# Risk Averse Decision Making under Catastrophic Risk

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**Abstract**

A nonstandard probabilistic setting for modeling of the risk of catastrophic events is presented. It allows random variables to take on infinitely large negative values with non-zero probability, which corresponds to catastrophic consequences unmeasurable in monetary terms, e.g. loss of human lives. Thanks to this extension, the safety-first principle is proved to be consistent with traditional axioms of rational behavior, such as monotonicity, continuity, and risk aversion. Also, a *robust* preference relation is introduced, and an example of a monotone robust preference relation, sensitive to catastrophic events in the sense of Chichilnisky [14], is provided. The suggested setting is demonstrated in evaluating nuclear power plant projects when the probability of a catastrophe is itself a random variable.

**Key Words:** decision making, catastrophic risk, safety-first principle, risk aversion, nonstandard probabilistic setting, nuclear power plant

## 1 Introduction

The theory of choice under uncertainty aims to provide a coherent framework of principles of *rational behavior* for analyzing and guiding[1] decision maker's attitudes toward potential losses/rewards. It is traditionally studied on a space of lotteries or random variables. While assessment of outcomes and corresponding probabilities of a random variable (r.v.) ultimately depends on decision maker's preferences, does the resulting order (ranking) of random variables adhere to principles of rational choice? For example, if an r.v. $X_1$ is preferred to an r.v. $X_2$ and the latter is preferred to an r.v. $X_3$, is $X_1$ preferred to $X_3$ (transitivity of a preference order)? The work of von Neumann and Morgenstern [51] is arguably the first fundamental study in the theory of choice that postulates axioms on a preference order: *completeness*, *transitivity*, *continuity*, and *independence* and shows that these four axioms admit a numerical representation in the form of expected utility function,[2] so that instead of verifying all four axioms, a decision maker merely needs to choose a utility function $u$ and to rank given random variables according to the expected value of $u$. Almost every decision theory views *risk aversion* as a cornerstone principle of rational behavior that states that given a choice between a random outcome $X$ and a sure payoff equal to the expected value of $X$, a risk averse agent always prefers the latter. In the framework of the expected utility theory (EUT), risk aversion implies that the utility function $u$ is concave and can be conveniently characterized by various measures through the derivatives of $u$, e.g. by the Arrow-Pratt measure of absolute risk-aversion $-u''/u'$. Using the ideas of Finetti [24] and von Neumann and Morgenstern [51], Savage [57] introduces somewhat similar four axioms[3] on a preference order: (a) transitivity and completeness (weak order), (b) the "sure-thing" principle, which parallels the independence axiom, (c) likelihood payoff independence (if an agent prefers getting the prize under event $A$ rather than (the same) prize under event $B$, then this choice does not depend on the size of the prize), and (d) Archimedean axiom (an agent preferences are robust with respect to low-probability events). Essentially, Savage's axioms replace objective probabilities by subjective ones, but still have a numerical representation in the form of expected utility. Hence, the resulting

[1] Individuals often correct their attitudes once inconsistencies of their choices are revealed to them; see [57].

[2] In fact, Daniel Bernoulli [9] used the form of expected utility as early as in 1738 to explain famous St. Petersburg's paradox.

[3] In fact, Savage [57] introduces seven axioms, but the other three are rather technical and have little relevance for the present discussion.

*subjective-probability expected utility theory* (SPEUT) retains most of the properties of the EUT and is often viewed as a version of the former. The intuitive appeal and mathematical simplicity of expected utility largely rests on the *independence* axiom ("sure-thing" principle), which implies linearity in probability. However, namely this axiom is widely acknowledged to contradict certain empirical/experimental evidence, commonly known as the Allais paradox[4] ("fanning out," common ratio and common consequence effects) [4, 47]. Similar to the Allais paradox, there is the Ellsberg paradox (*ambiguity aversion*) [20] that contradicts the "sure-thing" principle. This has motivated either changing or completely omitting the independence axiom and, as a result, has given rise to a variety of so-called non-expected utility theories, including weighted expected utility theory (WEUT) [25], rank-dependent (anticipated) utility theory (AUT) [59], prospect theory [41, 27], cumulative prospect theory (CPT) [42], regret theory [46], disappointment theory [8], dual utility theory [64, 60], etc.; see [47] for a detailed discussion of these and other theories. However, while resolving the Allais paradox, none of these theories are free from their own paradoxes. For example, in Yaari's theory [64], a dual independence axiom, replacing the independence axiom, implies that a dual utility function linearly depends on outcomes and, thus, leads to paradoxes "dual" to the Allais paradox and the common ratio effect, in which the role of outcomes and probabilities is reversed. Among these theories, the CPT [42] is arguably most sophisticated and the one that mimics individual's behavior most closely. It generalizes the prospect theory [41] and Quiggin and Yaari's theories [59, 64] and prescribes a decision maker to use an $S$-shaped evaluation function for outcomes (convex for losses and concave for gains) and to transform the linear cumulative probability distribution as a function of probability into an inverse $S$-shape (unlikely but extreme outcomes become overweight, whereas outcomes in the middle of the distribution become underweight). Chateauneuf and Wakker [12] show that for decision making under risk, the CPT is consistent with four axioms on a preference order: *weak ordering*, *continuity*, *stochastic dominance*, and *tradeoff consistency*. Remarkably, all the mentioned theories, whose title bears "utility," start from an axiomatic framework for a preference order and arrive at a corresponding utility function, whereas the prospect theory and the CPT originate from modeling of actual individual's behavior ("empirical realism") in terms of value function and probability and only then are "translated" into an axiomatic framework [12]. Nonetheless, the CPT is not immune to criticism: it assumes the existence of constant (non-random) reference point interpreted typically as current endowment and, thus, fails to work when agent's endowment is uncertain, i.e., when the agent owns a stock and is thinking about selling/exchanging it. Also, the CPT postulates that for lotteries to be compared, agents should ignore lotteries' common outcomes ("isolation effect"), which resembles the independence axiom of the EUT and, thus, incurs similar criticism [54].

While the discussed theories of choice display a steady progress toward understanding and modeling of individual's attitudes toward risk, there is a growing evidence questioning their applicability to decision making under *catastrophic risk* [55], which is characterized as rare events with extreme consequences, i.e. terrorist attacks, industrial accidents and natural/environmental disasters (floods, fires, earthquakes, oil spills, etc.). In fact, in all these theories, the axioms of rational behavior are designed from the perspective of a single investor [29] (or a group of investors [32, 34] through *collective risk aversion*), whose goal is to attain gains beyond a risk-free return and who, if desired, may limit or completely eliminate exposure to risky assets. Those axioms may not be adequate for evaluating structural safety of construction projects that have limited ability to coup with catastrophic events (so-called "black swans"); see [14]. For example, Buchholz and Schymura [10] report that for low degrees of risk aversion, the EUT almost neglects catastrophic events, whereas for moderate levels of risk aversion, it leads to a "tyranny of catastrophic risk;" see also [31, 2, 63]. Chichilnisky [14, 15] observes that for catastrophic events, neither the *continuity* axiom nor the traditional definition of *risk aversion* is applicable and introduces the "swan" axioms requiring subjective probabilities to be sensitive to both frequent and rare events. Moreover, the axioms of *continuity* and *risk aversion* are inconsistent with the *safety-first* principle [13], whose objective is to minimize the probability of a catastrophic event. The existing literature on structural engineering and safety offers several methods for estimating the severity and

[4] Savage [57] questions the validity of the Allais paradox: individuals usually correct themselves once they are shown that their choices fail to satisfy the independence axiom. This is where the theory of choice fulfills its educational role of being a guidance of rational behavior.

probability of catastrophic events [7], including the *extreme value theory*[5] [16, 53, 23] and *probabilistic safety assessment* (PSA) (also known as *probabilistic risk assessment* (PRA)) [18], whereas actuarial mathematics has long employed the *Cramer-Lundberg model*[6] for estimating likelihood of rare events; see, e.g. [22, 52]. However, these methods are not intended for making the analysis of the estimated losses and probabilities to conform to the principles of rational choice. Moreover, Volkanovskia and Cepin [62] noted that the PSA faces several uncertainties: in model, in parameters, and so-called completeness uncertainty. Using the PSA with Monte-Carlo simulation, they showed that because of those uncertainties, the *core damage frequency* (CDFr), characterizing the likelihood of damaging the core of a nuclear reactor and being a crucial safety characteristic for licensing nuclear power plants [50, 39, 40], has a substantial dispersion and, in fact, should be treated as a random variable with normal distribution. In economics, the idea that the probabilities of outcomes in question may themselves be unknown is often referred to as *Knightian uncertainty* [45] or *ambiguity*, whose significance remains a highly debated issue to this day [5, 20, 56, 6].[7] For example, Klibanoff et al. [44] and Nau [49] propose axiomatic models of choice under uncertainty that generalize the EUT by allowing to distinguish and to incorporate attitudes toward both "risk" and "ambiguity" ("uncertainty"). In those models, a decision maker, still being an expected utility maximizer, may exhibit different degrees of risk aversion toward "risk" and "ambiguity." In economics and actuarial science, ambiguity (uncertainty in probability) is closely related to the notion of *self-protection* [19].[8] In application to insurance problems, Alary et al. [3] use the model of Klibanoff et al. [44] to show that ambiguity aversion, which could be associated with "more pessimism" under the SPEUT, diminishes incentive to self-protect. However, it remains to be seen how ambiguity and self-protection are related in the context of catastrophic risk.

