Optimal Risk Sharing with General Deviation Measures

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Abstract

An optimal risk sharing problem for agents with utility functionals depending only on the expected value and a deviation measure of an uncertain payoff has been studied. The agents are assumed to have no initial endowments. A set of Pareto-optimal solutions to the problem has been characterized, and a particular solution from the set has been suggested. If an equilibrium exists, the suggested solution coincides with an equilibrium solution. As special cases, the optimal risk sharing problem in the form of expected gain maximization and the problem with a linear mean-deviation utility functional including averse and coherent risk measures have been addressed. In the case of expected gain maximization, the existence of an equilibrium has been shown.

1 Introduction

Optimal risk sharing, originated from the actuarial science in the context of reinsurance problems [3] and complementing the theory of collective risk, is now widely used in a variety of finance and economic applications dealing with cooperation of agents with different risk preferences; see [1]. In an abstract setting, an optimal risk sharing problem can be formulated as follows:

> Given an uncertain payoff X and m agents, divide X into (uncertain) shares Y_i, $i = 1,\dots,m$, such that $\sum_{i=1}^m Y_i = X$ with Y_i being acceptable for agent i, $i = 1,\dots,m$.

If risk-reward preferences of agent i are represented by a utility functional U_i, this problem is formulated by

$$\max_{Y\in\mathcal{A}(X)} \{U_1(Y_1),\dots,U_m(Y_m)\}, \tag{1}$$

where $\mathcal{A}(X) = \{Y = (Y_1,\dots,Y_m) | \sum_{i=1}^m Y_i = X\}$. The problem (1), being m-criteria optimization, is usually approached through *Pareto optimality*. A vector $Y = (Y_1,\dots,Y_m) \in \mathcal{A}(X)$ is called a Pareto-optimal division in (1), if there is no $Y' = (Y_1',\dots,Y_m') \in \mathcal{A}(X)$ such that $U_i(Y_i') \ge U_i(Y_i)$, $i = 1,\dots,m$, with at least one inequality being strict. The problem (1) has been extensively studied in the framework of the expected utility theory, i.e., for $U_i(Z) = E[u_i(Z)]$ with utility functions $u_i : \mathbb{R} \to \mathbb{R}$ assumed to be non-decreasing and concave; see [1] for the detailed discussion. In this case, the set of Pareto-optimal solutions has an explicit characterization, and a particular solution from this set can be chosen based on the notion of *equilibrium* defined as follows (see also [1]). Suppose the uncertain payoff X is aggregated agents' initial endowments X_i, i.e., $\sum_{i=1}^m X_i = X$ (for example, X_i may represent a portfolio of risky financial instruments of agent i), and suppose all agents agree on a linear pricing functional $\mathcal{P}$ (or certainty equivalent) for each uncertain payoff. A vector $Y = (Y_1,\dots,Y_m) \in \mathcal{A}(X)$ is called an equilibrium in (1), if each (uncertain) component Y_i in Y solves the optimization problem

$$\max_{Z_i} U_i(Z_i) \qquad \text{s.t.} \quad \mathcal{P}(Z_i) \le \mathcal{P}(X_i), \quad i = 1,\dots,m. \tag{2}$$

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For the case $U_i(Z) = E[u_i(Z)]$, different characterizations of equilibrium has been obtained, and conditions on equilibrium existence has been established; see [1]. However, the well-known criticism of the expected utility theory motivated studying (1) with U_i represented in non expected utility forms.

As an alternative approach, agents' utility functions can be replaced by risk measures $\mathcal{R}_i$, $i = 1,\ldots,m$, and optimal risk sharing can be formulated by

$$\min_{Y\in\mathcal{A}(X)} \{\mathcal{R}_1(Y_1),\ldots,\mathcal{R}_m(Y_m)\}. \tag{3}$$

For example, in the case of coherent risk measures (see [2]), Joini [11] and Burgert and Ruschendorf [4] showed that a vector $Y = (Y_1,\ldots,Y_m)$ is a Pareto-optimal solution to (3) if and only if it minimizes the infimal convolution

$$\mathcal{R}_S(X) = \min_{Y\in\mathcal{A}(X)} \left(\mathcal{R}_1(Y_1)+\ldots+\mathcal{R}_m(Y_m)\right). \tag{4}$$

Burgert and Ruschendorf [4] established that $\mathcal{R}_S(X)$ is a coherent risk measure, which could be considered as the risk measure of the coalition of m agents, and Landsberger and Meilijson [13] showed that components of an optimal solution to (4) are pairwise comonotone.[1] Also, Joini et al. [11] obtained a subdifferential characterization for the optimal solution and derived an explicit formula for $\mathcal{R}_S(X)$ when $\mathcal{R}_i(Y_i)$, $i = 1,\ldots,m$, are comonotone. In the case of a finite probability space, Dana et al. [6] derived conditions for equilibrium existence, based on which a particular solution from the Pareto-optimal set could be chosen. However, to the best of our knowledge, equilibrium conditions for a general probability space are unknown. We provide insight on this issue in a more general framework of optimal risk sharing with general deviation measures.

In this work, we address the problem (1) with utility functionals U_i in the form

$$U_i(Z) = V_i(EZ,\mathcal{D}_i(Z)), \quad i = 1,\ldots,m, \tag{5}$$

where EZ is the expected value of Z, and $\mathcal{D}_i$ is a general deviation measure. Rockafellar et al. [14] introduced general deviation measures as a generalization of the standard deviation and studied a Markowitz's type portfolio selection problem in which the standard deviation was replaced by a general deviation measure. In particular, Rockafellar et al. [16, 15, 17] derived optimality conditions for the portfolio selection problem in the form of *capital asset pricing model* (CAPM) like relations, generalized the classical one fund theorem, and established conditions on market equilibrium with investors using different deviation measures. Grechuk et al. [8] showed that compared to the mean-variance approach and coherent risk measures in the context of the theory of choice, the mean-deviation approach corresponds to more relaxed axioms on preference relations. Also, Grechuk et al. [9] showed that on a stock market, individual investors, whose preferences are represented by (5), are strictly better off to form a coalition and to invest in a cooperative portfolio. In this context, the following question arise: how to optimally divide cooperative portfolio's uncertain payoff among the participating investors under the assumption that the investors had *no* initial endowments X_i?

