Schur Convex Functionals: Fatou Property and Representation

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Abstract

The Fatou property for every Schur convex lower semicontinuous (l.s.c.) functional on a general probability space is established. As a result, the existing quantile representations for Schur convex l.s.c. positively homogeneous convex functionals, established on $\mathcal{L}^p(\Omega)$ for either $p = 1$ or $p = \infty$ and with the requirement of the Fatou property, are generalized for $\mathcal{L}^p(\Omega)$, $p \in [1,\infty]$, with no requirement of the Fatou property. In particular, the existing quantile representations for law invariant coherent risk measures and law invariant deviation measures on an atomless probability space are extended for a general probability space.

1 Introduction

In risk analysis and finance, the uncertainty inherent in a random variable (r.v.) is usually characterized by notions such as *risk*, *deviation*, and *error*. Since the seminal work of Artzner, Delbaen, Eber, and Heath [1], who introduced *coherent risk measures* based on four axioms: *subadditivity*, *positive homogeneity*, *constant translation*, and *monotonicity*, measuring risk has been closely linked to positively homogeneous convex functionals. Rockafellar et al. [12, 13] developed a coordinating theory of general deviation measures, error measures, and averse measures of risk. General deviation measures [12] generalize the notion of *standard deviation* to measure *"nonconstancy"* in an r.v., whereas *error measures* generalize the expected squared error to measure "nonzeroness" of an r.v. and are used in a generalized linear regression [13]. In this theory, risk, deviation and error measures satisfy closely related systems of axioms. While deviation and error measures are *nonnegative*, and risk measures are *constant translation invariant*, all these measures are *lower semicontinuous (l.s.c.) positively homogeneous convex* functionals to be called for brevity *basic functionals*.

Deviation measures and coherent risk measures enjoy dual representations in terms of so-called risk envelopes, see [12]. If in addition, a corresponding measure is *law invariant*, i.e. depends only on the distribution of an r.v., then on an atomless probability space, its dual representation can be transformed into a *quantile* form. Quantile representations for positively homogeneous convex functionals have long been a major modeling tool in decision problems. Kusuoka [7] constructed a quantile representation for law invariant coherent risk measures for bounded r.v.'s on an atomless probability space. This result was generalized in several directions. Leitner [8] noticed that Kusuoka's representation holds for a general probability space for coherent risk measures consistent with second-order stochastic dominance (SSD)[1], while Cherny and Madan [2] extended Kusuoka's representation for the case of unbounded r.v.'s. Quantile representations for law invariant deviation measures were obtained on an atomless probability space and played a pivotal role in generalizing the maximum entropy principle with general deviation measures [5]. This paper extends these quantile representations for a broader class of functionals: *Schur convex* (or *convex-order preserving*) basic functionals.

In the case of risk measures, *Schur convexity* is equivalent to *SSD-consistency*, and for an atomless probability space, it reduces simply to law invariance. A quantile representation for weak* l.s.c. basic functionals can be obtained from Dana's representation [3] for weak* upper-semicontinuous concave Schur concave functionals. The contribution and organization of this paper are as follows. Section 2 introduces Schur convex basic functionals. Section 3 generalizes Theorem 2.2 in [6], i.e., it shows that every such measure is weak* l.s.c for an arbitrary probability space. This result implies that Dana's representation holds for Schur convex basic functionals. Section 4 generalizes the existing quantile representations [3, Theorems 2.1 and 3.1] for Schur convex

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[1]Monotonicity of risk measures with respect to SSD was introduced and studied by Ogryczak and Ruszczynski, see [10].

basic functionals for $p \in [1,\infty]$ with no requirement of the Fatou property.[2] As a corollary, the quantile representations for coherent risk measures [2, 7] and deviation measures [5, Proposition 2.1], originally established on an atomless probability space, are extended for a general probability space.

2 Schur Convex Basic Functionals

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability space, with Ω, $\mathcal{M}$, and $\mathbb{P}$ being a set of elementary events, a σ-algebra over Ω, and a probability measure on $(\Omega, \mathcal{M})$, respectively. A random variable (r.v.) is any measurable function from Ω to $\mathbb{R}$. We restrict our attention to r.v.'s from $\mathcal{L}^p(\Omega) = \mathcal{L}^p(\Omega, \mathcal{M}, \mathbb{P})$, $p = [1,\infty]$, with norms $\|X\|_p = (E[|X|^p])^{1/p}$, $p < \infty$, and $\|X\|_\infty = \operatorname{ess\,sup}|X|$ and also introduce the space $\mathcal{L}^F(\Omega) \subset \mathcal{L}^\infty(\Omega)$ of r.v.'s which assume only a finite number of values. The relations between r.v.'s are understood to hold in the almost sure sense, e.g. $X = Y$ and $X \geq Y$ imply $\mathbb{P}[X = Y] = 1$ and $\mathbb{P}[X \geq Y] = 1$, respectively. Also, $\inf X$ means $\operatorname{ess\,inf} X$.

