Inverse Portfolio Problem with Mean-Deviation Model

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Abstract

A Markowitz-type portfolio selection problem is to minimize a deviation measure of portfolio rate of return subject to constraints on portfolio budget and on desired expected return. In this context, the inverse portfolio problem is finding a deviation measure by observing the optimal mean-deviation portfolio that an investor holds. Necessary and sufficient conditions for the existence of such a deviation measure are established. It is shown that if the deviation measure exists, it can be chosen in the form of a mixed CVaR-deviation, and in the case of n risky assets available for investment (to form a portfolio), it is determined by a combination of $(n + 1)$ CVaR-deviations. In the later case, an algorithm for constructing the deviation measure is presented, and if the number of CVaR-deviations is constrained, an approximate mixed CVaR-deviation is offered as well. The solution of the inverse portfolio problem may not be unique, and the investor can opt for the most conservative one, which has a simple closed-form representation.

Key Words: risk preferences, portfolio optimization, deviation measure, conditional value-at-risk.

1 Introduction

Understanding and modeling of individual risk preferences has long been a central venue in decision sciences, finance, and economics. A general approach is to introduce a system of axioms on preference relations that agree with "rational" behavior and then to construct a numerical equivalent (representation) to that system, so that given a choice of several random variables, a decision maker compares only their numerical equivalents. For example, the celebrated von Neumann and Morgenstern's theory [12] introduces axioms of completeness, transitivity, continuity, and independence and shows that the decision maker needs to deal only with expected utilities. A change in the system of axioms leads to another numerical equivalent that results in possibly different decisions. Since Neumann and Morgenstern's seminal work [12], similar choice theories such as the prospect theory [7], the dual utility theory [20], the regret theory, etc. have emerged. However, none of their axiomatic systems are in complete agreement with empirical evidence, which is known as axiom violations or paradoxes. Even if a particular decision maker does agree on a proposed system of axioms on preference relations, still there is a wide class of possible numerical equivalents (utility functions) that represent the system, so that finding a particular numerical equivalent amounts to a tedious questionnaire procedure. A different approach was proposed by Markowitz [10], who suggested that portfolios of risky assets can be ordered at least partially based on two quantities: mean and standard deviation of portfolio rate of return. The approach reduces mean-variance portfolio selection to a simple quadratic programming problem and also leads to a convenient capital asset pricing model (CAPM). However, the simplicity of this approach has proved to be both its advantage and disadvantage. The extensive body of empirical and theoretical research shows that standard deviation is hardly an appropriate measure of risk and that a market portfolio, predicted by the mean-variance approach to be held by all investors, is only a theoretical concept (a real market index does not follow from the mean-variance approach). General measures of deviation introduced by Rockafellar et al. [14, 15] address main shortcomings of standard deviation: they are not symmetrical with respect to ups and downs of a random variable and provide sufficient flexibility in customizing individual risk preferences. Let a random variable X represent the portfolio return. A Markowitz-type portfolio selection problem replaces standard deviation by a general deviation measure $\mathcal{D}$:

$$\min_{X \in \mathcal{V}} \mathcal{D}(X) \quad \text{subject to} \quad EX \geq \pi, \tag{1}$$

where π is a desired expected return, and $\mathcal{V}$ is a feasible set of portfolio returns. Rockafellar et al. [17, 16] generalized the one-fund theorem and CAPM, originally established for (1) with standard deviation, for an arbitrary deviation measure. Also, if each investor believes in the mean-deviation paradigm and has a corresponding deviation measure as a numerical representation of his/her risk preferences then there exists a market equilibrium (see [18]), and, moreover, a group of investors can form a cooperative portfolio that satisfies (1) with a "cooperative" deviation measure (see [5]). However, as in the aforementioned utility theories and in contrast to the Markowitz's mean-variance approach, a variety of deviation measures brings the question of choosing appropriate deviation measure for each investor. Partially, this question was answered in [3] by establishing a one-to-one correspondence between the class of alpha-concave distributions and the class of comonotone deviation measures via the maximum entropy principle. However, the resulting deviation measure reflected the "average" preferences (of the market as a whole) rather than preferences of an individual investor.

A different perspective to this question could be offered through an inverse approach. Instead of agonizing over idealized postulates of rational behavior, an investor can identify his/her risk preferences in the form of a set of benchmarks (portfolios, financial instruments, assets, etc.) that he/she is relatively satisfied with. This work solves the inverse portfolio problem: given a convex set of bounded portfolio returns $\mathcal{V}$ and a portfolio $X^* \in \mathcal{V}$, find a deviation measure $\mathcal{D}^*$ such that X^* is optimal for (1) with $\mathcal{D}^*$. The found deviation measure $\mathcal{D}^*$ can be interpreted as "the correct one" for the investor holding portfolio X^*. An approximate solution to this problem was given in [8], where a set of feasible deviation measures was confined to linear combinations of conditional value-at-risk (CVaR) deviations for certain risk-tolerance levels or, equivalently, to so-called *mixed CVaR-deviations*. However, an approximate $\mathcal{D}^*$ substantially depends on the choice of the risk-tolerance levels and on an error measure for residuals. Here, it is proved that "exact" $\mathcal{D}^*$ exists for every risk-averse investor,[1] and that, indeed, the set for $\mathcal{D}^*$ can be narrowed down to mixed CVaR-deviations. However, the found deviation measure may not be unique, and it is suggested that in this case, the investor takes the supremum over all possible deviation measures $\mathcal{D}$, such that X^* is optimal for (1) with $\mathcal{D}$. Remarkably, this supremum admits a simple closed-form representation in the form similar to a *worst-case mixed-CVaR deviation*. In addition, given T scenarios for asset returns, an algorithm for finding $\mathcal{D}^*$ in (1) in the form of mixed CVaR-deviations is provided, and if the number of CVaR-deviations is constrained, an alternative approach to constructing an approximate deviation measure is also presented.

This work is organized into seven sections. Section 2 reviews basic notions of general deviation measures. Section 3 formulates and solves the inverse portfolio problem. Section 4 addresses the inverse problem for scenario-based portfolio selection. Section 5 finds an approximate deviation measure in the form of mixed CVaR-deviation with specified number of risk-tolerance levels. Section 6 constructs a risk-envelope representation for a non law-invariant deviation measure provided that law-invariant one does not exists. Section 7 concludes the work.

2 Deviation Measures

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability space, where Ω denotes the designated space of future states ω, $\mathcal{M}$ is a field of sets in Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{M})$. A random variable (r.v.) is considered to be an element of $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, \mathbb{P})$. $F_X(x)$ and $q_X(\alpha) = \inf\{x | F_X(x) > \alpha\}$ will denote the cumulative distribution function (CDF) and quantile function of an r.v. X, respectively. Also, C will denote a con-

[1] An investor is risk-averse, if portfolio X^* he/she holds is undominated in the sense of second-order stochastic dominance (SSD); see Section 2 for the exact definition.

stant in the real numbers. The relations between r.v.'s are understood to hold in the almost sure sense, e.g., we write $X = Y$ if $\mathbb{P}[X = Y] = 1$ and $X \geq Y$ if $\mathbb{P}[X \geq Y] = 1$. The probability space Ω is called *atomless*, if there exists an r.v. with a continuous CDF. This implies existence of r.v.'s on Ω with all possible CDFs (see e.g. [2]).

Definition 1 *(general deviation measures). A* deviation measure *is any functional* $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ *satisfying*[2]

(D1) $\mathcal{D}(X) = 0$ *for constant X, but* $\mathcal{D}(X) > 0$ *otherwise* (nonnegativity)*,*

(D2) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ *for all X and all* $\lambda > 0$ (positive homogeneity)*,*

(D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ *for all X and Y* (subadditivity)*,*

(D4) set $\{X \in \mathcal{L}^2(\Omega) \big| \mathcal{D}(X) \leq C\}$ *is closed for all* $C < \infty$ (lower semicontinuity).

As shown in [14], axioms D1–D3 imply that

$$\mathcal{D}(X + C) = \mathcal{D}(X) \text{ for all constants } C \quad (\textit{constant translation invariance}). \tag{2}$$

In general, for two r.v.'s with the same CDF, a deviation measure may assume different values. This work is confined to law-invariant deviation measures [15], i.e., those which depend only on the CDF of an r.v.

Definition 2 *(law-invariant deviation measures). A deviation measure* $\mathcal{D}(X)$ *is called* law-invariant*, if* $\mathcal{D}(X_1) = \mathcal{D}(X_2)$ *for any two r.v.'s* X_1 *and* X_2 *yielding the same CDF on* $(-\infty; \infty)$.

Well-known examples of deviation measures include (see [15, 14]):

(i) deviation measures of $\mathcal{L}^p$ type $\mathcal{D}(X) = \|X - EX\|_p$, $p \in [1, \infty]$, for example, the standard deviation $\sigma(X) = \|X - EX\|_2$ and mean absolute deviation $\text{MAD}(X) = \|X - EX\|_1$, where $\|\cdot\|_p$ is the $\mathcal{L}^p$ norm;

(ii) deviation measures of semi-$\mathcal{L}^p$ type $\mathcal{D}_-(X) = \|[X - EX]_-\|_p$ and $\mathcal{D}_+(X) = \|[X - EX]_+\|_p$, $p \in [1, \infty]$, for example, *standard lower semideviation* $\sigma_-(X)$ and *standard upper semideviation* $\sigma_+(X)$ defined by

$$\sigma_\pm(X) = \|[X - EX]_\pm\|_2,$$

where $[X]_\pm = \max\{0, \pm X\}$;

(iii) range-based deviations, for example, lower range of X, defined as $\mathcal{D}(X) = EX - \inf X$, and its reflection, upper range of X, defined as $\mathcal{D}(X) = \sup X - EX$.

(iv) conditional value-at-risk (CVaR) deviation, defined for any $\alpha \in (0, 1)$ by

$$\text{CVaR}_\alpha^\Delta(X) \equiv EX - \frac{1}{\alpha} \int_0^\alpha q_X(\beta)\, d\beta. \tag{3}$$

For convenience, let $\text{CVaR}_0^\Delta(X) = EX - \inf X$ and $\text{CVaR}_1^\Delta(X) = \sup X - EX$.[3]

As generalizations of CVaR deviation, Rockafellar et al. [15, 14] introduced *mixed CVaR-deviation*

$$\mathcal{D}(X) = \int_0^1 \text{CVaR}_\alpha^\Delta(X)\, d\lambda(\alpha) \tag{4}$$

with some $\lambda(\alpha) \geq 0$ such that $\int_0^1 d\lambda(\alpha) = 1$, and *worst-case mixed-CVaR deviation*

$$\mathcal{D}(X) = \sup_{\lambda \in \Lambda} \int_0^1 \text{CVaR}_\alpha^\Delta(X)\, d\lambda(\alpha) \tag{5}$$

for some collection Λ of weighting nonnegative measures λ on $[0, 1]$[4] with $\int_0^1 d\lambda(\alpha) = 1$. All the above deviation measures are law invariant. See [15, 16] for other examples.