This work presents an equilibrium (or pricing) approach to solving the optimal risk sharing problem (1) with utility functionals (5) under the assumption of no initial endowments and is organized into four sections. Section 2 reviews well-known properties of general deviation measures. Section 3 develops the pricing approach to the problem (1) with (5), and Section 4 applies the developed approach to some well-known classes of utility functionals U_i in the form (5).

2 Deviation Measures

Let $(\Omega,\mathcal{M},\mathbb{P})$ be a probability space, where Ω denotes the designated space of future states ω, $\mathcal{M}$ is a field of sets in Ω, and $\mathbb{P}$ is a probability measure on $(\Omega,\mathcal{M})$. A random variable (r.v.) is considered to be an element

[1]On a probability space $(\Omega,\mathcal{M},\mathbb{P})$, two random variables $X:\Omega\to\mathbb{R}$ and $Y:\Omega\to\mathbb{R}$ are *comonotone*, if there exists a set $A\subseteq\Omega$ such that $\mathbb{P}[A]=1$ and $(X(\omega_1)-X(\omega_2))(Y(\omega_1)-Y(\omega_2))\geq 0$ for all $\omega_1,\omega_2\in A$.

of $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, \mathbb{P})$ with the cumulative distribution function (CDF) $F_X(x)$ and the quantile function $q_X(\alpha) = \inf\{x | F_X(x) > \alpha\}$. The relations between r.v.'s are understood to hold in the almost sure sense, e.g., we write $X = Y$ if $\mathbb{P}[X = Y] = 1$ and $X \geq Y$ if $\mathbb{P}[X \geq Y] = 1$. Throughout this work, we assume that the probability space Ω is *atomless*, i.e., there exists an r.v. with a continuous CDF. This implies existence of r.v.'s on Ω with all possible CDFs (see e.g. [7]). Also, C will denote a real-valued constant.

The following system of axioms defining general deviation measures was introduced by Rockafellar et al. [14, 19].

Definition 1 *(general deviation measures). A* deviation measure *is any functional* $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0; \infty]$ *satisfying the axioms*

(D1) $\mathcal{D}(X) = 0$ *for constant* X*, but* $\mathcal{D}(X) > 0$ *otherwise* (nonnegativity),

(D2) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ *for all* X *and all* $\lambda > 0$ (positive homogeneity),

(D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ *for all* X *and* Y (subadditivity),

(D4) set $\{X \in \mathcal{L}^2(\Omega) \big| \mathcal{D}(X) \leq C\}$ *is closed for all* $C < \infty$ (lower semicontinuity).[2]

As shown in [19], axioms D1–D3 imply that $\mathcal{D}$ is constant translation invariant, i.e.,

$$\mathcal{D}(X + C) = \mathcal{D}(X) \text{ for all constants } C. \tag{6}$$

A deviation measure $\mathcal{D}$ is lower range dominated if in addition to D1–D4, it satisfies

(D5) $\mathcal{D}(X) \leq EX - \inf X$ *for all* X (lower range dominance).

A *subgradient* of $\mathcal{D}$ at X is any $Q \in \mathcal{L}^2(\Omega)$ such that

$$\mathcal{D}(X') \geq \mathcal{D}(X) + E[(X' - X)Q] \quad \text{for all} \quad X' \in \mathcal{L}^2(\Omega), \tag{7}$$

and the *subdifferential* $\partial\mathcal{D}(X)$ is the set of all subgradients of $\mathcal{D}$ at X, see [15].

In general, for two r.v.'s with the same CDF, a deviation measure may assume different values. In this work, we consider only law-invariant deviation measures [14], i.e., those which depend only on the CDF of an r.v.

Definition 2 *(law-invariant deviation measures). A deviation measure* $\mathcal{D}(X)$ *is called* law-invariant*, if* $\mathcal{D}(X_1) = \mathcal{D}(X_2)$ *for any two r.v.'s* X_1 *and* X_2 *yielding the same CDF on* $(-\infty; \infty)$.

A functional $\mathcal{D}(X)$ is a law-invariant deviation measure if and only if

$$\mathcal{D}(X) = \sup_{g(\alpha) \in G} \int_0^1 g(\alpha) \cdot d(q_X(\alpha)), \tag{8}$$

where G is a collection of positive concave functions $g : (0,1) \to (0, \infty)$, see [10]. This collection is called *g-envelope* of the deviation measure $\mathcal{D}(X)$. Different g-envelopes can produce the same deviation measures in (8), but all of them are subsets of the *maximal g-envelope* [10], defined as

$$G_M = \left\{ g(\alpha) \,\middle|\, \int_0^1 g(\alpha) \cdot d(q_X(\alpha)) \leq \mathcal{D}(X) \quad \forall X \in \mathcal{L}^2(\Omega) \right\}. \tag{9}$$

For every $Q \in \partial\mathcal{D}(X)$, the function $g_Q(\alpha) = \int_0^\alpha (-q_Q(\beta)) d\beta$ belongs to the maximal g-envelope G_M of $\mathcal{D}$, see [9].

From now, all deviation measures in this work are assumed to be *finite*, i.e., $\mathcal{D}(X) < \infty$ for all X. It follows from [14, Proposition 2], that every finite deviation measure is continuous on $\mathcal{L}^2(\Omega)$.

Well-known examples of deviation measures include

[2]In [14, 19], axiom D4 was not included in the definition. Deviation measures satisfying D4 were called *lower semicontinuous* deviation measures.

