NEW EXAMPLES OF WILLMORE TORI IN $S^4$

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Abstract. Using the (generalized) Darboux transformation in the case of the Clifford torus, we construct for all Pythagorean triples $(p,q,n) \in \mathbb{Z}^3$ a $\mathbb{CP}^3$–family of Willmore tori in $S^4$ with Willmore energy $2n\pi^2$.

1. Introduction

Classical geometers like Bianchi, Darboux and Bäcklund used local transformations to obtain new examples of a particular class of surfaces out of simple known ones by geometric constructions. For instance, the Darboux transformation was classically [6] defined for isothermic surfaces, that is surfaces which allow a conformal curvature line parametrization: two immersions $f$ and $f^\#$ form a Darboux pair if there exists a sphere congruence which envelopes both surfaces $f$ and $f^\#$. In this case, both $f$ and $f^\#$ are isothermic.

In modern days, the Darboux transformation is used to study global properties of surfaces: relaxing the enveloping condition one obtains a (generalized) Darboux transformation for conformal immersions $f : M \to S^4$ of a Riemann surface into the 4–sphere. The existence of a Riemann surface worth of global solutions is, at least in the case when $M = T^2$ is a 2–torus, given by the link to the multiplier spectral curve of the conformal torus [3]. Here points on the multiplier spectral curve correspond to closed Darboux transforms of $f$.

In the case when the conformal immersion is given by a harmonicity condition, e.g. for constant mean curvature surfaces, Hamiltonian stationary Lagrangians or (constrained) Willmore surfaces, one obtains an associated family of flat connections, and one can construct Darboux transforms – so called $\mu$–Darboux transforms – by using parallel sections of these [5], [12], [11], [2]. In the spirit of classical surface theory, we study in this short note the $\mu$–Darboux transforms of the Clifford torus $f : M \to S^3$ to obtain new Willmore tori in $S^4$.

2. The Darboux transformation

We briefly recall the Darboux transformation on a conformal immersion $f : M \to S^4$ of a Riemann surface into the 4–sphere [3]. To this end, we consider the 4–sphere $S^4 = \mathbb{HP}^1$ as quaternionic projective line and identify $f : M \to S^4$ with the pull–back $L = f^*T$ of the tautological line bundle over $\mathbb{HP}^1$ by $f$, that is $L_p = f(p)$. The derivative of $f$ can be identified with the map $\delta = \pi d|_L$ where $\pi : V \to V/L$ is the canonical projection of the trivial $\mathbb{H}^2$ bundle $V$, and $d$ is the trivial connection on $V$. Moreover, $f$ is a conformal immersion if and only if there exists a complex structure $S \in \Gamma(\text{End}(V))$, $S^2 = -1$. 

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stabilizing $L$ such that
\[(2.1) \quad *\delta = S\delta = \delta S,\]
where $*$ denotes the negative Hodge star operator. Complex structures $S$ on $V$ can be, and will be in the following, identified with sphere congruences [4, Prop. 2]. The conformality condition $(2.1)$ means geometrically that the sphere congruence $S$ envelopes $f$, that is, $S$ passes through $f$ and the tangent planes of $f$ and $S$ coincide at corresponding points in an oriented way. In particular, two immersions $f, f^\# : M \to S^4$ are classical Darboux transforms of each other, if there exists a complex structure $S \in \Gamma(\text{End}(V))$ with $*\delta = S\delta = \delta S$ and $*\delta^\# = S\delta^\# = \delta^\# S$ where $\delta$ and $\delta^\#$ denote the derivatives of $f$ and $f^\#$ respectively.

To obtain the Darboux transformation for conformal immersions $f : M \to S^4$ one considers sphere congruences with a relaxed enveloping condition:

**Definition 2.1** ([3]). Let $f, \hat{f} : M \to S^4$ be conformal immersions. Then $\hat{f}$ is a Darboux transform of $f$ if there exists a sphere congruence enveloping $f$ and left-enveloping $\hat{f}$, that is if there exists a complex structure $S \in \Gamma(\text{End}(V))$, $S^2 = -1$, with $*\delta = S\delta = \delta S$ and $*\delta^\# = S\delta^\# = \delta^\# S$.

In the case when $M = T^2$ is a 2–torus there always exists a Riemann surface $\Sigma$ worth of Darboux transforms, and indeed $\Sigma$ is the multiplier spectral curve [3] of the conformal torus. We shortly recall the construction of Darboux transforms: since $f$ is a conformal immersion, that is in particular $*\delta = S\delta$, the complex structure $S$ induces a complex structure $J = S_{V/L} \in \Gamma(\text{End}(V/L))$, $J^2 = -1$, on the line bundle $V/L$.

**Lemma 2.2** ([3]). Let $f : M \to S^4$ be a conformal immersion and $J$ be the associated complex structure on $V/L$. Then $D\varphi := (\pi d\hat{\varphi})''$ defines a (quaternionic) holomorphic structure on $V/L$. Here $\hat{\varphi}$ is an arbitrary lift of $\varphi = \pi \hat{\varphi} \in \Gamma(V/L)$, and $\omega'' = \frac{1}{2}(\omega + J * \omega)$ denotes the $(0,1)$ part of a 1–form $\omega \in \Omega^1(V/L)$ with respect to the complex structure $J$.