We define an α-quantile of an r.v. X by $q_X(\alpha) = -\mathrm{VaR}_\alpha(X) = \inf\{z | \mathbb{P}[X \leq z] > \alpha\}$. If $q_X(\alpha) \equiv q_Y(\alpha)$ for two r.v.'s X and Y, we say that X and Y have the same distribution and write $X \overset{d}{\sim} Y$. The integral[3] $EX = \int_0^1 q_X(\alpha) d\alpha$ is the *expected value* of an r.v. X.

The probability space Ω is called *atomless* if there exists an r.v. with a continuous cumulative distribution function (CDF). On an atomless probability space, there exist r.v.'s with all possible distribution functions (see [4, Proposition A27]). Let $\overline{\mathbb{R}}$ denote the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Definition 1 (basic functionals) *A* basic functional *is any functional* $\mathcal{F} : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ *satisfying*

(F1) $\mathcal{F}(\lambda X) = \lambda \mathcal{F}(X)$ *for all* X *and all* $\lambda \geq 0$ *(here* $0\infty = 0$*) (*positive homogeneity*),*

(F2) $\mathcal{F}(X+Y) \leq \mathcal{F}(X) + \mathcal{F}(Y)$ *for all* X *and* Y *(*subadditivity*),*

(F3) $\{X \in \mathcal{L}^p(\Omega) \,|\, \mathcal{F}(X) \leq c\}$ *is closed in* $\mathcal{L}^p(\Omega)$ *for all* $c < \infty$ *(*lower semicontinuity*).*

Basic functionals include risk measures, deviation measures, and error measures. A *coherent risk measure* $\mathcal{R} : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ is a basic functional that satisfies two additional axioms

(R4) $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for constants C,

(R5) $\mathcal{R}(X) \leq 0$ for all $X \geq 0$.

A *deviation measure* $\mathcal{D} : \mathcal{L}^p(\Omega) \to [0,\infty]$ is a basic functional that satisfies an additional axiom

(D4) $\mathcal{D}(X) = 0$ for constant X, and $\mathcal{D}(X) > 0$ otherwise (*nonnegativity*).

Examples of deviation measures include standard deviation $\mathcal{D}(X) = \|X - EX\|_2$; *lower* and *upper semideviations* $\mathcal{D}(X) = \|[X - EX]_-\|_2$ and $\mathcal{D}(X) = \|[X - EX]_+\|_2$, respectively, where $[X]_- = \max\{0, -X\}$ and $[X]_+ = \max\{0, X\}$; and *conditional value-at-risk (CVaR) deviation* $\mathrm{CVaR}^\Delta_\beta(X) = -\frac{1}{\beta}\int_0^\beta (q_X(\alpha) - EX)\, d\alpha$ defined for any $\beta \in [0,1)$. For other examples of deviation measures, see [12, 13].

An *error measure* $\mathcal{E} : \mathcal{L}^p(\Omega) \to [0,\infty]$ is a basic functional that satisfies

(E4) $\mathcal{E}(X) > 0$ for nonzero X with $\mathcal{E}(C) < \infty$ for constant C.

Well-known examples of error measures are the expected squared error $\mathcal{E}(X) = \|X\|_2$ and error measures given by $\mathcal{E}(X) = \|a[X]_+ + b[X]_-\|_p$ with $a \geq 0$, $b \geq 0$ (excluding $a = b = 0$), and $1 \leq p \leq \infty$; see, e.g., [12].

An r.v. X dominates Y with respect to convex ordering, or $X \succcurlyeq_{cx} Y$, if $E[f(X)] \geq E[f(Y)]$ for every convex function $f : \mathbb{R} \to \mathbb{R}$, see [9]. This condition is equivalent to having $EX = EY$ and $\int_0^t q_X(\alpha) d\alpha \leq \int_0^t q_Y(\alpha) d\alpha$ for $t \in [0,1]$. In particular, $X \succcurlyeq_{cx} E[X|\mathcal{M}']$, i.e. every r.v. $X \in \mathcal{L}^p(\Omega)$ dominates its conditional expectation with respect to any σ-algebra $\mathcal{M}' \subset \mathcal{M}$ in the sense of convex order, see e.g. [4, Corollary 2.62].