(a) deviation measures of $\mathcal{L}^p$ type $\mathcal{D}(X) = \|X - E[X]\|_p$, $p \in [1,\infty]$, e.g., the standard deviation $\sigma(X) = \|X - E[X]\|_2$ and mean absolute deviation $\text{MAD}(X) = \|X - E[X]\|_1$, where $\|\cdot\|_p$ is the $\mathcal{L}^p$ norm;

(b) deviation measures of semi-$\mathcal{L}^p$ type $\mathcal{D}_-(X) = \|[X - E[X]]_-\|_p$ and $\mathcal{D}_+(X) = \|[X - E[X]]_+\|_p$, $p \in [1,\infty]$, e.g., *standard lower* and *upper semideviations*

$$\sigma_-(X) = \|[X - E[X]]_-\|_2, \qquad \sigma_+(X) = \|[X - E[X]]_+\|_2,$$

where $[X]_- = \max\{0, -X\}$ and $[X]_+ = \max\{0, X\}$;

(c) conditional Value-at-Risk (CVaR) deviation, defined for any $\beta \in (0,1)$ by[3]

$$\text{CVaR}^{\Delta}_{\beta}(X) \equiv EX - \frac{1}{\beta}\int_0^{\beta} q_X(\alpha)d\alpha. \tag{10}$$

All these deviation measures are law-invariant and are also finite provided that $p < \infty$ in examples (a) and (b). In particular, $\mathcal{D}(X) = \|[X - E[X]]_-\|_p$, $p \in [1,\infty]$, and $\mathcal{D}(X) = \text{CVaR}^{\Delta}_{\alpha}(X)$ are lower range dominated[4]; see [14, 15, 16, 17] for other examples.

For every deviation measure $\mathcal{D}$ and constant $\alpha > 0$, a functional $(\alpha\mathcal{D})(X) \equiv \alpha \cdot \mathcal{D}(X)$ is, obviously, a deviation measure. For law-invariant deviation measures $\mathcal{D}_i$, $i \in S = \{1,\dots,m\}$, a functional defined by

$$\mathcal{D}_S(X) = \inf_{Y \in \mathcal{A}(X)} \max\{\mathcal{D}_1(Y_1),\dots,\mathcal{D}_m(Y_m)\}, \tag{11}$$

is a law-invariant deviation measure, see [9], and is called a *deviation measure of the coalition S*. For deviation measures $\mathcal{D}_i$, $i \in S$, and an r.v. X, there exists a vector $Y^* \in \mathcal{A}(X) = \{Y = (Y_1,\dots,Y_m) |\ \sum_{i=1}^m Y_i = X\}$ at which the infimum in (11) is attained, see [9], and this vector is called *optimal risk sharing*[5] of X with respect to the deviation measures $\mathcal{D}_i$, $i \in S$. Grechuk et al. [9] showed that for optimal risk sharing Y^*,

$$\mathcal{D}_1(Y_1^*) = \ldots = \mathcal{D}_m(Y_m^*). \tag{12}$$

If G_i, $i = 1,\dots,m$, are the maximal g-envelopes of $\mathcal{D}_i$, $i = 1,\dots,m$, respectively, then the maximal g-envelope of $\mathcal{D}_S$ is given by (see [9])

$$G_S = \left\{g(\alpha) = \min_{i=1,\dots,m}\{\lambda_i g_i(\alpha)\} \middle| g_i \in G_i,\ \lambda_i \in (0,1),\ i = 1,\dots,m,\ \textstyle\sum_{i=1}^m \lambda_i = 1\right\}. \tag{13}$$

3 Optimal Risk Sharing

3.1 Problem Formulation

This section addresses optimal risk sharing of an uncertain future payoff X among m agents, whose risk-reward preferences are described by utility functionals U_i, $i \in S$, in the form (5). Each $U_i : \mathbb{R} \times [0,\infty] \mapsto [-\infty,\infty)$ is assumed to be increasing[6] in the first argument and nonincreasing in the second one. In this case, the problem (1) is formulated as

$$\max_{Y \in \mathcal{A}(X)} \{U_1(EY_1, \mathcal{D}_1(Y_1)),\dots,U_m(EY_m, \mathcal{D}_m(Y_m))\}. \tag{14}$$

A vector $Y = (Y_1,\dots,Y_m) \in \mathcal{A}(X)$ is called a *Pareto-optimal division* in (14), if there is no $Y' = (Y_1',\dots,Y_m') \in \mathcal{A}(X)$ such that $U_i(EY_i', \mathcal{D}_i(Y_i')) \geq U_i(EY_i, \mathcal{D}_i(Y_i))$, $i = 1,\dots,m$, with at least one inequality being strict. A

[3] $\text{CVaR}^{\Delta}_1(X) = -E[X] + E[X] = 0$ is not a deviation measure, since it vanishes for all r.v.'s (not only for constants).

[4] Indeed, $\|[X - E[X]]_-\|_p \leq \|[X - E[X]]_-\|_\infty = E[X] - \inf X$ for $p \in [1,\infty]$, and $\text{CVaR}^{\Delta}_{\alpha}(X) = E[X] - \text{CVaR}_{\alpha}(X) \leq E[X] - \inf X$.

[5] This definition will be justified in the next section.

[6] For $d \in [0,\infty]$ and $\mu_1 < \mu_2$, either $U_i(\mu_1, d) < U_i(\mu_2, d)$ or $U_i(\mu_1, d) = U_i(\mu_2, d) = -\infty$.

vector $Y=(Y_1,\dots,Y_m)\in\mathcal{A}(X)$ is called *weakly Pareto optimal*, if there is no $Y'=(Y'_1,\dots,Y'_m)\in\mathcal{A}(X)$ such that $EY'_i\ge EY_i$, $\mathcal{D}_i(Y'_i)\le\mathcal{D}_i(Y_i)$, $i=1,\dots,m$, with at least one inequality being strict. For example, if in (14), $U_i(EY_i,\mathcal{D}_i(Y_i))=EY_i$, $i=1,\dots,m$, then *every* $Y\in\mathcal{A}(X)$ is a Pareto-optimal division. We will call the problem (14) *non-degenerate* if every Pareto-optimal division in (14) is weakly Pareto optimal. In particular, the problem (14) is non-degenerate if the functions U_i are strictly decreasing in the second argument.

The following proposition establishes a necessary and sufficient condition for Y to be weakly Pareto optimal.