Indeed, $D$ is well-defined since $f$ is conformal and thus $(\pi d\psi)'' = \delta\psi'' = 0$ for $\psi \in \Gamma(L)$. A holomorphic structure is an elliptic operator [8, Sec. 2], and thus has finite dimensional kernel $\ker D =: H^0(V/L)$. To obtain a Riemann surface worth of Darboux transforms of a conformal torus, we have to use holomorphic sections with multiplier, that is $\varphi \in \ker D \subset \Gamma(V/L)$ with
\[\gamma^*\varphi = \varphi h_\gamma, \quad h_\gamma \in \mathbb{C}_*, \quad \gamma \in \pi_1(M),\]
where we denote by $\tilde{W}$ the pullback of a bundle $W$ to the universal cover $\tilde{M}$ of $M$.

**Lemma 2.3** (see [3]). Every holomorphic section with multiplier $\varphi \in H^0(V/L)$ of the canonical holomorphic bundle of a conformal immersion $f : M \to S^4$ has a unique lift $\hat{\varphi} \in \Gamma(V)$ such that
\[(2.2) \quad \pi d\hat{\varphi} = 0,\]
where $\pi : V \to V/L$ is the canonical projection. This unique lift $\hat{\varphi}$ is called the prolongation of $\varphi$. 

The conformality condition $(2.1)$ means geometrically that the sphere congruence $S$ envelopes $f$, that is, $S$ passes through $f$ and the tangent planes of $f$ and $S$ coincide at corresponding points in an oriented way. In particular, two immersions $f, f^\# : M \to S^4$ are classical Darboux transforms of each other, if there exists a complex structure $S \in \Gamma(\text{End}(V))$ with $*\delta = S\delta = \delta S$ and $*\delta^\# = S\delta^\# = \delta^\# S$ where $\delta$ and $\delta^\#$ denote the derivatives of $f$ and $f^\#$ respectively.
Note that the prolongation \( \hat{\varphi} \) has the same multiplier as \( \varphi \) so that, if \( \varphi \) has no zeros, \( \hat{f} = \hat{\varphi} : M \to S^4 \) defines a map from the Riemann surface \( M \) into the 4–sphere which turns out to be a Darboux transform of \( f \). In the case when \( \varphi \) has zeros, one obtains a conformal map \( \hat{f} \) away from the zeros of \( \varphi \), which is again a Darboux transform on its domain. Such a map \( \hat{f} \) is called a \textit{singular} Darboux transform.

**Lemma 2.4** ([3]). A branched conformal immersion \( \hat{f} : M \to S^4 \) is a (singular) Darboux transform of \( f \) if and only if \( \hat{f} \) is obtained by the non–constant prolongation of a holomorphic section \( \varphi \in H^0(\tilde{V}/L) \) with multiplier.

**Remark 2.5.** If we omit the closing condition that \( \varphi \) has a multiplier, we obtain Darboux transforms \( \hat{f} : \tilde{M} \to S^4 \) on the universal cover \( \tilde{M} \) of \( M \).

Given a complex structure \( S \), we decompose the derivative of \( S \)
\[
dS = 2(\ast Q - \ast A)
\]
into \((1,0)\) and \((0,1)\)–parts
\[
(dS)' = \frac{1}{2}(dS - S \ast dS) = -2 \ast A
\]
and
\[
(dS)'' = \frac{1}{2}(dS + S \ast dS) = 2 \ast Q
\]
respectively. The \textit{conformal Gauss map} of a conformal immersion \( f : M \to S^4 \) is a sphere congruence which envelopes \( f \) and has the same mean curvature vector \( \mathcal{H} \) as \( f \). In terms of the corresponding complex structure \( S \), this reads as [4, Thm. 2]
\[
(2.3) \quad \ast \delta = S \delta = \delta S \quad \text{and} \quad \text{im} A \subset \Omega^1(L).
\]
In this case, \( A, Q \) are called the \textit{Hopf fields} of \( f \). Since \( S^2 = -1 \) the Hopf fields satisfy
\[
(2.4) \quad \ast A = SA = -AS \quad \text{and} \quad \ast Q = -SQ = QS
\]
Let now \( f : M \to S^4 \) be a Willmore surface that is \( f \) is an immersion which is a critical point of the Willmore energy \( W(f) = \int_M \mathcal{H}^2 dA \) under variations with compact support. It is a well-known fact [7],[13] that \( f \) is Willmore if and only if the conformal Gauss map of \( f \) is harmonic. This can be expressed [4, Prop. 5] by the condition
\[
d \ast A = 0 \quad \text{or, equivalently,} \quad d \ast Q = 0.
\]