[2]In [3], the representations were obtained on $\mathcal{L}^p(\Omega)$ with either $p = 1$ or $p = \infty$ and with the requirement of the Fatou property.

[3]All integrals here are Lebesgue integrals.

Definition 2 (Schur convexity) *A functional $\mathcal{F} : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ is called* convex-order preserving, *or* Schur convex,[4] *if $X \succcurlyeq_{cx} Y$ implies $\mathcal{F}(X) \geq \mathcal{F}(Y)$.*

A set $A \subset \mathcal{L}^p(\Omega)$ is called *dominance-closed* if $X \in A$ and $X \succcurlyeq_{cx} Y$ implies $Y \in A$. A functional $\mathcal{F}$ is Schur convex if and only if $\{X \in \mathcal{L}^p(\Omega) \,|\, \mathcal{F}(X) \leq c\}$ is dominance-closed for all $c < \infty$.

Definition 3 (law invariance) *A functional $\mathcal{F} : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ is* law invariant, *if $X \overset{d}{\sim} Y$ implies $\mathcal{F}(X) = \mathcal{F}(Y)$.*

Every Schur convex functional is law invariant. The following result is due to Dana [3, Theorem 4.1].

Proposition 1 *On an atomless probability space, every law invariant basic functional $\mathcal{F}(X) : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ is Schur convex.*

Proof. By [6, Theorem 2.2], every law invariant basic functional on an atomless probability space is weak* l.s.c., and by [3, Theorem 4.1], every weak* l.s.c. law invariant basic functional is Schur convex. □

3 Dual Representation and Fatou Property

A functional $\mathcal{F} : \mathcal{L}^\infty(\Omega) \to \overline{\mathbb{R}}$ satisfies the *Fatou property* if $\mathcal{F}(X) \leq \liminf_{n\to\infty} \mathcal{F}(X_n)$ for any *bounded* sequence with $X_n \overset{a.s.}{\to} X$. For convex functionals, the Fatou property is equivalent to $\sigma(\mathcal{L}^\infty(\Omega), \mathcal{L}^1(\Omega))$-lower semicontinuity, i.e., lower semicontinuity with respect to weak* topology (see [14, Theorem 1.6]). The next result is due to Rudloff [14, Theorem 1.5, Theorem 1.6], see also Ruszczynski and Shapiro [15].

Proposition 2 *Let $\mathcal{F}(X) : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ satisfy F1–F3. For $p \in [1, \infty)$, $\mathcal{F}(X)$ can be represented by*

$$\mathcal{F}(X) = \sup_{Y \in \mathcal{Y}} E[XY], \qquad \mathcal{Y} \subset \mathcal{L}^q(\Omega), \quad 1/p + 1/q = 1. \tag{1}$$

For $p = \infty$, $\mathcal{F}(X)$ can be represented by (1) for some $\mathcal{Y} \subset \mathcal{L}^1(\Omega)$ if and only if it satisfies the Fatou property.

For a basic functional $\mathcal{F}(X)$, the set $\mathcal{Y}$ in (1) can be recovered by

$$\mathcal{Y} = \left\{ Y \in \mathcal{L}^q(\Omega) \;\middle|\; E[XY] \leq \mathcal{F}(X) \text{ for all } X \in \mathcal{L}^p(\Omega) \right\}, \tag{2}$$

and in this case, $\mathcal{Y}$ is nonempty closed and convex.[5] Thus, (1) and (2) establish a one-to-one correspondence between nonempty closed convex sets in $\mathcal{L}^q(\Omega)$ and basic functionals $\mathcal{F} : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$, which, in addition, satisfy the Fatou property for $p = \infty$.

Jouini et al. [6, Theorem 2.2] proved that on an atomless probability space every law invariant functional $\mathcal{F}(X) : \mathcal{L}^\infty(\Omega) \to \overline{\mathbb{R}}$, satisfying F1–F3, has the Fatou property. This, together with Proposition 2, implies that on an atomless probability space, *every* law invariant functional $\mathcal{F}(X) : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$, satisfying F1–F3, can be represented by (1). The next result generalizes Theorem 2.2 in [6] for an arbitrary probability space.