2.1 Dual Characterization

Deviation measures can be characterized in terms of risk envelopes $\mathcal{Q} \subset \mathcal{L}^2(\Omega)$ each of which, being a set of r.v.'s, satisfies

[2] In [14, 15], axiom D4 was not included in the definition. Deviation measures satisfying D4 were called *lower semicontinuous* deviation measures.

[3] Observe that $\lim_{\alpha \to 1-} \text{CVaR}_\alpha^\Delta(X) = 0$, whereas $\lim_{\alpha \to 1-} \frac{d}{d\alpha} \text{CVaR}_\alpha^\Delta(X) = EX - \sup X$, so that $\text{CVaR}_1^\Delta(X) = \sup X - EX$ is only a mathematical convention.

[4] In [15, 14], nonnegative measures λ are defined on $(0, 1)$, so that range-based deviations are not included into mixed CVaR-deviations.

(Q1) $\mathcal{Q}$ is a convex, closed set containing 1 (constant r.v.),

(Q2) $EQ = 1$ for every $Q \in \mathcal{Q}$,

(Q3) for every nonconstant $X \in \mathcal{L}^2(\Omega)$ there is a $Q \in \mathcal{Q}$ such that $E[XQ] < EX$.

Namely, there is one-to-one correspondence between deviation measures and risk envelopes given by Theorem 1 in [15]:

$$\begin{aligned} \mathcal{Q} &= \left\{ Q \in \mathcal{L}^2(\Omega) \,\middle|\, E[X(1-Q)] \le \mathcal{D}(X)\ \forall X \right\}, \\ \mathcal{D}(X) &= \sup_{1-Q \in \mathcal{Q}} E[XQ]. \end{aligned} \tag{6}$$

For each $X \in \mathcal{L}^2(\Omega)$, elements Q from the set

$$\mathcal{Q}_{\mathcal{D}}(X) = \left\{ Q \in \mathcal{Q} \,\middle|\, E[X(1-Q)] = \mathcal{D}(X) \right\}$$

are called the *risk identifiers* for X with respect to $\mathcal{D}$. They are closely related to the *subgradients* of $\mathcal{D}$ at X and along with $\mathcal{Q}$ are instrumental in formulating necessary optimality conditions for optimization problems involving $\mathcal{D}$, see [16].

The representation (6) shows that if $\mathcal{Q}$ is a risk envelope, then $\mathcal{D}(X) = EX - \inf_{Q \in \mathcal{Q}} E[XQ]$ is a deviation measure. If all elements of $\mathcal{Q}$ are nonnegative, then all $\mathbb{P}'$ such that $d\mathbb{P}' = Q\, d\mathbb{P}$, $Q \in \mathcal{Q}$, are probability measures that can be viewed as "distortions" of the original ("reference") probability measure $\mathbb{P}$. In this case, $\inf_{Q \in \mathcal{Q}} E[XQ] = \inf_{Q \in \mathcal{Q}} E_{\mathbb{P}'}[X]$ is the expected value of X under the "worst-case" probability distortion $\mathbb{P}'$, and $\mathcal{D}(X)$ has a simple meaning: it is the drop in the expected value of X under the "reference" and "worst-case" probability measures. In fact, all deviation measures that have nonnegative risk envelopes form a class of so-called *lower range dominated* deviation measures that satisfy $\mathcal{D}(X) \le EX - \inf X$ for all $X \in \mathcal{L}^2(\Omega)$ and have a one-to-one correspondence with *coherent averse measures of risk*; see [15].[5] Examples of lower range dominated deviation measures include lower semideviation, lower range deviation, CVaR-deviation, and mixed CVaR-deviation. Nonnegative risk envelopes offer a powerful modeling tool: given T scenarios for asset returns with historical probabilities $p_1, \ldots, p_T$, an investor may generate distorted probabilities $p_1 q_1, \ldots, p_T q_T$ according to his/her risk perception with each $Q = (q_1, \ldots, q_T)$ being such that $\sum_{k=1}^{T} p_k q_k = 1$ and $q_k \ge 0$, $k = 1, \ldots, T$. Then a convex hull of all such Q including element 1 forms a risk envelope $\widehat{\mathcal{Q}}$, and investor's deviation measure is uniquely determined by $\mathcal{D}(X) = EX - \inf_{Q \in \widehat{\mathcal{Q}}} E[XQ]$. In other words, if investor's risk preferences conform to the mean-deviation problem (1) with a lower range dominated deviation measure, then they can be expressed in the form of a certain set of probability measures that are distortions of the reference (historical) probability measure.

[5] Originally, in [15], averse measures of risk were called strictly expectation bounded risk measures.

2.2 Concave Ordering

An r.v. X dominates Y with respect to concave ordering, i.e. $X \succcurlyeq_c Y$, if $E[f(X)] \ge E[f(Y)]$ for every concave function $f : \mathbb{R} \to \mathbb{R}$, see [11]. Similarly X dominates Y with respect to second-order stochastic dominance (SSD), i.e. $X \succcurlyeq_{SSD} Y$, if $E[f(X)] \ge E[f(Y)]$ for every concave increasing function $f : \mathbb{R} \to \mathbb{R}$. Theorem 1.5.3 in [11] states that $X \succcurlyeq_c Y$ if and only if $X \succcurlyeq_{SSD} Y$ and $EX = EY$. A functional $\mathcal{F} : \mathcal{L}^p(\Omega) \to [-\infty, \infty]$ is called *Schur convex* if $X \succcurlyeq_c Y$ implies $\mathcal{F}(X) \le \mathcal{F}(Y)$. On an atomless probability space, every law invariant deviation measure $\mathcal{D}(X) : \mathcal{L}^p(\Omega) \to [0, \infty]$ is Schur convex, see [6]. In particular, this implies that $\mathrm{CVaR}_\alpha^\Delta(X) \le \mathrm{CVaR}_\alpha^\Delta(Y)$ for every $X \succcurlyeq_c Y$. In fact, this necessary condition is also sufficient.

Proposition 1 *$X \succcurlyeq_c Y$ if and only if $EX = EY$ and*

$$\mathrm{CVaR}_\alpha^\Delta(X) \le \mathrm{CVaR}_\alpha^\Delta(Y) \quad \textit{for all } \alpha \in [0,1]. \tag{7}$$

Proof Theorem 1.5.3 in [11] implies that $X \succcurlyeq_c Y$ if and only if $EX = EY$ and $X \succcurlyeq_{SSD} Y$. The last condition is equivalent to (7) (under $EX = EY$) by Theorem 2.58 in [2]. □

Two r.v.'s $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are said to be *comonotone*, if there exists a set $A \subseteq \Omega$ such that $P[A] = 1$ and $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \ge 0$ for all $\omega_1, \omega_2 \in A$. A deviation measure $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ is called *comonotone*, if $\mathcal{D}(X + Y) = \mathcal{D}(X) + \mathcal{D}(Y)$ for comonotone r.v.'s X and Y. The representation (3) implies that $\mathrm{CVaR}_\alpha^\Delta$ is comonotone for every $\alpha \in [0,1]$, and so is mixed CVaR-deviation (4). A deviation measure $\mathcal{D}$ is called *proper* if $\mathcal{D}(X) < \infty$ for some nonconstant X. It follows from [3, Proposition 2.4] that every proper comonotone law-invariant deviation measure $\mathcal{D}$, which can be defined on an atomless probability space (in particular, every mixed CVaR-deviation (4)), can be represented in the form

$$\mathcal{D}(X) = \int_0^1 g(\alpha)\, d(q_X(\alpha)), \tag{8}$$

for some positive concave function $g : (0,1) \to \mathbb{R}$. For mixed CVaR-deviation (4), Proposition 2.2 in [3] implies that $g(\alpha) = -\int_0^\alpha \phi(t)\,dt$, where ϕ is defined by conditions $d\lambda(\alpha) = \alpha\, d\phi(\alpha)$ and $\int_0^1 \phi(\alpha)\,d\alpha = 0$. Conversely, deviation measure (8) is a mixed CVaR-deviation if and only if $g'(1-) \equiv \lim_{\alpha\to 1} g'(\alpha) = -1$. The function g in (8) can be viewed as investor's risk profile and is closely related to a dual utility function [20, 19].

Example 1 *The mixed CVaR-deviation* (4) *with* $\lambda(\alpha) \equiv \alpha$, *i.e.*

$$\mathcal{D}(X) = \int_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, d\alpha, \tag{9}$$

corresponds to (8) *with* $g(\alpha) = -\alpha \ln \alpha$.

Detail. In this case, $\alpha\, d\phi(\alpha) = d\lambda(\alpha) = 1$ implies $\phi(\alpha) = \ln\alpha + C$, whereas $\int_0^1 \phi(\alpha)\,d\alpha = 0$ implies $C = 1$, so that $g(\alpha) = -\int_0^\alpha \phi(t)\,dt = -\alpha\ln\alpha$. □

Example 2 *A finite convex combination of CVaR-deviations*

$$\mathcal{D}_{\lambda,a}(X) = \sum_{i=1}^n \lambda_i\, \mathrm{CVaR}_{a_i}^\Delta(X) \tag{10}$$

for some $a = (a_1,\ldots,a_n)$ *such that* $a_1 < a_2 < \cdots < a_n$ *with* $a_i \in (0,1)$, $i = 1,\ldots,n$, *and weights* $\lambda = (\lambda_1,\ldots,\lambda_n)$ *such that* $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \ge 0$, $i = 1,\ldots,n$, *can be represented in the form (8) with a piecewise linear function g that joins the points* $(0,0)$, $(a_1, g(a_1))$, ..., $(a_n, g(a_n))$, $(1,0)$, *where*

$$g(a_i) = \sum_{j=1}^{i-1} \lambda_j + a_i \sum_{j=i}^n \frac{\lambda_j}{a_j} - a_i, \quad i = 1,\ldots,n. \tag{11}$$

Conversely, if $\mathcal{D}$ *is defined by (8) with a piecewise linear concave function g with* $g(0) = g(1) = 0$, $g(a_n) = 1 - a_n$, *then* $\mathcal{D} = \mathcal{D}_{\lambda,a}$, *where* $\lambda_1,\ldots,\lambda_n$ *are found from (11).*

Detail. $\mathcal{D}_{\lambda,a}$ can be represented in the form (4) with a discrete weighting measure $\lambda(\alpha)$, having atoms λ_i at points α_i, $i = 1,\ldots,n$, and assuming 0 otherwise. In this case, the condition $d(\lambda(\alpha)) = \alpha d(\phi(\alpha))$ implies that ϕ is a step function being c_i on intervals (a_{i-1}, a_i), $i = 1,\ldots,n+1$, where $a_0 = 0$, $a_{n+1} = 1$, and $c_{i+1} - c_i = \lambda_i/a_i$, $i = 1,\ldots,n$. Then the condition $0 = \int_0^1 \phi(\alpha)\,d\alpha = \sum_{i=1}^{n+1} c_i\,(a_i - a_{i-1})$ reduces to

$$\sum_{i=1}^{n+1} c_i\, a_i = \sum_{i=1}^{n+1} c_i\, a_{i-1} = \sum_{i=0}^{n} c_{i+1}\, a_i$$
$$= 0 + \sum_{i=1}^{n}\left(c_i + \frac{\lambda_i}{a_i}\right) a_i = \sum_{i=1}^n c_i\, a_i + 1,$$

which implies $c_{n+1} = 1$. Thus, $c_i = 1 - \sum_{j=i}^n \lambda_j/a_j$, $i = 1,\ldots,n+1$, and $g(\alpha) = -\int_0^\alpha \phi(t)\,dt$ is a piecewise linear function that joins the points $(0,0)$, $(a_1, g(a_1))$, ..., $(a_n, g(a_n))$, $(1,0)$ with $g(a_i)$ given by (11). □

In fact, every (not necessarily comonotone) Schur convex deviation measure $\mathcal{D}$ can be represented in the form

$$\mathcal{D}(X) = \sup_{g(\alpha)\in G} \int_0^1 g(\alpha)\, d(q_X(\alpha)), \tag{12}$$

where G is a collection of positive concave functions $g : (0,1) \to \mathbb{R}$. The representation (12) was first established for an atomless probability space in [3, Proposition 2.2], and then it was generalized to an arbitrary probability space in [6]. It implies the following result.

