Cooperative Games with General Deviation Measures

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Abstract

Cooperative games with players using different law-invariant deviation measures as numerical representations for their attitudes towards risk in investing to a financial market are formulated and studied. As a central result, it is shown that players (investors) form a coalition (cooperative portfolio) that behaves similar to a single player (investor) with a certain deviation measure. An explicit formula for that deviation measure is obtained. An approach to optimal risk sharing among investors is developed, and a "fair" division of the cooperative portfolio expected gain, belonging to the core of a corresponding cooperative game, is suggested.

Key Words: deviation measures, cooperative games, portfolio theory, risk sharing.

1 Introduction

For the past decade, optimal risk sharing has become one of the central venues in studying the interaction of agents with different risk preferences. It is formulated as follows: given an uncertain outcome X and m agents, how can X be partitioned into random Y_j's, $j = 1,\dots,m$, such that $\sum_{j=1}^{m} Y_j = X$ and such that each Y_j is acceptable for agent j? Optimal risk sharing is a particular multi-agent model which finds its application in the actuarial science and corporate finance. From the perspective of the investment science, in particular portfolio theory, the interaction of agents, however, involves not only their risk preferences but also their reward preferences and can be studied in the framework of cooperative games. The rationale for pursuing this research venue is that an investor alone may not be able to form a desired portfolio/instrument, since only finite number of assets with although uncertain but not arbitrary returns are available for investment. For example, two investors may find different aspect of the same asset to be attractive, and as a result, they may buy the asset and divide its uncertain return optimally based on their risk-reward preferences.

This work formulates and studies cooperative games with players using different deviation measures as numerical representations for their attitudes towards risk. General measures of deviation were introduced by Rockafellar, Uryasev, and Zabarankin (2006a, 2002) to measure variability of a random variable based on four axioms: *nonnegativity*,[1] *subadditivity*, *positive homogeneity*, and *lower semicontinuity*. Examples of deviation

[1] A deviation measure is strictly positive for nonconstant random variables and vanishes for constants.

measures include standard deviation, *lower and upper semideviations*, *conditional value-at-risk (CVaR) deviation*, *mixed CVaR-deviation*, *mean absolute deviation*, etc. (see Rockafellar, Uryasev, and Zabarankin, 2006a). Rockafellar, Uryasev, and Zabarankin (2006a) investigated the relationship between deviation measures and *averse measures of risk* and generalized Markowitz's portfolio selection problem with standard deviation for an arbitrary deviation measure (Rockafellar, Uryasev, and Zabarankin, 2006c,b). In their problem, investor i measures the risk of the future (uncertain) portfolio return X by a deviation measure $\mathcal{D}_i(X)$ rather than by the standard deviation and minimizes $\mathcal{D}_i(X)$ subject to a constraint on desired portfolio expected return π_i:

$$\min_{X \in \mathcal{F}} \mathcal{D}_i(X) \quad \text{subject to} \quad EX \geq \pi_i, \tag{1.1}$$

where $\mathcal{F}$ is a feasible set of uncertain X. The criticism of Markowitz's mean-variance model includes the following well-known arguments: (i) the standard deviation, or equivalently variance, penalizes ups and downs of X equally, (ii) the model provides no customization in expressing investor's risk preferences, and as a result, it assumes the same risk preferences for all investors, and (iii) the mean-variance approach does not conform with the expected utility theory unless X is normally distributed or a utility function is quadratic. In contrast to the standard deviation, a general deviation measure is no longer symmetric with respect to ups and downs of a random variable, and it is flexible enough to capture variability in individual risk preferences (Rockafellar, Uryasev, and Zabarankin, 2006a). Along with these advantages, the problem (1.1) also preserves all benefits of Markowitz's model: concentrating on just mean and risk (deviation) keeps the model simple and computationally tractable. Rockafellar, Uryasev, and Zabarankin (2006c,b) derived optimality conditions for (1.1) and generalized several results well-known for Markowitz's model: the one fund theorem (Rockafellar, Uryasev, and Zabarankin, 2006c,b), the capital asset pricing model (CAPM) (Rockafellar, Uryasev, and Zabarankin, 2006c), market equilibrium with investors using a diversity of deviation measures (Rockafellar, Uryasev, and Zabarankin, 2007), and error decomposition in a generalized linear regression (Rockafellar, Uryasev, and Zabarankin, 2008).

In general, the mean EX and a deviation measure $\mathcal{D}(X)$ of a random outcome X can be combined in a utility function $V(EX, \mathcal{D}(X))$ increasing in the first argument and decreasing in the second. Grechuk, Molyboha, and Zabarankin (2011) showed that the mean-deviation functional $V(EX, \mathcal{D}(X))$ is a numerical representation of the following system of axioms on preference relations: completeness, transitivity, monotonicity, continuity, risk aversion, and some restricted versions of uncertainty aversion and constant risk aversion. In other words, if a preference relation $\succeq$ satisfies this system of axioms then $X \succeq Y$ **only if** $V(EX, \mathcal{D}(X)) \geq V(EY, \mathcal{D}(Y))$ for some deviation measure $\mathcal{D}$. Also, Yaari's dual utility theory (Yaari, 1987) and coherent risk measures[2] (Artzner, Delbaen, Eber, and Heath, 1999) are shown to be special cases of the mean-deviation model.

Suppose there are m investors, each of which solves the portfolio problem (1.1). Since the investors view risk differently (by using possibly different deviation measures $\mathcal{D}_1, \ldots, \mathcal{D}_m$), they may find different aspects of the same financial asset to be attractive, and consequently, investing individually may no longer be the best strategy. For example, given only a single asset on the market, none of the investors may find it attractive to buy the asset alone, whereas they can buy the asset together and then divide the asset payoff based on their risk-reward preferences. In general, this opens up a possibility for the investors to form a coalition or a joint portfolio in which the share of investor i is preferred over the optimal portfolio that investor i can form alone. We pose the following questions:

1. How should a cooperative portfolio selection problem be formulated and solved?
2. How then should the cooperative portfolio future payoff X be divided among the investors?

Answering these questions reduces to finding a Pareto-optimal solution and then to finding an optimal division of the Pareto-optimal solution, which are the subject of this work.

[2] Theorem 2 in Rockafellar, Uryasev, and Zabarankin (2006a) and Proposition 8 in Grechuk and Zabarankin (2011) imply that every law-invariant coherent risk measure $\mathcal{R}(X) \not\equiv -EX$ can be represented in the form $\mathcal{R}(X) = \mathcal{D}(X) - EX$ for some *lower range dominated* deviation measure $\mathcal{D}$.

Suppose investors invest into a cooperative portfolio with uncertain future payoff $X \in \mathcal{F}$, where $\mathcal{F}$ represents a set of all feasible portfolio payoffs due to the limited total budget, position restrictions, etc., and then offer uncertain payoff (share) Y_i to investor i. A vector $Y = (Y_1, \dots, Y_m)$ is called a feasible division if $\sum_{i=1}^m Y_i = X \in \mathcal{F}$. It is called a *Pareto-optimal* division if there is no feasible division $Y' = (Y_1', \dots, Y_m')$ such that $\mathcal{D}(Y_i') \le \mathcal{D}(Y_i)$ and $EY_i' \ge EY_i$ for $i = 1, \dots, m$, with at least one of these inequalities being strict. As a central result, we prove that for every coalition S, there exists a deviation measure $\mathcal{D}_S$ such that $Y = (Y_1, \dots, Y_m)$ is a Pareto-optimal division if and only if cooperative portfolio payoff $X_S = \sum_{i=1}^m Y_i$ is a solution to the portfolio selection problem with $\mathcal{D}_S$:

$$\max_{X \in \mathcal{F}} EX \quad \text{subject to} \quad \mathcal{D}_S(X) \le 1. \tag{1.2}$$

In this context, $\mathcal{D}_S$ can be considered as the deviation measure of the coalition S. Given investors' deviation measures $\mathcal{D}_1$, ..., $\mathcal{D}_m$, we derive an explicit formula for $\mathcal{D}_S$ and then solve (1.2) in the framework of the portfolio theory with general deviation measures (Rockafellar, Uryasev, and Zabarankin, 2006c,b, 2007). Finally, in the game theoretic setting, we identify a solution from a set of Pareto-optimal solutions and prove that this solution belongs to the core of a corresponding cooperative game.