At this point, it is instructive to analyze the existing decision making approaches to assessing nuclear power plant projects under the risk of core damage. In nuclear safety, a great deal of analysis effort is devoted to two tasks: (i) identifying *postulated initiating events (PIEs)*[9] and corresponding *damage states* and (ii) estimating *likelihood* and *consequences* of the identified damage states through the PSA; see [39, 50, 62, 43]. Typically, the PSA provides two characteristics, specific to each nuclear plant: the aforementioned CDFr and *large early release frequency (LERFr)*, which is the probability of an accident resulting in a considerable early release of radioactivity into the environment;[10] see [50, 40]. For existing nuclear power plants, the CDFr is required to be less than $10^{-4}$ = 1E-04 reactor-years (ry) [50, 62], i.e. the probability that at least one catastrophe will occur within next 100 years should be less than 1%, whereas the LERFr should not exceed one tenth of the corresponding CDFr (see [50, 40]). Once the CDFr and corresponding consequences of the core damage states are evaluated/modeled and objectives (cost, safety, etc.) and constraints (acceptance criteria[11]) for decision

[5] The extreme value theory is a branch of statistics which deals with assessing probabilities of extremely rare events; see [16, 53, 23]. To estimate the probability of catastrophic events with this theory, one needs to make assumptions about the tail distribution, which are usually hard to verify, see [21].

[6] In the ruin theory, the Cramer-Lundberg model, also known as the compound-Poisson risk model, is a closed-form formula for the probability of the ultimate ruin of an insurance company provided that customers' claims arrive according to a Poisson process. There is a generalization of this model for the case when inter-arrival time of claims has an arbitrary distribution.

[7] Knight [45, p. 233] states that the difference between risk and uncertainty is that in the case of risk, the distribution of random outcomes is known, whereas in the case of uncertainty, it is not. This distinction, however, is a subject of extensive debate, see e.g. [5, 20, 56, 6] and references therein.

[8] Self-protection is an investment/action that an agent makes to *reduce the probability* of an undesired event, e.g. getting vaccination, installing additional protective devices and security systems, reinforcing and fortifying structures, etc. However, in contrast to insurance, self-protection is often ignored for one of the reasons that in the case of such an event, it will be an additional loss. An infamous example of self-protection ignorance is the failure to reinforce and enlarge the levee system of New Orleans, LA, devastated by hurricane Katrina in August 29, 2005 mainly by flooding.

[9] PIEs are divided into two groups: internal events and external events. The first group includes component failure and human error, e.g. loss of electrical power supply, insertion of excess reactivity, loss of flow, loss of coolant, erroneous handling or failure of equipment, etc., whereas the second group includes natural hazards, e.g. fires, earthquakes, tsunami, flooding, etc., and man-made events, e.g. terrorist acts and airplane crash. PIEs are identified by constructing fault trees; see [39].

[10] Late release of radioactivity has considerably lesser impact, since before it occurs, the public in the vicinity of a nuclear power plant could already be evacuated.

[11] Acceptance criteria can be basic and specific. Basic acceptance criteria are limits set by a regulatory agency on maximum allowed doses for the public, on fuel failures, etc., whereas specific acceptance criteria are set by a designer on maximum cladding temperature,

variables are established, decision theories come to play. A decision approach that selects projects based solely on the CDFr criterion (so-called *safety-first principle*) is well justified but has obvious limitations. For example, any project with lower CDFr is considered to be better regardless of its cost, whereas the profit that a project can generate in case of no major core-damaging accident and moderate losses due to less severe but more frequent events are largely ignored. An alternative approach that accounts for these additional factors instead of meeting a specified CDFr-level target is to minimize the expected lifecycle cost, $E[LC]$; see, e.g., [33]. The *minimum* $E[LC]$ *criterion* is, however, not free from shortcomings. It assumes that decision making is *risk neutral*, i.e. indifferent between, say, \$100 sure loss, and a small chance (0.0001%) of a huge \$$10^8$ loss, whereas a safe choice under uncertainty is a synonym to risk aversion. In application to structural safety, Rosenblueth [35] proposes to replace the minimum $E[LC]$ criterion by the EUT, whereas Goda & Hong [28] and Cha & Ellingwood [11] suggest to use the CPT instead. However, as discussed, the EUT is not appropriate for decision making under catastrophic risk [14, 15, 10], and the fact that the CPT preserves continuity of preference order [12] and that the continuity axiom is incompatible with catastrophic events [14, 15] implies that the CPT will be inadequate similarly to the EUT in this respect. Thus, on the theoretical side, only the axiomatization of Chichilnisky seems to be relevant for analyzing catastrophic risks since it is designed specifically for this purpose, whereas on the practical side, the extreme value theory, PCA and Cramer-Lundberg model mainly deal with estimating and modeling of potential catastrophic losses rather than provide a decision framework.

However, Chichilnisky's axiomatization [14, 15], along with other existing decision theories, assumes possible outcomes to be *finite-valued* random variables. In particular, the loss under a catastrophic event is typically assigned some large negative but finite constant value $C$. While in the finance industry, all losses are monetary and, thus, can be estimated reasonably well, in non-finance industries, such as health care, construction sector, nuclear safety, etc., catastrophic events often result in the loss of human lives, which can hardly be assigned any monetary value. On the other hand, even in the finance industry, variations or estimation errors of the constant $C$ can dramatically alter final decisions. In the EUT, the sensitivity to large negative outcomes, e.g. to $C$, can be limited by choosing a utility function *bounded from below*.[12] However, Chichilnisky [14] points out that namely such utility functions are insensitive to rare events and, thus, should be avoided in dealing with catastrophic risk. But in the EUT, CPT and Chichilnisky's axiomatization, decision making with any utility function *unbounded from below* will inevitably be affected by the choice of $C$.

This work identifies theoretical criteria for analyzing relevance of principles/axioms of rational behavior such as monotonicity, robustness, risk aversion and sensitivity to rare events for finance and non-finance applications under the risk of catastrophic events and for examining the existing theories of choice in this context. It introduces a novel *probabilistic setting*, which allows random variables to take on infinitely large values ($\pm\infty$) with non-zero probability and shows that this setting (called *nonstandard*) is naturally suited for modeling of catastrophic events: a catastrophic loss can simply be assigned $-\infty$. Remarkably, this simple idea resolves several inconsistencies of the existing decision theories in application to catastrophic events:

1. A large negative (often arbitrarily chosen) constant $C$ is no longer required and, thus, cannot alter final decisions.
2. In the nonstandard probabilistic setting, the safety-first principle is consistent with traditional axioms of rational behavior, such as continuity and risk aversion. This is in sharp contrast with the standard setting, in which the safety-first principle is known to be both discontinuous and not risk averse.
3. Decision making is *robust* with respect to small variations of the catastrophe probability, which cannot be estimated exactly.

In the nonstandard probabilistic setting, a decision making approach satisfying all of the identified criteria is suggested and is compared to the existing theories in evaluating nuclear power plant projects in terms of cost and safety.

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maximum heat flux, flow conditions, etc.; see [40].

[12] Indeed, if $\lim_{x\to-\infty} u(x) = u_0$, then a decision maker assigns the catastrophic loss the "utility weight" $u(C) \approx u_0$, and the choice of (large negative) constant $C$ has no "utility" impact.

This work is organized into seven sections. Section 2 introduces basic definitions and notations and reviews modeling of catastrophic events and safety-first principle in the standard probabilistic setting. Section 3 introduces a nonstandard probabilistic setting and shows that, in this setting, the safety-first principle is consistent with traditional axioms of rational behavior, namely with continuity and risk aversion. Section 4 shows the importance of a preference relation to be *robust* and offers a new approach for decision making, which identifies a preference relation being simultaneously monotone, robust, and sensitive to catastrophic events. Sections 5 and 6 demonstrate the suggested approach in evaluating nuclear power plant projects. Section 7 concludes the work.