Proposition 2 *Let* $\mathcal{D}$ *be a proper Schur convex deviation measure. Then* $\mathcal{D}(X) < \infty$ *for* every *bounded r.v.* X.

Proof For proper $\mathcal{D}$, functions $g \in G$ in (12) are uniformly bounded by some constant M [3, Proposition 3.2], and, consequently, $\mathcal{D}(X) \le M(\sup X - \inf X) < \infty$ for *every* bounded r.v. X. □

3 Mean-Deviation Framework

3.1 Portfolio Problem Formulation

Let $\mathcal{V}$ be a feasible set of *bounded* portfolio returns. It is assumed that the set $\mathcal{V}$ contains a constant r_0 corresponding to the rate of return of a risk-free asset, and that $\mathcal{V}$ is convex: if $X_1 \in \mathcal{V}$ and $X_2 \in \mathcal{V}$ then

$\lambda X_1+(1-\lambda)X_2 \in \mathcal{V}$ for any $\lambda \in [0,1]$. A set $\mathcal{V}$ is called *arbitrage-free* if there exists a probability measure $\mathbb{P}'$ on $(\Omega, \mathcal{M})$, equivalent to $\mathbb{P}$, such that $E_{\mathbb{P}'}[X] = r_0$ for all $X \in \mathcal{V}$, i.e. $E[Q^*X] = r_0$ for all $X \in \mathcal{V}$, where $Q^* = \mathbb{P}'/\mathbb{P}$ is the Radon-Nikodym derivative of $\mathbb{P}'$ with respect to $\mathbb{P}$. Obviously, $EQ^* = 1$, and $Q^* \geq 0$. Such Q^* will be called a *risk-neutral* r.v.

Example 3 *Let*

$$\mathcal{V} = \left\{ X = \sum_{j=0}^{n} r_j x_j \;\middle|\; \sum_{j=0}^{n} x_j = 1,\ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \right\}, \tag{13}$$

where r_0, r_1, $\ldots$, r_n are the rates of return of $n+1$ assets, of which asset 0 *is assumed to be risk-free, x_j is the fraction of capital invested into asset j with $x_j < 0$ corresponding to short selling, and $\sum_{j=0}^{n} x_j = 1$ is the budget constraint. $\mathcal{V}$ is called arbitrage-free if $X \geq r_0$ for $X \in \mathcal{V}$ implies $X = r_0$. By the fundamental theorem of asset pricing (see, e.g., [2, Theorem 1.6]), this is equivalent to the definition of arbitrage-free $\mathcal{V}$ stated above. An r.v. $Q^* \in \mathcal{L}^2(\Omega)$ is risk-neutral if and only if $Q^* \geq 0$, $EQ^* = 1$, and $E[Q^* r_i] = r_0$, $i = 1, \ldots, n$.*

Suppose an investor formulates the Markowitz-type portfolio selection problem (1) with some set of feasible portfolios $\mathcal{V}$. Theorem 4 in [16] shows that the necessary and sufficient optimality condition for (1) with finite $\mathcal{D}$ and with $\mathcal{V}$ in the form (13) amounts to the existence of a risk identifier $Q^* \in \mathcal{Q}_{\mathcal{D}}(X^*)$ such that

$$E\left[(Er_i - r_i)Q^*\right] = \lambda\left[Er_i - r_0\right], \quad i = 1, \ldots, n, \tag{14}$$

where $\lambda = \mathcal{D}(X^*)/\pi$. In fact, the condition (14) can be reformulated in the form of a generalized *capital asset pricing model (CAPM)*: $Er_i - r_0 = \beta_i\,(Er_M - r_0)$, where $\beta_i = \mathrm{covar}(-r_i, Q_M)/\mathcal{D}(r_M)$ is the generalized beta, r_M is the rate of return of a *master fund* that has zero weight in the risk-free asset, and Q_M is a risk identifier of r_M with respect to $\mathcal{D}$, i.e. $Q_M \in \mathcal{Q}_{\mathcal{D}}(r_M)$, see [17, 16].

3.2 Inverse Portfolio Problem

Suppose an investor believes that his/her risk preferences are consistent with the mean-deviation approach, i.e. he/she prefers instruments' rates of return with higher means and lower deviations, and suppose the investor is relatively satisfied with a given rate of return X^*, so that X^* can be assumed to represent the rate of return of investor's optimal portfolio. This is a typical situation at a pension fund, when the investor, being presented with several predetermined portfolios (investment "schemes"), choses just one of them based on his/her risk preferences. Now, the question is what deviation measure $\mathcal{D}$ produces X^* through the portfolio problem (1). In addition, $\mathcal{D}$ is desired to be law invariant and to be defined for all possible distributions, which is equivalent to $\mathcal{D}$ being Schur convex. This is an inverse portfolio problem that can be formulated as follows.

Problem I *Given a convex set $\mathcal{V}$ of bounded feasible portfolio returns and given $X^* \in \mathcal{V}$, find a proper Schur convex deviation measure $\mathcal{D}$, such that X^* is a solution of* (1).

Observe that if $\mathcal{V}$ contains no constant C^* such that $C^* \geq EX^*$, the deviation measure $\mathcal{D}(X) \equiv \infty$ with $X \neq C$ is a trivial solution, so that $\mathcal{D}^*$ is required to be proper. Once the investor determines $\mathcal{D}$ from Problem I, he/she can use it in (1) with a different set $\mathcal{V}$, e.g. with different instruments and different constraints on portfolio weights.

First, we characterize portfolio rates of return that cannot be optimal in (1) and then establish necessary and sufficient conditions for the existence of a solution to Problem I.

Definition 3 *(super-domination). An r.v. X super-dominates another r.v. Y, or $X >_s Y$ if*

$$\mathrm{CVaR}_\alpha^\Delta(X) < \mathrm{CVaR}_\alpha^\Delta(Y) \quad \textit{for all } \alpha \in [0,1]. \tag{15}$$

Equivalently, $X >_s Y$ for nonconstant r.v.'s X and Y if and only if

$$\phi_{XY}(\alpha) < 1 \quad \text{for all } \alpha \in [0,1], \tag{16}$$

where

$$\phi_{XY}(\alpha) := \frac{\mathrm{CVaR}_\alpha^\Delta(X)}{\mathrm{CVaR}_\alpha^\Delta(Y)} \quad \text{for all } \alpha \in [0,1]. \tag{17}$$

If both X and Y are bounded, (3) implies that $\lim_{\alpha\to 0+} \phi_{XY}(\alpha)$ and $\lim_{\alpha\to 1-} \phi_{XY}(\alpha)$ exist, are finite and determined by

$$\lim_{\alpha\to 0+} \phi_{XY}(\alpha) = \frac{EX - \inf X}{EY - \inf Y} = \phi_{XY}(0),$$
$$\lim_{\alpha\to 1-} \phi_{XY}(\alpha) = \lim_{\alpha\to 1-} \frac{\frac{d}{d\alpha}\left(EX - \frac{1}{\alpha}\int_0^\alpha q_X(t)\,dt\right)}{\frac{d}{d\alpha}\left(EY - \frac{1}{\alpha}\int_0^\alpha q_Y(t)\,dt\right)} \tag{18}$$
$$= \frac{\sup X - EX}{\sup Y - EY} = \phi_{XY}(1),$$

so that $\phi_{XY}(\alpha)$ is a continuous function on $[0, 1]$.

The following proposition shows that there is no deviation measure for which a super-dominated portfolio is optimal in (1).

Proposition 3 *For nonconstant bounded r.v.'s X and Y, the following two statements are equivalent*

(i) $X \succ_s Y$, and

(ii) $\mathcal{D}(X) < \mathcal{D}(Y)$ for every proper Schur convex deviation measure $\mathcal{D}$.

Proof Obviously, (ii) implies (i). Let us show that (i) yields (ii). Suppose that $X \succ_s Y$. Since the function $\phi_{XY}(\alpha)$ defined by (17) is continuous on $[0, 1]$, it attains its maximum on $[0, 1]$, and (16) implies that $\phi_{XY}(\alpha) < 1 - \epsilon$ for all $\alpha \in [0, 1]$ and some $\epsilon > 0$. Then by Proposition 1, the r.v. $X/(1 - \epsilon) + E[Y - X/(1 - \epsilon)]$ dominates Y with respect to concave ordering, so that $\mathcal{D}(Y) \geq \mathcal{D}(X/(1 - \epsilon)) = \mathcal{D}(X)/(1 - \epsilon)$. Because X and Y are bounded and nonconstant, it follows that $\infty > \mathcal{D}(Y) > \mathcal{D}(X) > 0$. □

An r.v. $Y \in \mathcal{V}$ is called $\succ_s$-undominated, if there is no $X \in \mathcal{V}$ such that $EX \geq EY$ and $X \succ_s Y$. Proposition 3 implies that every optimal solution in (1) must be $\succ_s$-undominated. Next we show that the converse also holds.

Proposition 4 *Let $\mathcal{V}$ be a convex set of bounded feasible portfolio returns, and let $X^* \in \mathcal{V}$ be $\succ_s$-undominated. Then there exist a proper Schur convex deviation measure $\mathcal{D}$ and constant π, such that X^* is a solution of (1). Moreover, $\mathcal{D}$ can be chosen in the form of mixed CVaR-deviation* (4).