Proposition 3 *Every Schur convex l.s.c. functional $\mathcal{F}(X) : \mathcal{L}^\infty(\Omega) \to \overline{\mathbb{R}}$ has the Fatou property.*

[4]The notion of Schur convexity in this sense was introduced by Ostrowski [11]. For the detailed discussion of other definitions of Schur convexity, the reader may refer to [3].

[5]Observe that if in (1), one starts with arbitrary $\mathcal{Y} \subset \mathcal{L}^q(\Omega)$, then (2) determines the closed convex hull of $\mathcal{Y}$.

Proof. Let X_k be a bounded sequence of r.v.'s, such that $X_k \stackrel{a.s.}{\to} X$. We need to prove that $\mathcal{F}(X) \leq A$, where $A = \liminf_{k\to\infty} \mathcal{F}(X_k)$. Without loss of generality, we can assume that $A = \lim_{k\to\infty} \mathcal{F}(X_k)$ (otherwise instead of X_k, we can consider a subsequence of X_k converging to A).

Let $C = \|X\|_\infty$, and let $\Omega_i = \{\omega \in \Omega | \frac{i}{2^n} C \leq X(\omega) < \frac{i+1}{2^n} C\}$ for fixed $n \in N$ and $i = -2^n, \dots, -1, 0, 1, \dots, 2^n - 1$. Then $\mathcal{P}_n = \{\Omega_i | -2^n \leq i \leq 2^n - 1\}$ is a partition of Ω, and $X_k \succcurlyeq_{cx} Y_{k,n}$ for $Y_{k,n} = E[X_k|\mathcal{P}_n]$. Consequently, $\mathcal{F}(X_k) \geq \mathcal{F}(Y_{k,n})$.

Let $I = \{i : \mathbb{P}[\Omega_i] > 0\}$. For every $i \in I$, the r.v. $Y_{k,n}$ is constant on Ω_i, i.e. $Y_{k,n}(\omega) = y^i_{k,n}$, $\omega \in \Omega_i$, where $y^i_{k,n} = \int_{\Omega_i} X_k d\omega \big/ \mathbb{P}[\Omega_i]$. Convergence of a bounded sequence of r.v.'s a.s. implies $\mathcal{L}^1$ convergence, and consequently, $\lim_{k\to\infty} y^i_{k,n} = x^i_n$, $i \in I$, where $x^i_n = \int_{\Omega_i} X d\omega \big/ \mathbb{P}[\Omega_i]$. Since I is a finite set, for $\varepsilon = C/2^n$ there exists $k(n)$ such that $|y^i_{k,n} - x^i_n| < \varepsilon$ for all $k \geq k(n)$. Then

$$\sup_{\Omega_i} |X - Y_{k(n),n}| \leq \sup_{\Omega_i} |X - x^i_n| + |x^i_n - y^i_{k,n}| < C/2^n + \varepsilon = 2\varepsilon$$

for every $i \in I$, where the second inequality follows from the definition of Ω_i. Thus, $\sup_{\Omega_i} |X - Y_{k(n),n}| < 2\varepsilon$ for all $i \in I$, whence $\|X - Y_{k(n),n}\|_\infty < 2\varepsilon$. This implies that $Y_{k(n),n} \to X$ in $\mathcal{L}^\infty(\Omega)$ for $n \to \infty$, and lower semicontinuity of $\mathcal{F}(X)$ implies $\mathcal{F}(X) \leq \liminf_{n\to\infty} \mathcal{F}(Y_{k(n),n}) \leq \liminf_{n\to\infty} \mathcal{F}(X_{k(n)}) = A$, which concludes the proof. □

Proposition 3 does not require Schur convex l.s.c. functionals to satisfy F1 and F2. However, the l.s.c. condition cannot be omitted. For example, the functional $\mathcal{F}(X)$ defined as $\mathcal{F}(X) = 0$ for $EX < 1$ and $\mathcal{F}(X) = 1$ for $EX \geq 1$ is Schur convex but not l.s.c. The Fatou property does not hold for this functional.

For convex functionals, the Fatou property is equivalent to $\sigma(\mathcal{L}^\infty(\Omega), \mathcal{L}^1(\Omega))$-lower semicontinuity, and Proposition 3 has the following corollary.