# 2 Standard Probabilistic Setting

## 2.1 Risk Averse Decision Making

In the existing theories of choice, uncertain outcomes are usually modeled by *finite-valued* random variables (r.v.'s), i.e. measurable functions from a probability space $\Omega = (\Omega, \mathcal{M}, P)$ to the real line $\mathbb{R}$ ($\mathcal{M}$ is a sigma algebra of sets in $\Omega$, and $P$ is a probability measure on $(\Omega, \mathcal{M})$), which will be referred to as a *standard probabilistic setting*. Agent's preference relation is then represented by a weak relation $\succeq$ on an arbitrary set $\mathcal{A}$ of r.v.'s, where $X \succeq Y$ means that either an r.v. $X$ is preferred over an r.v. $Y$ ($X \succ Y$) or $X$ and $Y$ are equally preferable ($X \sim Y$). Usually, the attention is restricted to r.v.'s from $\mathcal{L}^1(\Omega) = \mathcal{L}^1(\Omega, \mathcal{M}, P)$, i.e. such that $E|X| < \infty$, where $E[\cdot]$ denotes the expected value. Let $F_X(x) = \mathbb{P}[X \le x]$ be the cumulative distribution function (CDF) of $X$. If $F_X(x) \equiv F_Y(x)$ for two r.v.'s $X$ and $Y$, we say that $X$ and $Y$ have the same distribution and write $X \overset{d}{\sim} Y$. Let $\mathcal{F}$ be the set of all possible CDFs, i.e. the set of non-decreasing right-continuous functions $F$ with $\lim\limits_{x\to-\infty} F(x) = 0$ and $\lim\limits_{x\to+\infty} F(x) = 1$. The probability space $\Omega$ is assumed to be *atomless*, i.e. there exists an r.v. with a continuous CDF. For any r.v.'s $X$ and $Y$ and every $\lambda \in [0,1]$, an r.v. $Z$ with the CDF $F_Z(z) = \lambda F_X(z) + (1-\lambda) F_Y(z)$ is called $\lambda$-*lottery* of $X$ and $Y$ and is denoted by $Z = \lambda X \oplus (1-\lambda) Y$. Also, any constant $C$ corresponds to a constant r.v. $X_C$ such that $\mathbb{P}[X_C = C] = 1$.

A preference relation $\succeq$ is called *rational* if it satisfies a certain set of principles (axioms) of rational behavior that traditionally includes the axioms of *completeness*, *monotonicity*, and *continuity*. Often, $\succeq$ is also assumed to be *risk averse*, i.e. a sure outcome $C$ is preferred to any lottery with the expected payoff of $C$.

(i) *Completeness*: $\succeq$ defines a total order on $\mathcal{A}$, namely, $\succeq$ is antisymmetric ($X \succeq Y$ and $Y \succeq X$ imply that $X \sim Y$), transitive ($X \succeq Y$ and $Y \succeq Z$ imply that $X \succeq Z$) and total ($X \succeq Y$ or $Y \succeq X$ for every $X$ and $Y$).

(ii) *Monotonicity*: $X \succeq Y$ when $\mathbb{P}[X \ge Y] = 1$. If, in addition, $X \succ Y$ when $\mathbb{P}[X > Y] > 0$, then $\succeq$ is called *strictly monotone*.

(iii) *Continuity*: sets $\{Y \in \mathcal{A} | Y \succeq X\}$ and $\{Y \in \mathcal{A} | X \succeq Y\}$ are closed,[13] where "closedness" is defined in a topology specified for each application. If $\mathcal{A} = \mathcal{L}^1(\Omega)$, then continuity with respect to $\mathcal{L}^1$ norm is a natural choice. The continuity axiom guarantees that infinitely small variations in an r.v. cannot drastically change preference relations.

(iv) *Risk aversion*: In the existing literature, risk aversion has several equivalent definitions. Here, we follow the one due to Rothschild and Stiglitz [36] stating that adding independent noise always increases risk.

**Definition 1 (risk averse preference relation)** *Let $X$ and $Z$ be r.v.'s such that $E[Z|X = x] = 0$ for all $x$. If $Y \overset{d}{\sim} X + Z$, we say that $Y$ can be obtained from $X$ by* mean-preserving spread. *Then $\succeq$ is* risk averse *if $X \succeq Y$ when $Y$ is obtainable from $X$ by mean-preserving spread.*

By Definition 1, $X \overset{d}{\sim} Y$ implies $X \sim Y$, i.e. $\succeq$ depends only on the CDFs' of $X$ and $Y$. In fact, such $\succeq$ is called *law invariant*. Also, it follows from Definition 1 that $X_{EY} \succeq Y$ for all $Y$ (risk aversion).

A monotone and law-invariant $\succeq$ is also called consistent with the *first-order stochastic dominance* (FSD), while monotone and risk averse $\succeq$ is called consistent with the *second-order stochastic dominance* (SSD).

[13] Equivalently, $X \succ Y$ implies $X_n \succ Y_n$ for large enough $n$ when $X_n \to X$ and $Y_n \to Y$ as $n \to \infty$.

## 2.2 Modeling of Catastrophic Risk

A catastrophic event is an event with a (usually) small probability of realization but with an extremely large loss. In the standard probabilistic setting, such an event is modeled as $\{X \le C\}$ with $X$ representing an uncertain revenue and $C$ being a large negative constant. Examples of catastrophic events can be found in virtually all spheres of human activity ranging from finance to health care sector.

**Example 1 (finance)** *Let $X$ be the uncertain revenue from a financial investment $D$. A catastrophic event is usually associated with the loss of all the money, i.e. with the event $\{X \le -D\}$.*

**Example 2 (civil infrastructure)** *Let $X$ be the uncertain revenue from a civil infrastructure project exposed to earthquakes. Then a catastrophic event is modeled as $\{X \le -C\}$, where $C$ is an estimate of possible damages due to an earthquake and where not only financial, but also human losses should be taken into account [11].*

**Example 3 (health care)** *A breast cancer screening has four possible outcomes: true positive (TP), false positive (FP), true negative (TN) and false negative (FN) [1]. Let patient's wellbeing after screening be modeled by an r.v. $X$ with outcomes $X_{TP}$, $X_{FP}$, $X_{TN}$ and $X_{FN}$ and corresponding probabilities. Because the false negative outcome will result in untreated cancer, it will likely lead to death. In this case, a catastrophic event (death) is $\{X \le X_{FN}\}$, and $X_{FN}$ represents a "death" threshold.*

Examples 2–3 show that purely financial losses are often not the main concern and that an agent would try to avoid a catastrophic event *at any cost*. This behavior is regarded as a *safety-first principle*.

**Definition 2 (safety-first principle)** *Let $X$ and $Y$ be r.v.'s, and let $\alpha(X)$ and $\alpha(Y)$ be the probabilities of catastrophic events associated with $X$ and $Y$, respectively. A preference relation $\succeq$ is* consistent with the safety-first principle*, if $X \succ Y$ when $\alpha(X) < \alpha(Y)$.*

In the literature, the safety-first principle is usually formulated in the non-strict form: $X \succeq Y$ when $\alpha(X) \le \alpha(Y)$. Definition 2 precludes some degenerate cases, e.g. indifference among *every* outcome.

In the standard probabilistic setting, a catastrophic event for an r.v. $X$ is modeled as $\{X \le C\}$ for some $C \in \mathbb{R}$. Then $\alpha(X) = \mathbb{P}[X \le C]$, and the safety-first principle is formulated as $X \succ Y$ when $\mathbb{P}[X \le C] < \mathbb{P}[Y \le C]$. It is well-known, however, that this definition of the safety-first principle is inconsistent with some basic axioms of rational behavior, namely with continuity and risk aversion.

**Proposition 1 (safety-first principle in the standard probabilistic setting)** *Let $\succeq$ be consistent with the safety-first principle in the standard probabilistic setting, i.e. $X \succ Y$ when $\mathbb{P}[X \le C] < \mathbb{P}[Y \le C]$ for some $C \in \mathbb{R}$. Then*

*(i) $\succeq$ is not continuous (in any topology in which $X_{C+1/n} \to X_C$ as $n \to \infty$).*

*(ii) $\succeq$ is not risk averse.*

**Proof** (i) The set $\{Y \in \mathcal{A} | Y \succeq X^*\}$ with $X^* = 1/2X_C \oplus 1/2X_{C+1}$ is not closed. Indeed, a sequence of constants $\{X_{C+1/n}\}_{n=1}^{\infty}$ converges to $X_C$, and $X_{C+1/n} \succ X^*$ for any $n \in \mathcal{N}$, but $X^* \succ X_C$. (ii) $Y^* \succ X_{EY^*}$ for $Y^* = 1/2X_{C-1} \oplus 1/2X_{C+1}$. □

Observe that the proofs of (i) and (ii) rely on the conditions that $\{X_{C+1/n}\}_{n=1}^{\infty} \to X_C$ and $\mathbb{P}[X < C] > 0$, respectively. In Example 1, these conditions have obvious financial interpretation since $D$ is a natural threshold, whereas in Examples 2–3, they do not make much sense since $C$ and $X_{FN}$ are artificial constants. We believe that the inconsistency of the safety-first principle with the axioms of rational behavior is attributed to inadequacy of the standard probabilistic setting for modeling of catastrophic events.

Another disadvantage of the safety-first principle is that it often leads to a non-cardinal preference relation. A preference relation $\succeq$ on the set $\mathcal{A}$ is *cardinal*, if there exists a utility functional $U : \mathcal{A} \to \mathbb{R}$ such that $X \succeq Y \Leftrightarrow U(X) \ge U(Y)$.

