}} = 0$. Then

$$J_-\psi_- = -\psi < \beta, Sb >$$

defines a complex structure on L_- and f_- is a Frenet curve [LP03] with canonical complex structure S_- given by

$$S_- = \begin{pmatrix} J_- & B \\ 0 & S \end{pmatrix},$$

where $B \in \Gamma(\mathrm{Hom}(V, L))$ is defined by $B = 2\psi\beta$. Since the 1–step Bäcklund transform of f with respect to ω is given by a projection of f_- Lemma 2.23 shows that the 1–step Bäcklund transform is Frenet, too. □

To conclude that the $(n+1)$–step Bäcklund transform is Frenet, we have to assure that the 1–step Bäcklund transform of a Willmore curve is Willmore.

Lemma 3.17. *Under the assumptions of the previous Lemma, the 1–step Bäcklund transform of f is Willmore.*

Proof. We adapt the proof of [BFL$^+$02, Prop. 16] for the 1–step Bäcklund transform of a Willmore surface in S^4 to the case of Willmore curves in $\mathbb{HP}^n$. By the previous lemma the 1–step Bäcklund transform $\tilde{f}$ of f is a Frenet curve so that it is enough to show, see Corollary 3.6, that

$$(\tilde{d} * \tilde{A})|_{\tilde{e}\mathbb{H}} = 0\,,$$

where $\tilde{d}$ is the flat connection on $\tilde{V}$, and $\tilde{A}$ the Hopf field of the canonical complex structure $\tilde{S}$ of $\tilde{f}$. Moreover, $\tilde{e} \in \mathbb{H}^{n+1}$ such that

$$\tilde{V}_{n-1} \oplus \tilde{e}\mathbb{H} = \tilde{V}\,.$$

Note that we can identify $\tilde{V}_k = \tilde{L} \oplus V_{k-1}$, where V_k are the flag spaces of f. In particular, $\tilde{e}\mathbb{H} = L_{n-1}$. We decompose $\tilde{S}$ in the splitting $\tilde{L} \oplus V_{n-1} = \tilde{V}$ as

$$\tilde{S} = \begin{pmatrix} \tilde{J} & \tilde{B} \\ 0 & \hat{S} \end{pmatrix}.$$

We will show that

$$\tilde{\delta}_0(2\nabla\tilde{B} + *\tilde{A})|_{L_{n-1}} = *A\delta_{n-1}\,. \tag{3.5}$$

Let R be the parallel homomorphism $R \in \Gamma(\mathrm{Hom}(b\mathbb{H}, \tilde{L}))$ given by

$$Rb = \tilde{\psi}\,,$$

where $< \tilde{\alpha}, \tilde{\psi} >= 1$ and $\tilde{\alpha} \in (\mathbb{H}^{n+1})^*$ is a hyperplane at infinity such that $\ker\tilde{\alpha} = V_{n-1}$. Since $*Ab = \tilde{\delta}_0\tilde{\psi}$ the equation (3.5) implies that

$$*\tilde{A}|_{L_{n-1}} = R\delta_{n-1} - 2(\nabla\tilde{B})|_{L_{n-1}}\,.$$

Therefore, since $L_{n-1} = \tilde{e}\mathbb{H}$ is a constant bundle with respect to the connection $\tilde{d}$ on $\tilde{V}$, we have

$$(\tilde{d} * \tilde{A})|_{L_{n-1}} = \tilde{d}(R\delta_{n-1}) + 2(\tilde{d}\nabla\tilde{B})|_{L_{n-1}} = 0\,, \tag{3.6}$$

where we used that R is parallel with respect to the induced connections and $d\delta_{n-1} = 0$ by the flatness of d.