Corollary 1 *Every convex Schur convex l.s.c. functional $\mathcal{F}(X) : \mathcal{L}^\infty(\Omega) \to \overline{\mathbb{R}}$ is $\sigma(\mathcal{L}^\infty(\Omega), \mathcal{L}^1(\Omega))$-l.s.c., i.e., l.s.c. with respect to weak* topology. Equivalently, any convex dominance-closed $||.||_\infty$-closed subset of $\mathcal{L}^\infty(\Omega)$ is $\sigma(\mathcal{L}^\infty(\Omega), \mathcal{L}^1(\Omega))$-closed.*

Propositions 2 and 3 imply that every Schur convex basic functional $\mathcal{F}(X) : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ has the dual representation (1).

4 Quantile Representation

This section generalizes Dana's representation [3] for Schur convex functionals.

By Proposition 3.1 in [3], every Schur convex functional $\mathcal{F}(X) : \mathcal{L}^\infty(\Omega) \to \overline{\mathbb{R}}$, satisfying F1–F3 and the Fatou property, can be represented in the form

$$\mathcal{F}(X) = \sup_{Y \in \mathcal{Y}} \int_0^1 q_Y(\alpha) q_X(\alpha) \, d\alpha,$$

where $\mathcal{Y}$ is a subset of $\mathcal{L}^1(\Omega)$.

Proposition 3 shows that the requirement of the Fatou property in [3, Proposition 3.1] can be omitted. To generalize this result for $\mathcal{F}(X) : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ with an arbitrary $p \in [1, \infty]$, we prove the following statement.

Proposition 4 *For every r.v. $X \in \mathcal{L}^p(\Omega)$, there exists a sequence $X_n \in \mathcal{L}^F(\Omega)$, $n = 1, 2, \dots$, such that: (i) every X_n is comonotone[6] with X; (ii) $X \succcurlyeq_{cx} X_n$ for all n; and (iii) $X_n \to X$ in $\mathcal{L}^p(\Omega)$.*

[6] Two r.v.'s $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are *comonotone*, if there exists a set $A \subseteq \Omega$ such that $\mathbb{P}[A] = 1$ and $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in A$.

Proof. Let $\Omega_i = \{\omega \in \Omega | \frac{i}{2^n} \le X(\omega) < \frac{i+1}{2^n}\}$ for fixed $n \in N$ and $i = -2^{2n}, \dots, -1, 0, 1, \dots, 2^{2n}-1$, and let $\Omega_*^- = \{\omega \in \Omega | X(\omega) < -2^n\}$, $\Omega_*^+ = \{\omega \in \Omega | X(\omega) \ge 2^n\}$. Then $\mathcal{P}_n = \{\Omega_i | -2^{2n} \le i \le 2^{2n}-1\} \cup \{\Omega_*^-\} \cup \{\Omega_*^+\}$ is a partition of Ω, and the sequence $X_n = E[X|\mathcal{P}_n] \in \mathcal{L}^F(\Omega)$ satisfies all the conditions of the proposition. Indeed, (i) follows from the definition of $\mathcal{P}_n$, (ii) follows from [4, Corollary 2.62], and it is left to prove (iii).

Let $\Omega_* = \Omega_+ \cup \Omega_-$. Then $\Omega_* = \emptyset$ for $p = \infty$ and $n > \log_2 \|X\|_\infty$, and thus, $\sup |X - X_n| \le 2^{-n}$. For $p < \infty$,

$$\int_\Omega |X - X_n|^p d\omega = \int_{\Omega_*} |X - X_n|^p d\omega + \int_{\Omega/\Omega_*} |X - X_n|^p d\omega \le \int_{\Omega_*} |2X|^p d\omega + \left(2^{-n}\right)^p.$$

The fact $\mathcal{P}(\Omega_*) \to 0$ as $n \to \infty$ implies that $X_n \to X$ in $\mathcal{L}^p(\Omega)$ as $n \to \infty$, and (iii) is proved. □

Now, it is straightforward to extend Theorems 2.1 and 3.1 from [3].

Proposition 5 *Let $X_0 \in \mathcal{L}^p(\Omega)$ and $Y \in \mathcal{L}^q(\Omega)$ with $1/p + 1/q = 1$, then*

$$\sup_{X \in \mathcal{L}^p(\Omega): X_0 \succcurlyeq_{cx} X} E[XY] = \int_0^1 q_Y(\alpha)\, q_{X_0}(\alpha)\, d\alpha. \tag{3}$$

Proof. The cases $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$ are covered by Theorem 2.1 in [3].[7] In general,

$$E[XY] \le \int_0^1 q_Y(\alpha)\, q_X(\alpha)\, d\alpha \le \int_0^1 q_Y(\alpha)\, q_{X_0}(\alpha)\, d\alpha \quad \text{for all } X \in \mathcal{L}^p(\Omega) \text{ such that } X_0 \succcurlyeq_{cx} X,$$

where the first and second inequalities are due to Hardy-Littlewood [4, Theorem 2.1] and Dana [3, Lemma 2.1], respectively.