**Lemma 2.6** ([8, Lemma 6.3]). Let \( f : M \to S^4 \) be a conformal immersion with conformal Gauss map \( S \) and Hopf field \( A \). Then \( f \) is Willmore if and only if the family of complex connections
\[
d^\mu = d + \ast A(S(a - 1) + b)
\]
is flat for all \( \mu \in \mathbb{C}_+ \). Here \( \mathbb{C} = \text{Span}\{1, I\} \) where \( I \) is the complex structure on \( V \) given by right multiplication by the imaginary quaternion \( i \), and
\[
a = \frac{\mu + \mu^{-1}}{2}, \quad b = I \frac{\mu^{-1} - \mu}{2}.
\]

**Proof.** Since \( d \) is the trivial connection and \([I, S] = 0\), the curvature of \( d^\mu \) is given by
\[
R^\mu = (d \ast A)(S(a - 1) + b)
\]
where we used that \( Q \wedge A = 0 \) by type considerations. Therefore, \( S \) is harmonic if and only if \( d^\mu \) is flat. \( \square \)
We consider now parallel sections of $d\mu$ with multiplier that is $d\mu \hat{\varphi} = 0$ and $\gamma^* \hat{\varphi} = \varphi h_\gamma$, $h_\gamma \in \mathbb{C}_\times$, $\gamma \in \pi_1(M)$. Denoting the projection of $\hat{\varphi}$ to $V/L$ by $\varphi = \pi \hat{\varphi} \in \Gamma(V/L)$ and recalling (2.3) that $\ast A(S\hat{\varphi}(a-1) + \hat{\varphi} b) \in \Gamma(L)$, we obtain
\[ \pi d\mu \hat{\varphi} = 0. \]

In particular, $\varphi$ is a holomorphic section with multiplier, and $\hat{\varphi}$ is the prolongation of $\varphi$. In other words, every $d\mu$–parallel section with multiplier gives rise to a Darboux transform of $f$.

**Definition 2.7.** A Darboux transform $\hat{f} : M \to S^4$ which is given by a parallel section of $d\mu$ is called a $\mu$–Darboux transform of $f$.

Although in general the Darboux transforms of a Willmore torus are not necessary Willmore [1], the $\mu$–Darboux transforms are [2].

### 3. The Clifford torus

In this paper we shall compute all $\mu$–Darboux transforms of the Clifford torus
\[ f : \mathbb{C}/\Gamma \to S^3, \quad u + iv \mapsto \frac{1}{\sqrt{2}}(e^{iu} + je^{iv}), \]
where $\Gamma = 2\pi \mathbb{Z} + 2\pi i \mathbb{Z}$ is the lattice in $\mathbb{C}$. Since the multiplier spectral curve of the Clifford torus has genus zero, parallel sections of the family of flat connections $d\mu$ can be computed explicitly. Note that though $f$ maps into the 3–sphere, the $\mu$–Darboux transforms will be conformal immersions into the 4–sphere. Therefore, we will consider a map $f : M \to S^3$ into the 3–sphere with the inclusions
\[ S^3 \hookrightarrow \mathbb{R}^4 = \mathbb{H} \quad \text{and} \quad \mathbb{H} \hookrightarrow \mathbb{HP}^1, x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix} \]
as a map into the 4–sphere. The associated line bundle of $f$ is given by $L = \psi \mathbb{H}$ where
\[ \psi = \begin{pmatrix} f \\ 1 \end{pmatrix}. \]
The derivative of $L$ is given by
\[ \delta \psi = \pi \begin{pmatrix} df \\ 0 \end{pmatrix} \]
so that $f$ is conformal if and only if there exists left and right normals $N, R : M \to S^2$ with $\ast df = N df = -df R$. The mean curvature vector $\mathcal{H}$ of a conformal immersion $f$ is given [4, Sec. 7.2] by
\[ \mathcal{H} = -N \bar{H} \]
where $H$ is defined by $df H = (dN)'$. Here $'$ denotes the $(1,0)$ part with respect to the complex structure given by left multiplication by $N$, that is
\[ \omega' = \frac{1}{2} (\omega - N \ast \omega). \]
In particular, the conformal Gauss map of $f$ is given by $S = GS_0G^{-1}$ where
\[ G = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S_0 = \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix}, \]
\[ (3.1) \]
and the Hopf field \( A = GA_0G^{-1} \) by
\[
* A_0 = \frac{1}{4} \left( dH + H * df + R * dH - H * dN - dR + R * dR \right).
\]

Let us now turn to the case when \( f : \mathbb{C}/\Gamma \to S^3 \) is the Clifford torus. Then \( f \) is a conformal immersion with left and right normal
\[
N(u,v) = je^{i(u-v)} \quad \text{and} \quad R(u,v) = je^{i(v+u)},
\]
and mean curvature vector \( \mathcal{H} = -N\bar{H} \) where
\[
H = \frac{\sqrt{2}}{2} (e^{-iu} + je^{iv}).
\]

Moreover \( f \) satisfies the following fundamental symmetries
\[
(i) \ R = Hf, \quad N = fH \\
(ii) \ H \text{ is conformal with } * dH = -RdH = dHN.
\]

Therefore, the Hopf field \( A = GA_0G^{-1} \) is given by
\[
* A_0 = \frac{1}{4} \left( \begin{array}{cc} 0 & 0 \\ dH & 2dHf \end{array} \right)
\]
where we also used that \( RH = HN \), see [4, Sec. 7.2].