Proposition 1 *A vector $Y\in\mathcal{A}(X)$ with $\mathcal{D}_i(Y_i)=d_i$, $i=1,\dots,m$, is a weakly Pareto-optimal solution to (14), if and only if Y is an optimal risk sharing of X with respect to deviation measures $\mathcal{D}_i(Y_i)/d_i$, $i=1,\dots,m$, or, equivalently, if and only if $\mathcal{D}_S(X)=1$, where the deviation measure $\mathcal{D}_S$ is defined by*

$$\mathcal{D}_S(Z)=\inf_{Y\in\mathcal{A}(Z)}\max\{\mathcal{D}_1(Y_1)/d_1,\dots,\mathcal{D}_m(Y_m)/d_m\}. \tag{15}$$

Proof By definition, $\mathcal{D}_S$ is a deviation measure of the coalition of m agents with deviation measures $\mathcal{D}'_i(Z)=\mathcal{D}_i(Z)/d_i$. It follows from $Y\in\mathcal{A}(X)$ and $\mathcal{D}'_i(Y_i)=1$, $i=1,\dots,m$, that $\mathcal{D}_S(X)\le 1$, and consequently, by definition, Y is an optimal risk sharing of X if and only if $\mathcal{D}_S(X)=1$. Then $\mathcal{D}_S(X)<1$ implies that $\mathcal{D}'_i(Z_i)/d_i<1$, $i=1,\dots,m$, for some $(Z_1,\dots,Z_m)\in\mathcal{A}(X)$, and thus, Y is not weakly Pareto optimal. Finally, let $\mathcal{D}_S(X)=1$, and let $Z=(Z_1,\dots,Z_m)\in\mathcal{A}(X)$ be such that $EZ_i\ge EY_i$, $\mathcal{D}_i(Z_i)\le\mathcal{D}_i(Y_i)=d_i$, $i=1,\dots,m$. Then it follows from $\sum_{i=1}^m EZ_i=EX=\sum_{i=1}^m EY_i$ that $EZ_i=EY_i$, $i=1,\dots,m$. Because $\mathcal{D}_S(X)=1$, Z is an optimal risk sharing of X, and hence, (12) implies that $\mathcal{D}'_1(Z_1)=\ldots=\mathcal{D}'_m(Z_m)=\mathcal{D}_S(X)=1$. Consequently, $\mathcal{D}'_i(Z_i)=\mathcal{D}'_i(Y_i)$, $i=1,\dots,m$, and thus, Y is a weakly Pareto-optimal solution. □

Proposition 1 explains the definition "optimal risk sharing of X" for the vector Y and also yields the following characterization for weakly Pareto-optimal solutions of the problem (14).

Proposition 2 *Let $\mathcal{D}_i$, $i=1,\dots,m$, be continuous law-invariant deviation measures. A vector $Y\in\mathcal{A}(X)$ with $\mathcal{D}_i(Y_i)=d_i$, $i=1,\dots,m$ is a weakly Pareto-optimal solution of the problem (14) if and only if there exist $\lambda_i>0$, $i=1,\dots,m$, and an r.v. Z_0 such that $\lambda_i Z_0\in\partial\mathcal{D}_i(Y_i)/d_i$, $i=1,\dots,m$.*

Proof This result follows from Proposition 1 and the characterization of optimal risk sharing given by [9, Proposition 12]. □

Proposition 1 along with (6) implies that if Y is a weakly Pareto-optimal solution then so is $(Y_1+C_1,\dots,Y_m+C_m)$ for any constants $C_1,\dots,C_m$ with $\sum_{i=1}^m C_i=0$. We will call two weakly Pareto-optimal solutions Y and Z *deviation equivalent*, if $\mathcal{D}_i(Y_i)=\mathcal{D}_i(Z_i)$, $i=1,\dots,m$, and *equivalent*, if, in addition, $EY_i=EZ_i$, $i=1,\dots,m$. Every weakly Pareto-optimal solution Y generates a set of deviation-equivalent solutions, which can be written in the form $(Y_1-EY_1+y_1,\dots,Y_m-EY_m+y_m)$, where constants $y_1,\dots,y_m$ with $\sum_{i=1}^m y_i=EX$ are independent of the choice of Y within the deviation-equivalent class. Thus, choosing a weakly Pareto-optimal solution of (14) can be summarized in the form of a two-step algorithm:

(i) Choose constants $d_i>0$, $i=1,\dots,m$, such that $\mathcal{D}_S(X)=1$ for $\mathcal{D}_S$ defined by (15). In other words, assign risk d_i to every agent i.

(ii) Choose constants $y_1,\dots,y_m$ with $\sum_{i=1}^m y_i=EX$. In other words, assign premium y_i to agent i for risk d_i.

As a result of this algorithm, we obtain a division in the form $(Y_1-EY_1+y_1,\dots,Y_m-EY_m+y_m)$, where $Y=(Y_1,\dots,Y_m)$ is any optimal risk sharing for $\mathcal{D}_S$, and the condition $\mathcal{D}_S(X)=1$ guarantees that this division is weakly Pareto optimal. The next section introduces the notion of equilibrium in obtaining "fair" premiums y_i, $i=1,\dots,m$.

3.2 Equilibrium

Suppose all agents agree on a pricing functional (or certainty equivalent) for each uncertain payoff $Y \in \mathcal{L}^2(\Omega)$ defined as follows.

Definition 3 *A* pricing functional *is any functional* $\mathcal{P} : \mathcal{L}^2(\Omega) \to \mathbb{R}$ *satisfying*

(P1) $\mathcal{P}(C) = C$ *for constant* C,

(P2) $\mathcal{P}(Z_1 + Z_2) = \mathcal{P}(Z_1) + \mathcal{P}(Z_2)$ *for all* Z_1 *and* Z_2,

(P3) $\mathcal{P}$ *is continuous on* $\mathcal{L}^2(\Omega)$.