**Example 4** *A preference relation $\succeq$ on $\mathcal{L}^1(\Omega)$ such that (i) $X \succ Y$ when $\alpha(X) < \alpha(Y)$ and (ii) $X \succeq Y \Leftrightarrow EX \geq EY$ if $\alpha(X) = \alpha(Y)$ is not cardinal.*

**Detail.** Let $\mathcal{A} \subset \mathcal{L}^1(\Omega)$ be the set of r.v.'s $X$ with $EX \in [0,1]$. Each $X \in \mathcal{A}$ is considered as a point $(\alpha, m) \in [0,1] \times [0,1]$ such that $\alpha = \alpha(X)$ and $m = EX$. Then $\succeq$ on $\mathcal{A}$ introduces a lexicographical order on $[0,1] \times [0,1]$, which is not cardinal; see Example 2.7 in [37]. □

# 3 Non-standard Probabilistic Setting

## 3.1 Risk Aversion and Safety-First Principle

We suggest to model random outcomes with possibly catastrophic realizations as r.v.'s which can assume infinitely large values $\pm\infty$. Specifically, let an r.v. be any measurable functions from the probability space $\Omega$ to the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Here, we accept the convention that $0 \cdot (-\infty) = 0 \cdot (+\infty) = 0$ and $\infty - \infty = 0$, which allows one to define addition, subtraction, and multiplication of r.v.'s.[14] The relations between r.v.'s are understood to hold in the almost sure sense, namely, we write $X = Y$ if $\mathbb{P}[X = Y] = 1$ and $X \geq Y$ if $\mathbb{P}[X \geq Y] = 1$, even if the r.v. $X - Y$ assumes $-\infty$ with zero probability. The CDF $F_X(x) = \mathbb{P}[X \leq x]$, $x \in \mathbb{R}$, is a non-decreasing right-continuous function with $\lim\limits_{x \to -\infty} F_X(x) = \alpha \geq 0$ and $\lim\limits_{x \to +\infty} F_X(x) = \beta \leq 1$, where $\alpha = \mathbb{P}[X = -\infty]$ and $\beta = 1 - \mathbb{P}[X = +\infty]$. If $\alpha > 0$ and $\beta < 1$, the expectation $EX$ is considered to be undefined. In this case, an r.v. $X$ is called *proper* if $\beta > \alpha$. For every proper r.v., $F^*_X(x) \equiv (F_X(x) - \alpha)/(\beta - \alpha)$ and $E^*X = \int_{-\infty}^{+\infty} x\, F^*_X(x) dx$ denote the conditional CDF and expectation of $X$, respectively, provided that $X$ is finite. For any r.v. $X$, $q_X(t) = \inf\{x | F_X(x) > t\}$ defines the quantile function $q_X(t) : [0,1] \to \overline{\mathbb{R}}$, so that $EX = \int_0^1 q_X(t)\, dt$ and $E^*X = \int_\alpha^\beta q_X(t)\, dt$ when $EX$ or $E^*X$ are defined.

To model uncertain outcomes with possibly catastrophic realizations, we restrict our attention to r.v.'s from $\mathcal{L}^1_-(\Omega) = \{X \in \mathcal{L}^1_*(\Omega) | \mathbb{P}[X = +\infty] = 0\}$, where $\mathcal{L}^1_*(\Omega) = \{X | E^*|X| < \infty\}$. A catastrophic realization of $X$ is then the event $\{X = -\infty\}$, and its probability is given by $\alpha(X) = \mathbb{P}[X = -\infty]$. In contrast to the standard probabilistic setting, there is no need for an artificial negatively large constant corresponding to a catastrophic loss. Observe that the constant $X_{-\infty} \notin \mathcal{L}^1_-(\Omega)$, and consequently, $\alpha(X) \in [0,1)$ for every $X \in \mathcal{L}^1_-(\Omega)$. For any $X \in \mathcal{L}^1_-(\Omega)$, the expected value of $X$ is defined as follows: $EX = E^*X$ if $\alpha(X) = 0$ and $EX = -\infty$ otherwise.

For a preference relation $\succeq$ defined on $\mathcal{L}^1_-(\Omega)$, the definitions of completeness, monotonicity, continuity in $\mathcal{L}^1$ norm, law invariance, risk aversion, and consistence with the safety-first principle remain unchanged. Continuity requires a sequence $X_n \in \mathcal{L}^1_-(\Omega)$ to converge to a limit $X \in \mathcal{L}^1_-(\Omega)$ if $E|X - X_n|$ is defined for all $n > N$ (for some $N$) and to vanish otherwise. In Definition 1, $E[Z|X = x] = 0$ implies that $Z$ is finite with probability 1. In Definition 2, $\alpha(X)$ is now $\mathbb{P}[X = -\infty]$.

Let $\succeq^*$ be an arbitrary law invariant preference relation on $\mathcal{L}^1(\Omega)$. In this case, we write $F \succeq^* G$ for any functions $F, G \in \mathcal{F}$, if $X \succeq^* Y$ for r.v.'s $X$ and $Y$ with the CDFs $F$ and $G$, respectively. A preference relation $\succeq$ on $\mathcal{L}^1_-(\Omega)$ is *induced* by $\succeq^*$, if it is consistent with the safety-first principle and if $X \succeq Y \Leftrightarrow F^*_X(x) \succeq^* F^*_Y(x)$ for every $X, Y \in \mathcal{L}^1_-(\Omega)$ such that $\alpha(X) = \alpha(Y)$. Obviously, every law invariant $\succeq^*$ on $\mathcal{L}^1(\Omega)$ induces *unique* $\succeq$ on $\mathcal{L}^1_-(\Omega)$.

**Proposition 2** *For every continuous and risk averse $\succeq^*$ on $\mathcal{L}^1(\Omega)$, the induced preference relation $\succeq$ on $\mathcal{L}^1_-(\Omega)$ is continuous, risk averse, and consistent with the safety-first principle.*

**Proof** Let $Y_n \to Y$ and $Y_n \succeq X$ for a sequence $Y_n \in \mathcal{L}^1_-(\Omega)$, $n \in \mathcal{N}$, and r.v.'s $X, Y \in \mathcal{L}^1_-(\Omega)$. Then $\alpha(Y_n) = \alpha(Y)$ for all $n \geq N$ and some $N \in \mathcal{N}$. Hence, $\alpha(X) \geq \alpha(Y_N) = \alpha(Y)$. If $\alpha(X) > \alpha(Y)$, then $Y \succeq X$ by the safety-first principle. If $\alpha(X) = \alpha(Y)$, then $Y \succeq X$ follows from the continuity of $\succeq^*$.

[14] The convention $0 \cdot (-\infty) = 0 \cdot (+\infty) = 0$ is standard in the probability and measure theory (see [58, p. xi]) and corresponds to the fact that zero-probability events can be neglected, whereas the convention $\infty - \infty = 0$ means that if both $X$ and $Y$ leads to a catastrophe ($X = Y = -\infty$), then a decision maker does not distinguish between them ($X - Y = 0$).

Next, suppose $Y$ is obtained from $X$ by mean-preserving spread. Then $\alpha(X) = \alpha(Y)$ by Definition 1, and consequently, $X \succeq Y$ by risk aversion of $\succeq^*$. □

Proposition 1 states that in the standard probabilistic setting, every $\succeq$ consistent with the safety-first principle fails to satisfy both continuity and risk aversion. On the contrary, Proposition 2 shows that in the nonstandard probabilistic setting, $\succeq$ consistent with the safety-first principle *can be both* continuous and risk averse. Moreover, the following result holds.

**Proposition 3** *Every strictly monotone and risk averse $\succeq$ on $\mathcal{L}^1_-(\Omega)$ is consistent with the safety-first principle.*

**Proof** Suppose $X, Z \in \mathcal{L}^1(\Omega)$ are two r.v.'s with $EX = EZ$. If there exists $x_0 \in \mathbb{R}$ such that satisfies two conditions: (i) $F_X(x) \le F_Z(x)$ for $x < x_0$ and (ii) $F_X(x) \ge F_Z(x)$ for $x \ge x_0$, then $X$ dominates $Z$ in concave order (see [38, Theorem 3]), or, equivalently, that $Z$ can be obtained from $X$ by mean-preserving spread. Obviously, this condition applies to r.v.'s $X, Z \in \mathcal{L}^1_-(\Omega)$ such that $\alpha(X) = \alpha(Z)$ and $E^*X = E^*Z$.

Let $X, Y \in \mathcal{L}^1_-(\Omega)$ be such that $\alpha(X) < \alpha(Y)$. Choose any $\alpha^* \in (\alpha(X), \alpha(Y))$, and let $z = q_X(\alpha^*)$. For any $t \le z$, let $Z(t)$ be an r.v. with the CDF $F_Z$ defined as $F_Z(x) = F_X(x)$ for $x < t$; $F_Z(x) = \alpha^*$ for $t \le x < z$; and $F_Z(x) = \min\{F_X(x), F_Y(x)\}$ for $z \le x$. Since $E^*[Z(z)] \ge E^*X$ and $E^*[Z(t)] \to -\infty$ as $t \to -\infty$, there exists $t^*$ such that $E^*[Z(t^*)] = E^*X$. Then conditions (i) and (ii) are satisfied for $x_0 = z$, and risk aversion of $\succeq$ implies that $X \succeq Z(t^*)$. Next, $\mathbb{P}[Z(t^*) \ge Y] = 1$ and $\mathbb{P}[Z(t^*) > Y] \ge \alpha(Y) - \alpha^* > 0$ by construction, and consequently, $Z(t^*) \succ Y$ by strict monotonicity. Thus, $X \succ Y$, i.e. $\succeq$ is consistent with the safety-first principle. □

## 3.2 Relative Utility Functional

This section addresses the issue of non-cardinality of $\succeq$ consistent with the safety-first principle. The main advantage of cardinal $\succeq$ is that a choice between any two alternatives reduces to comparing their utilities: $X \succeq Y$ if and only if $U(X) - U(Y) \ge 0$. This suggests the following generalization.