It remains to show (3.5). We use the fact that the complex structure $\hat{S}$ on V_{n-1} given by $\tilde{S}$ via the splitting $\tilde{L} \oplus V_{n-1} = \tilde{V}$ also occurs when decomposing the canonical complex structure S of f with respect to the splitting $V_{n-1} \oplus b\mathbb{H} = V$:

$$S = \begin{pmatrix} \hat{S} & B \\ 0 & \hat{J} \end{pmatrix}.$$

The conditions (1.35) and (1.33) that $\tilde{S}$ and S are the canonical complex structures of $\tilde{f}$ and f respectively give

$$2\tilde{\delta}_0\tilde{B} = \hat{S} * \hat{\nabla}\hat{S} - \hat{\nabla}\hat{S} \tag{3.7}$$

and

$$2B\delta_{n-1} = \hat{S} * \hat{\nabla}\hat{S} + \hat{\nabla}\hat{S}\,. \tag{3.8}$$

Differentiating (3.7) and (3.8) and comparing the results, we obtain

$$-\tilde{\delta}_0 \wedge \nabla\tilde{B} = \nabla B \wedge \delta_{n-1}\,,$$

or, using the identification $\omega \wedge \eta = \omega * \eta - *\omega\eta$ of a 2–form with its quadratic form,

$$-\tilde{\delta}_0(*\nabla\tilde{B} - \tilde{J}\nabla\tilde{B}) = ((\nabla B)\hat{J} - *\nabla B)\delta_{n-1}\,. \tag{3.9}$$

We denote by $\tilde{\eta} = 4 * \tilde{A}|_{V_{n-1}}$ so that

$$\tilde{\eta} = -2(\nabla\tilde{B})' + \tilde{B} * \tilde{\delta}_0\tilde{B} + \tilde{B} * \hat{\nabla}\hat{S}\,.$$

Since $\tilde{S}$ is a complex structure, i.e. $\tilde{S}^2 = -1$, we have

$$\tilde{J}\tilde{B} + \tilde{B}\hat{S} = 0\,.$$

Together with (3.7) we get

$$\tilde{J}\tilde{\eta} = -\tilde{J}\nabla\tilde{B} - *\nabla\tilde{B} - \tilde{B}\tilde{\delta}_0\tilde{B} - \tilde{B}\hat{\nabla}\hat{S}\,,$$

so that

$$\begin{aligned} \hat{S}\tilde{\delta}_0(2\nabla\tilde{B} + \tilde{\eta}) &= \tilde{\delta}(2\tilde{J}\nabla\tilde{B} - \tilde{J}\tilde{\eta}) \\ &= ((\nabla B)\hat{J} - *\nabla B)\delta_{n-1} - \tilde{\delta}_0(\tilde{\delta}_0 + \hat{\nabla}\hat{S}) \\ &= ((\nabla B)\hat{J} - *\nabla B)\delta_{n-1} - \frac{1}{4}(\hat{S} * \hat{\nabla}\hat{S} - \hat{\nabla}\hat{S})(\hat{S} * \hat{\nabla}S + \hat{\nabla}\hat{S})\,. \end{aligned} \tag{3.10}$$

On the other hand, if we decompose the trivial connection d in the splitting $V_{n-1} \oplus b\mathbb{H} = V$ as

$$d = \begin{pmatrix} \hat{\nabla} & 0 \\ \hat{\delta}_{n-1} & \nabla^b \end{pmatrix},$$

then the Hopf field A of S is given in this splitting as

$$-4 * A = \begin{pmatrix} -4 * A^{\hat{S}} & 2(\nabla B)' - B * \hat{\delta}_{n-1}B - B * \nabla^b\hat{J} \\ 0 & 0 \end{pmatrix},$$

and

$$\hat{\delta}_{n-1}B = 2 * A^{\hat{J}}$$

holds. Note that $\hat{\delta}_{n-1}|_{L_{n-1}} = \delta_{n-1}$. As before, $S^2 = -1$ implies $\hat{S}B + B\hat{J} = 0$, and differentiating this equation

$$-\hat{S}\nabla B - B\nabla^b \hat{J} = (\nabla B)\hat{J} + (\hat{\nabla}\hat{S})B\,.$$