### 4. \( \mu \)-Darboux transforms

To compute \( \mu \)-Darboux transforms of the Clifford torus \( f \) we have to find parallel sections \( \hat{\varphi} \in \Gamma(V) \) of the family of flat connections \( d^\mu \) on the trivial \( \mathbb{H}^2 \) bundle \( V \). We solve the differential equation \( d^\mu \hat{\varphi} = 0 \) that is with (2.4)
\[
d\hat{\varphi} = -A\hat{\varphi}(a - 1) - * A\hat{\varphi}b.
\]

Putting \( \phi := G^{-1}\hat{\varphi} \) we can equivalently find solutions of
\[
(4.1) \quad d\phi = -A_0\phi(a - 1) - * A_0\phi b - (dG)\phi,
\]
where we use that \( G^{-1}dG = dG \). Since the connections \( d^\mu \) are complex, this leads to a system of complex differential equations: Writing \( \phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \) and decomposing \( \alpha = \alpha_1 + j\alpha_2, \beta = \beta_1 + j\beta_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma(\mathbb{C}) \) with respect to the splitting \( \mathbb{H} = \mathbb{C} + j\mathbb{C} \), we consider
\[
\phi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} \in \Gamma(\mathbb{C}^4)
\]
as a section of the trivial \( \mathbb{C}^4 \) bundle. After a lengthy but straightforward computation [9] we obtain the system of linear partial differential equation with non-constant coefficients:
\[
(4.2) \quad \phi_u = U\phi, \quad \phi_v = V\phi,
\]
where we denote by \( (\cdot)_u \) and \( (\cdot)_v \) the partial derivatives with respect to \( u \) and \( v \) respectively, and

\[
U(u, v) = \frac{1}{4\sqrt{2}} \begin{pmatrix}
0 & 0 & -4ie^{iu} & 0 \\
0 & 4\sqrt{2}i & 0 & 4i \\
ie^{iv}(a - 1) & -ib & \sqrt{2}i((a - 1) + b) & \sqrt{2}i(a - 1 - b) \\
ib & i(a - 1) & \sqrt{2}i((a - 1) + b) & -\sqrt{2}i(a - 1 - b)
\end{pmatrix}
\]

\[
V(u, v) = \frac{1}{4\sqrt{2}} \begin{pmatrix}
0 & 0 & -4ie^{iv} & 0 \\
0 & 4\sqrt{2}i & 0 & 4i \\
-ib & ie^{iu}(a - 1) & \sqrt{2}i(a - 1 - b) & -\sqrt{2}i(a - 1 + b) \\
i(b) & -ie^{iu}(a - 1) & \sqrt{2}i(a - 1 - b) & \sqrt{2}i(a - 1 + b)
\end{pmatrix}
\]

Lemma 4.1. A section \( \hat{\varphi} \in \Gamma(V) \) is parallel with respect to \( d\mu \) if and only if

\[ \eta := e^{D}G^{-1}\hat{\varphi}, \quad D(u, v) := \text{diag}(iv, iu, i(u + v), 0), \]

solves

\[ (4.3) \quad \eta_u = \tilde{U}\eta, \quad \eta_v = \tilde{V}\eta, \]

where

\[ \tilde{U} = \frac{1}{4\sqrt{2}} \begin{pmatrix}
0 & 0 & -4i & 0 \\
0 & 4\sqrt{2}i & 0 & 4i \\
i(b) & i(a - 1) & \sqrt{2}i((a - 1) + b) & \sqrt{2}i(a - 1 - b) \\
i(b) & i(a - 1) & \sqrt{2}i((a - 1) + b) & \sqrt{2}i(a - 1 - b)
\end{pmatrix}, \]

\[ \tilde{V} = \frac{1}{4\sqrt{2}} \begin{pmatrix}
4\sqrt{2}i & 0 & 0 & -4i \\
0 & 0 & -4i & 0 \\
i(a - 1) & -ib & \sqrt{2}i((a - 1) + b) & -\sqrt{2}i(a - 1 + b) \\
ib & -i(a - 1) & -\sqrt{2}i(a - 1 + b) & -\sqrt{2}i(a - 1 - b)
\end{pmatrix}. \]

are constant. In particular, \( \tilde{U} \) and \( \tilde{V} \) are commuting matrices.