Next, we prove the reverse inequality. By Proposition 4, there exists a sequence X_n such that X_n and X_0 are comonotone, $X_0 \succcurlyeq_{cx} X_n$ and $X_n \to X_0$ in $\mathcal{L}^p(\Omega)$. Convergence $X_n \to X_0$ and comonotonicity of X_n and X_0 imply that $q_{X_n}(\alpha) \to q_{X_0}(\alpha)$ in $\mathcal{L}^p(0,1)$, whence $I_n \to I$, where $I_n = \int_0^1 q_Y(\alpha)\, q_{X_n}(\alpha)\, d\alpha$ and $I = \int_0^1 q_Y(\alpha)\, q_{X_0}(\alpha)\, d\alpha$. Let $\varepsilon > 0$, and let n be such that $|I - I_n| < \varepsilon/2$, then it follows from [3, Theorem 2.1] for $X_n \in \mathcal{L}^F(\Omega) \subset \mathcal{L}^\infty(\Omega)$ and $Y \in \mathcal{L}^q(\Omega) \subset \mathcal{L}^1(\Omega)$ that there exists $X \in \mathcal{L}^\infty(\Omega)$ such that $X_n \succcurlyeq_{cx} X$ and $|I_n - E[XY]| < \varepsilon/2$. Now, since $\varepsilon > 0$ is arbitrary, $X \in \mathcal{L}^\infty(\Omega) \subset \mathcal{L}^p(\Omega)$ and $X_n \succcurlyeq_{cx} X_0 \succcurlyeq_{cx} X$, which finishes the proof. □

Proposition 6 *A basic functional $\mathcal{F}(X) : \mathcal{L}^p(\Omega) \to \overline{\mathbb{R}}$ is Schur convex if and only if the following representation holds*

$$\mathcal{F}(X) = \sup_{Y \in \mathcal{Y}} \int_0^1 q_Y(\alpha)\, q_X(\alpha)\, d\alpha, \tag{4}$$

where $\mathcal{Y}$ is given by (2).

Proof. If $\mathcal{F}(X)$ is Schur convex, then by Proposition 3, it satisfies the Fatou property, and consequently, (1) holds. Now Proposition 5 implies that

$$\mathcal{F}(X) = \sup_{Z: X \succcurlyeq_{cx} Z} \mathcal{F}(Z) = \sup_{Z:\, X \succcurlyeq_{cx} Z} \Big(\sup_{Y:\, Y \in \mathcal{Y}} E[YZ] \Big) = \sup_{Y:\, Y \in \mathcal{Y}} \Big(\sup_{Z:\, X \succcurlyeq_{cx} Z} E[YZ] \Big) = \sup_{Y \in \mathcal{Y}} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha.$$

The "if" part follows directly from [3, Lemma 2.1]. □

For coherent risk measures, (4) generalizes Kusuoka's representation for a general probability space and requires no Fatou property, while for deviation measures, (4) generalizes a similar representation in [5] established originally for an atomless probability space.

[7] Theorem 2.1 in [3] states that the infimum of $E[XY]$ is given by $\int_0^1 q_Y(1-\alpha)\, q_{X_0}(\alpha)\, d\alpha$, which is equivalent to (3) in view of the relation $q_Y(1-\alpha) = -q_{-Y}(\alpha)$, which holds for almost all α.

5 Conclusions

It has been shown that every Schur convex lower semicontinuous functional $\mathcal{F}(X):\mathcal{L}^\infty(\Omega)\to\overline{\mathbb{R}}$ is weak* lower semicontinuous. This fact generalizes Theorem 2.2 in [6] for an arbitrary probability space and extends Dana's quantile representation [3] for Schur convex basic functionals for $\mathcal{L}^p(\Omega)$, $p\in[1,\infty]$, with no requirement of the Fatou property. In particular, the obtained quantile representations extend the existing representations for law invariant coherent risk measures [2, 7] and law invariant deviation measures [5, Proposition 2.1] for an arbitrary probability space.

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