Proof Without loss of generality, we may assume that $EX \geq EX^*$ for every $X \in \mathcal{V}$, otherwise r.v.'s with $EX < EX^*$ may be safely excluded from $\mathcal{V}$. Let $\mathcal{S}$ be the vector space of all continuous functions $[0, 1] \to \mathbb{R}$. The convexity of $\mathcal{V}$ and of $\mathrm{CVaR}^\Delta_\alpha$ implies that the set $\mathcal{U} \subset \mathcal{S}$ defined as $\mathcal{U} = \{\phi | \exists X \in \mathcal{V} : \phi(\alpha) \geq \phi_{XX^*}(\alpha) \text{ for all } \alpha \in [0, 1]\}$ is convex. Because $\phi_{X^*X^*}(\alpha) = 1$ for all α, and $X^* \in \mathcal{V}$ is $\succ_s$-undominated, the function $\phi^* \equiv 1$ belongs to $\mathcal{U}$, but not to the interior of $\mathcal{U}$. Thus, there exists a supporting functional $\mathcal{L}$ for the set $\mathcal{U}$ at the point ϕ^*: a non-zero linear functional $\mathcal{L} : \mathcal{S} \to \mathbb{R}$ which attains its supremum on $\mathcal{U}$ at ϕ^*; see e.g. [21, Corollary 1.1.4]. Then a functional $\mathcal{D}$ defined as $\mathcal{D}(X) = -\mathcal{L}(\phi_{XX^*}(\alpha))$ attains its infimum on $\mathcal{V}$ at X^*.

For any nonnegative function ϕ ($\phi(\alpha) \geq 0$ for all $\alpha \in [0, 1]$), $\phi^* + \phi \in \mathcal{U}$, so that $\mathcal{L}(\phi^* + \phi) \leq \mathcal{L}(\phi^*)$, which implies $\mathcal{L}(\phi) \leq 0$, or $(-\mathcal{L})(\phi) \geq 0$. Thus, $(-\mathcal{L})$ is a positive linear functional, and $(-\mathcal{L})(\phi) = \int_0^1 \phi(\alpha)\,d\mu(\alpha)$ for some Borel regular measure μ on $[0, 1]$. Consequently,

$$\mathcal{D}(X) = -\mathcal{L}(\phi_{XX^*}(\alpha)) = \int_0^1 \frac{\mathrm{CVaR}^\Delta_\alpha(X)}{\mathrm{CVaR}^\Delta_\alpha(X^*)} d\mu(\alpha). \tag{19}$$

Then the functional $\mathcal{D}'(X) = \mathcal{D}(X)/C$, where $C = \int_0^1 \frac{d\mu(\alpha)}{\mathrm{CVaR}^\Delta_\alpha(X^*)}$, is a mixed CVaR-deviation (4) with measure λ given by $d\lambda(\alpha) = \frac{1}{C}\frac{d\mu(\alpha)}{\mathrm{CVaR}^\Delta_\alpha(X^*)}$. □

Propositions 3 and 4 imply that $X^* \in \mathcal{V}$ is optimal in (1) for a proper Schur convex deviation measure $\mathcal{D}$ if and only if it is $\succ_s$-undominated, and, in this case, the corresponding deviation measure can always be chosen in the form of mixed CVaR-deviation (4).

A solution to Problem I is obviously not unique. For example, if $\mathcal{D}$ is a solution, then so is $(C\mathcal{D})$ defined as $(C\mathcal{D})(X) \equiv C\mathcal{D}(X)$ for any constant $C > 0$. Also, $\mathcal{D}'(X) \equiv \max\{\mathcal{D}(X), \mathcal{D}_1(X)\}$ is a solution, where $\mathcal{D}_1$ is any deviation measure with $\mathcal{D}_1(X^*) = \mathcal{D}(X^*)$. However, the following result holds.

Proposition 5 *Let Ω be an atomless probability space and let $\mathcal{D}^* : \mathcal{L}^2(\Omega) \to [0, \infty]$ be a mixed CVaR-deviation* (4). *Then there exist a convex set $\mathcal{V}$ of bounded portfolio returns and $X^* \in \mathcal{V}$, such that $\mathcal{D}^*$ is the unique solution of Problem I in the form* (4).

Proof Let X^* be an arbitrary r.v. with a continuous CDF, and let $\mathcal{V}$ be a set of all bounded r.v.'s X such that $EX = EX^*$, $\mathcal{D}^*(X) \geq \mathcal{D}^*(X^*)$, and $X = f(X^*)$ for some non-decreasing function $f : \mathbb{R} \to \mathbb{R}$. Then all r.v.'s in $\mathcal{V}$ are pairwise comonotone, which implies

$$\mathcal{D}^*(\lambda X + (1 - \lambda)Y) = \lambda\mathcal{D}^*(X) + (1 - \lambda)\mathcal{D}^*(Y)$$
$$\geq \mathcal{D}^*(X^*) \quad \text{for all } X \text{ and } Y \text{ in } \mathcal{V}.$$

Consequently, $\mathcal{V}$ is a convex set. Obviously, $X^* \in \mathcal{V}$, and $\mathcal{D}^*$ is a solution to Problem I.

Now, let $\mathcal{D}_1$ be another solution in the form of (4). Then $\mathcal{D}_1$ is comonotone, and $\mathcal{D}_2(X) = \frac{\mathcal{D}^*(X^*)}{\mathcal{D}_1(X^*)}\mathcal{D}_1(X)$ is another comonotone solution with $\mathcal{D}_2(X^*) = \mathcal{D}^*(X^*)$. Proposition 2.4 in [3] shows that every comonotone deviation measure $\mathcal{D}$ can be represented in the form (8) for some positive concave function $g : (0,1) \to \mathbb{R}$. Let g^* and g_2 be the corresponding functions in (8) for $\mathcal{D}^*$ and $\mathcal{D}_2$, respectively, such that $g_2(\alpha) < g^*(\alpha)$ for some $\alpha \in (0,1)$. For the function f defined by $f(x) = 0$ for $x \le q_{X^*}(\alpha)$ and by $f(x) = 1$ for $x > q_{X^*}(\alpha)$, the r.v. $Y = f(X^*)$ takes values 0 and 1 with probabilities α and $1-\alpha$, respectively, so that $\mathcal{D}_2(Y) = g_2(\alpha) < g^*(\alpha) = \mathcal{D}^*(Y)$. Thus, the r.v. $Y' = \frac{\mathcal{D}^*(X^*)}{\mathcal{D}^*(Y)}(Y - EY) + EX^*$ is in $\mathcal{V}$. However, $\mathcal{D}_2(Y') < \mathcal{D}^*(X^*) = \mathcal{D}_2(X^*)$, which contradicts the optimality of X^*. Consequently, $g_2(\alpha) \ge g^*(\alpha)$ for all $\alpha \in (0,1)$. Now, if the last inequality is strict for some $\alpha \in (0,1)$, concavity of g_2 and g^* implies that it is strict on some interval with non-zero measure. Thus, $\mathcal{D}_2(X^*) > \mathcal{D}^*(X^*)$ by (8) and by continuity of the distribution of X^*, which is a contradiction. Consequently, $g_2(\alpha) = g^*(\alpha)$ for all $\alpha \in (0,1)$, which implies $\mathcal{D}_2(X) \equiv \mathcal{D}^*(X)$, or $\mathcal{D}_1(X) \equiv C \cdot \mathcal{D}^*(X)$. Because both $\mathcal{D}_1$ and $\mathcal{D}^*$ are mixed CVaR-deviations (4), $C = 1$ and $\mathcal{D}_1(X) \equiv \mathcal{D}^*(X)$. □

However, Problem I may have other solutions not necessarily in the form (4). More importantly, for a given set $\mathcal{V}$, more than one mixed CVaR-deviation can solve it. In this case, which one to chose?

Proposition 4 shows that if $X^* \in \mathcal{V}$ is $\succ_s$-undominated, Problem I always has a solution in the form of mixed CVaR-deviation (4). If $\mathcal{V}$ is given by (13), this implies that there exists a weighting measure λ such that (4) satisfies the optimality conditions (14), which can be restated as

$$\int_0^1 f_i(\alpha)\, d\lambda(\alpha) = 0, \qquad i = 1, \ldots, n,$$

where

$$f_i(\alpha) = (Er_i - r_0)\,\mathrm{CVaR}_\alpha^\Delta(X^*) + \pi\,\mathrm{covar}\,(r_i, Q_\alpha(X^*))$$

with $Q_\alpha(X^*)$ being the risk identifier of X^* with respect to $\mathcal{D} = \mathrm{CVaR}_\alpha^\Delta$. Observe that $f_1(\alpha), \ldots, f_n(\alpha)$ are linearly dependent. Indeed, if $x_0^*, x_1^*, \ldots, x_n^*$ are optimal portfolio weights that correspond to X^* in (13), i.e. $X^* = \sum_{i=0}^n r_i\, x_i^*$, then $\sum_{i=1}^n x_i^*\, f_i(\alpha) \equiv 0$. Thus, with $x_n^* \neq 0$, only conditions for $i = 1, \ldots, n-1$ should be used.

If λ is not unique, then the investor may take the supremum over all corresponding mixed CVaR-deviations. In other words, he/she finds the worst-case mixed-CVaR deviation from the variational problem

$$\begin{aligned} \mathcal{D}(X) = \sup_{\lambda(\alpha)\ge 0} &\int_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, d\lambda(\alpha) \\ \text{s.t.} &\int_0^1 f_i(\alpha)\, d\lambda(\alpha) = 0,\ i = 1, \ldots, n-1, \\ &\int_0^1 d\lambda(\alpha) = 1. \end{aligned} \tag{20}$$

Let $\xi_1, \ldots, \xi_{n-1}$ and u be Lagrange multipliers corresponding to the constraints in (20), then with the Lagrangian

$$L(X;\lambda;\xi,u) = \int_0^1 \left(\mathrm{CVaR}_\alpha^\Delta(X) + \sum_{i=1}^{n-1} \xi_i f_i(\alpha) - u \right) d\lambda(\alpha) + u$$

where $\xi = (\xi_1, \ldots, \xi_{n-1})$, (20) takes the form

$$\mathcal{D}(X) = \sup_{\lambda(\alpha)\ge 0}\ \inf_{\xi,\,u}\ L(X;\lambda;\xi,u). \tag{21}$$

Since $L(X;\lambda;\xi,u)$ is linear with respect to λ, ξ, and u, it has a saddle point, so that "sup" and "inf" in (21) can be exchanged, and (21) simplifies to

$$\begin{aligned} \mathcal{D}(X) = \inf_{\xi_1,\ldots,\xi_{n-1},\,u}\ & u \\ \text{s.t.}\ & \mathrm{CVaR}_\alpha^\Delta(X) + \sum_{i=1}^{n-1} \xi_i f_i(\alpha) \le u \\ & \text{for all } \alpha \in [0,1], \end{aligned}$$

which can be recast as

$$\mathcal{D}(X) = \inf_{\xi_1,\ldots,\xi_{n-1}}\ \sup_{\alpha\in[0,1]} \left(\mathrm{CVaR}_\alpha^\Delta(X) + \sum_{i=1}^{n-1} \xi_i\, f_i(\alpha) \right). \tag{22}$$

For $X^* \in \mathcal{V}$ being $\succ_s$-undominated, the right-hand side in (20) has always a feasible solution, whereas for bounded X, (22) and (20) are both finite and, consequently, define the same deviation measure $\mathcal{D}$.