As similar computation as before together with the above equation gives for $\eta = 4 * A|_{b\mathbb{H}}$

$$\hat{S}\eta = (\hat{\nabla}\hat{S})B + (\nabla B)\hat{J} - *\nabla B - B\hat{\delta}_{n-1}B$$

so that

$$\begin{aligned}\hat{S}\eta\delta_{n-1} &= ((\nabla B)\hat{J} - *\nabla B)\delta_{n-1} + (\hat{\nabla}\hat{S} - B\hat{\delta}_{n-1})B\hat{\delta}_{n-1} \\ &= ((\nabla B)\hat{J} - *\nabla B)\delta_{n-1} - \frac{1}{4}(\hat{S} * \hat{\nabla}\hat{S} - \hat{\nabla}\hat{S})(\hat{S} * \hat{\nabla}S + \hat{\nabla}\hat{S})\,.\end{aligned}$$

Comparing to (3.10) this yields (3.5), and we can conclude that the 1–step Bäcklund transform $\tilde{f}$ is a Willmore curve since (3.6) and Corollary 3.6 show that

$$\tilde{d} * \tilde{A} = 0\,.$$

□

Corollary 3.18. *The $(n+1)$–step Bäcklund transforms $\tilde{f}$ and $\hat{f}$ of f are Frenet curves.*

Combining the Corollary with the result in [Les] we obtain:

Theorem 3.19. *The $(n+1)$–step forward and backward Bäcklund transforms $\tilde{f} : M \to \mathbb{HP}^n$ and $\hat{f} : M \to \mathbb{HP}^n$ of a Willmore curve $f : M \to \mathbb{HP}^n$ are Willmore curves in $\mathbb{HP}^n$ with Willmore energies*

$$\mathcal{W}(\tilde{f}^\dagger) = \mathcal{W}(f) \quad \text{and} \quad W(\hat{f}) = \mathcal{W}(f^\dagger)\,. \tag{3.11}$$

3.3.2 Finite sequences

In what follows we only consider $(n+1)$–step Bäcklund transforms as these are globally defined objects. For simplicity of notation we write "$\tilde{f}$ is a forward Bäcklund transform" when referring to a $(n+1)$–step transform forward Bäcklund transform given by ker A.

We now discuss the case when the sequence of successive forward Bäcklund transforms is finite, that is that after the k^{th} Bäcklund transform $\tilde{f}^k : M \to \mathbb{HP}^n$ of a Willmore curve $f : M \to \mathbb{HP}^n$ the $k+1^{\text{st}}$ Bäcklund transform does not exist as a full curve in $\mathbb{HP}^n$.

One reason for this to happen is that the Hopf field $\tilde{A}^k$ of $\tilde{f}^k$ vanishes identically. This could happen in the first step, in which case f itself has Hopf field $A \equiv 0$, and f is the twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$. Otherwise, i.e., for $k \geq 1$ we know that the backward Bäcklund transform $\widehat{\tilde{f}^k} = \tilde{f}^{k-1} : M \to \mathbb{HP}^n$ is a full curve in $\mathbb{HP}^n$ where we denote by $\tilde{f}^0 = f$ the original Willmore curve in $\mathbb{HP}^n$.

We denote for a Frenet curve $f : M \to \mathbb{HP}^n$ by

$$\delta^k = \delta_{k-1} \circ \ldots \circ \delta_0 \in H^0(K^k \operatorname{Hom}(L, V_k/V_{k-1}))\,, \quad 1 \leq k \leq n\,,$$

the complex holomorphic section which is given by the composition of the derivatives of the successive Frenet spaces.

Lemma 3.20. *Let $f : M \to \mathbb{HP}^n$ be a Willmore curve with canonical complex structure S such that the backward Bäcklund transform $\hat{f} : M \to \mathbb{HP}^n$ given by $\hat{L} = \operatorname{im} Q$ is a full curve in $\mathbb{HP}^n$. If*

$$A\hat{\delta}^k Q = 0$$

for all $k = 1, \ldots, n-1$ then $-S$ is the canonical complex structure of $\hat{L}$.

Proof. If $AQ \equiv 0$ then $\hat{L} = \operatorname{im} Q \subset \ker A$ so that Q and A stabilize $\hat{L}$. [Les] shows that this implies

$$*\hat{\delta}_0 = -\hat{S}\hat{\delta}_0 = -S\hat{\delta}_0 S\,.$$

By applying this argument successively for all flag spaces, we see that $-S$ is an adapted complex structure. But the Hopf field $\bar{A}$ of $-S$ is given by $\bar{A} = Q$ so that

$$\operatorname{im} \bar{A} = \operatorname{im} Q \subset \hat{L}$$

and $-S$ is the canonicacl complex structure of $\hat{f}$. □

A special case when the assumptions of the previous lemma are satisfied is when the Hopf field of f vanishes identically, i.e., $A \equiv 0$, that is when f is the twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^{2n+1}$.