Properties P1 and P2 follow from the no-arbitrage assumption provided that the riskless rate of return is zero. Indeed, if $P(C) < C$ (or $P(C) > C$) then $C - P(C) > 0$ (or $P(C) - C > 0$) is a sure profit. Also, if $\mathcal{P}(Z_1 + Z_2) > \mathcal{P}(Z_1) + \mathcal{P}(Z_2)$ for some Z_1 and Z_2, then Z_1 and Z_2 can be purchased separately and then $Z_1 + Z_2$ can be sold for $\mathcal{P}(Z_1 + Z_2)$ resulting in a sure profit $\mathcal{P}(Z_1 + Z_2) - \mathcal{P}(Z_1) - \mathcal{P}(Z_2) > 0$ (similarly if $\mathcal{P}(Z_1 + Z_2) < \mathcal{P}(Z_1) + \mathcal{P}(Z_2)$). Properties P2 and P3 imply homogeneity $\mathcal{P}(\lambda Z) = \lambda \mathcal{P}(Z)$ for all Z and $\lambda \in \mathbb{R}$, and consequently, $\mathcal{P}$ satisfying P1–P3 is a linear functional. By the Riesz representation theorem, every linear continuous functional on $\mathcal{L}^2(\Omega)$ can be represented by $\mathcal{P}(Z) = E[Q_0 Z]$ for some $Q_0 \in \mathcal{L}^2(\Omega)$, and this r.v. Q_0 is known as *state-price deflator* [1]. Establishing existence of a pricing functional $\mathcal{P}$, which all agents agree on, is the subject of this section.

Since the agents have no initial endowments, and $\mathcal{P}(0) = 0$, the best share for agent i in an uncertain payoff X that all agents agree to divide among themselves, is an uncertain share $Y_i^* \in \mathcal{L}^2(\Omega)$ having a nonpositive price and solving the problem

$$\max_{Y_i} U_i(EY_i, \mathcal{D}_i(Y_i)) \qquad \text{s.t.} \quad \mathcal{P}(Y_i) \le 0, \quad Y_i \in \mathcal{L}^2(\Omega), \quad i = 1, \ldots, m. \tag{16}$$

Obviously, the constraint in (16) is always active, i.e., $\mathcal{P}(Y_i^*) = 0$. In an *equilibrium*, the sum of desirable shares of all agents is equal to the payoff X, i.e., $\sum_{i=1}^m Y_i^* = X$.

Definition 4 $\mathcal{P} : \mathcal{L}^2(\Omega) \to \mathbb{R}$ *is called an* equilibrium pricing functional *for m agents with utility functionals* $U_i(EY_i, \mathcal{D}_i(Y_i))$, $i = 1, \ldots, m$, *if there exists a solution* Y_i^* *of (16),* $i = 1, \ldots, m$, *such that*

$$\sum_{i=1}^m Y_i^* = X. \tag{17}$$

A vector $Y^* = (Y_1^*, \ldots, Y_m^*) \in \mathcal{A}(X)$ *will be called* equilibrium division *for the problem (14).*

The conditions $\mathcal{P}(Y_i^*) = 0$, $i = 1, \ldots, m$, and (17) imply that, in an equilibrium, $\mathcal{P}(X) = 0$.

Next, we prove that an equilibrium division is Pareto optimal.

Proposition 3 *Let* $Y^* = (Y_1^*, \ldots, Y_m^*) \in \mathcal{A}(X)$ *be an equilibrium division for the problem (14). Then* Y^* *is a Pareto-optimal solution for (14).*

Proof Let $U_i(EY_i, \mathcal{D}_i(Y_i)) \ge U_i(EY_i^*, \mathcal{D}_i(Y_i^*))$, $i = 1, \ldots, m$, for some division $Y = (Y_1, \ldots, Y_m) \in \mathcal{A}(X)$. Because $\mathcal{P}(Y_i - \mathcal{P}(Y_i)) = 0$, by definition of Y^*, $U_i(EY_i^*, \mathcal{D}_i(Y_i^*)) \ge U_i(EY_i - \mathcal{P}(Y_i), \mathcal{D}_i(Y_i))$, $i = 1, \ldots, m$, which implies $\mathcal{P}(Y_i) \ge 0$, $i = 1, \ldots, m$. Thus, it follows from $\sum_{i=1}^m \mathcal{P}(Y_i) = \mathcal{P}(X) = 0$ that $\mathcal{P}(Y_i) = 0$, $i = 1, \ldots, m$, whence $U_i(EY_i, \mathcal{D}_i(Y_i)) \le U_i(EY_i^*, \mathcal{D}_i(Y_i^*))$, $i = 1, \ldots, m$, i.e., Y^* is a Pareto-optimal division for (14). □

The equilibrium Y^* is a "fair" division from the set of Pareto-optimal divisions for (14) provided that the pricing functional $\mathcal{P}$, which all agents participating in sharing X agree on, exists. Next, we establish some properties of an equilibrium division and equilibrium pricing functional.

If the problem (14) is non-degenerate, Propositions 3 and 1 imply that Y^* is an optimal risk sharing of X with respect to the deviation measure $\mathcal{D}_S$ given by (15) with $d_i = \mathcal{D}_i(Y_i^*)$, $i = 1, \ldots, m$. The next proposition characterizes an equilibrium pricing functional in terms of $\mathcal{D}_S$.

Proposition 4 *Every pricing functional $\mathcal{P} : \mathcal{L}^2(\Omega) \to \mathbb{R}$ can be represented by $\mathcal{P}(Z) = E[Q_0 Z]$ for some $Q_0 \in \mathcal{L}^2(\Omega)$. If $\mathcal{P}$ is an equilibrium pricing functional, and (14) is non-degenerate, then*

$$\mathcal{P}(Z) = E[(1 - Q \cdot EX)Z] = EZ - E[QZ]EX, \tag{18}$$

where $Q \in \partial \mathcal{D}_S(X)$ for $\mathcal{D}_S$ defined by (15) with $d_i = \mathcal{D}_i(Y_i^*)$, $i = 1, \ldots, m$.

Proof Since $\mathcal{P}$ is a linear functional, by the Riesz representation theorem, $\mathcal{P}(Z) = E[Q_0 Z]$ for some $Q_0 \in \mathcal{L}^2(\Omega)$, and P1 implies that $EQ_0 = 1$. We need to prove (18), i.e., to show that $Q \in \partial \mathcal{D}_S(X)$ for $Q = (1 - Q_0)/EX$ or, equivalently, that $E[QX] = \mathcal{D}_S(X)$ and $E[QZ] \le \mathcal{D}_S(Z)$ for all $Z \in \mathcal{L}^2(\Omega)$.