The deviation measure (20) solves Problem I in the form of worst-case mixed-CVaR deviation (5). However, it may not be a unique solution in this form and in deviation measures that do not admit the representation (5). In this case, the investor may take the supremum over *all* possible solutions of Problem I. Since any solution $\mathcal{D}$ generates a family of equivalent solutions $(C\mathcal{D}), C > 0$, we may assume $\mathcal{D}(X^*) = \pi$. Let $\mathcal{W}$ be the set of all solutions $\mathcal{D}$ to Problem I such that $\mathcal{D}(X^*) = \pi$.

Proposition 6 *If non-empty, the set $\mathcal{W}$ contains the unique "maximal" element, i.e. there exists a unique deviation measure $\mathcal{D}^* \in \mathcal{W}$ such that $\mathcal{D}^*(X) \geq \mathcal{D}(X)$ for all $X \in \mathcal{L}^2(\Omega)$ and $\mathcal{D} \in \mathcal{W}$. In fact, $\mathcal{D}^*$ is the supremum of all elements of $\mathcal{W}$:*

$$\mathcal{D}^*(X) = \sup_{\mathcal{D}\in\mathcal{W}} \mathcal{D}(X) \quad \textit{for all } X \in \mathcal{L}^2(\Omega), \tag{23}$$

which admits an equivalent representation

$$\mathcal{D}^*(X) = \pi \sup_{\alpha\in[0,1]} \frac{\mathrm{CVaR}_\alpha^\Delta(X)}{\mathrm{CVaR}_\alpha^\Delta(X^*)} \quad \textit{for all } X \in \mathcal{L}^2(\Omega). \tag{24}$$

Proof The deviation (23), being the supremum of Schur convex deviations, is Schur convex. Also, $\mathcal{D}^*(X^*) = \sup_{\mathcal{D}\in\mathcal{W}} \mathcal{D}(X^*) = \sup_{\mathcal{D}\in\mathcal{W}} \pi = \pi$ and $\mathcal{D}^*(X) = \sup_{\mathcal{D}\in\mathcal{W}} \mathcal{D}(X) \geq \sup_{\mathcal{D}\in\mathcal{W}} \pi = \pi$ for every $X \in \mathcal{V}$ with $EX \geq \pi$, so that $\mathcal{D}^* \in \mathcal{W}$, and (23) implies that $\mathcal{D}^*(X) \geq \mathcal{D}(X)$ for all $X \in \mathcal{L}^2(\Omega)$ and $\mathcal{D} \in \mathcal{W}$.

Now, let $\mathcal{D}^{**}(X)$ be the right-hand side in (24). $\mathcal{D}^{**}$, being the supremum of Schur convex deviations, is Schur convex, and, obviously, $\mathcal{D}^{**}(X^*) = \pi$. Since $\mathcal{W}$ is non-empty, X^* is $\succ_s$-undominated by Proposition 3, which implies that $\mathcal{D}^*(X) \geq \pi$ for every $X \in \mathcal{V}$ with $EX \geq \pi$, i.e. $\mathcal{D}^{**} \in \mathcal{W}$. Let $\mathcal{D}$ be another proper Schur convex deviation measure such that $\mathcal{D}(X^*) = \pi$, and let $X \in \mathcal{L}^2(\Omega)$. By Proposition 3.3 in [3], the supremum in (12) is attained for some positive concave function $g_X : (0,1) \to [0,\infty]$, so that

$$\mathcal{D}(X) = \int_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, d\lambda_X(\alpha) \tag{25}$$

for some nonnegative weighting measure λ_X on $[0,1]$. Consequently,

$$\begin{aligned}\mathcal{D}(X) &\leq \int_0^1 \frac{\mathcal{D}^{**}(X)}{\pi} \mathrm{CVaR}_\alpha^\Delta(X^*)\, d\lambda_X(\alpha)\\ &\leq \frac{\mathcal{D}^{**}(X)}{\pi} \mathcal{D}(X^*) = \mathcal{D}^{**}(X),\end{aligned}$$

which implies that $\mathcal{D}^{**}$ is the maximal element of $\mathcal{W}$, and thus, $\mathcal{D}^* = \mathcal{D}^{**}$ by the uniqueness of the maximal element. □

Remark 1 *The deviation measure* (24) *is finite for any bounded X and is $+\infty$ for any unbounded X.*

Remark 2 *In contrast to* (22)*, the deviation measure* (24) *does not depend on the set $\mathcal{V}$, which is a substantial advantage. For example, if no-shorting constraints are added to* (13)*, then the optimality conditions* (14) *may no longer hold, and as a result, the deviation* (22) *that follows from* (14) *through* (20) *may be meaningless.*

The deviation measure (24) can be represented in the form

$$\begin{aligned}\mathcal{D}^*(X) = \pi\ &\inf_u u\\ &\text{s.t. } \mathrm{CVaR}_\alpha^\Delta(X) \leq u\, \mathrm{CVaR}_\alpha^\Delta(X^*)\\ &\quad \text{for all } \alpha \in [0,1].\end{aligned} \tag{26}$$

Let X be bounded. Then with a Lagrange multiplier $\eta(\alpha) \geq 0$ and the Lagrangian

$$\widetilde{L}(X;u;\eta) = u + \int_0^1 \eta(\alpha) \left(\mathrm{CVaR}_\alpha^\Delta(X) - u\, \mathrm{CVaR}_\alpha^\Delta(X^*)\right) d\alpha,$$

(26) can be rewritten as

$$\mathcal{D}^*(X) = \pi \inf_u \sup_{\eta(\alpha)\geq 0} \widetilde{L}(X;u;\eta).$$

Since $\widetilde{L}(X;u;\eta)$ is linear with respect to u and $\eta(\alpha)$, it has a saddle point, so that "inf" and "sup" in the last problem can be exchanged, and (26) simplifies to

$$\begin{aligned}\mathcal{D}^*(X) = \pi &\sup_{\eta(\alpha)\geq 0} \int_0^1 \eta(\alpha)\, \mathrm{CVaR}_\alpha^\Delta(X)\, d\alpha\\ &\text{s.t. } \int_0^1 \eta(\alpha)\, \mathrm{CVaR}_\alpha^\Delta(X^*)\, d\alpha = 1.\end{aligned} \tag{27}$$

Observe that since $\mathrm{CVaR}_\alpha^\Delta(X^*) > 0$ for $\alpha \in [0,1)$, $\mathrm{CVaR}_\alpha^\Delta(X) < \infty$ for $\alpha \in (0,1)$ and $X \in \mathcal{L}^2(\Omega)$, and $\lim\limits_{\alpha\to 1-} \mathrm{CVaR}_\alpha^\Delta(X)/\mathrm{CVaR}_\alpha^\Delta(X^*)$ exists for bounded X, both (26) and (27) have feasible solutions. The representation (27) shows that (24) is similar to the worst-case mixed-CVaR deviation (5) with $d\lambda(\alpha) = \eta(\alpha)\, d\alpha$.

4 Application to Scenario-based Portfolio Optimization

Let $\mathcal{V}$ be given by (13). Suppose that there are $T \geq 2$ equiprobable scenarios with the excess return $a_{ji} = r_{ji} - r_0$ under scenario i. Elements a_{ji} form an $n \times T$ matrix A, and the columns of A are assumed to be linearly independent. In this model, each $Y \in \mathcal{V}$ is a discrete r.v. assuming values $y_1, \ldots, y_T$ with equal probabilities.

Proposition 7 *Let $\mathcal{V}$ be given by (13), and let $X, Y \in \mathcal{V}$. Then $X \succ_s Y$ if and only if*

$$\mathrm{CVaR}_\alpha^\Delta(X) < \mathrm{CVaR}_\alpha^\Delta(Y), \quad \alpha = i/T, \quad 1 \leq i \leq T-1. \tag{28}$$

Proof For discrete r.v.'s X and Y, $\mathrm{CVaR}_\alpha^\Delta(X)$ and $\mathrm{CVaR}_\alpha^\Delta(Y)$ are piecewise-linear functions of α on $(0,1)$. Consequently, it suffices to check the inequality $\mathrm{CVaR}_\alpha^\Delta(X) < \mathrm{CVaR}_\alpha^\Delta(Y)$ only at the joints of the segments. Moreover, for any r.v. $Z \in \mathcal{V}$,

$$\begin{aligned}\mathrm{CVaR}_{1/T}^\Delta(Z) &= EZ - T\int_0^{1/T} q_Z(\beta)\, d\beta \\ &= EZ - T\int_0^{1/T} (\inf Z)\, d\beta \\ &= EZ - \inf Z = \mathrm{CVaR}_0^\Delta(Z)\end{aligned}$$

and

$$\begin{aligned}\mathrm{CVaR}_{\frac{T-1}{T}}^\Delta(Z) &= EZ - \frac{T}{T-1}\int_0^{\frac{T-1}{T}} q_Z(\beta)\, d\beta \\ &= EZ - \frac{T}{T-1}\left(EZ - \int_{\frac{T-1}{T}}^1 (\sup Z)\, d\beta\right) \\ &= \frac{\sup Z - EZ}{T-1} = \frac{\mathrm{CVaR}_1^\Delta(Z)}{T-1}.\end{aligned}$$

Consequently, for $\alpha = 0$ and $\alpha = 1$, (15) follows from (28) for $i = 1$ and $i = T-1$, respectively. □

The proof of Proposition 7 implies that in the discrete case, the portfolio problem (1) with the restored deviation measure (24) in the form (26) can be solved with the constraint in (26) taken at $\alpha = i/T$, $i = 1, \ldots, T-1$, i.e.

$$\begin{aligned}\min_{X \in \mathcal{V},\, u}\ & u \\ \text{s.t.}\ & \mathrm{CVaR}_{i/T}^\Delta(X) \leq u\, \mathrm{CVaR}_{i/T}^\Delta(X^*), \quad i = 1, \ldots, T-1, \\ & EX \geq \widetilde{\pi}.\end{aligned}$$

With the formula [13]

$$\mathrm{CVaR}_\alpha^\Delta(X) = \min_{c \in \mathbb{R}} \left(EX - c + \frac{1}{\alpha} E[\max\{0, c - X\}] \right), \tag{29}$$

this problem can be rewritten as

$$\begin{aligned}\min_{X \in \mathcal{V},\, u,\, c_i,\, Z_i}\ & u \\ \text{s.t.}\ & EX - c_i + \frac{T}{i} EZ_i \leq u\, \mathrm{CVaR}_{i/T}^\Delta(X^*), \\ & Z_i \geq c_i - X, \quad Z_i \geq 0, \quad i = 1, \ldots, T-1, \\ & EX \geq \widetilde{\pi}.\end{aligned} \tag{30}$$

If $\mathcal{V}$ is a set of linear constraints for portfolio weights, then (30) admits a linear programming (LP) formulation.

Similarly, through the formula (29), the portfolio problem (1) with the worst-case mixed-CVaR deviation in the form (22) and with $\mathcal{V}$ being a set of linear constraints for portfolio weights can be reduced to an LP problem.

For discrete random variables, the proof of Proposition 4 yields the following result.