**Definition 3 (relative utility functional)** *Let $\mathcal{A}$ be an arbitrary set of r.v.'s. A functional $V : \mathcal{A} \times \mathcal{A} \to \overline{\mathbb{R}}$ is a* relative utility functional *if two conditions hold: (i) $V(X,Y) = -V(Y,X)$ for all $X, Y \in \mathcal{A}$, and (ii) $V(X,Y) + V(Y,Z) = V(X,Z)$ for all $X, Y, Z \in \mathcal{A}$ provided that $V(X,Y) + V(Y,Z)$ is defined (i.e. does not lead to $\infty - \infty$).*

Every relative utility functional $V$ induces a complete $\succeq$ on $\mathcal{A}$ by $X \succeq Y \Leftrightarrow V(X,Y) \ge 0$. A relative utility functional is *proper*, if for every $X \in \mathcal{A}$ and $\varepsilon > 0$, there exist r.v.'s $Y, Z \in \mathcal{A}$ such that $-\varepsilon < V(X,Y) < 0 < V(X,Z) < \varepsilon$. The preference relation induced by a proper $V$ is continuous if and only if $V$ is continuous.

**Proposition 4** *Let $\succeq$ be induced by a proper relative utility functional $V$. Then $\succeq$ is continuous in a topology $\tau$ if and only if $V(X, X_n) \to 0$ when $X_n \to X$ in $\tau$.*

**Proof** Suppose $V(X, X_n) \not\to 0$ for some $X_n \to X$, i.e. for infinitely many $n$, either $V(X, X_n) > \delta$ or $V(X, X_n) < -\delta$ for some $\delta > 0$. In the first case, $X \succ Y$ but $Y \succ X_n$ for $Y$ such that $0 < V(X,Y) < \delta$, whereas in the second, $Y \succ X$ but $X_n \succ Y$ for $Y$ such that $-\delta < V(X,Y) < 0$. Conversely, if $V(X, X_n) \to 0$ when $X_n \to X$, then $X \succ Y \Leftrightarrow V(X,Y) > 0$ implies that $V(X_n, Y_n) > 0 \Leftrightarrow X_n \succ Y_n$ for large enough $n$ when $X_n \to X$ and $Y_n \to Y$. □

Every cardinal $\succeq$ is represented by a relative utility functional $V(X,Y) = U(X) - U(Y)$. However, a non-cardinal preference relation can often be represented by a relative utility functional as well.

**Example 5** *Let $\succeq$ be a preference relation on $\mathcal{L}^1_-(\Omega)$ such that $X \succ Y$ if $\alpha(X) < \alpha(Y)$ and $X \succeq Y \Leftrightarrow E^*X \ge E^*Y$ if $\alpha(X) = \alpha(Y)$. Then $\succeq$ can be represented by the proper relative utility functional*

$$V(X,Y) = \int_0^1 (q_X(t) - q_Y(t))\, dt, \tag{1}$$

*with convention $q_X(t) - q_Y(t) = 0$ if $q_X(t) = q_Y(t) = -\infty$.*

Similarly, for every cardinal $\succeq^*$ on $\mathcal{L}^1(\Omega)$ represented by a utility functional $U$, the induced preference relation $\succeq$ on $\mathcal{L}^1_-(\Omega)$ corresponds to the relative utility functional

$$V(X,Y) = \begin{cases} U(X) - U(Y), & \text{when } \alpha(X) = \alpha(Y), \\ +\infty, & \text{when } \alpha(X) > \alpha(Y), \\ -\infty, & \text{when } \alpha(X) < \alpha(Y). \end{cases}$$

# 4 Robust Decision Making

## 4.1 Risk Aversion versus Robustness

Proposition 3 states that every monotone and risk averse $\succeq$ is conservative. This is also true in the standard probabilistic setting.

**Example 6** *Suppose a rich agent with assets worth of* $\$10^9$ *is under the risk of imprisonment with all assets being confiscated, the probability of which is estimated to be* $p = 20\%$. *A lawyer can reduce* $p$ *to 19% for the premium of* $\$10^7$ *in case of success. If agent's preference relation is monotone and risk averse, the agent agrees to the deal.*

**Detail.** For every $x \in \mathbb{R}$,

$$\int_{-\infty}^{x} F_X(t)\,dt \leq \int_{-\infty}^{x} F_Y(t)\,dt, \tag{2}$$

where $F_X$ and $F_Y$ are CDFs of r.v.'s $X = 0.19\,X_{-10^9} \oplus 0.81\,X_{-10^7}$ and $Y = 0.2\,X_{-10^9} \oplus 0.8\,X_0$, respectively. Consequently, Theorem 2.58 in [37] implies that $X$ dominates $Y$ by SSD, or, equivalently, that $X$ is preferred to $Y$ by any agent with monotone and risk averse preferences. $\square$

In the economic literature, actions (payments) that an agent undertakes (makes) to decrease the probability of loss, e.g. as in Example 6, are known as *self-protection* [19]. However, self-protection is not viewed similarly to insurance, and the results in [17] suggest that a rational agent would not necessary self-protect. Partially, this could be explained by the fact that agents tend to *round* up probabilities (see [41]) and, thus, are unlikely to pay $\$10^7$ to decrease the probability from 20% to 19%, which are perceived to be the same. Also, Kahneman and Tversky [41] argue that the majority of agents, while being risk averse for gains, are risk seeking for losses. This suggests that for *catastrophic* losses, the axiom of risk aversion, at least as defined in this work, does not agree with typical behavior of a rational agent, and, consequently, can be replaced by the one of *robustness*: small variations of the underlying probability measure should not drastically alter a preference relation.

Let $Q_\varepsilon$ be a probability measure close to the underlying probability measure $P$ with respect to a distance measure $\rho(P, Q_\varepsilon)$ to be defined below. For any r.v. $X$, let $\widetilde{X}^{(\varepsilon)}$ be an r.v. with the CDF $F_{\widetilde{X}^{(\varepsilon)}}(x) = \mathbb{P}\left[\widetilde{X}^{(\varepsilon)} \leq x\right] = Q_\varepsilon[X \leq x]$. Let also $\Delta_\varepsilon$ be an r.v. (error) such that $\sup_\Omega |\Delta_\varepsilon| \leq \varepsilon$.

**Definition 4 (robust preference relation)** *A preference relation* $\succeq$ *is* robust, *if* $X \succ Y$ *implies that there exists* $\varepsilon > 0$ *such that* $\widetilde{X}^{(\varepsilon)} + \Delta_\varepsilon \succ \widetilde{Y}^{(\varepsilon)}$ *for any* $\Delta_\varepsilon$.

Obviously, a robust preference relation depends on $\rho(P, Q_\varepsilon)$. However, not every distance measure is appropriate. For example, if $\rho(P, Q_\varepsilon) = \sup_{A \in \mathcal{M}} |Q_\varepsilon(A) - P(A)|$, then robustness of $\succeq$ in Definition 4 with $\rho(P, Q_\varepsilon) \leq \varepsilon$ is equivalent to continuity of $\succeq$ with respect to convergence in probability, which, however, makes $\succeq$ insensitive to catastrophic events as shown by Chichilnisky [14]. If $\succeq$ is desired to be robust and sensitive to catastrophic events simultaneously, how to define $\rho(P, Q_\varepsilon)$?

**Definition 5 (insensitivity to catastrophic events)** *A preference relation* $\succeq$ *is insensitive to catastrophic events, if* $X \succ Y$ *implies that there exists* $\varepsilon > 0$ *such that* $X^{(\varepsilon)} \succ Y^{(\varepsilon)}$, *where* $X^{(\varepsilon)}$ *and* $Y^{(\varepsilon)}$ *are any r.v.'s such that* $\mathbb{P}\left[X = X^{(\varepsilon)}\right] \geq 1 - \varepsilon$ *and* $\mathbb{P}\left[Y = Y^{(\varepsilon)}\right] \geq 1 - \varepsilon$.

Insensitivity to catastrophic events means that low probability events are ignored regardless of how large the losses are, and thus, an agent with such a preference relation would never buy an insurance against catastrophic events. Observe that in Example 6, the probabilities of 20% and 19% should indeed be considered as close, but the probabilities of 1% and 0% should not. Thus, robustness of $\succeq$ should be defined with respect to a *relative* distance for $P$ and $Q_\varepsilon$. We postulate that $\rho(P,Q_\varepsilon) = \sup_{A\in\mathcal{M}} |Q_\varepsilon(A) - P(A)|/P(A)$ with convention that $0/0 = 1$.[15] In this case, $\rho(P,Q_\varepsilon) \le \varepsilon$ is equivalent to $1-\varepsilon \le Q_\varepsilon(A)/P(A) \le 1+\varepsilon$ for any event $A$. Note that $\rho(P,Q_\varepsilon)$ should not be confused a metric. The postulated $\rho(P,Q_\varepsilon)$ is not symmetric with respect to $P$ and $Q_\varepsilon$ and does not satisfy the triangle inequality.