This work is organized into five sections. Section 2 formulates a portfolio selection problem for a cooperative game with general deviation measures. Section 3 reduces finding Pareto-optimal solutions of this problem to a portfolio problem with a single deviation measure $\mathcal{D}_S$ and studies properties of $\mathcal{D}_S$. Section 4 solves the cooperative portfolio problem and indicates which solution should be chosen from the Pareto-optimal set. Section 5 concludes the work.

2 Problem Formulation

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability space, where Ω denotes the designated space of future states ω, $\mathcal{M}$ is a field of sets in Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{M})$. For the purposes of this work, a random variable (r.v.) will be an element of $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, \mathbb{P})$. $F_X(x)$ and $q_X(\alpha) = \inf\{x | F_X(x) > \alpha\}$ will denote the cumulative distribution function (CDF) and quantile function of an r.v. X, respectively. Also, C will denote a constant in the real numbers. The relations between r.v.'s are understood to hold in the almost sure sense, e.g., we write $X = Y$ if $\mathbb{P}[X = Y] = 1$ and $X \ge Y$ if $\mathbb{P}[X \ge Y] = 1$. Throughout the paper, we assume that the probability space Ω is *atomless*, i.e., there exists an r.v. with a continuous CDF. This implies existence of r.v.'s on Ω with all possible CDFs (see e.g. Föllmer and Schied, 2004).

2.1 Deviation Measures

This section introduces law-invariant deviation measures.

Definition 2.1 *(general deviation measures). A* deviation measure *is any functional* $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0; \infty]$ *satisfying*[3]

(D1) $\mathcal{D}(X) = 0$ *for constant* X*, but* $\mathcal{D}(X) > 0$ *otherwise* (nonnegativity),

(D2) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ *for all* X *and all* $\lambda > 0$ (positive homogeneity),

(D3) $\mathcal{D}(X + Y) \le \mathcal{D}(X) + \mathcal{D}(Y)$ *for all* X *and* Y (subadditivity),

(D4) set $\{X \in \mathcal{L}^2(\Omega) \big| \mathcal{D}(X) \le C\}$ *is closed for all* $C < \infty$ (lower semicontinuity).

[3] In Rockafellar, Uryasev, and Zabarankin (2006a, 2002), axiom D4 was not included in the definition. Deviation measures satisfying D4 were called *lower semicontinuous* deviation measures.

As shown by Rockafellar, Uryasev, and Zabarankin (2002), axioms D1–D3 imply that

$$\mathcal{D}(X+C)=\mathcal{D}(X) \text{ for all constants } C \text{ (\textit{constant translation invariance})}. \tag{2.1}$$

In addition, a deviation measure $\mathcal{D}$ is lower range dominated if it satisfies

(D5) $\mathcal{D}(X) \leq EX - \inf X$ *for all* X (lower range dominance).

In general, for two r.v.'s with the same CDF, a deviation measure may assume different values. In this work, we consider only law-invariant deviation measures (Rockafellar, Uryasev, and Zabarankin, 2006a), i.e., those which depend only on the CDF of an r.v.

Definition 2.2 *(law-invariant deviation measures). A deviation measure $\mathcal{D}(X)$ is called* law-invariant*, if $\mathcal{D}(X_1)=\mathcal{D}(X_2)$ for any two r.v.'s X_1 and X_2 yielding the same CDF on $(-\infty;\infty)$.*

Well-known examples of deviation measures include (Rockafellar, Uryasev, and Zabarankin, 2006a, 2002)

(i) deviation measures of $\mathcal{L}^p$ type $\mathcal{D}(X)=\|X-EX\|_p$, $p\in[1,\infty]$, for example, the standard deviation $\sigma(X)=\|X-EX\|_2$ and mean absolute deviation $\mathrm{MAD}(X)=\|X-EX\|_1$, where $\|\cdot\|_p$ is the $\mathcal{L}^p$ norm;

(ii) deviation measures of semi-$\mathcal{L}^p$ type $\mathcal{D}_-(X)=\|[X-EX]_-\|_p$ and $\mathcal{D}_+(X)=\|[X-EX]_+\|_p$, $p\in[1,\infty]$, for example, *standard lower* and *upper semideviations*

$$\sigma_-(X)=\|[X-EX]_-\|_2, \qquad \sigma_+(X)=\|[X-EX]_+\|_2,$$

where $[X]_-=\max\{0,-X\}$ and $[X]_+=\max\{0,X\}$;

(iii) conditional value-at-risk (CVaR) deviation, defined for any $\alpha\in(0,1)$ by[4]

$$\mathrm{CVaR}^{\Delta}_{\alpha}(X)\equiv EX-\frac{1}{\alpha}\int_0^{\alpha} q_X(\beta)\,d\beta. \tag{2.2}$$

As generalizations of CVaR deviation, Rockafellar, Uryasev, and Zabarankin (2006a, 2002) introduced *mixed CVaR-deviation*

$$\mathcal{D}(X)=\int_0^1 \mathrm{CVaR}^{\Delta}_{\alpha}(X)\,d\lambda(\alpha), \qquad \int_0^1 d\lambda(\alpha)=1, \quad \lambda(\alpha)\geq 0,$$

and *worst-case mixed-CVaR deviation*

$$\mathcal{D}(X)=\sup_{\lambda\in\Lambda}\int_0^1 \mathrm{CVaR}^{\Delta}_{\alpha}(X)\,d\lambda(\alpha)$$

for some collection Λ of weighting nonnegative measures λ on $(0,1)$ with $\int_0^1 d\lambda(\alpha)=1$. In these two examples, the weighting measure $\lambda(\alpha)$ can be considered as agent's risk profile and is closely related to Yaari's dual utility function (Yaari, 1987).

All the above deviation measures are law invariant. See Rockafellar, Uryasev, and Zabarankin (2006a,c,b, 2007) for other examples.

[4] $\mathrm{CVaR}^{\Delta}_{1}(X)=-EX+EX=0$ is not a deviation measure, since it vanishes for all r.v.'s (not only for constants).

2.2 Portfolio Selection

Suppose a financial market has $n+1$ assets with the rates of return r_0, r_1, …, r_n. Asset 0 is assumed to be riskless, but the other assets are risky with nonconstant r_1, …, r_n, whose expected values are denoted by $\bar{r}_1$, …, $\bar{r}_n$. Suppose there are m investors, who can either invest in the market individually or form a cooperative portfolio. Investor i, $i = 1,\ldots,m$, has initial investment capital $C_i \geq 0$ and risk-reward preferences represented by a pair $(\mathcal{D}_i, \pi_i)$ with a deviation measure $\mathcal{D}_i$ and desired expected gain $\pi_i \geq 0$ over the guaranteed riskless revenue $C_i(1+r_0)$ (in other words, π_i is the premium for taking the risk).