Proposition 8 *Let $\mathcal{V}$ be given by (13), and let $X^* \in \mathcal{V}$ be $\succ_s$-undominated. Then there exist non-negative weights $\lambda_1, \ldots, \lambda_{T-1}$ with $\sum_{i=1}^{T-1} \lambda_i = 1$ and such that X^* is a solution of (1) for the deviation measure*

$$\mathcal{D}(X) = \sum_{i=1}^{T-1} \lambda_i\, \mathrm{CVaR}_{i/T}^\Delta(X). \tag{31}$$

Proof As in the proof of Proposition 4, it is assumed that $EX \geq EX^*$ for all $X \in \mathcal{V}$. The convexity of $\mathcal{V}$ and of $\mathrm{CVaR}_\alpha^\Delta$ implies that the set $\mathcal{U} = \{u = (u_1, \ldots, u_{T-1}) \mid \exists X \in \mathcal{V} : u_i \geq \mathrm{CVaR}_{i/T}^\Delta(X),\ i = 1, \ldots, T-1\} \subset \mathbb{R}^{T-1}$ is convex. Because $X^* \in \mathcal{V}$ is $\succ_s$-undominated, the vector u^* with components $u_i^* = \mathrm{CVaR}_{i/T}^\Delta(X^*), i = 1, \ldots, T-1$, belongs to the boundary of $\mathcal{U}$. Thus, by the supporting hyperplane theorem, there exists a non-zero vector $\lambda = (\lambda_1, \ldots, \lambda_{T-1})$, such that $\sum_i \lambda_i u_i \geq \sum_i \lambda_i u_i^*$ for all $u \in \mathcal{U}$. Because $u^* + e^i \in \mathcal{U}$, $i = 1, \ldots, T-1$, where e^i has components $e_j^i = 0$, $j \neq i$, and $e_i^i = 1$, this implies $\lambda_i \geq 0, i = 1, \ldots, T-1$. Normalizing λ if necessary, we can assume that $\sum_{i=1}^{T-1} \lambda_i = 1$, and, consequently, λ defines a deviation measure $\mathcal{D}$ via (31). Finally, for every $X \in \mathcal{V}$,

$$\mathcal{D}(X) = \sum_{i=1}^{T-1} \lambda_i \, \mathrm{CVaR}^{\Delta}_{i/T}(X) \geq \sum_{i=1}^{T-1} \lambda_i \, u_i^* = \mathcal{D}(X^*),$$

as required. □

The optimality conditions for the portfolio problem (1) with the mixed CVaR-deviation (31) can be formulated in the form (see [16, 8])

$$\sum_{i=1}^{T-1} c_{ij} \, \lambda_i = 0, \quad j = 1, \ldots, n, \tag{32}$$

with

$$c_{ij} = \left(E[r_j] - r_0\right) \mathrm{CVaR}^{\Delta}_{\alpha_i}(X^*) + (E[X^*] - r_0) \, \mathrm{covar}\left(r_j, Q_i^*\right) \tag{33}$$

for $i = 1, \ldots, T-1$, $j = 1, \ldots, n$, and with

$$\sum_{i=1}^{T-1} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \ldots, T-1, \tag{34}$$

where Q_i^* is a risk identifier such that $E\left[Q_i^*\right] = 1$ and $0 \leq Q_i^* \leq 1/\alpha_i$ and satisfies $\mathrm{CVaR}^{\Delta}_{\alpha_i}(X^*) = \mathrm{covar}\left(-X^*, Q_i^*\right)$. If X^* assumes values X_k with equal probabilities $1/T$ for $k = 1, \ldots, T$, then $Q_i^* = \left\{q_{ik}^*\right\}_{k=1}^{T}$, where $q_{i1}^*, \ldots, q_{iT}^*$ are found from the optimization problem

$$\min_{q_{ik}} \sum_{k=1}^{T} q_{ik} \, X_k \quad \text{s.t.} \quad \sum_{k=1}^{T} q_{ik} = T, \; 0 \leq q_{ik} \leq 1/\alpha_i, \quad k = 1, \ldots, K. \tag{35}$$

In this case, $\mathrm{CVaR}^{\Delta}_{\alpha_i}(X^*) = \frac{1}{T} \sum_{k=1}^{T} \left(1 - q_{ik}^*\right) X_k$.

The system (32)–(34) has $T-1$ unknowns, $n+1$ linear equations, and $T-1$ linear inequalities. Observe that since X^* is the rate of return of an optimal portfolio, the equations in (32) are linearly dependent. (Indeed, let x_j^*, $j = 1, \ldots, n$, be optimal portfolio weights with $\sum_{j=1}^{n} x_j^* = 1$, then multiplying equation j in (32) by x_j^* and adding all equations, we obtain an identity.) If $T - 1 < n$, the system (32)–(34) can be solved approximately by minimizing the total quadratic error in satisfying the equations in (32) with respect to $\lambda_1, \ldots, \lambda_{T-1}$:

$$\min_{\lambda_1, \ldots, \lambda_{T-1}} \sum_{j=1}^{n} \left(\sum_{i=1}^{T-1} c_{ij} \, \lambda_i \right)^2 \quad \text{s.t.} \quad \sum_{i=1}^{T-1} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \ldots, T-1.$$

A problem similar to this one was solved for the S&P500 index in [8].

The results obtained in this work imply that

- If $X^* \in \mathcal{V}$ is not $\succ_s$-undominated, then the system (32)–(34) has no solution even if $T - 1 > n$.
- If $X^* \in \mathcal{V}$ is $\succ_s$-undominated, then by Proposition 8, the system (32)–(34) always has a solution even if $n > T - 1$.

In real-life portfolio management, $T \gg n$, in which case, the following result is useful.

Proposition 9 *Let $\mathcal{V}$ be defined by* (13)*, and let $X^* \in \mathcal{V}$ be $\succ_s$-undominated. Then there exist $a_1, \ldots, a_{n+1}$, $a_i \in (0, 1)$, and non-negative weights $\lambda_1, \ldots, \lambda_{n+1}$, $\sum_{i=1}^{n+1} \lambda_i = 1$, such that X^* is a solution of (1) with the deviation measure*

$$\mathcal{D}(X) = \sum_{i=1}^{n+1} \lambda_i \, \mathrm{CVaR}^{\Delta}_{a_i}(X). \tag{36}$$

Proof Let $T-1 > n+1$, otherwise the statement of this proposition follows from Proposition 8. It is sufficient to prove that the system (32)–(34) has a solution with at most $n+1$ non-zero λ_i. Proposition 8 implies that the system has *some* solution $\lambda = (\lambda_1, \ldots, \lambda_{T-1})$. Let λ have at least $n+2$ non-zero entries with the corresponding indices to be $K = \{k_1, \ldots, k_{n+2}\}$. The system

$$\sum_{i \in K} z_i = 0, \quad \sum_{i \in K} c_{ij} \, z_i = 0, \quad j = 1, \ldots, n,$$

with $n+1$ equations and $n+2$ unknowns has a non-zero solution $(z_{k_1}^*, \ldots, z_{k_{n+2}}^*)$. Let $y = (y_1, \ldots, y_{T-1})$ be such that $y_i = z_i^*$ for $i \in K$ and $y_i = 0$ otherwise. Then, for positive and small enough ϵ, the vector $\lambda' = \lambda + \epsilon \, y$ satisfies the system (32)–(34). With maximal ϵ such that the inequalities $\lambda_1' \geq 0, \ldots, \lambda_{T-1}' \geq 0$ still hold, the solution $\lambda' = (\lambda_1', \ldots, \lambda_{T-1}')$, compared to λ, has strictly smaller number of non-zero entries. The process is then repeated. □

In other words, Proposition 9 claims that there exists a solution in the form (8) with a piecewise linear function g with at most $n + 2$ segments.

Observe that if $X^* \in \mathcal{V}$ is not $\succ_s$-undominated, it can be dominated by some $Y \in \mathcal{V}$ with respect to SSD. This indicates that the investor is *not* risk averse. On the other hand, the mean-deviation portfolio problem (1) is inappropriate for risk seeking investors. Several algorithms are available for testing of whether solutions to portfolio optimization with n assets and T scenarios are SSD-efficient; see e.g. [9]. Thus, the inverse portfolio Problem I can be solved by the following algorithm.

(i) Test whether X^* is dominated by SSD within $\mathcal{V}$. If so, stop. The investor is *not* risk averse, and investor's risk preferences are incompatible with the mean-deviation portfolio problem (1).

(ii) If $X^* \in \mathcal{V}$ is not dominated by SSD within $\mathcal{V}$, it is also $\succ_s$-undominated, and thus, investor's deviation measure has the form (31). Also, this implies that the system (32)–(34) has a solution even if $n > T - 1$.

   (a) If $n + 1 \geq T - 1$, solve the system (32)–(34) for weights λ to be used in (31).

   (b) If $n + 1 < T - 1$, use the procedure in the proof of Proposition 9 to find a simpler solution in the form (36).

(iii) Construct the risk profile g in the representation (8) by (11).

The portfolio problem (1) with the "restored" deviation measure (31) takes the form

$$\min_{X \in \mathcal{V}} \sum_{i=1}^{T-1} \lambda_i \, \mathrm{CVaR}^{\Delta}_{i/T}(X) \quad \text{s.t.} \;\; EX \geq \widetilde{\pi}.$$

Using the formula (29), we can rewrite it as

$$\begin{aligned} \min_{X \in \mathcal{V},\, c_i,\, Z_i} \; & \sum_{i=1}^{T-1} \lambda_i \left(EX - c_i + \frac{T}{i} EZ_i \right) \\ \text{s.t.} \;\; & Z_i \geq c_i - X, \;\; Z_i \geq 0, \;\; i = 1, \ldots, T-1, \\ & EX \geq \widetilde{\pi}. \end{aligned} \tag{37}$$

If $\mathcal{V}$ is a set of linear constraints for portfolio weights, then (37) is an LP problem, which can be solved with popular optimization packages such as CPLEX or Portfolio Safeguard.

Example 4 (numerical illustration) *We consider a risk-free asset with zero rate of return and 91 assets from the FTSE 100 index*[6] *with weekly rates of return from 3-Jan-2011 to 4-Mar-2013. The set $\mathcal{V}$ is assumed to be in the form* (13)*. The algorithm in [9] shows that the index is dominated by SSD on $\mathcal{V}$. In this case, one may construct an SSD-efficient portfolio either by the algorithm in [9] starting from the index or as described in [22]. Yet another way is to find an optimal mean-variance portfolio, which will be SSD-efficient. We proceed with the latter approach. With X^* being the rate of return of the optimal mean-variance portfolio consisting of the specified 91 assets, the system (32)–(34) is solved to find weights $\lambda_1, \ldots, \lambda_{T-1}$ in (31). Figure 1 shows the risk profile g, which is determined by* (11) *and plays the role of a dual utility function [20, 19] in the representation* (8)*. Once $\lambda_1, \ldots, \lambda_{T-1}$ in the mixed CVaR-deviation* (31) *are determined, we can now solve the portfolio problem* (1)*, which in the discrete case and with $\mathcal{V}$ being a set of linear constraints for portfolio weights reduces to the LP problem* (37)*. However, the found $\lambda_1, \ldots, \lambda_{T-1}$ are not unique, and "to be safe," we can use either the worst-case deviation measure in the form* (26) *or the worst-case mixed-CVaR deviation* (22) *with given X^*.*

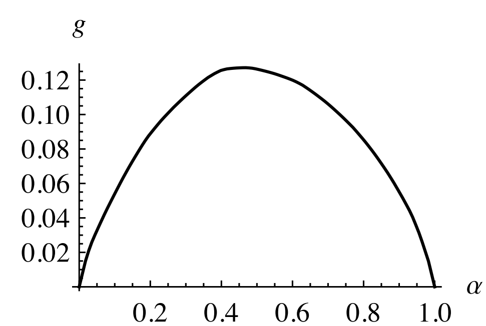

Figure 1: The function g for the found deviation measure in Example 4.