Corollary 3.21. *If $f : M \to \mathbb{HP}^n$ is the twistor projection of a holomorphic cure in $\mathbb{CP}^{2n+1}$ and has a backward Bäcklund transform $\hat{L} = \operatorname{im} Q$ as full curve in $\mathbb{HP}^n$ then $\hat{f}$ has $\hat{Q} \equiv 0$, and is the dual curve of the twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$.*

Applying this result to the case of the sequence of forward Bäcklund transforms we get

Corollary 3.22. *If the k^{th} forward Bäcklund transform $\tilde{f}^k$ of f is the twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$ then $0 \leq k \leq 1$, and f or $f^\dagger$ is the twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$. Moreover,*

$$\mathcal{W}(f) \in 4\pi\mathbb{N}\,.$$

Now, assume that $\tilde{A}^j, \tilde{Q}^j \not\equiv 0$ for all $0 \leq j \leq k$. Then $\ker \tilde{A}^k$ does not define a full curve in $\mathbb{HP}^n$ if and only if $\ker \tilde{A}^k$ contains a constant subbundle $W \subset \ker \tilde{A}^k$, see Example 1.18.

By dualization, i.e., switching to $f^\dagger$ instead of f, we can use the backward sequence instead of the forward one. Moreover, if $\hat{L} \subset W \subset V$ where W is a proper subspace of V with $\dim W \neq 1$ then we consider the sequence of $\hat{f}$ instead of the sequence of f. Note that $\mathcal{W}(f) - \mathcal{W}(\hat{f}) \in 4\pi\mathbb{N}$ so that f has integer Willmore energy if $\hat{f}$ has $\mathcal{W}(\hat{f}) \in 4\pi\mathbb{N}$.

So, we are left to discuss sequences of Bäcklund transforms which terminate because the backward Bäcklund transform is a constant point in $\mathbb{HP}^n$.

Assume that $f : M \to \mathbb{HP}^n$ with $n \geq 2$ has constant backward Bäcklund transform $\hat{L} = W$. Then the projection $\hat{\pi} : V \to V/\hat{L} = \check{V}$ defines a Willmore curve in $\mathbb{HP}^{n-1}$, see Proposition 3.10, with canonical complex structure

$$\check{S}\hat{\pi} = \hat{\pi} S \,.$$

In particular, $\check{Q}\hat{\pi} = \hat{\pi} Q = 0$ so that $\check{f}$ is the dual curve of a twistor projection of a holomorphic curve in $\mathbb{CP}^{2n-1}$. In particular,

$$\mathcal{W}(f) = \mathcal{W}(\check{f}) \in 4\pi N \,.$$

In case $f : M \to S^4$ has constant backward Bäcklund transform $\hat{f}$ the standard argument [BFL+02, Sec. 11.2] shows that f is a minimal surface in $\mathbb{R}^4$ when choosing the point at infinity as $\hat{L}$: all mean curvature spheres of f pass through $\hat{L}$.

We summarize:

Theorem 3.23. *Let $f : M \to \mathbb{HP}^n$ be a Willmore surface such that the sequence of Bäcklund transforms is finite where we allow the Bäcklund transforms to be full curves in lower dimensional $\mathbb{HP}^k$'s. Then f has Willmore energy*

$$\mathcal{W}(f) \in 4\pi\mathbb{N} \,.$$

3.3.3 Willmore spheres and Willmore tori

We show that in the case of Willmore spheres and Willmore tori the sequence of Bäcklund transforms is finite.