Since $E[Q_0 X] = \mathcal{P}(X) = 0$, we have $E[QX] = E\left[\frac{1-Q_0}{EX} X\right] = 1 - \frac{E[Q_0 X]}{EX} = 1 = \mathcal{D}_S(X)$, where the last equality follows from Proposition 1. Next, we assume that $E[QZ] > \mathcal{D}_S(Z)$ for some $Z \in \mathcal{L}^2(\Omega)$, i.e., $E[QZ] > \mathcal{D}_i(Y_i)/\mathcal{D}_i(Y_i^*)$ or $\mathcal{D}_i\left(\frac{Y_i}{E[QZ]}\right) < \mathcal{D}_i(Y_i^*)$, $i = 1, \ldots, m$, for some $Y \in \mathcal{A}(Z)$. Thus, for $Z_i = \frac{Y_i}{E[QZ]} - \frac{\mathcal{P}(Y_i)}{E[QZ]}$, $i = 1, \ldots, m$, we obtain $\mathcal{D}_i(Z_i) < \mathcal{D}_i(Y_i^*)$ and $\mathcal{P}(Z_i) = 0$. Consequently, optimality of Y_i^* in (16) implies $EZ_i < EY_i^*$, $i = 1, \ldots, m$, and hence, $\sum_{i=1}^m EZ_i < \sum_{i=1}^m EY_i^*$. Since $\sum_{i=1}^m EZ_i = \frac{EZ - \mathcal{P}(Z)}{E[QZ]}$ and $\sum_{i=1}^m EY_i^* = EX$, we obtain $\frac{EZ - \mathcal{P}(Z)}{E[QZ]} < EX$ or, equivalently, $EZ - \mathcal{P}(Z) < E[QZ]EX$, which with $\mathcal{P}(Z) = E[Q_0 Z]$ and $Q = (1 - Q_0)/EX$ reduces to contradiction $EZ - E[Q_0 Z] < E\left[\frac{1-Q_0}{EX} Z\right] EX$. □

Now, let us return to the algorithm in the end of Section 3.1. Suppose at step (i), a deviation-equivalent class of Pareto-optimal solutions is chosen. Proposition 4 implies that if an equilibrium exists, $0 = \mathcal{P}(Y_i - EY_i + y_i) = EY_i - E[QY_i]EX - EY_i + y_i$, which suggests the following formula for the premiums $y_1, \ldots, y_m$ at step (ii) in the algorithm:

$$y_i = E[Q\overline{Y_i}]EX, \quad Q \in \partial \mathcal{D}_S(X), \quad \overline{Y_i} = Y_i - EY_i \quad i = 1, \ldots, m. \tag{19}$$

The formula (19) is one of the main results of this work because it shows which solution to choose from the set of deviation-equivalent weakly Pareto-optimal solutions. The suggested solution coincides with an equilibrium division when an equilibrium exists, but can be used in case of no equilibrium as well.

4 Special Cases

This section applies the developed pricing approach to optimal risk sharing with some important special cases of utility functionals U_i, $i = 1, \ldots, m$.

4.1 Expected Gain Maximization

Let U_i, $i = 1, \ldots, m$, in (14) be given by

$$U_i(\mu, d) = \begin{cases} \mu, & d \le d_i \\ -\infty, & d > d_i, \end{cases} \tag{20}$$

for some positive constants $d_1, \ldots, d_m$. This case corresponds to expected gain maximization. We show that in this case, if the problem (14) is non-degenerate, the equilibrium division exists and, thus, coincides with (19).

Let $\mathcal{D}_S$ be a deviation measure given by

$$\mathcal{D}_S(Z) = \inf_{Y \in \mathcal{A}(Z)} \max\{\mathcal{D}_1(Y_1)/d_1, \ldots, \mathcal{D}_m(Y_m)/d_m\}, \tag{21}$$

where constants $d_1, \ldots, d_m$ are defined in (20).

Next, we argue that the case $\mathcal{D}_S(X) = 1$ is *the only* interesting case for the problem (14) with (20). As an illustration, suppose that m agents agree to choose a total risky payoff X from some set $\mathcal{F}$ of all feasible payoffs and then to divide chosen X. For example, X can represent payoff of a portfolio of risky instruments, payoff of a venture, etc. The case $\mathcal{D}_S(X) > 1$ implies that for any feasible division $Y \in \mathcal{A}(X)$, there exists i such that $\mathcal{D}_i(Y_i) > d_i$, whence $U_i(EY_i, \mathcal{D}_i(Y_i)) = -\infty$, i.e., there is always at least one unsatisfied agent. When, in contrast, $\mathcal{D}_S(X) < 1$ and Y is an optimal risk sharing of X, we have $\mathcal{D}_i(Y_i) < d_i$ for all i, i.e., there exists a division when every agent is willing to accept higher risk, which means that $X \in \mathcal{F}$ is not optimally chosen. Consequently, to determine optimal X, the agents solve the optimization problem

$$\max_{X \in \mathcal{F}} EX \qquad \text{s.t.} \quad \mathcal{D}_S(X) \leq 1. \tag{22}$$

If $\mathcal{F}$ is a cone, then for any optimal solution to (22), the constraint in (22) holds as equality, and consequently, $\mathcal{D}_S(X) = 1$ always holds for chosen X.

In the rest of this section, we assume that $\mathcal{D}_S(X) = 1$, which implies that the problem (14) is non-degenerate.

Proposition 5 *Let U_i in (14) be defined by (20), and let $\mathcal{D}_S(X) = 1$, where $\mathcal{D}_S$ is defined by (21). Then the problem (14) is non-degenerate.*

Proof According to (21), $\mathcal{D}_S$ is the deviation measure of the coalition S of agents with deviation measures $\mathcal{D}'_i = \mathcal{D}_i/d_i$. Let $Z = (Z_1, \ldots, Z_m) \in \mathcal{A}(X)$ be an optimal risk sharing of X. Proposition 2 implies that $\mathcal{D}'_1(Z_1) = \ldots = \mathcal{D}'_m(Z_m) = \mathcal{D}_S(X) = 1$.