We begin with a single-period portfolio selection problem for an *individual* investor. Investor i invests the capital C_i into a portfolio $(x_{i0}, x_{i1}, \ldots, x_{in})$, where x_{ij} is the amount of money invested into asset j, $j = 0,1,\ldots,n$, so that $C_i = \sum_{j=0}^{n} x_{ij}$. The portfolio gain X_i is then defined as the difference between the portfolio future value $\sum_{j=0}^{n}(1+r_j)x_{ij}$ and the risk free payoff $(1+r_0)C_i$, i.e., $X_i = \sum_{j=1}^{n}(r_j - r_0)x_{ij}$. Since borrowing and short selling are allowed, x_{i0} can be expressed from the budget constraint as $x_{i0} = C_i - \sum_{j=1}^{n} x_{ij}$ with no additional restrictions (x_{i0} can be negative or greater than C_i), and consequently, the budget constraint can be omitted from consideration. The portfolio selection problem is then bi-criteria optimization in which EX_i is desired to be maximized while $\mathcal{D}_i(X_i)$ is desired to be minimized. These conflicting objectives can be combined in a utility function $U_i(EX_i, \mathcal{D}_i(X_i))$ non-decreasing in the first argument and non-increasing in the second one, so that investor i solves

$$\max_{X_i \in \mathcal{F}} U_i(EX_i, \mathcal{D}_i(X_i)), \tag{2.3}$$

where $\mathcal{F} = \{X \,|\, X = \sum_{j=1}^{n}(r_j - r_0)x_{ij},\ (x_{i1},\ldots,x_{in}) \in \mathbb{R}^n\}$ is a feasible set of the portfolio gain.

The problem (2.3) has two special cases:

(i) minimization of the deviation measure with a constraint on the expected gain:

$$\min_{X_i \in \mathcal{F}} \mathcal{D}_i(X_i) \qquad \text{subject to} \quad EX_i \geq \pi_i, \tag{2.4}$$

which corresponds to (2.3) with U_i defined by $U_i(m,d) = -d$ for $m \geq \pi_i$ and $U_i(m,d) = -\infty$ otherwise; and

(ii) maximization of the expected gain with a constraint on the deviation (risk):

$$\max_{X_i \in \mathcal{F}} EX_i \qquad \text{subject to} \quad \mathcal{D}_i(X_i) \leq d_i, \tag{2.5}$$

which corresponds to (2.3) with U_i given by $U_i(m,d) = m$ for $d \leq d_i$ and $U_i(m,d) = -\infty$ otherwise.

As an alternative to individual investment, m investors can form a coalition S and buy a cooperative portfolio $(x_{S0}, x_{S1}, \ldots, x_{Sn})$, where x_{Sj} is the amount of money invested into asset j, $j = 0,1,\ldots,n$. The coalition's investment capital is $C_S = \sum_{i=1}^{m} C_i$, and thus, $\sum_{j=0}^{n} x_{Sj} = C_S$. The cooperative portfolio gain X_S is defined as the difference between the portfolio future value $\sum_{j=0}^{n}(1+r_j)x_{Sj}$ and the risk free capital gain $(1+r_0)C_S$, i.e., $X_S = \sum_{j=0}^{n}(r_j - r_0)x_{Sj}$.

Before discussing how to formulate a cooperative portfolio problem, we first explain the reason of why forming a coalition is preferred to individual investment. Suppose, at the present moment, all investors in the coalition agree to divide the cooperative portfolio gain into shares Y_i, $i = 1,\ldots,m$, so that $X_S = \sum_{i=1}^{m} Y_i$. Obviously, investor i joins the coalition only if Y_i is preferred to an optimal solution X_i^* of (2.3). In terms of the risk-reward preferences $(\mathcal{D}_i, \pi_i)$, this is guaranteed if $\mathcal{D}_i(Y_i) \leq \mathcal{D}_i(X_i^*)$ and $EY_i \geq EX_i^*$. We will write $Y_i \succ X_i^*$ if both $\mathcal{D}_i(Y_i) \leq \mathcal{D}_i(X_i^*)$ and $EY_i \geq EX_i^*$ hold with at least one inequality being strict. Consequently, m investors form the coalition S only if there is a division $Y = (Y_1,\ldots,Y_m)$ of the cooperative portfolio gain X_S such that

$$\textstyle\sum_{i=1}^{m} Y_i = X_S, \qquad Y_i \succ X_i^*, \qquad i = 1,\ldots,m. \tag{2.6}$$

At first glance, (2.6) seems never to hold: the coalition portfolio gain X_S is chosen from the same set $\mathcal{F}$, in which X_i^* is optimal for investor i. The point is that shares Y_i are no longer restricted by the feasible set of portfolio gains, i.e., $Y_i \notin \mathcal{F}$. An investor alone may not be able to form a desired portfolio, since although uncertain but not arbitrary returns are available for investment. In other words, the coalition allows the investors to have financial instruments that do not exist on the market. The following example illustrates this idea.

Example 2.1 *Suppose there are only two assets on the market: one is risk free with the rate of return r_0, and the other is risky with the rate of return r_0+r, where r is uniformly distributed on $[-1,2]$. Also, suppose there are two investors who solve (2.4) with the pairs $\mathcal{D}_1(X) = EX - \inf X$ and π_1 and $\mathcal{D}_2(X) = \sup X - \inf X$[5] and π_2, respectively. If*

$$\pi_1 < \pi_2 < 3\pi_1, \tag{2.7}$$

then forming a cooperative portfolio is strictly preferable for the both investors.

Detail. In the case of individual investment, let investor i invest x_i in the risky asset. Then $X_i = x_i r$ with $EX_i = x_i Er = x_i/2$ and $\mathcal{D}_1(X_1) = x_1 \mathcal{D}_1(r) = 3x_1/2$ and $\mathcal{D}_2(X_2) = x_2 \mathcal{D}_2(r) = 3x_2$. The problem (2.4) has the optimal solution $x_i^* = 2\pi_i$, for which $EX_i^* = \pi_i$ and $\mathcal{D}_1(X_1^*) = 3\pi_1$ and $\mathcal{D}_2(X_2^*) = 6\pi_2$.

In the case of forming a coalition, the cooperative portfolio gain is $X_S = 2(\pi_1+\pi_2)r$. The division (Y_1,Y_2) of X_S should satisfy (2.6). Let $Y_1 = 2(\pi_1+\pi_2)\max\{r, Er\} - 3\pi_1/4 - 7\pi_2/4$ and $Y_2 = 2(\pi_1+\pi_2)\min\{r, Er\} - \pi_1/4 + 3\pi_2/4$. Obviously, $Y_1 + Y_2 = X_S$ and $EY_i = \pi_i$, $i = 1,2$. Then $Y_i \succ X_i^*$, $i = 1,2$, reduce to $\mathcal{D}_1(Y_1) = 3(\pi_1+\pi_2)/4 < 3\pi_1$ and $\mathcal{D}_2(Y_2) = 3(\pi_1+\pi_2) < 6\pi_2$, which hold provided (2.7).