Proof. The systems of linear differential equations (4.2) and (4.3) are equivalent for

\[ \tilde{U} = e^{D}(D_u + U)e^{-D} \quad \text{and} \quad \tilde{V} = e^{D}(D_v + V)e^{-D}. \]

One easily verifies

\[ e^{D}Ue^{-D} = \frac{1}{4\sqrt{2}} \begin{pmatrix}
0 & 0 & -4i & 0 \\
0 & 0 & 0 & 4i \\
i(b) & i(a - 1) & \sqrt{2}i(a - 1 + b) & \sqrt{2}i(a - 1 - b) \\
i(b) & i(a - 1) & \sqrt{2}i(a - 1 - b) & \sqrt{2}i(a - 1 + b)
\end{pmatrix} \]

so that \( \tilde{U} \) is given by (4.4), and a similar computation gives \( \tilde{V} \). Finally, since \( \tilde{U} \) and \( \tilde{V} \) are constant, the compatibility condition \( \eta_{uv} = \eta_{vu} \) shows that \( \tilde{U} \) and \( \tilde{V} \) are commuting. \( \square \)

Since \( \tilde{U} \) and \( \tilde{V} \) are simultaneously diagonalizable, all solutions of (4.3) are of the form

\[ \eta(u, v) = Ce^{D_1u + D_2v}, \quad c \in \mathbb{C}^4. \]

where \( C \) is a common basis auf eigenvectors of \( \tilde{U} \) and \( \tilde{V} \), and \( D_1, D_2 \) are the corresponding diagonal matrices of eigenvalues.
Lemma 4.2.  

(i) The spectra of $\tilde{U}$ and $\tilde{V}$ coincide, and 

$$\text{spec}(\tilde{U}) = \{ \lambda_k \mid k \in \mathbb{Z}_4 \}, \quad \lambda_k := \lambda(i^k x).$$

Here we put $x := e^{\frac{i}{4}\log(\mu)}$, where $\log$ is the main branch of the logarithm, and 

$$\lambda(y) = \frac{(1+i)(y+1)(y+i)}{4y},$$

that is 

\begin{align*}
\lambda_0 &= \frac{(1+i)(x+1)(x+i)}{4x}, \\
\lambda_1 &= -\frac{(1-i)(x+1)(x-i)}{4x}, \\
\lambda_2 &= -\frac{(1+i)(x-1)(x-i)}{4x}, \\
\lambda_3 &= \frac{(1-i)(x-1)(x+i)}{4x}.
\end{align*}

(ii) Let 

$$w(y) = \begin{pmatrix} \frac{1}{\sqrt{2}} \xi(y) \\ \frac{1}{\sqrt{2}} i \xi(y) \lambda(y) \\ i(i - \lambda(y)) \end{pmatrix}$$

with 

$$\xi(y) := \frac{\mu - i}{y + i},$$

and define $w_k := w(i^k x)$ and $\xi_k = \xi(i^k x)$ where again $x = e^{\frac{i}{4}\log(\mu)}$.

- For $\mu \neq \pm 1$ the eigenvalues of $\tilde{U}$ (and $\tilde{V}$) are pairwise distinct. The eigenspaces of $\tilde{U}$ and $\tilde{V}$ are spanned by 

$$E_{\lambda_k}(\tilde{U}) = \text{span}\{w_k\}.$$ 

- For $\mu = 1$ the eigenvalues $\lambda_0 = \lambda_1 = i, \lambda_2 = \lambda_3 = 0$ coincide, and the complex 2-dimensional eigenspaces are given by 

$$E_{\lambda_0}(\tilde{U}) = \lim_{\mu \to 1} E_{\lambda_0}(\tilde{U}) \oplus E_{\lambda_1}(\tilde{U}),$$

$$E_{\lambda_i}(\tilde{U}) = \lim_{\mu \to 1} E_{\lambda_2}(\tilde{U}) \oplus E_{\lambda_3}(\tilde{U}).$$

- For $\mu = -1$ the eigenvalues are $\lambda_0 = \frac{1+i\sqrt{2}}{2} i, \lambda_2 = \frac{1-i\sqrt{2}}{2} i$ and $\lambda_1 = \lambda_3 = \frac{1}{2} i$. The eigenspaces are given by $E_{\lambda_k}(\tilde{U}) = \text{span}\{w_k\}, k = 0, 2$, and 

$$E_{\lambda_{-\frac{1}{2}}}(\tilde{U}) = \lim_{\mu \to -1} E_{\lambda_1}(\tilde{U}) \oplus E_{\lambda_3}(\tilde{U}),$$

where the latter is again complex 2-dimensional.

(iii) Let $\lambda_k \in \text{spec}(\tilde{U})$ be an eigenvalue of $\tilde{U}$, and define 

$$\epsilon_k := \xi_k \lambda_k = \lambda_{k+1}, \quad k \in \mathbb{Z}_4.$$ 

Then $\epsilon_k$ is an eigenvalue of $\tilde{V}$, and 

$$E_{\lambda_k}(\tilde{U}) = E_{\epsilon_k}(\tilde{V}).$$

We skip the computational proof [9] and remark that the group $< \zeta_4 >= < i >$ acts on the spectrum by 

$$\lambda(\sqrt[4]{\mu}) \mapsto \lambda(i\sqrt[4]{\mu})$$

for some fourth root $\sqrt[4]{\mu}$ of $\mu$. For the subgroup $< \zeta_2 >= < -1 >$ the action can be described by 

$$\lambda(-\sqrt[4]{\mu}) = i - \lambda(\sqrt[4]{\mu}) \quad \text{resp.} \quad \lambda_{k+2} = i - \lambda_k, \quad k \in \mathbb{Z}_4.$$
Furthermore we see that the eigenvalues are discontinuous in $\mu \in \mathbb{C}^*$ but are continuous on the 4 : 1-covering $\mathbb{C}^* \to \mathbb{C}^*$ given by $x \mapsto x^4 = \mu$. The group $< \zeta_4 >$ acts as deck transformations of this covering.