If $\succeq$ is induced by a proper relative utility functional $V$, Proposition 4 implies that $\succeq$ is robust if and only if $V(X, X+\Delta_\varepsilon) \to 0$ and $V\left(X, \widetilde{X}^{(\varepsilon)}\right) \to 0$ as $\varepsilon \to 0$, and $\succeq$ is insensitive to catastrophic events if and only if $V\left(X, X^{(\varepsilon)}\right) \to 0$ as $\varepsilon \to 0$.

## 4.2 An Approach to Decision Making on $\mathcal{L}^1_-(\Omega)$

Let $\succeq$ be a von Neumann-Morgenstern preference relation induced by the expected value of a non-decreasing continuous utility function $u$. The corresponding relative utility functional is given by

$$V(X,Y) = E[u(X)] - E[u(Y)] = \int_0^1 \left( \int_{q_Y(t)}^{q_X(t)} v(x)dx \right) dt,$$

where $v(x) = u'(x) \ge 0$ is defined almost everywhere. Thus, $v(x)$ can be interpreted as a measure of satisfaction when $q_X(t) \ge x \ge q_Y(t)$, or, equivalently, when $\mathbb{P}[X \le x] \le t \le \mathbb{P}[Y \le x]$. This interpretation suggests that $v$ can be generalized to depend also on $t$.

A non-negative *bounded* function $v(x,t)$, defined on $\mathbb{R}\times(0,1)$, introduces a preference relation on $\mathcal{L}^1_-(\Omega)$ such that $X \succeq Y$ if and only if $V(X,Y) \ge 0$ with

$$V(X,Y) = \int_0^1 \left( \int_{q_Y(t)}^{q_X(t)} v(x,t)dx \right) dt. \tag{3}$$

**Proposition 5** *For any non-negative bounded function $v(x,t)$, the integral in* (3) *is a well-defined proper relative utility functional.*

**Proof** Let r.v.'s $X, Y \in \mathcal{L}^1_-(\Omega)$, and let $\alpha(X) \le \alpha(Y)$. Then $V(X,Y) = \int_0^{\alpha(X)} f(t)\,dt + \int_{\alpha(X)}^{\alpha(Y)} f(t)\,dt + \int_{\alpha(Y)}^1 f(t)\,dt$, where $f(t) = \int_{q_Y(t)}^{q_X(t)} v(x,t)dx$. Observe that $\int_0^{\alpha(X)} f(t)\,dt = 0$, then $\int_{\alpha(X)}^{\alpha(Y)} f(t)\,dt$ is either finite or $+\infty$ since $v(x,t) \ge 0$, and finally, $\int_{\alpha(Y)}^1 f(t)\,dt$ is finite because $X, Y \in \mathcal{L}^1_-(\Omega)$ and $v(x,t)$ is bounded. Consequently, $V(X,Y) \in \mathbb{R}\cup\{+\infty\}$. Similarly, if $\alpha(X) \ge \alpha(Y)$, then $V(X,Y) \in \mathbb{R}\cup\{-\infty\}$. Thus, $V : \mathcal{L}^1_-(\Omega)\times\mathcal{L}^1_-(\Omega) \to \overline{\mathbb{R}}$ is a well-defined functional. Verification of properness and conditions (i)–(ii) in Definition 3 is straightforward. □

We call $v(x,t)$ in (3) *value-chance utility function* and $\succeq$ induced by (3) *value-chance preference relation*.

The representation (3) has several special cases. Namely,

(a) If $v(x,t) \equiv 1$, (3) simplifies to (1) and represents a preference relation consistent with the safety-first principle.

(b) If $v(x,t)$ is independent of $t$, (3) is a von Neumann-Morgenstern expected utility functional.

(c) If $v(x,t)$ is independent of $x$, (3) corresponds to a Yaari's dual utility functional [64].

(d) If $v(x,t) = u(x)p(t)$, (3) generalizes a utility functional in the prospect theory [41].

Next we derive conditions under which the functional (3) is consistent with axioms (i)–(iv) of rational behavior under the risk of catastrophic events. Obviously, (3) induces a complete, law invariant and monotone preference relation, which is also strictly monotone if and only if $v(x,t) > 0$ almost surely. Let $\phi(t) = \int_{-\infty}^0 v(x,t)\,dx$. Then $\phi(t)$ is a function from $(0,1)$ to $[0,+\infty]$ such that $\phi(t)\,dt$ represents a dis-utility if the probability of a catastrophic event changes from $t$ to $t+dt$. It will be used to characterize robustness and sensitivity to catastrophic events.

[15] In other words, $1-\varepsilon \le Q_\varepsilon(A)/P(A) \le 1+\varepsilon$ for any event $A$ such that $P(A) > 0$, and $P(A) = 0$ if and only if $Q_\varepsilon(A) = 0$.

**Proposition 6** *A strictly monotone preference relation, induced by* (3)*, is robust (in the sense of Definition 4), if and only if for any* $\varepsilon > 0$,

$$\int_{\varepsilon}^{1-\varepsilon} \phi(t)\,dt < +\infty. \tag{4}$$

**Proof** The condition (4) can be restated as: $\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \phi(t)\,dt \to 0$ as $\varepsilon \to 0$ for every $\alpha \in (0,1)$. If it does not hold for some $\alpha^* \in (0,1)$, i.e. $\int_{\alpha^*-\varepsilon}^{\alpha^*+\varepsilon} \phi(t)\,dt \geq \delta > 0$ for any $\varepsilon > 0$, then either $V\left(\widetilde{X}_1^{(\varepsilon)}, X\right) \geq \delta/2$ or $V\left(\widetilde{X}_2^{(\varepsilon)}, X\right) \geq \delta/2$ holds for the r.v.'s $X = \alpha^* X_{-\infty} \oplus (1-\alpha^*)X_0$, $\widetilde{X}_1^{(\varepsilon)} = (\alpha^* - \varepsilon)X_{-\infty} \oplus (1-\alpha^*+\varepsilon)X_0$, and $\widetilde{X}_2^{(\varepsilon)} = (\alpha^*+\varepsilon)X_{-\infty} \oplus (1-\alpha^*-\varepsilon)X_0$, i.e. $\succeq$ is not robust.

Suppose (4) holds, and $\widetilde{X}^{(\varepsilon)}$ is as in Definition 4. If $V\left(\widetilde{X}^{(\varepsilon)}, X\right) \to 0$ as $\varepsilon \to 0$, then $\succeq$ is robust. If $\alpha(X) = 0$, then this condition follows from boundedness of $v(x,t)$, so we can assume $\alpha(X) \in (0,1)$. Let $\alpha(X) \leq \alpha\left(\widetilde{X}^{(\varepsilon)}\right)$. Then

$$V\left(\widetilde{X}^{(\varepsilon)}, X\right) = \int_{\alpha(X)}^{\alpha\left(\widetilde{X}^{(\varepsilon)}\right)} \phi(t)\,dt + \int_{\alpha(X)}^{\alpha\left(\widetilde{X}^{(\varepsilon)}\right)} \left(\int_0^{q_X(t)} v(x,t)dx\right) dt + \int_{\alpha\left(\widetilde{X}^{(\varepsilon)}\right)}^1 \left(\int_{q_{\widetilde{X}^{(\varepsilon)}}(t)}^{q_X(t)} v(x,t)dx\right) dt.$$

By definition of $\widetilde{X}^{(\varepsilon)}$, $\alpha\left(\widetilde{X}^{(\varepsilon)}\right) \to \alpha(X)$ as $\varepsilon \to 0$. Consequently, in the last equation, the first term vanishes as $\varepsilon \to 0$ by virtue of (4), and the last two vanish by boundedness of $v(x,t)$. The case $\alpha(X) \geq \alpha\left(\widetilde{X}^{(\varepsilon)}\right)$ is proved similarly. □

**Proposition 7** *A strictly monotone robust preference relation, induced by* (3)*, is sensitive to catastrophic events (in the sense that Definition 5 does not hold), if and only if for any* $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(t)\,dt = +\infty. \tag{5}$$

**Proof** Suppose (5) holds. Then $X_0 \succ \varepsilon X_{-\infty} \oplus (1-\varepsilon)X_1$ for any $\varepsilon > 0$, which contradicts Definition 5. If (5) does not hold, then it follows from (4) that $\int_0^{1-\varepsilon} \phi(t)\,dt < +\infty$ for any $\varepsilon > 0$. Since $v(x,t)$ is bounded, this implies that $V\left(X^{(\varepsilon)}, X\right) \to 0$ as $\varepsilon \to 0$ for any $X \in \mathcal{L}^1_-(\Omega)$, where $X^{(\varepsilon)}$ is any r.v. such that $\mathbb{P}\left[X = X^{(\varepsilon)}\right] \geq 1-\varepsilon$, i.e. $\succeq$ is insensitive to catastrophic events. □

**Example 7** *A preference relation, induced by* (3) *with* $v(x,t) = (1+|x|)^{t^2-t-1}$*, is strictly monotone, robust, and sensitive to catastrophic events.*

**Detail.** In this case, $v(x,t) \in (0,1]$, and $\phi(t) = \int_{-\infty}^0 v(x,t)dx = 1/(t-t^2)$, so that (4) and (5) hold. □