Observe that as financial instruments, the shares Y_1 and Y_2 do not exist on this market, and only by forming the coalition, the investors are able to have them. ☐

Example 2.2 *If, in Example 2.1, $\mathcal{D}_1(X) = \mathcal{D}_2(X) = \mathcal{D}(X)$ then forming a cooperative portfolio cannot be strictly preferable for the both investors.*

Detail. As in Example 2.1, the problem (2.4) has the optimal solution $X_i^* = 2\pi_i r$ with $\mathcal{D}(X_i^*) = 2\pi_i \mathcal{D}(r)$. Let (Y_1,Y_2) be a division of $X_S = 2(\pi_1+\pi_2)r$. If $\mathcal{D}(Y_i) < \mathcal{D}(X_i^*)$, $i = 1,2$, then $\mathcal{D}(Y_1) + \mathcal{D}(Y_2) < 2(\pi_1+\pi_2)\mathcal{D}(r) = \mathcal{D}(X_S) = \mathcal{D}(Y_1+Y_2)$ which contradicts D3. ☐

Example 2.1 demonstrates that the investors with different preferences can benefit from a cooperative investment by creating financial instruments not existing on the market, whereas Example 2.2 shows that this is not the case for the investors with the same preferences. This fact has a simple explanation: the investors may benefit from forming a coalition only if they find different aspects of the cooperative portfolio to be attractive, or in other words, if they have different preferences.

In Example 2.1, the suggested division (Y_1,Y_2) of the joint portfolio is strictly preferable to the individual investment but is not optimal. Constructing an optimal portfolio for a coalition and dividing the resulting portfolio among the investors optimally is the subject of this work.

In this setting, a portfolio selection problem for the coalition of m investors is m-criteria optimization

$$\max_{Y_i} (U_1(EY_1, \mathcal{D}_1(Y_1)), U_2(EY_2, \mathcal{D}_2(Y_2)), \ldots, U_m(EY_m, \mathcal{D}_m(Y_m))) \quad \text{subject to} \quad \textstyle\sum_{i=1}^m Y_i \in \mathcal{F}. \tag{2.8}$$

A vector $Y = (Y_1,\ldots,Y_m)$ is a feasible division if $\sum_{i=1}^m Y_i \in \mathcal{F}$. For two feasible divisions $Y = (Y_1,\ldots,Y_m)$ and $Y' = (Y_1',\ldots,Y_m')$, we write $Y \succeq Y'$ if $Y_i \succeq Y_i'$ for $i = 1,\ldots,m$, i.e., $\mathcal{D}(Y_i) \le \mathcal{D}(Y_i')$ and $EY_i \ge EY_i'$. Also, a division Y is strictly preferable to Y' ($Y \succ Y'$), if at least one of these inequalities is strict.

[5] $\sup X$ and $\inf X$ are understood to be essential supremum and essential infimum of X, respectively.

Definition 2.3 *A feasible division* $Y = (Y_1, \dots, Y_m)$ *is* Pareto optimal *if there is no feasible division* $Y' = (Y_1', \dots, Y_m')$ *such that* $Y' \succ Y$.

Further, we address the following problems:

1. Finding all Pareto-optimal divisions Y.
2. Formulating conditions for the existence of Pareto-optimal divisions Y which strictly dominate the individual investment.
3. Among all Pareto-optimal divisions Y, choosing a "fair" division.

The set of points $(y_1, \dots, y_m, d_1, \dots, d_m) \in \mathbb{R}^{2m}$ with $EY_i = y_i$ and $\mathcal{D}_i(Y_i) = d_i$, $i = 1, \dots, m$, for some Pareto-optimal $Y = (Y_1, \dots, Y_m)$, will be called *efficient frontier*. For $d = (d_1, \dots, d_m)$, where $d_i > 0$, $i = 1, \dots, m$, let $\mathcal{P}_d$ be the set of r.v.'s $X \in \mathcal{F}$ such that $X = \sum_{i=1}^m Y_i$ and $\mathcal{D}_i(Y_i) \le d_i, i = 1, \dots, m$ for some $Y = (Y_1, \dots, Y_m)$.

The next proposition characterizes all Pareto-optimal divisions.

Proposition 2.1 *Let* $d = (d_1, \dots, d_m)$*, where* $d_i > 0$*,* $i = 1, \dots, m$*. If* $Y = (Y_1, \dots, Y_m)$ *with* $\mathcal{D}_i(Y_i) = d_i$*,* $i = 1, \dots, m$*, is a Pareto-optimal division, then an r.v.* $X_S = \sum_{i=1}^m Y_i$ *solves the problem*

$$\max_{X \in \mathcal{P}_d} EX. \tag{2.9}$$

Moreover, if X_S *is not a solution to (2.9), then* $Z_i \succ Y_i$*,* $i = 1, \dots, m$*, for some feasible division* $Z = (Z_1, \dots, Z_m)$*.*

Proof Let X solve (2.9), and let $\sum_{i=1}^m EY_i = EX_S < EX$. Then $\sum_{i=1}^m y_i = EX$ and $EY_i < y_i$ for $y_i = EY_i \frac{EX}{EX_S}$, $i = 1, \dots, m$. Since $X \in \mathcal{P}_d$, there exists $Y^* = (Y_1^*, \dots, Y_m^*)$ such that $X = \sum_{i=1}^m Y_i^*$ and $\mathcal{D}_i(Y_i^*) \le d_i$, $i = 1, \dots, m$. Then $X = \sum_{i=1}^m Z_i$, $\mathcal{D}_i(Z_i) = \mathcal{D}_i(Y_i^*) \le d_i$ and $EZ_i = y_i$, $i = 1, \dots, m$, for $Z = (Z_1, \dots, Z_m)$ with $Z_i = Y_i^* - EY_i^* + y_i$, $i = 1, \dots, m$, and thus, $Z_i \succ Y_i$ for all i. □

Proposition 2.1 reduces finding Pareto-optimal divisions to one-parameter optimization problem (2.9). Also, it provides a sufficient condition for the investors to participate in the coalition.

Proposition 2.2 *Let* $X^* = (X_1^*, \dots, X_m^*)$*, where* X_i^* *are solutions to* (2.4)*, and let* $d = (d_1, \dots, d_m)$ *with* $d_i = \mathcal{D}_i(X_i^*)$*,* $i = 1, \dots, m$*. If* $X_S = \sum_{i=1}^m X_i^*$ *is not a solution to (2.9), then* X^* *is not a Pareto-optimal division, and moreover,* $Z_i \succ X_i^*$*,* $i = 1, \dots, m$*, for some feasible division* $Z = (Z_1, \dots, Z_m)$*.*

Proof Proof follows from Proposition 2.1 with $Y = X^*$. □

The next sections develop an approach to solving (2.9) and also prove that the converse to Proposition 2.1 also holds, i.e., if for a feasible division $Y = (Y_1, \dots, Y_m)$ with $\mathcal{D}_i(Y_i) = d_i$, $i = 1, \dots, m$, the r.v. $X_S = \sum_{i=1}^m Y_i$ solves (2.9), then Y is a Pareto-optimal division.

3 Deviation Measure of Coalition

This section characterizes Pareto-optimal divisions. It shows that (2.9) can be reformulated as a portfolio optimization problem with some deviation measure $\mathcal{D}_S$, which will be referred to as the deviation measure of the coalition S.

3.1 Definition

Throughout this section, the vector $d = (d_1, \dots, d_m)$ is assumed to be fixed. Moreover, without loss of generality, we can assume $d_1 = \dots = d_m = 1$, since otherwise the deviation measure $\mathcal{D}_i(X)$ of investor i can be replaced by $\mathcal{D}_i'(X) = \mathcal{D}_i(X)/d_i$.