We summarize:

**Proposition 4.3.** For each $\mu \in \mathbb{C}^*$ the fundamental parallel sections $\hat{\phi}_k := G\phi_k, k = 0, \ldots, 3$, span the space of $d^4$–parallel sections where

\[
\phi_k := e^{-D}C e^{D_1 u + D_2 v} e_k.
\]

Here $e_k \in \mathbb{C}^4$ is the $(k + 1)$-th standard basis vector,

\[
D = \text{diag}(iv, iu, i(u + v), 0)
\]

\[
D_1 = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)
\]

\[
D_2 = \text{diag}(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)
\]

and the columns of $C$ are the corresponding basis of eigenvectors of $\hat{U}$. In particular, for $\mu \neq 1$ we get

\[
\phi_k = \left( \frac{1}{\sqrt{2}}(\xi_k e^{-iv} + je^{-iu}) \right) e^{\lambda_k u + \epsilon_k v}
\]

and for $\mu = 1$

\[
\phi_0 = \begin{pmatrix} f \\ -1 \end{pmatrix}, \quad \phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} j \\ 0 \end{pmatrix}, \quad \phi_2 = \phi_0 j, \quad \phi_3 = \phi_1 ij.
\]

We now obtain all $\mu$–Darboux transforms on the universal cover $\hat{M} = \mathbb{C}$ of the Clifford torus:

**Theorem 4.4.** Every $\mu$–Darboux transform $\hat{f} : \mathbb{C} \to S^4$ of the Clifford torus, $\mu \neq 1$, is given by

\[
\hat{f}(u, v) = \frac{1}{\sqrt{2}} (g_1(u, v) e^{iu} + j g_2(u, v) e^{iv}),
\]

where

\[
g_1(u, v) = \sum_{k,l=0}^{3} \frac{(-i - \lambda_k)(\xi_k e^{-iv})}{(\lambda_k + \xi_k e^{-iv}) e^{(\lambda_k + \xi_k e^{-iv}) u + (\epsilon_k + \pi) v} s_k s_l}
\]

\[
g_2(u, v) = \sum_{k,l=0}^{3} \frac{(-i - \lambda_k)(\xi_k e^{-iv})}{(\lambda_k + \xi_k e^{-iv}) e^{(\lambda_k + \xi_k e^{-iv}) u + (\epsilon_k + \pi) v} s_k s_l}
\]

with $s_k \in \mathbb{C}$.

**Proof.** Let $\mu \neq 1$ and $\phi = \sum_{k=0}^{3} \phi_k s_k$ be a parallel section of $d^4$ where $s_k \in \mathbb{C}$ and $\phi_k$ are the fundamental solutions (4.7). Then $\hat{f} = f + \alpha \beta^{-1}$ is the $\mu$–Darboux transform given by $\phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, and the claim follows by a straightforward computation. $\square$

**Remark 4.5.** In [2, Thm. 2.5] it is shown that all immersed $\mu$–Darboux transforms of a Willmore surface are again Willmore. In particular, the $\mu$–Darboux transforms obtained above are Willmore surfaces in $S^4$. 
NEW EXAMPLES OF WILLMORE TORI IN $S^4$

So far, we considered the $\mu$–Darboux transformation on the universal cover $\mathbb{C}$ of the 2–torus $T^2 = \mathbb{C}/\Gamma$. To obtain tori we have to find parallel sections with multiplier. Since $\dot{\phi} = G^{-1} \phi$, and $G$ is defined (3.1) on $T^2 = \mathbb{C}/\Gamma$, it is enough to find solutions $\phi$ of (4.2) with multiplier.

**Theorem 4.6.** Let $f : \mathbb{C}/\Gamma \to S^3$ be the Clifford torus.

(i) A fundamental solution $\phi_k$ is a parallel section of $d\mu$ with multiplier, and the $\mu$–Darboux transform given by $\phi_k$, $\mu \neq 1$, is obtained by rotating and scaling $f$. For $\mu = 1$ all $\mu$-Darboux transforms are constant.

(ii) Let $\mu \neq 1$ and $\hat{f} : \mathbb{C}/\Gamma \to S^4$ be a closed $\mu$-Darboux transform of $f$. Then there exists a fundamental solution $\hat{\phi}_k = G\phi_k$ with

$$\hat{f} = \hat{\phi}_k \mathbb{H}.$$

In particular, every non-constant $\mu$-Darboux transform $\hat{f} : \mathbb{C}/\Gamma \to S^4$ of $f$ is the Clifford torus.