Let $Y = (Y_1, \ldots, Y_m)$ be a Pareto-optimal solution to (14). First, suppose $U_i(EY_i, \mathcal{D}(Y_i)) = -\infty$ for some i. Then $U_j(EY_j, \mathcal{D}(Y_j)) \leq EY_j = U_j(EZ^*_j, \mathcal{D}(Z^*_j))$, $j \neq i$, and $U_i(EY_i, \mathcal{D}(Y_i)) = -\infty < U_i(EZ^*_i, \mathcal{D}(Z^*_i))$ for $Z^* = (Z^*_1, \ldots, Z^*_m) \in \mathcal{A}(X)$ with $Z^*_j = Z_j - EZ_j + EY_j$, $i = 1, \ldots, m$, which contradicts Pareto optimality of Y. Thus, $U_i(EY_i, \mathcal{D}(Y_i)) = EY_i$, and $\mathcal{D}(Y_i) \leq d_i$, $i = 1, \ldots, m$.

If $EY'_i \geq EY_i$ and $\mathcal{D}_i(Y'_i) \leq \mathcal{D}_i(Y_i) \leq d_i$, $i = 1, \ldots, m$, for some $Y' = (Y'_1, \ldots, Y'_m) \in \mathcal{A}(X)$, then, by definition, Y' is an optimal risk sharing of X, and consequently, by Proposition 2, $\mathcal{D}_i(Y'_i) = d_i = \mathcal{D}_i(Y_i)$, $i = 1, \ldots, m$. Also, $\sum_{i=1}^m EY'_i = EX = \sum_{i=1}^m EY_i$ implies $EY'_i = EY_i$, $i = 1, \ldots, m$. Thus, Y is weakly Pareto optimal, and the problem is non-degenerate. □

With U_i given by (20), the problem (16) takes the form

$$\max_{Y_i \in \mathcal{L}^2(\Omega)} EY_i \qquad \text{s.t.} \quad \mathcal{P}(Y_i) = 0, \quad \mathcal{D}_i(Y_i) \leq d_i, \quad i = 1, \ldots, m. \tag{23}$$

Proposition 6 *An equilibrium pricing functional $\mathcal{P}$ for (23) exists and is determined by (18) with $Q \in \partial\mathcal{D}_S(X)$ and $\mathcal{D}_S$ defined by (21).*

Proof According to (21), $\mathcal{D}_S$ is the deviation measure of the coalition S of agents with $\mathcal{D}'_i = \mathcal{D}_i/d_i$. Since $Q \in \partial\mathcal{D}_S(X)$, the function $g_Q(\alpha) = \int_0^\alpha (-q_Q(\beta))d\beta$ belongs to the maximal g-envelope G_M of $\mathcal{D}_S$. Hence, (13) implies that $g_Q(\alpha) = \min_{i \in S}\{\lambda_i g_i(\alpha)\}$ for some $\lambda_i \in (0,1)$, $i = 1, \ldots, m$, with $\sum_{i=1}^m \lambda_i = 1$, where g_i, $i = 1, \ldots, m$, are elements of the maximal g-envelopes of $\mathcal{D}'_i$, $i = 1, \ldots, m$. Consequently, for every $Z \in \mathcal{L}^2(\Omega)$, we have

$$E[QZ] \leq \int_0^1 g_Q(\alpha)d(q_Z(\alpha)) \leq \lambda_i \int_0^1 g_i(\alpha)d(q_Z(\alpha)) \leq \lambda_i \mathcal{D}'_i(Z), \quad i = 1, \ldots, m. \tag{24}$$

where the first inequality is due to Hardy-Littlewood [7, Theorem A.24].

Let $Y^* = (Y^*_1, \ldots, Y^*_m)$ be an optimal risk sharing of X such that $EY^*_i = \lambda_i EX$, $i = 1, \ldots, m$. Then (12) implies $\mathcal{D}'_i(Y^*_i) = 1$, $i = 1, \ldots, m$, while (24) implies $E[QY^*_i] \leq \lambda_i \mathcal{D}'_i(Y^*_i) = \lambda_i$. Since $\sum_{i=1}^m E[QY^*_i] = E[QX] = \mathcal{D}_S(X) = 1 = \sum_{i=1}^m \lambda_i$, we conclude that $E[QY^*_i] = \lambda_i$, $i = 1, \ldots, m$. Thus, $EY^*_i = E[QY^*_i]EX$, whence $\mathcal{P}(Y^*_i) = 0$, $i = 1, \ldots, m$.

For every $Z \in \mathcal{L}^2(\Omega)$ with $\mathcal{P}(Z) = 0$ and $\mathcal{D}_i(Z) \le d_i$, (24) implies $EZ = E[QZ]EX \le \lambda_i \mathcal{D}_i'(Z)EX \le \lambda_i EX = E[QY_i^*]EX = EY_i^*$. Consequently, Y_i^*, $i = 1,\dots,m$, are solutions to (23), and the condition $\sum_{i=1}^m Y_i^* = X$ for equilibrium division holds. □

Thus, in the equilibrium division, agent i's expected gain is $y_i = E[Y_i - \mathcal{P}(Y_i)] = EY_i - (EY_i - E[QY_i]EX) = E[QY_i]EX$, which coincides with the expected gain division (19).

4.2 Mean-deviation Approach

Another important case of optimal risk sharing is (14) with a *mean-deviation* model $U_i(EY_i, \mathcal{D}_i(Y_i)) = \rho_i EY_i - \mathcal{D}_i(Y_i)$:

$$\min_{Y \in \mathcal{A}(X)} \{\mathcal{D}_1(Y_1) - \rho_1 EY_1, \dots, \mathcal{D}_m(Y_m) - \rho_m EY_m\}, \tag{25}$$

where $\rho_i > 0$ is a constant coefficient.

The following proposition characterizes the set of Pareto-optimal solutions for (25).