Let $\mathcal{H} = \{Y = (Y_1, \dots, Y_m), Y_i \in \mathcal{L}^2(\Omega)\}$ be a Hilbert space with the inner product $\langle Y, Z\rangle = \sum_{i=1}^m E[Y_i Z_i]$, and let $\mathcal{A}(X) = \{Y \in \mathcal{H} : X = \sum_{i=1}^m Y_i\}$ be the set of divisions of the uncertain payoff X among investors $1, 2, \dots, m$. We proceed with the following definition.

Definition 3.1 *Let risk preferences of investor i be expressed by a law-invariant deviation measure $\mathcal{D}_i$, and let m investors form a coalition S. Then a functional defined by*

$$\mathcal{D}_S(X) = \inf_{Y \in \mathcal{A}(X)} \max\{\mathcal{D}_1(Y_1), \dots, \mathcal{D}_m(Y_m)\} \tag{3.1}$$

is called a deviation measure of the coalition *S*.

We need to show that (3.1) is indeed a deviation measure and that (2.9) is equivalent to the portfolio optimization problem (1.2) with $\mathcal{D}_S$.

An r.v. X is said to dominate Y with respect to convex ordering, or $X \succcurlyeq_{cx} Y$, if $EX = EY$ and $\int_{-\infty}^x F_X(t)dt \geq \int_{-\infty}^x F_Y(t)dt$ for all $x \in \mathbb{R}$ (Dana, 2005). Theorem 4.1 in Dana (2005) implies that on an atomless probability space a deviation measure is law-invariant if and only if it is consistent with convex ordering, i.e., $X \succcurlyeq_{cx} Y$ implies $\mathcal{D}(X) \geq \mathcal{D}(Y)$.

First we establish that the set of all divisions in (3.1) can be reduced to the set with only comonotone divisions. A division $Y \in \mathcal{A}(X)$ is called *comonotone*, if Y_i and Y_j are comonotone[6] for all $1 \leq i < j \leq m$. Let $\mathcal{C}(X)$ be the set of all comonotone divisions from $\mathcal{A}(X)$.

Proposition 3.1 *For any law-invariant deviation measures $\mathcal{D}_1$, …, $\mathcal{D}_m$, $\mathcal{D}_S$ takes the form*

$$\mathcal{D}_S(X) = \inf_{Y \in \mathcal{C}(X)} \max\{\mathcal{D}_1(Y_1), \dots, \mathcal{D}_m(Y_m)\}. \tag{3.2}$$

Proof The proof follows from consistency of $\mathcal{D}_i$ with convex ordering and from the fact that for any division $Y \in \mathcal{A}(X)$, there exists a comonotone division $Y' \in \mathcal{C}(X)$ such that $Y_i \succcurlyeq_{cx} Y_i'$, $i = 1, \dots, m$. The latter fact was first established for a finite probability space by Landsberger and Meilijson (1994), and then it was proved for a general case by Ludkovskia and Ruschendorf (2008). □

Next we prove that the infimums in (3.1) and (3.2) are attained.

Proposition 3.2 *Let $\mathcal{D}_1, \dots, \mathcal{D}_m$ be law-invariant deviation measures. Then for some $Y^* \in \mathcal{C}(X)$, $\mathcal{D}_S(X)$ can be represented by*

$$\mathcal{D}_S(X) = \max\{\mathcal{D}_1(Y_1^*), \dots, \mathcal{D}_m(Y_m^*)\} \tag{3.3}$$

Proof Property (2.1) implies that without loss of generality, we can put $EX = EY_1 = \dots = EY_m = 0$ in (3.2). Since $X \succcurlyeq_{cx} Y_i$ for every $Y_i \in \mathcal{C}(X)$, $i = 1, \dots, m$, $\sigma(Y_i) \leq \sigma(X)$ or, equivalently, $\|Y_i\|_2 \leq \|X\|_2$. Thus, the set $\mathcal{C}'(X) = \{Y \in \mathcal{C}(X) : EY_1 = \dots = EY_m = 0\}$ is convex, closed (and hence is weakly closed), and bounded in the Hilbert space. Consequently, it is weakly compact. Since all deviation measures $\mathcal{D}_i(Y_i)$ are convex and lower semicontinuous, the functional $\max\{\mathcal{D}_1(Y_1), \dots, \mathcal{D}_m(Y_m)\}$ is also convex and lower semicontinuous.

[6] Two r.v.'s $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are said to be *comonotone*, if there exists a set $A \subseteq \Omega$ such that $P[A] = 1$ and $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in A$.

Consequently, this functional is weakly lower semicontinuous and attains its infimum on the weakly compact set $\mathcal{C}'(X)$ at some $Y^* \in \mathcal{C}'(X) \subset \mathcal{C}(X)$. □

Observe that for arbitrary constants C_i, $i = 1, \ldots, m$, with $\sum_{i=1}^m C_i = 0$, $(Y_1^* + C_1, \ldots, Y_m^* + C_m)$ is also a solution to (3.3). This suggests that through (3.3), $(Y_1^*, \ldots, Y_m^*)$ represents "risk" sharing of X.

Definition 3.2 *A vector $Y^* \in \mathcal{C}(X)$ satisfying (3.3) will be called* optimal risk sharing *of X among investors with deviation measures $\mathcal{D}_1, \ldots, \mathcal{D}_m$.*

Solving the cooperative game problem can be decomposed into three steps: (i) finding optimal cooperative portfolio with the gain X; (ii) finding optimal risk sharing Y of uncertain $X - EX$; and (iii) determining constants C_i, $i = 1, \ldots, m$, with $\sum_{i=1}^m C_i = EX$, which will be called "fair" division of the cooperative portfolio expected gain. As a solution of the game, investor i is offered $Y_i - EY_i + C_i$.

Pairwise comonotonicity of the components of an optimal risk sharing Y^* implies that $Y_i = f_i(X)$, $i = 1, \ldots, m$ for some functions $f_1, \ldots, f_m$.

The next proposition proves that (3.1) is a deviation measure.

Proposition 3.3 *For arbitrary law-invariant deviation measures $\mathcal{D}_1, \ldots, \mathcal{D}_m$, the functional $\mathcal{D}_S(X)$, defined by (3.1), is a law-invariant deviation measure.*

Proof See Appendix A. □

Now the following result is straightforward.

Proposition 3.4 *The problem (2.9) with $d_1 = \ldots = d_m = 1$ is equivalent to the portfolio selection problem* (1.2) *with $\mathcal{D}_S$.*

Proof By definition, $X \in \mathcal{P}_d$ if and only if $X \in \mathcal{F}$, and there exists $Y \in \mathcal{A}(X)$ such that $\mathcal{D}_i(Y_i) \leq 1$, $i = 1, \ldots, m$. Definition 3.1 and Proposition 3.2 imply that the last condition is equivalent to $\mathcal{D}_S(X) \leq 1$. Consequently, the feasible solution sets of the problems (2.9) and (1.2) coincide. □

Proposition 3.4 shows that the coalition S can be viewed as a single investor with preferences, represented by the deviation measure $\mathcal{D}_S$. The following proposition establishes that in Definition 3.1, the investors can be replaced by coalitions of investors.