Let $f : M \to \mathbb{HP}^n$ be a Willmore curve with canonical complex structure S. Since

$$\delta^i = \delta_{i-1} \circ \ldots \circ \delta_0 \in H^0(K^i \operatorname{Hom}_+(L, V_{i+1}/V_i))$$

is complex holomorphic, we have

$$\operatorname{ord} \delta^n = n \deg K + \deg(V/V_{n-1}) - \deg L \,.$$

The Hopf fields A and Q are holomorphic sections in the appropriate bundles and so is [Les]

$$AQ \in H^0(K^2 \operatorname{Hom}_+(V/V_{n-1}, L))\,.$$

Therefore, if $AQ \not\equiv 0$ then

$$0 \leq \operatorname{ord}(AQ) = 2\deg K + \deg L - \deg(V/V_{n-1}) = (n+2)\deg K - \operatorname{ord}\delta^n\,.$$

In the case of sphere, i.e., $\deg K < 0$, this implies $AQ \equiv 0$ since otherwise

$$\operatorname{ord}(AQ) = (n+2)\deg K - \operatorname{ord}\delta^n < 0\,.$$

In the case of a torus, i.e., $\deg K = 0$, the above inequality gives for $AQ \not\equiv 0$, that A and Q are nowhere vanishing, and

$$\operatorname{ord}\delta^n = 0$$

In particular, a Willmore torus $f : T^2 \to \mathbb{HP}^n$ with $AQ \not\equiv 0$ is unramified.

If $AQ \equiv 0$ and the backward Bäcklund transform $\hat{f}$ is not a point in $\mathbb{HP}^n$, i.e., $\hat{\delta} \not\equiv 0$, then it is shown in [Les] that

$$*\hat{\delta} = -S\hat{\delta} = -\hat{\delta}S\,,$$

and thus

$$A\hat{\delta}Q \in H^0(K^3 \operatorname{Hom}_+(V/V_{n-1}, L))\,.$$

As before this implies $A\hat{\delta}Q \equiv 0$ in case of a Willmore sphere. Moreover, for a Willmore torus with $A\hat{\delta}Q \not\equiv 0$ we again see that $\operatorname{ord}\delta^n = 0$. In particular, if we assume that the backward Bäcklund transform of a Willmore sphere $f : S^2 \to \mathbb{HP}^n$ is a full curve in $\mathbb{HP}^n$, we see by proceeding inductively that $A \equiv 0$. In the case of a Willmore torus $f : T^2 \to \mathbb{HP}^n$ with $\hat{f} : T^2 \to \mathbb{HP}^n$ full, we have $A \equiv 0$ or $f : T^2 \to \mathbb{HP}^n$ is unramified.

Lemma 3.24. *If $f : S^2 \to \mathbb{HP}^n$ is a Willmore sphere in $\mathbb{HP}^n$ then the forward and backward Bäcklund transforms are not full curves in $\mathbb{HP}^n$ unless f or $f^\dagger$ is the twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$.*

If the Willmore torus $f : T^2 \to \mathbb{HP}^n$ (or its dual $f^\dagger$) is not obtained by a twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$ and if the backward (or forward) Bäcklund transform $\hat{f} : T^2 \to \mathbb{HP}^n$ is a full curve in $\mathbb{HP}^n$ then $f : T^2 \to \mathbb{HP}^n$ is unramified.

Moreover, in the case of a Willmore sphere $f : S^2 \to \mathbb{HP}^n$ with backward Bäcklund transform $\hat{f} : M \to \mathbb{HP}^k$ in $\mathbb{HP}^k, k < n$, the above computations show that the canonical complex structure of $\hat{f}$ is given by $-S$. This allows to extend the argument used for a constant backward Bäcklund transform $\hat{f}$ to Bäcklund transforms with $\hat{L} \subset W \subset V$ since W is then S–stable. Again, f projects onto a dual curve of a twistor projection of a holomorphic curve in $\mathbb{CP}^{2m+1}$ for $m = n - k$:

Theorem 3.25 (see [Les])**.** *A Willmore sphere $f : S^2 \to \mathbb{HP}^n$ has integer Willmore energy*

$$\mathcal{W}(f) \in 4\pi\mathbb{N}\,.$$

If $n \neq 1$ then f comes from the twistor projection of a holomorphic curve $h : S^2 \to \mathbb{CP}^{2k+1}$ for some $k \leq n$. If $n = 1$ and f does not arise from a twistor projection of a holomorphic curve $h : S^2 \to \mathbb{CP}^3$ then, after choosing a suitable point at infinity, f is a minimal sphere in $\mathbb{R}^4$ with planar ends.