**Proof.**

(i) If

$$\phi_k = \left( \frac{1}{\sqrt{2}}(\xi_k e^{-iv} + je^{-iu}) \right) e^{\lambda_k u + \epsilon_k v}.$$

is a fundamental solution, then the corresponding $\mu$-Darboux transform is

$$\hat{f} = \frac{1}{\sqrt{2}}(r_1 e^{iu} + r_2 e^{iv}),$$

where

$$r_1 = \frac{|\epsilon_k|^2 + |i - \lambda_k|^2 - i\xi_k \epsilon_k + i(i - \lambda_k)}{|\epsilon_k|^2 + |i - \lambda_k|^2},$$

$$r_2 = \frac{|\epsilon_k|^2 + |i - \lambda_k|^2 - i\xi_k \epsilon_k - \xi_k i(i - \lambda_k)}{|\epsilon_k|^2 + |i - \lambda_k|^2}. $$
One easily verifies with \( \epsilon_k = \xi_k \lambda_k \) and \( i - \lambda_k = -\xi_k (i - \epsilon_k) \) that
\[
\frac{r_1}{r_2} = -\frac{\xi_k}{\xi_k} \in S^1,
\]
so that \( r_2 = r_1 e^{\theta} \) for a \( \theta \in \mathbb{R} \) and \( \hat{f}(u, v) = f(u, v + \theta)r_1 \).

Proposition 4.3 implies that \( \hat{\phi}_k = G\phi_k \) is constant for \( \mu = 1 \), and thus an arbitrary solution \( \phi = \sum_k \phi_k s_k \) gives a constant Darboux transform \( \hat{f} = G\phi^H = \text{const.} \)

(ii) Let \( \hat{f} \) be given by the section \( \phi = G^{-1}\hat{\phi} \) and suppose that \( \phi \) is not a fundamental solution, i.e. \( \phi = \sum_k \phi_k s_k \) and \( s_k, s_l \neq 0 \) for some \( k \neq l \). The monodromy condition implies that
\[
\phi(u + 2\pi, v) = \phi(u, v) h_1 \quad \text{and} \quad \phi(u, v + 2\pi) = \phi(u, v) h_2
\]
with \( h_1, h_2 \in \mathbb{C} \). Since the fundamental solutions
\[
\phi_k = \left( \frac{1}{\sqrt{2}}(\xi_k e^{-iuv} + j e^{-iu}) \right) e^{\lambda_k u + \epsilon_k v}.
\]
are linearly independent over \( \mathbb{C} \), it follows that
\[
h_1 = e^{2\pi \lambda_k} = e^{2\pi \lambda_l} \quad \text{and} \quad h_2 = e^{2\pi \epsilon_k} = e^{2\pi \epsilon_l},
\]
that is
\[
\lambda_k - \lambda_l \in i\mathbb{Z} \quad \text{and} \quad \epsilon_k - \epsilon_l = \lambda_{k+1} - \lambda_{l+1} \in i\mathbb{Z}.
\]
From (4.6) we see that
\[
\lambda_0 - \lambda_1 = \frac{x^2 - 1}{2x}, \quad \lambda_0 - \lambda_3 = \frac{i(x^2 + 1)}{2x}
\]
and the remaining differences \( \lambda_k - \lambda_l \) can be computed by using \( \Sigma_{k=0}^{3} (-1)^k \lambda_k = 0. \) Then it is easy to show that (4.8) is satisfied only if \( x \in \{ \pm 1, \pm i \} \) which contradicts \( \mu = x^4 \neq 1. \)

\[ \square \]

5. The spectral curve

As we have seen, for \( \mu \neq \pm 1 \) the holonomy matrix \( H^\mu \) of the complex connection \( d^\mu \) has four distinct eigenvalues. Since \( H^\mu \) depends holomorphically on \( \mu \), the set of eigenvalues
\[
\text{Eig} = \{ (\mu, h) \mid \exists \hat{\phi} : d^\mu \hat{\phi} = 0 \text{ and } \gamma^* \hat{\phi} = \hat{\phi} h \}
\]
is an analytic set. We denote by \( \Sigma_h \) the normalization of Eig, the so–called spectral curve of the Clifford torus (compare [10]). Note that the map \( \text{Eig} \rightarrow \mathbb{C}_*, (\mu, h) \mapsto \mu \) gives a 4–fold covering \( \Sigma_h \rightarrow \mathbb{C}_* \). Moreover, a fundamental solution \( \phi_k \) defines a holomorphic line bundle \( \mathcal{E}_k^\mu := G\phi_k \mathbb{C} 
→ \Sigma_h \) over the spectral curve. We show that \( \mathcal{E}_k^\mu \) extends holomorphically to \( 0, \infty \), and thus the spectral curve \( \Sigma_h \) can be compactified.