Proposition 3.5 *Let $S = \{1, 2, \ldots, m\}$ be a coalition of m investors with law-invariant deviation measures $\mathcal{D}_1, \ldots, \mathcal{D}_m$, respectively, and let S be partitioned into k nonempty disjoint subcoalitions $S_1, S_2, \ldots, S_k$, with deviation measures $\mathcal{D}_S, \mathcal{D}_{S_1}, \ldots, \mathcal{D}_{S_k}$, each of which defined by* (3.1). *Then*

$$\mathcal{D}_S(X) = \inf_{Y=(Y_1,\ldots,Y_k):\ \sum_{i=1}^k Y_i = X} \max\{\mathcal{D}_{S_1}(Y_1), \ldots, \mathcal{D}_{S_k}(Y_k)\}. \tag{3.4}$$

Proof Let $Y = (Y_1, \ldots, Y_k)$ and $Z = (Z_1, \ldots, Z_m)$, then

$$\begin{aligned}\mathcal{D}_S(X) &= \inf_{Z:\ \sum_{i=1}^m Z_i = X} \left(\max_{i \in \{1,\ldots,m\}} \mathcal{D}_i(Z_i) \right) = \inf_{Y: \sum_{i=1}^k Y_i = X} \left(\inf_{Z:\ \sum_{i \in S_j} Z_i = Y_j \forall j} \left(\max_{j \in \{1,\ldots,k\}} \left\{ \max_{i \in S_j} \mathcal{D}_i(Z_i) \right\} \right) \right) \\ &= \inf_{Y:\ \sum_{i=1}^k Y_i = X} \left(\max_{j \in \{1,\ldots,k\}} \left\{ \inf_{Z:\ \sum_{i \in S_j} Z_i = Y_j} \left(\max_{i \in S_j} \mathcal{D}_i(Z_i) \right) \right\} \right) = \inf_{Y:\ \sum_{i=1}^k Y_i = X} \left(\max_{j \in \{1,\ldots,k\}} \{\mathcal{D}_{S_j}(Y_j)\} \right).\end{aligned}$$

□

Thus, finding Pareto-optimal divisions in (2.8) is reduced to the portfolio selection problem (1.2) for a single imaginary investor, whose preferences are represented by the deviation measure $\mathcal{D}_S$. Given $\mathcal{D}_S$, (1.2) can be solved by the approach developed by Rockafellar, Uryasev, and Zabarankin (2006c).

To characterize Pareto-optimal solutions in (2.8), it is left to address the following issues:

1. Given $\mathcal{D}_1, \ldots, \mathcal{D}_m$, evaluate $\mathcal{D}_S$;
2. Given an optimal solution to (1.2), obtain a Pareto-optimal division $Y = (Y_1, \ldots, Y_m)$ in an explicit form.

These issues are the subject of next sections.

3.2 Comonotone Deviation Measures

This section derives explicit formulas for an optimal division $Y = (Y_1, Y_2, \ldots, Y_m)$ and the deviation measure $\mathcal{D}_S$ when $\mathcal{D}_i$, $i = 1, \ldots, m$, are comonotone. A functional $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0;\infty]$ is called *comonotone*, if $\mathcal{D}(X+Y) = \mathcal{D}(X) + \mathcal{D}(Y)$ for comonotone r.v.'s X and Y. A deviation measure $\mathcal{D}$ is called *proper*, if $\mathcal{D}(X) < \infty$ for some r.v. $X \neq C$.

The next two representations for law invariant deviation measures were obtained by Grechuk, Molyboha, and Zabarankin (2009).

A functional $\mathcal{D}(X)$ is a law-invariant deviation measure if and only if

$$\mathcal{D}(X) = \sup_{g(\alpha)\in G} \int_0^1 g(\alpha) \cdot d(q_X(\alpha)), \tag{3.5}$$

where G is a collection of positive concave functions $g : (0,1) \to (0,\infty)$. This collection is called *g-envelope* of a deviation measure $\mathcal{D}(X)$; and a functional $\mathcal{D}(X)$ is a proper comonotone law-invariant deviation measure if and only if

$$\mathcal{D}(X) = \int_0^1 g(\alpha) \cdot d(q_X(\alpha)), \tag{3.6}$$

for some positive concave function $g : (0,1) \to (0,\infty)$.

In (3.5) and (3.6), the integrals are evaluated over a nonnegative measure $\mu_X = d(q_X(\alpha))$ on $(0,1)$, which may be infinite for unbounded r.v.'s. For any measurable set $T \subset (0,1)$, $\mu_X(T)$ will denote the measure of T.

Proposition 3.6 *Let S be a coalition of m investors with comonotone law-invariant deviation measures $\mathcal{D}_i$, $i = 1, \ldots, m$, and let $Y = (Y_1, \ldots, Y_m) \in \mathcal{C}(X)$ be an optimal risk sharing of $X \in \mathcal{L}^2(\Omega)$ with $\mathcal{D}_S(X) < \infty$. Then $\mathcal{D}_S(X) = \mathcal{D}_1(Y_1) = \ldots = \mathcal{D}_m(Y_m)$.*

Proof We assume that $X \neq C$, otherwise the statement is trivial. For any optimal risk sharing Y of X, let $I_Y \subset \{1,2,\ldots,m\}$ be the set of indices i such that $\mathcal{D}_S(X) = \mathcal{D}_i(Y_i)$. Let Y be such that $|I_Y|$ is minimal, then we need to show that $|I_Y| = m$. By Proposition 3.2, $|I_Y| \geq 1$. Suppose $|I_Y| < m$. Then $\mathcal{D}_S(X) = \mathcal{D}_i(Y_i) > \mathcal{D}_j(Y_j)$ for some $i, j \in \{1,2,\ldots,m\}$. Let (Z_1, Z_2) be an arbitrary pair of comonotone r.v.'s such that Z_1 is bounded and nonconstant, and $Z_1 + Z_2 = Y_i$. Then (3.6) implies that $\mathcal{D}_j(Z_1) < \infty$, and thus, $\mathcal{D}_j(Y_j + \varepsilon Z_1) \leq \mathcal{D}_j(Y_j) + \varepsilon \mathcal{D}_j(Z_1) < \mathcal{D}_S(X)$ for some $\varepsilon > 0$. On the other hand, $\mathcal{D}_i((1-\varepsilon)Z_1 + Z_2) < \mathcal{D}_i(Z_1) + \mathcal{D}_i(Z_2) = \mathcal{D}_i(Y_i) = \mathcal{D}_S(X)$. Since $(Y_j + \varepsilon Z_1) + ((1-\varepsilon)Z_1 + Z_2) = Y_i + Y_j$, this contradicts the minimality of $|I_Y|$. □

Let Λ be the set of $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_i \in (0,1)$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \lambda_i = 1$. The next proposition characterizes optimal risk sharing Y for an arbitrary X and provides an explicit formula for the g-envelope of $\mathcal{D}_S$.

Proposition 3.7 *Let S be a coalition of m investors with proper comonotone law-invariant deviation measures $\mathcal{D}_i$, $i = 1, \ldots, m$, which can be represented by (3.6) with g_i, $i = 1, \ldots, m$, respectively, and let $g_\lambda(\alpha) = \min_i \lambda_i g_i(\alpha)$ for $\lambda \in \Lambda$.*

(a) $Y \in \mathcal{C}(X)$ is an optimal risk sharing of $X \in \mathcal{L}^2(\Omega)$ with $\mathcal{D}_S(X) < \infty$ if and only if the following conditions hold

(i) $\mathcal{D}_1(Y_1) = \mathcal{D}_2(Y_2) = \ldots = \mathcal{D}_m(Y_m)$,

(ii) $\mu_{Y_i}(\{\alpha : \lambda_i g_i(\alpha) > g_\lambda(\alpha)\}) = 0$, $i = 1, \ldots, m$, *for some* $\lambda \in \Lambda$.

(b) $\mathcal{D}_S$ *can be represented in the form (3.5) with the g-envelope* $G = \{g_\lambda(\alpha) | \lambda \in \Lambda\}$.