[6] There are 94 assets in the index, and three assets EVRAZ, Glencore Intl, and Polymetal Intl are excluded because of the lack of data.

5 Approximate Deviation Measure

As in Proposition 5, let Problem I have a unique solution D^* in the form (4). If the corresponding weighting measure λ^* in (4) is not simple, an *approximate* solution to Problem I can be sought in the form (10).

Determining an approximate solution requires a metric on the space of deviation measures, which can be introduced by

$$d(\mathcal{D}_1, \mathcal{D}_2) = \sup_{X \in \mathcal{L}^2(\Omega):\, |X| \leq 1} |\mathcal{D}_1(X) - \mathcal{D}_2(X)|. \quad (38)$$

Proposition 10 *A set of proper Schur convex deviation measures $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ with the distance (38) forms a metric space, i.e. for all $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$,* (38) *satisfies the metric properties: (i) $d(\mathcal{D}_1, \mathcal{D}_2) < \infty$; (ii) $d(\mathcal{D}_1, \mathcal{D}_2) = 0$ if and only if $\mathcal{D}_1 = \mathcal{D}_2$; (iii) $d(\mathcal{D}_1, \mathcal{D}_2) = d(\mathcal{D}_2, \mathcal{D}_1)$; and (iv) $d(\mathcal{D}_1, \mathcal{D}_3) \leq d(\mathcal{D}_1, \mathcal{D}_2) + d(\mathcal{D}_2, \mathcal{D}_3)$.*

Proof (i) follows from Proposition 2. Then $d(\mathcal{D}_1, \mathcal{D}_2) = 0$ implies that $\mathcal{D}_1(X) = \mathcal{D}_2(X)$ for all $|X| \leq 1$. Positive homogeneity D2 yields $D_1(X) = \mathcal{D}_2(X)$ for all bounded X, so that $\mathcal{D}_1(X) = \mathcal{D}_2(X)$ for all $X \in \mathcal{L}^2(\Omega)$ by lower semicontinuity D4, so that (ii) holds. Properties (iii) and (iv) follow from (38). □

Now, the question of finding an approximate solution to Problem I in the form (10) can be stated as two alternative problems: let $\mathcal{D}^*$ be given by (4) with the weighting measure λ^* and let n be fixed, then find λ and a such that

$$\min_{\lambda, a} d\left(\mathcal{D}^*, \mathcal{D}_{\lambda, a}\right), \quad (39)$$

or, given a, find λ such that

$$\min_{\lambda} d\left(\mathcal{D}^*, \mathcal{D}_{\lambda, a}\right). \quad (40)$$

Next proposition is instrumental in solving (39) and (40).

Proposition 11 *Let Ω be an atomless probability space, and let $\mathcal{D}_1 : \mathcal{L}^2(\Omega) \to [0, \infty]$ and $\mathcal{D}_2 : \mathcal{L}^2(\Omega) \to [0, \infty]$ be deviation measures in the form (8) with g_1 and g_2, respectively. Then*

$$d(\mathcal{D}_1, \mathcal{D}_2) = 2 \sup_{\alpha \in (0,1)} |g_1(\alpha) - g_2(\alpha)|. \quad (41)$$

Proof For $\epsilon > 0$, let $\alpha^* \in (0, 1)$ be such that $\sup_{\alpha \in [0,1]} |g_1(\alpha) - g_2(\alpha)| \leq |g_1(\alpha^*) - g_2(\alpha^*)| + \epsilon$. Then

$$\begin{aligned} d(\mathcal{D}_1, \mathcal{D}_2) &\geq |\mathcal{D}_1(X^*) - \mathcal{D}_2(X^*)| = |2g_1(\alpha^*) - 2g_2(\alpha^*)| \\ &= 2 \sup_{\alpha \in [0,1]} |g_1(\alpha) - g_2(\alpha)| - 2\epsilon, \end{aligned}$$

where X^* is an r.v. such that $\mathbb{P}[X^* = -1] = \alpha^*$ and $\mathbb{P}[X^* = 1] = 1 - \alpha^*$. Since ϵ is arbitrary, "≥" part in (41) follows. Conversely,

$$\begin{aligned} |\mathcal{D}_1(X) - \mathcal{D}_2(X)| &= \left| \int_0^1 (g_1(\alpha) - g_2(\alpha))\, dq_X(\alpha) \right| \\ &\leq \sup_{\alpha \in (0,1)} \left(|g_1(\alpha) - g_2(\alpha)| \int_0^1 dq_X(\alpha) \right) \end{aligned}$$

for any $X \in \mathcal{L}^2(\Omega)$. If $|X| \leq 1$, then $\int_0^1 d(q_X(\alpha)) \leq 2$, and "≤" part in (41) holds. □

Proposition 11 and Example 2 reduce the problems (39) and (40) to piecewise linear approximation of a convex function on $(0, 1)$ with the uniform error measure.

Example 5 *The problem (39) with $\mathcal{D}^*$ in the form (9) and with $n = 1$ has the unique solution $\mathcal{D}(X) = \mathrm{CVaR}_a^\Delta(X)$, where $a \approx 0.515$ is the only zero of $1 - a + a \ln a - \exp(-1/a)$ on $(0, 1)$.*

Detail. For $n = 1$, g in Example 2 is defined by $g_a(\alpha) = \alpha(1 - a)/a$ for $\alpha \in (0, a]$ and by $g_a(\alpha) = 1 - \alpha$ for $\alpha \in [a, 1)$, where $a = a_1$. Then $\min_a \max_\alpha |g_a(\alpha) - g^*(\alpha)|$ for $g^*(\alpha) = -\alpha \ln \alpha$ (see Example 1) yields the statement. □

6 Inverse Portfolio Problem via Risk Envelope

6.1 General Solution

If $X^* \in \mathcal{V}$ is not $\succ$-undominated, Problem I has no solution. This section does not require deviation $\mathcal{D}$ in Problem I to be Schur convex.

Problem II *Given an arbitrage-free set $\mathcal{V}$ of feasible portfolio rates of returns and $X^* \in \mathcal{V}$, find a proper deviation measure $\mathcal{D}$, such that X^* is a solution to* (1).

If $\mathcal{D}$ solves Problem II, then so does $C\mathcal{D}$ defined as $(C\mathcal{D})(X) \equiv C \cdot \mathcal{D}(X)$ for $C > 0$. Thus, without loss of generality, $\mathcal{D}$ can be assumed to satisfy $\mathcal{D}(X^*) = \pi$.

Proposition 12 *If $\mathcal{V}$ is arbitrage-free and there exists a risk neutral $Q^* \in \mathcal{Q}_{\mathcal{D}}(X^*)$, then $X^* \in \mathcal{V}$ is an optimal solution in* (1) *for some $\mathcal{D}$. In addition, if $\mathcal{V}$ is given by* (13)*, $\mathcal{D}$ is finite, and $\mathcal{D}(X^*) = \pi$, then arbitrage-free $\mathcal{V}$ and the existence of a risk neutral $Q^* \in \mathcal{Q}_{\mathcal{D}}(X^*)$ are also necessary conditions for X^* to be optimal.*

Proof Let $\mathcal{V}$ be arbitrage-free, and let there exist a risk neutral $Q^* \in \mathcal{Q}_{\mathcal{D}}(X^*)$, then $\mathcal{D}(X) \geq E[(1 - Q^*)X] = EX - r_0 \geq EX^* - r_0 = E[(1 - Q^*)X^*] = \mathcal{D}(X^*)$ for every $X \in \mathcal{L}^2(\Omega)$ such that $EX \geq EX^*$, so that $X^* \in \mathcal{V}$ is an optimal solution in (1) for $\pi = EX^*$ and $\mathcal{D}$. Conversely, if $\mathcal{V}$ is given by (13), then $\mathcal{D}$ is finite, and (14) implies that $E[Q^* r_i] = r_0$, $i = 1, \ldots, n$, so that Q^* is risk-neutral. □

The next proposition solves Problem II for any arbitrage-free $\mathcal{V}$ and any $X^* \in \mathcal{V}$ such that $EX^* > r_0$.

Proposition 13 *Let $\mathcal{V}$ be arbitrage-free. Then any $X^* \in \mathcal{V}$ such that $EX^* > r_0$ is an optimal solution in (1) for $\pi = EX^*$ and for a deviation measure $\mathcal{D}^*$ with the risk envelope*

$$\mathcal{Q} = \Big\{ Q \in \mathcal{L}^2(\Omega) \,\Big|\, Q \geq 0,\ EQ = 1, \quad E[X^* Q] \geq E[X^* Q^*] \text{ for all } X \Big\}, \tag{42}$$

where $Q^* \in \mathcal{L}^2(\Omega)$ is a risk-neutral r.v.

Proof The set $\mathcal{Q}$ defined by (42) is a risk envelope. Indeed, convexity and closedness in property Q1 are straightforward, and $EX^* > r_0$ implies that $1 \in \mathcal{Q}$. Property Q2 follows directly from (42). Finally, let $Q_\epsilon \in \mathcal{L}^2(\Omega)$ be defined by $Q_\epsilon(\omega) = 1 + \epsilon$ if $X(\omega) < EX$, by $Q_\epsilon(\omega) = 1 - \epsilon\, p_2/p_1$ if $X(\omega) > EX$, and by $Q_\epsilon(\omega) = 1$ if $X(\omega) = EX$, where $X \in \mathcal{L}^2(\Omega)$ is nonconstant, $p_1 = \mathbb{P}[X > EX] > 0$ and $p_2 = \mathbb{P}[X < EX] > 0$. Then $EQ_\epsilon = 1$ and $E[XQ_\epsilon] < EX$ for all $\epsilon > 0$. On the other hand, $\lim_{\epsilon \to 0} E[X^* Q_\epsilon] = EX^* > r_0 = E[X^* Q^*]$, whence $E[X^* Q_\epsilon] \geq E[X^* Q^*]$, or $Q_\epsilon \in \mathcal{Q}$ for some $\epsilon > 0$, which proves Q3. Thus, the second representation in (6) with $\mathcal{Q}$ given by (42) defines a deviation measure $\mathcal{D}^*$, and (42) implies that Q^* is a risk identifier of X^*. Consequently, X^* is optimal in (1) by Proposition 12. □

6.2 Schur Convex Solution via Risk Envelope

Let $\mathcal{V}$ be be given by (13). This section constructs a solution of Problem I in the dual form (6).