For a Willmore curve $f : M \to \mathbb{HP}^n$ the Hopf field $A \in H^0(K \operatorname{Hom}_+(\tilde{V}/\tilde{V}_{n-1}, L))$ is a holomorphic section where $\tilde{L}$ is the forward Bäcklund transform of f given by $\ker A$ and $\tilde{V}_i$ denotes the Frenet flag of $\tilde{L}$, in particular, we have

$$0 \leq \operatorname{ord} A\tilde{\delta}^n = \deg(K^{n+1} \operatorname{Hom}_+(\tilde{L}, L)) = (n+1) \deg K + \deg L - \deg \tilde{L}\,.$$

Now, let $f : T^2 \to \mathbb{HP}^n$ be a Willmore torus, and assume that there exists an infinite series of successive Bäcklund transforms $\tilde{f}^m$. We telescope the above inequality and obtain, using $\deg K = 0$,

$$0 \leq \deg L - \deg \tilde{L}^m \tag{3.12}$$

The degree of $\tilde{L}^m$ is given by the Plücker formula (1.59) as

$$\deg \tilde{L}^m = \frac{1}{4\pi(n+1)}(\mathcal{W}(\tilde{f}^m) - \mathcal{W}((\tilde{f}^m)^\dagger))\,,$$

since the Bäcklund transforms $\tilde{f}^m$ are all unramified, see Lemma 3.24, and thus $\operatorname{ord} \tilde{H}^m = 0$. Using (3.11) we have $\mathcal{W}(\tilde{f}^{m-1}) = \mathcal{W}((\tilde{f}^m)^\dagger)$ so that

$$0 \leq \deg L - \frac{1}{4\pi(n+1)}(\mathcal{W}(\tilde{f}^m) - \mathcal{W}(\tilde{f}^{m-1}))$$

Telescoping this inequality again

$$0 \quad \leq \quad m \deg L - \frac{1}{4\pi(n+1)}(W(\tilde{f}^m) - W(f))$$

we finally get

$$-\frac{1}{4\pi(n+1)} W(f) \leq m \deg L\,.$$

Since we can assume without loss of generality that $\deg L < 0$ (otherwise consider the dual Willmore curve $L^\dagger$ which has (1.58) in case of an unramified torus $\deg L^\dagger = -\deg L$), this shows that the sequence of a Willmore torus with $\deg L \neq 0$ is finite. Theorem 3.23 thus shows

Theorem 3.26. *Every unramified Willmore torus $f : T^2 \to \mathbb{HP}^n$ with $\deg L \neq 0$ has integer Willmore energy*

$$\mathcal{W}(f) \in 4\pi\mathbb{N}\,.$$

Remark 3.27. A Willmore torus $f : T^2 \to S^4$ in the 4–sphere is given by complex holomorphic data: If f has trivial normal bundle then f has a spectral curve of finite genus, and is given by theta functions on the spectral curve [Sch02], [FPPS92].

If f has non–trivial normal bundle then f comes from the twistor projection of a holomorphic curve $h : T^2 \to \mathbb{CP}^3$, or is a minimal torus in $\mathbb{R}^4$ with planar ends after choosing an appropriate point at infinity.

To obtain a similar result for Willmore tori in $\mathbb{HP}^n$, $n \geq 2$, with $\deg L \neq 0$, one has to gain control on the canonical complex structure of a Bäcklund transform to "reconstruct" the Willmore torus from its Bäcklund transform $\hat{f}$ even if $\hat{f}$ is not a full curve in $\mathbb{HP}^n$.

Moreover, we conjecture that Willmore surfaces $f : M \to \mathbb{HP}^n$ with large degree of L (in terms of the genus of M) have integer Willmore energy, and are given, at least in case $n = 1$, as twistor projections of holomorphic curves in complex projective space or are minimal surfaces in $\mathbb{R}^4$. To prove such a result, one has to find sharper estimates on the vanishing orders of the involved holomorphic sections A, Q and δ^n.

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