**Example 8** *A preference relation, induced by* (3) *in any special case (a)–(d), cannot simultaneously be strictly monotone, robust, and sensitive to catastrophic events.*

**Detail.** If $v(x,t) = u(x)p(t)$ then $\phi(t) = p(t)\int_{-\infty}^0 u(x)dx$. Since $p(t)$ is bounded, (5) implies that $\int_{-\infty}^0 u(x)dx = +\infty$, which contradicts (4). □

Finally, we address the question of how to estimate agent's "utility" $v(x,t)$. For any $0 < \beta_1 < \beta_2 < 1$, let $\psi(\beta_1,\beta_2)$ be a number $C$ such that $\beta_1 X_{-\infty} \oplus (1-\beta_1)X_0 \sim \beta_2 X_{-\infty} \oplus (1-\beta_2)X_C$, i.e. $\psi(\beta_1,\beta_2)$ is a fair cost for the increase of the probability of a catastrophe from $\beta_1$ to $\beta_2$. If such $C$ does not exist, $\psi(\beta_1,\beta_2) = +\infty$. Then $\psi(\beta_1,\beta_2) < +\infty$ implies that $V(\beta_1 X_{-\infty} \oplus (1-\beta_1)X_0, \beta_2 X_{-\infty} \oplus (1-\beta_2)X_C) = 0$, or, equivalently, that

$$\int_{\beta_1}^{\beta_2} \phi(t)\,dt - \int_{\beta_2}^1 \left(\int_0^{\psi(\beta_1,\beta_2)} v(x,t)dx\right) dt = 0 \quad \text{for any} \quad 0 < \beta_1 < \beta_2 < 1. \tag{6}$$

The function $\psi(\beta_1,\beta_2)$ can be estimated from an empirical study, similar to those for determining a utility function in the EUT, and then $v(x,t)$ can be found from the integral equation (6).

Since $\mathcal{L}^1(\Omega) \subset \mathcal{L}^1_-(\Omega)$, a preference relation, induced by (3), can be considered also for r.v.'s on $\mathcal{L}^1(\Omega)$ (in the standard probabilistic setting), and thus, it can be used not only in the context of catastrophic events.

# 5 Comparison of Decision Making Approaches

This section compares six decision making approaches: the EUT, the CPT, Chichilnisky's axiomatization, safety-first principle in both standard and nonstandard probabilistic settings, and the robust approach, developed in Section 4, in evaluating four nuclear power plant projects.

In the nuclear safety literature, the *core damage frequency* (CDFr), i.e. the likelihood of a significant accident that damages reactor's core, is arguably the most critical safety characteristic; see e.g. [39, 40, 50, 43, 62]. Kamyab and Nematollahi [43] identify six *core damage states*, which can occur under various circumstances, and estimate the CDFrs for each of those states for Tehran Research Reactor. Their approach has two steps: (i) identify the most probable *postulated initiating events (PIEs)* and estimate their frequencies, and (ii) evaluate the CDFrs of the core damage states by constructing appropriate event trees and fault trees through a risk assessment tool software; see [43] for details and Table 1 for the CDFrs for each of the six states. The cumulative CDFr for the reactor core damage is bounded above by the sum of the CDFrs for the six core damage states (it is slightly lower than the sum, since some of the states can occur simultaneously).

Table 1: Core damage frequencies (CDFrs) for Tehran Research Reactor [43].

| Core damage state | Frequency per year (CDFr) |
|---|---|
| Heat removal system failure | 2.758E-08 |
| Fuel channel blockage accident | 6.797E-06 |
| Reactivity accident, heat removal works normally | 1.047E-07 |
| Reactivity accident, heat removal fails | 1.038E-09 |
| Failure of the pool isolation system | 1.706E-10 |
| Spurious opening of the flapper accident | 1.438E-06 |
| Cumulative CDFr | 8.343E-06 |

We examine the six decision making approaches in evaluating the following four projects.

(S) *Sensitivity:* A project for improving the fuel channel safety is suggested: a system that controls the maximum fuel temperature during a channel blockage accident averts core damage with probability $\beta\%$ [48]. Should the system be installed, provided that its cost is reasonable? *Comment: the probability of the fuel channel blockage accident is the largest contributor to the cumulative CDFr, and rejecting such a project would imply insensitivity to catastrophic risk.*

(R) *Robustness:* A project for improving the pool isolation system is suggested: a system that reduces the total oxidation of the cladding far beyond the acceptable level [61] reduces the probability of failure of the pool isolation system by $\gamma\%$. Should the project be rejected, provided that its cost is considerably high? *Comment: the probability of a pool isolation system failure has a negligible contribution to the cumulative CDFr, and any agent with robust preferences would reject this project.*

(C) *Continuity:* A project for improving the fuel channel safety by $\beta\%$ at price $\$x$ has been preliminary agreed on. A more careful investigation shows that an extra cost of $\$\varepsilon$ is needed. Should the project be still accepted, provided that $\varepsilon$ is small enough?

(CP) *Continuum of projects:* Suppose amount $\$x$ can be invested in the annual maintenance of the entire reactor. The larger $x$ is, the lower the risk of an accident is. How much should be invested?

## 5.1 Standard Probabilistic Setting: EUT, CPT, and Chichilnisky's Axiomatization

The standard probabilistic setting requires translating the loss in a catastrophic event including the loss of human lives into although very large but still finite cost $C > 0$. In this case, the outcome of the project with cost $x$ and catastrophe probability $\alpha$ is modeled as an r.v. $X$ assuming values $-C-x$ and $-x$ with the probabilities

$\alpha$ and $1-\alpha$, respectively. The EUT then prescribes to choose a project with the maximal expected utility $E[u(X)] = \alpha\, u(-C-x) + (1-\alpha)u(-x)$, where $u$ is a strictly increasing, continuous, and unbounded utility function. Similarly, the CPT recommends choosing a project which maximizes $\pi_1 v(-C-x) + \pi_2 v(-x)$ for some strictly increasing, continuous, and unbounded value function $v$ with $v(0) = 0$ and with decision weights $\pi_1, \pi_2$.

- (S) *Sensitivity:* This depends on (artificial) constant $C$. If $C$ is sufficiently small, catastrophic events with small probabilities are ignored, and the project is rejected.
- (R) *Robustness:* This also depends on $C$. For sufficiently large $C$, the project is accepted, so that robustness fails.
- (C) *Continuity:* The project is accepted.
- (CP) *Continuum of projects:* This also depends on $C$. For any $x^* \geq 0$, there exists $C$ such that annual maintenance $x = x^*$ is optimal.

Chichilnisky's axiomatization [14] includes the axioms of *continuity* and *sensitivity* and assigns a catastrophic loss a large negative (arbitrarily chosen) constant $C$. Consequently, its recommendations on projects (R) and (CP) coincide with those of the EUT.

## 5.2 Safety-First Principle in the Standard Probabilistic Setting

Let the loss be modeled by the r.v. $X$ as defined above. The safety-first principle dictates to choose a project with minimal $\mathbb{P}[X \leq -C]$.

- (S) *Sensitivity:* The project decreases the probability of a catastrophe and, hence, is accepted.
- (R) *Robustness:* Since only safety is taken into account, the project is accepted (negative decision).
- (C) *Continuity:* The project may be rejected (continuity fails) by virtue of Proposition 1.
- (CP) *Continuum of projects:* A project with cost $C-0$ is optimal. Indeed, this project decreases the probability of a catastrophe more that any project with lower cost. On the other hand, for $x \geq C$, the probability of the catastrophe is $\mathbb{P}[X \leq -C] = 1$.

## 5.3 Safety-First Principle in the Nonstandard Probabilistic Setting

In the nonstandard probabilistic setting, the loss of a catastrophic event is assigned $-\infty$. Now, the outcome of the project with cost $x$ and catastrophe probability $\alpha$ can be modeled as an r.v. $X$ assuming values $-\infty$ and $-x$ with the probabilities $\alpha$ and $1-\alpha$, respectively. An optimal project corresponds to minimal $\mathbb{P}[X = -\infty]$.

- (S) *Sensitivity:* The project decreases the probability of a catastrophe and, hence, is accepted.
- (R) *Robustness:* Since only safety is taken into account, the projected is accepted (negative decision).
- (C) *Continuity:* The project is accepted by virtue of Proposition 2.
- (CP) *Continuum of projects:* There is no optimal solution. Smaller $\alpha$ (and, hence, larger $x$) is preferred.

In the nonstandard probabilistic setting, the safety-first principle resolves the issue of continuity but not the ones of *robustness* and *continuum of projects*.

## 5.4 Value-Chance Preference Approach from Section 4

As in the case of the safety-first principle in the nonstandard probabilistic setting, the project with cost $x$ and catastrophe probability $\alpha$ is associated with the r.v. $X$ that assumes values $-\infty$ and $-x$ with the probabilities $\alpha$ and $1-\alpha$, respectively. Now the decision is made based on the relative utility functional (3) with $v(x,t)$ as in Example 7 (or with any other $v(x,t)$ which makes the corresponding preference relation strictly monotone, robust, and sensitive to catastrophic events): choose $X$ such that $V(X,Y) \geq 0$ for any other $Y$.