Proposition 7 *Let $Y \in \mathcal{A}(X)$ be a Pareto-optimal solution to (25). Then for any $y = (y_1, \dots, y_m)$ with $\sum_{i=1}^m y_i = EX$, a vector Z with components $Z_i = Y_i - EY_i + y_i$ is also a Pareto-optimal solution to (25).*

Proof Suppose Z is not a Pareto-optimal solution to (25), i.e., $\mathcal{D}_i(Z_i') - \rho_i EZ_i' \le \mathcal{D}_i(Z_i) - \rho_i EZ_i$, $i = 1,\dots,m$, for some $Z' \in \mathcal{A}(X)$ with at least one inequality being strict. Then $Y' \in \mathcal{A}(X)$ for $Y' = (Y_1', \dots, Y_m')$ with $Y_i' = Z_i' - EZ_i + EY_i$, $i = 1,\dots,m$, and $\mathcal{D}_i(Y_i') - \rho_i EY_i' = \mathcal{D}_i(Z_i') - \rho_i(EZ_i' - EZ_i + EY_i) \le \mathcal{D}_i(Z_i) - \rho_i EY_i = \mathcal{D}_i(Y_i) - \rho_i EY_i$, $i = 1,\dots,m$, with at least one inequality being strict, i.e., Y is not a Pareto-optimal solution to (25). □

For Pareto-optimal Y, the set of points $(a_1, \dots, a_m) \in \mathbb{R}^m$ with $\mathcal{D}_i(Y_i) - \rho_i EY_i = a_i$, $i = 1,\dots,m$, will be called *efficient frontier* for (25). Proposition 7 implies that the efficient frontier has the form $\sum_{i=1}^m a_i/\rho_i = C$ for some constant C.

Next, we address the problem of choosing a division from the Pareto-optimal set described in Proposition 7. Obviously, the problem (25) is non-degenerate, and consequently, an equilibrium pricing functional, if it exists, is given by (18) with $Q \in \partial\mathcal{D}_S(X)$ for $\mathcal{D}_S$ defined in (15) with $d_i = \mathcal{D}_i(\overline{Y}_i)$, $i = 1,\dots,m$. In this case, the equilibrium division Y^* is given by

$$Y_i^* = Y_i - EY_i + y_i, \quad y_i = E[Q\overline{Y}_i]EX, \quad i = 1,\dots,m, \qquad Q \in \partial\mathcal{D}_S(X), \tag{26}$$

which can be regarded as a "fair" division from the Pareto-optimal set of (25).

4.3 Risk Measure Approach

As another approach, optimal risk sharing can be formulated in the form (3) with risk measures $\mathcal{R}_i$, $i = 1,\dots,m$, to be defined axiomatically. First, we formally introduce two classes of risk measures: *averse measures of risk* due to Rockafellar et al. [19, 18] and coherent risk measures due to Artzner et al. [2].

Definition 5 *(averse measures of risk). By an* averse measure of risk *will be meant any functional $\mathcal{R} : \mathcal{L}^2(\Omega) \to (-\infty; \infty]$ satisfying*

(R1) $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C (constant translation),

(R2) $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda\mathcal{R}(X)$ for all X and all $\lambda > 0$ (positive homogeneity),

(R3) $\mathcal{R}(X_1 + X_2) \le \mathcal{R}(X_1) + \mathcal{R}(X_2)$ for all X_1 and X_2 (subadditivity),

(R4) $\mathcal{R}(X) > E[-X]$ for all nonconstant X (risk aversion).

Axiom R4 requires an additional explanation.[7] By definition of risk aversion, EX is always preferred over uncertain $X \neq C$. Consequently, for a risk averse agent, $\mathcal{R}(EX) < \mathcal{R}(X)$. With axiom R1, it is equivalent to $\mathcal{R}(0) - EX < \mathcal{R}(X)$, and axiom R2 reduces it to $\mathcal{R}(X) > E[-X]$.

Artzner et al. [2] introduced *coherent risk measures* by a different system of axioms, namely, $\mathcal{R}(X)$ is a *coherent risk measure* if it satisfies axioms R1–R3 and

(R5) $\mathcal{R}(X) \le \mathcal{R}(X_0)$ *when* $X \ge X_0$ *(monotonicity).*

Remarkably, on an atomless probability space, law-invariant coherent risk measures are a subclass of law-invariant averse measures of risk.

Proposition 8 *On an atomless probability space, every law-invariant coherent risk measure* $\mathcal{R}(X) \not\equiv -EX$ *satisfies axiom R4.*

Proof On an atomless probability space, every law-invariant coherent risk measure can be represented by

$$\mathcal{R}(X) = -\inf_{Q \in \mathcal{Q}} \int_0^1 q_X(t) q_Q(1-t)\, dt, \tag{27}$$

where $\mathcal{Q}$ is a *closed convex and nonempty* set of r.v.'s such that $EQ = 1$ and $Q \ge 0$ for all $Q \in \mathcal{Q}$, and for every $X \neq C$ there is some $Q \in \mathcal{Q}$ with $E[XQ] < EX$. A different form of (27) was first established by Kusuoka [12], and then (27) was proved by Dana [5]. If $\mathcal{Q}$ consists of a single element $Q \equiv 1$ then $\mathcal{R}(X) \equiv -EX$. Otherwise, there is some nonconstant $Q \in \mathcal{Q}$, and consequently, $\mathcal{R}(X) \ge -\int_0^1 q_X(t) q_Q(1-t)\, dt > -EX\, EQ = -EX$ for any $X \neq C$. □

Theorem 2 in [19] establishes a one-to-one correspondence between the classes of averse measures of risk $\mathcal{R}$ and deviation measures $\mathcal{D}$ through the relationships $\mathcal{D}(X) = \mathcal{R}(X - EX)$ and $\mathcal{R}(X) = \mathcal{D}(X) - EX$. This fact implies that (25) for $\rho_i = 1$, $i = 1, \dots, m$, is equivalent to optimal risk sharing (3) with averse measures of risk, and consequently, all the results obtained in Section 4.2 for (25) with $\rho_i = 1$, $i = 1, \dots, m$, can be restated in terms of averse measures of risk, and in view of Proposition 8, they will hold for (3) with coherent risk measures.

It is known that a solution to (3) with coherent risk measures is a minimizer of $\mathcal{R}_S$ given by (4) and that the Pareto-optimal set consists of vectors Z with components $Z_i = Y_i - EY_i + y_i$, where $Y \in \mathcal{A}(X)$ is a Pareto-optimal solution, and $y_1, \dots, y_m$ are arbitrary constants with $\sum_{i=1}^m y_i = EX$, see [4, 11]. For a finite probability space, a solution to (3) from the Pareto-optimal set can be specified based on the equilibrium conditions obtained in [6], whereas this work shows that for an atomless probability space, such a solution can be chosen in the form (26) disregarding of equilibrium existence and which is the equilibrium division if an equilibrium exists.

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