Proof See Appendix B. □

Let us illustrate Proposition 3.7 for $\mathcal{D}_i(X) = \mathrm{CVaR}^\Delta_{\beta_i}(X)$.

Example 3.1 *Let S be a coalition of two investors with* $\mathcal{D}_1(X) = \mathrm{CVaR}^\Delta_{\beta_1}(X)$ *and* $\mathcal{D}_2(X) = \mathrm{CVaR}^\Delta_{\beta_2}(X)$, *where* $\beta_1 < \beta_2$. *Then*

(a) For an r.v. $X \in \mathcal{L}^2(\Omega)$ *with* $EX = 0$, *the infimum in (3.1) is attained at a comonotone* $Y = (Y_1, Y_2)$ *with* $Y_1 = [X - a]_+$, $Y_2 = -[X - a]_- + a$ *for some constant a.*

(b)

(3.7)
$$\mathcal{D}_S(X) = \sup_{\beta \in [\beta_1, \beta_2]} \left(\frac{\beta(1 - \beta_2)}{\beta - 2\beta\beta_2 + \beta_2} \mathrm{CVaR}^\Delta_\beta(X) \right).$$

Detail. (a) Obviously, there is a constant a such that $\mathrm{CVaR}^\Delta_{\beta_1}(Y_1) = \mathrm{CVaR}^\Delta_{\beta_2}(Y_2)$ for Y_1 and Y_2 defined above. For $\alpha_0 = F_X(a)$, $q_{Y_1}(\alpha)$ and $q_{Y_2}(\alpha)$ are constant on $(0, \alpha_0)$ and $(\alpha_0, 1)$, respectively. $\mathrm{CVaR}^\Delta_{\beta_i}(X)$, $i = 1, 2$, can be represented in the form (3.6) with

(3.8)
$$g_i(\alpha) = \begin{cases} (\frac{1}{\beta_i} - 1)\alpha & \alpha \le \beta_i, \\ 1 - \alpha & \alpha \ge \beta_i \end{cases}$$

(see Grechuk, Molyboha, and Zabarankin, 2009). In this case, the function $g_1(\alpha)/g_2(\alpha)$ is nonincreasing, and thus, there exists $\lambda > 0$ such that the sets $\{\alpha : g_1(\alpha)/g_2(\alpha) > \lambda\}$ and $\{\alpha : g_1(\alpha)/g_2(\alpha) < \lambda\}$ reduce to $\{\alpha : \alpha < \alpha_0\}$ and $\{\alpha : \alpha > \alpha_0\}$. It follows from Proposition 3.7(a) that the infimum in (3.1) for $Y = (Y_1, Y_2)$ is attained.

(b) For $g_1(\alpha)$ and $g_2(\alpha)$ determined by (3.8), and for any $\lambda_0 \le \lambda \le 1/2$, where $\lambda_0 = \frac{1-\beta_2}{\beta_2/\beta_1 - 2\beta_2 + 1}$, the function $g_\lambda(\alpha) = \min\{\lambda g_1(\alpha), (1-\lambda) g_2(\alpha)\}$ takes the form

$$g_\lambda(\alpha) = \begin{cases} (1-\lambda)(\frac{1}{\beta_2} - 1)\alpha & \text{when} \quad \alpha \le \frac{\lambda}{(1-\lambda)/\beta_2 + 2\lambda - 1}, \\ \lambda(1 - \alpha) & \text{when} \quad \alpha \ge \frac{\lambda}{(1-\lambda)/\beta_2 + 2\lambda - 1}, \end{cases}$$

which corresponds to $\lambda \mathrm{CVaR}^\Delta_\beta$ with $\beta = \frac{\lambda}{(1-\lambda)/\beta_2 + 2\lambda - 1}$ through (3.6). If $\lambda > 1/2$ then $g_\lambda(\alpha) = (1-\lambda) g_2(\alpha) \le g_{1/2}(\alpha)$ for all α, and if $\lambda < \lambda_0$ then $g_\lambda(\alpha) = \lambda g_1(\alpha) \le g_{\lambda_0}(\alpha)$ for all α, and consequently, both these cases can be excluded from G, and (3.7) follows. □

For the coalition S of two investors with the deviation measures $\mathcal{D}_1(X) = EX - \inf X = \lim_{\beta \to 0} \mathrm{CVaR}^\Delta_\beta(X)$ and $\mathcal{D}_2(X) = \mathrm{CVaR}^\Delta_{\beta_0}(X)$, (3.7) implies

(3.9)
$$\mathcal{D}_S(X) = \sup_{\beta \in (0, \beta_0)} \left(\frac{\beta(1 - \beta_0)}{\beta - 2\beta\beta_0 + \beta_0} \mathrm{CVaR}^\Delta_\beta(X) \right).$$

Statement (a) of Example 3.1 holds true for any pair of comonotone deviation measures $\mathcal{D}_1$ and $\mathcal{D}_2$ such that the ratio $g_1(\alpha)/g_2(\alpha)$ for $g_1(\alpha)$ and $g_2(\alpha)$, corresponding to $\mathcal{D}_1$ and $\mathcal{D}_2$ through (3.6), is nonincreasing. In particular, it holds for the pair $\mathcal{D}_1(X) = EX - \inf X$ and $\mathcal{D}_2(X) = \sup X - EX$. The next example is considered from the mathematical point of view only, since $\mathcal{D}(X) = \sup X - EX$ represents reward preferences rather than risk preferences.

Example 3.2 *Let S be a coalition of two investors with $\mathcal{D}_1(X) = EX - \inf X$ and $\mathcal{D}_2(X) = \sup X - EX$. For an r.v. $X \in \mathcal{L}^2(\Omega)$ with $EX = 0$, the infimum in (3.1) is attained for a comonotone $Y = (Y_1, Y_2)$ with $Y_1 = [X - a]_+$, $Y_2 = -[X - a]_- + a$ and some constant a. In this case,*

$$\mathcal{D}_S(X) = \frac{1}{2}\mathrm{MAD}(X). \tag{3.10}$$

Detail. Through (3.6), the functions $g_1(\alpha) = 1 - \alpha$ and $g_2(\alpha) = \alpha$ correspond to $\mathcal{D}_1(X) = EX - \inf X$ and $\mathcal{D}_2(X) = \sup X - EX$, respectively, and the ratio $g_1(\alpha)/g_2(\alpha) = (1 - \alpha)/\alpha$ is a nonincreasing function. Proposition 3.7(b) implies that $\mathcal{D}_S$ can be represented by (3.5) with the g-envelope $G = \{g_\lambda | \lambda \in \Lambda\}$, where $g_\lambda(\alpha) = \min\{\lambda(1-\alpha), (1-\lambda)\alpha\}$. Example 2 in Grechuk, Molyboha, and Zabarankin (2009) provides the formula for the g-envelope of $\frac{1}{2}\mathrm{MAD}(X)$:

$$G_{\mathrm{MAD}} = \left\{ g(\alpha) \in \mathcal{G} \,\middle|\, g(0+) = g(1-) = 0,\ g'(0+) - g'(1-) \leq 1 \right\}. \tag{3.11}$$

Thus, $G \subset G_{\mathrm{MAD}}$, but $g(\alpha) \leq g_\lambda(\alpha)$ for every $g \in G_{\mathrm{MAD}}$ and some λ. Consequently, G and G_{MAD} are the g-envelopes of the same deviation measure, which proves (3.10). □

Jouini, Schachermayer, and Touzi (2008) obtained similar results for optimal risk sharing with utility functions and with coherent risk measures.