(S) *Sensitivity:* The preference relation is sensitive to catastrophic events (Proposition 7), and the project is accepted.

(R) *Robustness:* The preference relation is robust (Proposition 6), and the project is rejected (positive decision).

(C) *Continuity:* Since $v(x,t)$ in Example 7 is continuous, the project is accepted.

(CP) *Continuum of projects:* There exists one optimal solution that depends on $v(x,t)$.

### 5.5 Summary

Table 2 summarizes the results of Sections 5.1–5.4. In all decision making approaches based on the standard probabilistic setting, i.e. in the EUT, CPT, and Chichilnisky's axiomatization, the optimal decisions depend on $C$. Given that $C$ is an arbitrarily large negative constant, which is hard to estimate in practice, those decisions are likely to be questionable. Although in the standard and nonstandard probabilistic settings, the safety-first principle resolves subsequently the issues of sensitivity and continuity of preference relations, it still lacks robustness, i.e. it is cost sensitive to a negligibly small reduction in the probability of a nuclear power plant to withstand a major catastrophe. On the other hand, the approach suggested in Section 4 introduces a preference relation which is continuous, robust, and sensitive to catastrophic events.

Table 2: Comparison of the investigated decision approaches

| Decision making approach | Sensitivity | Robustness | Continuity | Continuum |
|---|---|---|---|---|
| Expected utility theory (EUT) | Depends on $C$ | Depends on $C$ | Yes | Depends on $C$ |
| Cumulative prospect theory (CPT) | Depends on $C$ | Depends on $C$ | Yes | Depends on $C$ |
| Chichilnisky's axiomatization | Yes | Depends on $C$ | Yes | Depends on $C$ |
| Safety-first principle (standard) | Yes | No | No | Depends on $C$ |
| Safety-first principle (nonstandard) | Yes | No | Yes | No solution |
| Suggested approach | Yes | Yes | Yes | Solution exists |

## 6 Evaluation of Projects under Uncertain Catastrophe Probability

By now, the uncertain revenue from any project has been modeled as a random variable $X$ with known (fixed) probability distribution function (PDF). In other words, probabilities of all events of the type $\{X \leq C\}$, including those of a catastrophe, have been treated as known. In this section, we argue that this assumption is often unrealistic and extend our analysis to a more general case when the PDF of $X$ (in particular, the probability of a catastrophe) is itself a random variable.

This section is based on the example from Volkanovskia and Cepin [62], who show that the *core damage frequency* (CDFr) can often be estimated only with significant dispersion and should be treated as a random variable with normal distribution. In particular, the last line of Table 4 in [62] shows that the resulting CDFr has the PDF close to normal with mean $m$ = 1.08E-04 ry and standard deviation $\sigma$ = 4.43E-05 ry. The rest of the section addresses the following questions: (i) Should a project with these parameters be accepted? and (ii) If not, how much safer (in terms of CDFr) should it become in order to be accepted?

### 6.1 The Mean Value Approach

Volkanovskia and Cepin [62] note that "The current risk-informed decision making is based on the comparison of mean values, which is conceptually simple and consistent with existing NRC guidelines, including the Commissions Safety Goals (U.S. NRC, 1986)." With this approach, the project is rejected because $m$ = 1.08E-04 ry exceeds the threshold of 1E-04 ry.[16] However, if $m$ is decreased by 8E-06 ry, the improved project (whose CDFr is normal with $m$ = 1E-04 ry and $\sigma$ = 4.43E-05 ry) would be accepted regardless of uncertainty and the fact that CDFr > 1E-04 ry with probability 50%.

## 6.2 The Percentile Measures

In view of the shortcomings of the mean value approach, Volkanovskia and Cepin [62] propose to use percentile measures. In particular, CDFr $\leq$ 1E-04 ry holds with the 95% confidence if $m$ is decreased by 9.5E-05 ry (which is almost 12 times as much as that with the mean value approach).

This approach, however, as any threshold-based safety-first approach, has its own drawbacks. Most importantly, the criterion "accept if and only if $\mathbb{P}$[CDFr $\leq$ 1E-04 ry] $\geq$ 0.95" takes into account neither project cost nor project expected profit in case of no catastrophe, i.e. as in Example 6, the project with the lowest CDFr is preferred regardless of its cost. Also, in addition to a major core-damaging catastrophe, there may be less severe events with moderate losses but with much higher probability of occurrence. Such events are simply ignored by the proposed percentile measure.

## 6.3 Value-Chance Preference Approach

The value-chance preference approach (3) can be readily extended to the case of uncertain PDFs. Now, every project is associated with a *collection* of r.v.'s $X_s$, where parameter $s$ belongs to another probability space $\Omega' = (\Omega', \mathcal{M}', P')$. The probability of any event of the form $\{X_s \leq C\}$, including the major catastrophe probability $P[X_s = -\infty]$, is a random variable in $\Omega'$. Project $X_s$ is preferred to project $Y_s$ if and only if $E_{P'}[u(V(X_s, Y_s))] \geq 0$ for some utility function $u$. With $V(X_s, Y_s)$ given by (3), this condition takes the form

$$E_{P'}[V(X_s, Y_s)] = E_{P'}\left[u\left(\int_0^1 \left(\int_{q_{Y_s}(t)}^{q_{X_s}(t)} v(x,t)dx\right) dt\right)\right] \geq 0. \tag{7}$$

In particular, if only core-damage catastrophe is taken into account, each r.v. $X_s$ takes values $-\infty$ and $0$ with probabilities $\alpha(s)$ and $1-\alpha(s)$, respectively. In Section 6.1, the CRFr of $\alpha(s)$ is itself an r.v. With respect to a benchmark $Y_s$, the best choice is the project with constant CDFr $\alpha^*$ = 1E-04, i.e. $Y_u$ takes values $-\infty$ and $0$ with probabilities 1E-04 and 1.0−1E-04, respectively, for all $s$. In this case, the acceptance criterion (7) becomes

$$E_{P'}\left[u\left(\int_{\alpha(s)}^{\alpha^*} \phi(t)\, dt\right)\right] \geq 0, \tag{8}$$

where $\phi(t) = \int_{-\infty}^{0} v(x,t)dx$. With the exponential utility function $u(p) = 1 - e^{-p}$ and with $v(x,t)$ as in Example 7, $\phi(t) = 1/(t-t^2)$, and consequently, (8) is determined by

$$\begin{aligned} 0 \leq E_{P'}\left[1 - \exp\left(-\int_{\alpha(s)}^{\alpha^*} \frac{dt}{t-t^2}\right)\right] &= E_{P'}\left[1 - \exp\left(-\ln\left[\frac{\alpha^*}{1-\alpha^*}\frac{1-\alpha(s)}{\alpha(s)}\right]\right)\right] \\ &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1 - \frac{\alpha}{1-\alpha}\frac{1-\alpha^*}{\alpha^*}\right)\exp\left(-\frac{(\alpha - m(d))^2}{2\sigma^2}\right)d\alpha, \end{aligned} \tag{9}$$

which holds for $m \leq$ 9.84E-05 ry that corresponds to the improvement of $m$ by $d \geq$ 9.6E-06 ry.

In the above example, the r.v.'s $X_s$ take only on values $-\infty$ and $0$. This is because only the CDFr, i.e. the probability of the most severe catastrophe, is taken into account. The suggested approach can readily handle an arbitrary distribution of $X_s$, which may take into account, for example, project cost and less severe events with moderate losses but with higher frequency of occurrence.

[16] See Volkanovskia and Cepin [62] for CDFr acceptance guidelines and for choices of the CDFr thresholds under different circumstances.

## 7 Conclusions

This work has introduced a nonstandard probabilistic setting for decision making under the risk of catastrophic events, in which an r.v. $X : \Omega \to \mathbb{R} \cup \{-\infty\}$ can take on infinitely large negative values corresponding to the loss in a catastrophic event. In this setting, the safety-first principle conforms to the traditional axioms of rational behavior, including continuity and risk aversion. Moreover, any strictly monotone and risk averse preference relation $\succeq$ necessarily agrees with the safety-first principle. A *robust* preference relation has been defined, which, on the one hand, is stable with respect to the relative error in estimating the underlying probability measure, and, on the other hand, is sensitive to catastrophic events in the sense of Chichilnisky [14]. Also, the binary utility functional (3) has been introduced and shown to induce preference relations that are strictly monotone, robust, and sensitive to catastrophic events. It generalizes utility functionals of the expected utility theory and dual utility theory and can also be applied to ordering random outcomes in the standard probabilistic setting in a wide range of risk modeling applications. Finally, several decision making approaches have been compared in evaluating nuclear power plant projects in terms of cost and safety.

A challenging problem for the future research is to develop a methodology for identifying "utility function" $v(x,t)$ in (3) for each particular application. This problem is similar to the one of identifying a utility function for an individual investor, which, to this day, remains a practically critical task in the investment science. The latter problem can be solved with the *mean-deviation model* [30], which "restores" investor's risk-reward preferences based on investor's existing portfolio of financial assets. Thus, one of the approaches to identifying the "utility function" in (3) is to extend the results of [30] based on the available historical data including previously made decisions.

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