Proposition 14 *A deviation measure $\mathcal{D}$ is Schur convex if and only if the corresponding risk envelope $\mathcal{Q}$ is $\succeq_c$-closed, i.e. $Y \in \mathcal{Q}$ and $Z \succeq_c Y$ imply that $Z \in \mathcal{Q}$. In this case, $-Q$ is comonotone with $X \in \mathcal{L}^2(\Omega)$ for all $Q \in \mathcal{Q}(X)$.*

Proof The first statement follows from [1, Corollary 3.3], let us prove the second one. By contradiction, let $X \in \mathcal{L}^2(\Omega)$ and $Q \in \mathcal{Q}(X)$, and let $\mathcal{Q}$ be $\succeq_c$-closed. Now suppose there exist sets $A_1 \subset \Omega$ and $A_2 \subset \Omega$ such that $p_i = \mathbb{P}(A_i) > 0$, $i = 1, 2$, $\epsilon = \inf_{\omega \in A_1} X(\omega) - \sup_{\omega \in A_2} X(\omega) > 0$ and $\delta = \inf_{\omega \in A_1} Q(\omega) - \sup_{\omega \in A_2} Q(\omega) > 0$. Then $Q^* \succeq_c Q$, where the r.v. Q^* is defined by $Q^*(\omega) = Q(\omega) - \delta p_2/(p_1 + p_2)$ for $\omega \in A_1$, by $Q^*(\omega) = Q(\omega) + \delta p_1/(p_1 + p_2)$ for $\omega \in A_2$, and by $Q^*(\omega) = Q(\omega)$ for $\omega \notin A_1 \cup A_2$. Consequently, $Q^* \in \mathcal{Q}$, and $\mathcal{D}(X) \geq E[(1 - Q^*)X] \geq E[(1 - Q)X] + \epsilon\, \delta p_1 p_2/(p_1 + p_2) > E[(1 - Q)X]$, which contradicts to $Q \in \mathcal{Q}(X)$. □

Proposition 15 *Let $\mathcal{V}$ be given by (13), and let $Er_i \neq r_0$ for some i. An r.v. X^* is $\succ_s$-undominated if and only if there exists a risk-neutral r.v. $Q^* \in \mathcal{L}^2(\Omega)$, such that $-Q^*$ is comonotone with X^*. In this case, X^* is a solution of (1) with $\pi = EX^*$ and with a proper Schur convex deviation measure $\mathcal{D}^*$ with the risk envelope*

$$\mathcal{Q} = \{\, Q \in \mathcal{L}^2(\Omega) \,|\, Q \succeq_c Q^* \}, \tag{43}$$

and $\mathcal{D}^*$ can be represented in the form

$$\mathcal{D}^*(X) = \int_0^1 q_{1-Q^*}(\alpha)\, q_X(\alpha)\, d\alpha. \tag{44}$$

Proof If X^* is $\succ_s$-undominated, Proposition 4 implies that it is optimal in (1) with $\pi = EX^*$ and with a deviation measure $\mathcal{D}$ in the form (4). In particular, $\mathcal{D}$ is finite, and, by Proposition 12, there exists a risk neutral $Q^* \in \mathcal{Q}_{\mathcal{D}}(X^*)$. Then $-Q^*$ is comonotone with X^* by Proposition 14.

Conversely, let $Q^* \in \mathcal{L}^2(\Omega)$ be risk neutral and such that $-Q^*$ is comonotone with X^*. Then $\mathcal{Q}$ defined in (42) satisfies the three properties (Q1, Q2, and Q3) of risk envelope. Indeed, Proposition 1 and convexity of $\mathrm{CVaR}_\alpha^\Delta$ imply $1 = EQ^* \succeq_c Q^*$, and

$\lambda Q_1 + (1-\lambda)Q_2 \succeq_c Q^*$ whenever $Q_1 \succeq_c Q^*$ and $Q_2 \succeq_c Q^*$, and Q1 follows. Then $Q \succeq_c Q^*$ implies $EQ = EQ^* = 1$ for all $Q \in \mathcal{Q}$, which is Q2. Because $Er_i \neq r_0$ for some i, $Q^* \neq 1$, and, by Proposition 1, $Q_\epsilon \succeq_c Q^*$ for some $\epsilon > 0$, where $Q_\epsilon \in \mathcal{L}^2(\Omega)$ is defined as in the proof of Proposition 13, namely, $Q_\epsilon(\omega) = 1 + \epsilon$ if $X(\omega) < EX$, $Q_\epsilon(\omega) = 1 - \epsilon\, p_2/p_1$ if $X(\omega) > EX$, and $Q_\epsilon(\omega) = 1$ if $X(\omega) = EX$, where $X \in \mathcal{L}^2(\Omega)$ is nonconstant, $p_1 = \mathbb{P}[X > EX] > 0$ and $p_2 = \mathbb{P}[X < EX] > 0$. Since $E[XQ_\epsilon] < EX$ for all $\epsilon > 0$, Q3 follows. Thus, the second representation in (6) with $\mathcal{Q}$ given by (43) determines a deviation measure $\mathcal{D}^*$, which is Schur convex by Proposition 14.

Next, for every $Q \in \mathcal{Q}$, the condition $Q^* \succeq_c Q$ is equivalent to $1 - Q^* \succeq_c 1 - Q$ by (43), which implies

$$\begin{aligned} E[(1-Q)X] &\le \int_0^1 q_X(\alpha)\, q_{1-Q}(\alpha)\, d\alpha \\ &\le \int_0^1 q_X(\alpha)\, q_{1-Q^*}(\alpha)\, d\alpha \le \mathcal{D}^*(X) \\ &\quad \text{for all } X \in \mathcal{L}^2(\Omega), \end{aligned}$$

where the first, second, and third inequalities are due to Hardy-Littlewood [2, Theorem A.24], Dana [1, Lemma 2.1], and Grechuk and Zabarankin [6, Proposition 4.3], respectively, and (44) follows.

Since $1 - Q^*$ is comonotone with X^*, $Y = E[1 - Q^*|X]$ is an increasing function of X, and consequently,

$$\begin{aligned} E[(1-Q^*)X^*] &= E[YX^*] \\ &= \int_0^1 q_{X^*}(\alpha)\, q_{1-Q^*}(\alpha)\, d\alpha = \mathcal{D}^*(X^*), \end{aligned}$$

where the second equality follows from [2, Theorem A.24]. Thus, $Q^* \in \mathcal{Q}_{\mathcal{D}^*}(X^*)$, and, by Proposition 12, X^* is optimal in (1) for $\mathcal{D}^*$. In particular, with Proposition 3, this implies that X^* is $\succ_s$-undominated. □

6.3 Constructing a Deviation Measure via Risk Envelope

If $\mathcal{V}$ is arbitrage-free and $EX^* > r_0$, a solution to Problem II is given by Proposition 13. Suppose there are T scenarios for the rates of return of the risky assets with historical probabilities $p_1, \ldots, p_T$. In this case, $Q^* = (q_1^*, \ldots, q_T^*)$ is risk-neutral if $Q^* \ge 0$, $EQ^* = 1$ and $E[Q^* r_i] = r_0$, $i = 1, \ldots, n$, i.e. if

$$\sum_{j=1}^T q_j^* p_j = 1, \quad q_j^* \ge 0, \quad j = 1, \ldots, T,$$
$$\sum_{j=1}^T q_j^* p_j r_{ij} = r_0, \quad i = 1, \ldots, n,$$

so $q_1^*, \ldots, q_T^*$ can be found from this system of linear equations and inequalities. If $\mathcal{V}$ is arbitrage-free, the solution is guaranteed to exist by the fundamental theorem of asset pricing. In fact, as in Proposition 9, one can show that there exists a risk-neutral r.v. Q^* with at most $n+1$ non-zero entries, which is useful if $T \gg n$. The risk envelope of $\mathcal{D}^*$ can then be found from (42), i.e. $Q = (q_1, \ldots, q_T) \in \mathcal{Q}$ if and only if $q_j \ge 0$, $j = 1, \ldots, T$, $\sum_{j=1}^T q_j p_j = 1$ and $\sum_{j=1}^T q_j p_j x_j^* \ge \sum_{j=1}^T q_j^* p_j x_j^*$, where $X^* = \left(x_1^*, \ldots, x_T^*\right)$.

Proposition 15 implies that Problem I has a solution if and only if there exists a risk-neutral r.v. $Q^* \in \mathcal{L}^2(\Omega)$, such that $-Q^*$ is comonotone with X^*. Suppose that $p_1 = \cdots = p_T = 1/T$ and that there exists a permutation $(j_1, \ldots, j_T)$ of the set $(1, \ldots, T)$ such that $x_{j_1}^* < x_{j_2}^* < \cdots < x_{j_T}^*$. Then the condition "$-Q^*$ is comonotone with X^*" can be written as $q_{j_1}^* \ge q_{j_2}^* \ge \cdots \ge q_{j_T}^*$, in which case, q_j^* can also be found from the above system of linear equations and inequalities, and, by Proposition 15, the solution exists if and only if X^* is $\succ_s$-undominated. Once $q_1^*, \ldots, q_T^*$ are found, then for any $X = (x_1, \ldots, x_T)$, $\mathcal{D}^*(X)$ is determined by (44): $\mathcal{D}^*(X) = \sum_{i=1}^T \left(1 - q_{j_i}^* p_{j_i}\right) x_{k_i}$, where $(k_1, \ldots, k_T)$ is a permutation of the set $(1, \ldots, T)$ such that $x_{k_1} \le x_{k_2} \le \cdots \le x_{k_T}$.

7 Conclusions

The problem of identifying investor's deviation measure in the mean-deviation portfolio problem (1) based on observing the rate of return of investor's portfolio has been solved. This inverse problem assumes that investor's portfolio is an optimal solution to (1) for some deviation measure. A necessary and sufficient condition for the inverse problem to have a law-invariant solution has been established. Proposition 4 shows that if a given portfolio is $\succ_s$-undominated, then a solution always exists and has the form of mixed CVaR-deviation (4); otherwise, the inverse problem has no law-invariant solution. If the solution is not unique, the investor may choose the worst-case solution given by (24). Also, Proposition 5 proves that in general, if the inverse problem is required to have a solution, mixed CVaR-deviations (4) represent the smallest feasible set of deviation measures. However, if the portfolio problem (1) has n assets with T scenarios for assets' rates of return, a solution to the inverse problem can always be chosen in the form of a mixed CVaR-deviation with no more than $\min\{T-1, n+1\}$ CVaR-deviations. When the number of CVaR-deviations is constrained, an approximate mixed CVaR-deviation is provided as well. Finally, if a given portfolio is not $\succ_s$-undominated, and law-invariant solution does not exists, a non law-invariant deviation measure can be represented in the dual form (6) with the risk-envelope Q determined by (42).

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