Proposition 5.1. The line bundle \( \mathcal{E}_k^\mu \rightarrow \mathbb{C}_* \) extends holomorphically at \( 0, \infty \), and \( \Sigma_h \rightarrow \mathbb{CP}^1 \) is a 4–fold covering branched at \( 0, \infty \). In particular, the compactified spectral curve \( \bar{\Sigma}_h \) has genus zero, and the Clifford torus \( f \) is the limit
\[
f = \lim_{\mu \rightarrow \infty} \mathcal{E}_k^\mu \mathbb{H} = \lim_{\mu \rightarrow 0} \mathcal{E}_k^\mu \mathbb{H}.
\]
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Figure 2. $\mu$–Darboux transform with $(p, q, n) = (3, 4, 5)$

Proof. Let $\tilde{\varphi}_k = G\phi_k$ be a fundamental solution with

$$\phi_k = \left( \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{2}}(\xi_k e^{-iu} + je^{-iv}) \\ i\xi_k e^{-i(u+v)} + ji(i - \lambda_k) \end{array} \right) e^{\lambda_k u + \epsilon_k v}.$$

Then $\tilde{f} = f + T_k$ with $T_k = \alpha_k \beta_k^{-1}$ and

$$T_k = \frac{1}{\sqrt{2}}(\xi e^{-iv} + je^{-iu})(i\epsilon e^{-i(u+v)} + ji(i - \lambda))^{-1} \to 0$$

since

$$\lambda_k = (1 + i)(i^k x + 1)(i^k x + i) \to \infty,$$

and $|\xi| \to 1$ for $\mu \to 0$ and $\mu \to \infty$. This shows that $\tilde{f} \to f$ for $\mu \to 0$ und $\mu \to \infty$, and

$$E^{\mu}_{\tilde{f}} H = G\phi_k H \to G\left( \begin{array}{c} 0 \\ 1 \end{array} \right) H = \left( \begin{array}{c} f \\ 1 \end{array} \right) H.$$

\box

6. New Willmore tori in $S^4$

As we have seen in Theorem 4.6 the only $\mu$–Darboux transforms of the Clifford torus on $\mathbb{C}/\Gamma$ are obtained by fundamental solutions $\tilde{\varphi}_k$, and in this case the $\mu$–Darboux transform is the reparametrized and scaled Clifford torus $f$. To obtain new examples, we consider an $n$-fold covering $f : \mathbb{C}/\Gamma_n \to S^3$, $u + iv \mapsto \frac{1}{\sqrt{2}}(e^{iu} + je^{iv})$ of the Clifford torus with lattice $\Gamma_n = 2\pi n \mathbb{Z} + 2\pi ni \mathbb{Z}$, and contemplate the $\mu$–Darboux transforms of $f$.

Lemma 6.1. Let $f : \mathbb{C}/\Gamma_n \to S^3$ be the $n$-fold covering of the Clifford torus, and $\mu = x^4 \in \mathbb{C}_n$. Then every $\mu$–Darboux transform $\tilde{f} : \mathbb{C}/\Gamma_n \to S^3$ is $\Gamma_n$ periodic if and only if $x = \frac{p+iq}{n} \in S^1$, where $(p, q) \in \mathbb{Z}^2 \setminus \{0\}$. In this case the multiplier $h : \Gamma_n \to \mathbb{C}^*$ is trivial, i.e. $h \equiv 1$.

Proof. Let $\phi = \sum_k \phi_k s_k$ be a parallel section of $d\mu$, $\mu = x^4$, with $s_k \neq 0$ for all $k$, where $\phi_k$ are the fundamental solutions (4.7). Then

$$\phi(u + 2\pi n, v) = \phi(u, v) \iff h = e^{2\pi n \lambda_k} \quad \text{for all} \quad k = 0, 1, 2, 3.$$
This implies $n(\lambda_k - \lambda_l) \in i\mathbb{Z}$ for all $k, l$, and as in the proof of Theorem 4.6 it is enough to consider

$$n(\lambda_0 - \lambda_1) = \frac{n(x^2 - 1)}{2x} = ip \quad \text{and} \quad n(\lambda_0 - \lambda_3) = \frac{in(x^2 + 1)}{2x} = iq$$

for some $p, q \in \mathbb{Z}$. Using (4.9) we see that these equations can be satisfied if and only if $p^2 + q^2 = n^2$, that is $x = \frac{p+iq}{n} \in S^1$. In this case

$$\lambda_k = \frac{i(\pm p \pm q + n)}{2n}$$

for all $k$.

For an arbitrary Pythagorean triple $(p, q, n)$ it is known that $p \pm q$ and $n$ are both odd, so that $\pm p \pm q + n$ is even so that $h = e^{2\pi n \lambda_k} = 1$. Since $\epsilon_k = \lambda_{k+1}$ we also see that the $v$–periods close.

Since the Darboux transformation preserves the geometric genus of the spectral curve and the Willmore energy [3] we have shown:

**Theorem 6.2.** For all Pythagorean triple $(p, q, n)$ there exists a $\mathbb{C}P^3$ family of Willmore tori $\hat{f} : \mathbb{C}^2/\Gamma_1 \to S^4$ of spectral genus zero with Willmore energy $W(\hat{f}) = 2\pi^2 n$.

### References

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