Next sections generalize Proposition 3.7 to the case of non-comonotone finite deviation measures.

3.3 Characterization of Optimal Risk Sharing

This section characterizes optimal risk sharing, i.e., for given $X \in \mathcal{L}^2(\Omega)$, it finds a minimizer $Y \in \mathcal{C}(X)$ in (3.1). From this moment, $\mathcal{D}_i$, $i = 1, \ldots, m$, are assumed to be law invariant and *finite*, i.e., $\mathcal{D}_i(X) < \infty$ for all X. The following result is due to Rockafellar, Uryasev, and Zabarankin (2006a).

Proposition 3.8 *A deviation measure $\mathcal{D}$ is finite if and only if it is continuous everywhere on $\mathcal{L}^2(\Omega)$.*

Proof The "only if" part follows from Proposition 2 in Rockafellar, Uryasev, and Zabarankin (2006a), and the "if" part follows from D2. Indeed, if $\mathcal{D}(X) = \infty$ for some X, then $\mathcal{D}(\lambda X) = \infty$ for every $\lambda > 0$, which contradicts the continuity of $\mathcal{D}$ at 0. □

The necessary and sufficient optimality conditions for (1.2) with a *continuous* deviation measure were established by Rockafellar, Uryasev, and Zabarankin (2006c).

Proposition 3.9 *If at least one of $\mathcal{D}_1, \ldots, \mathcal{D}_m$ is continuous, then $\mathcal{D}_S$ is continuous.*

Proof Let $\mathcal{D}_1$ be continuous on $\mathcal{L}^2(\Omega)$. By Proposition 3.8, $\mathcal{D}_1$ is finite. Since $X = \underbrace{X + 0 + \ldots + 0}_{m}$ for every X, (3.1) implies that $\mathcal{D}_S(X) \leq \max\{\mathcal{D}_1(X), \mathcal{D}_2(0) \ldots, \mathcal{D}_m(0)\} < \infty$, whence $\mathcal{D}_S$ is finite and, therefore, is continuous. □

Example 3.2 shows that the converse of Proposition 3.9 is not true. Indeed, $\mathcal{D}_S(X) = \frac{1}{2}\mathrm{MAD}(X)$ is continuous (because it is finite), while both $\mathcal{D}_1(X) = EX - \inf X$ and $\mathcal{D}_2(X) = \sup X - EX$ are discontinuous.

A *subgradient* of $\mathcal{D}$ at X is any $Q \in \mathcal{L}^2(\Omega)$ such that

$$\mathcal{D}(X') \geq \mathcal{D}(X) + E[(X' - X)Q] \quad \text{for all} \quad X', \tag{3.12}$$

and the *subdifferential* $\partial\mathcal{D}(X)$ of $\mathcal{D}$ is the set of all subgradients (see Rockafellar, Uryasev, and Zabarankin, 2006c). The following proposition characterizes optimal risk sharing $Y = (Y_1, \ldots, Y_m)$ in terms of subdifferentials of $\mathcal{D}_i$, $i = 1, \ldots, m$.

Proposition 3.10 *Let $\mathcal{D}_i$, $i = 1,\dots,m$, be continuous law-invariant deviation measures. Then $Y \in \mathcal{C}(X)$ is an optimal risk sharing of nonconstant $X \in \mathcal{L}^2(\Omega)$ if and only if the following conditions hold*

(i) $\mathcal{D}_1(Y_1) = \ldots = \mathcal{D}_m(Y_m)$,

(ii) $\lambda_i Z_0 \in \partial\mathcal{D}_i(Y_i)$, $\lambda_i > 0$, $i = 1,\dots,m$, *for some r.v.* Z_0.

Proof See Appendix C. □

For the well-known deviation measures, subdifferential formulas can be found in Rockafellar, Uryasev, and Zabarankin (2006c). Together with Proposition 3.10, they can be used to determine an optimal risk sharing $Y = (Y_1,\dots,Y_m) \in \mathcal{C}(X)$ and $\mathcal{D}_S(X)$ for nonconstant $X \in \mathcal{L}^2(\Omega)$ and $\mathcal{D}_i$, $i = 1,\dots,m$. Namely, Proposition 3.10 implies that Y can be found from the following system of equations

$$\begin{cases} Z_i \in \partial\mathcal{D}_i(Y_i),\ i = 1,\dots,m \\ Z_i = \lambda_i Z_m,\ i = 1,\dots,m-1 \\ \mathcal{D}_i(Y_i) = \mathcal{D}_m(Y_m),\ i = 1,\dots,m-1 \\ Y_1 + \ldots + Y_m = X. \end{cases} \tag{3.13}$$

The next example demonstrates this approach.

Example 3.3 *Let S be a coalition of two investors with $\mathcal{D}_1(X) = \sigma(X)$ and $\mathcal{D}_2(X) = \sigma_-(X)$. Then for every r.v. X with $EX = 0$, the infimum in (3.1) is attained at $Y = (Y_1,Y_2)$ with $Y_1 = X - Y_2$ and*

$$Y_2 = \begin{cases} X - \lambda C & X - \lambda C > 0, \\ \frac{X-\lambda C}{\lambda+1} & X - \lambda C \le 0, \end{cases} \tag{3.14}$$

where the constants λ and C are found from the conditions $\sigma(Y_1) = \sigma_-(Y_2)$ and $C = E[Y_2]_-$. Also, $\mathcal{D}_S(X) = \sigma(Y_1) = \sigma_-(Y_2)$.

Detail. Proposition 3.2 implies that the infimum in (3.1) is attained at some $Y = (Y_1,Y_2)$, and we can assume $EY_1 = EY_2 = 0$. It is known that $\partial\sigma(Y_1) = Y_1/\sigma(Y_1)$ and $\partial\sigma_-(Y_2) = (E[Y_2]_- - [Y_2]_-)/\sigma_-(Y_2)$ (see Rockafellar, Uryasev, and Zabarankin, 2006c). From (3.13), we have $Y_1/\sigma(Y_1) = \lambda(E[Y_2]_- - [Y_2]_-)/\sigma_-(Y_2)$ for some λ, and $\sigma(Y_1) = \sigma_-(Y_2)$, whence $Y_1 = \lambda(C - [Y_2]_-)$, where $C = E[Y_2]_-$. This together with $X = Y_1 + Y_2$ implies (3.14). □

The following example demonstrates that the continuity assumption in Proposition 3.10 cannot be omitted.

Example 3.4 *Let S be a coalition of two investors with law-invariant deviation measures $\mathcal{D}_1(X) = EX - \inf X$ and*

$$\mathcal{D}_2(X) = \sup_{\beta\in(0,1)} \left(\sqrt{\beta}\cdot \mathrm{CVaR}^{\Delta}_{\beta}(X)\right). \tag{3.15}$$

Then $\mathcal{D}_S(X) = 2$ for an r.v. X with the quantile function

$$q_X(\alpha) = \begin{cases} -\frac{1}{\sqrt{\alpha}} & 0 < \alpha < \frac{1}{4}, \\ \frac{4}{3} & \frac{1}{4} \le \alpha < 1, \end{cases} \tag{3.16}$$

and the infimum in (3.1) is attained at every $Y = (Y_1,Y_2) \in \mathcal{C}(X)$ with $EY_1 = 0$ and $|Y_1| \le 2$, in particular at $Y_1 = 0$, $Y_2 = X$. In this case, $\mathcal{D}_2(Y_2) > \mathcal{D}_1(Y_1)$, i.e. the statement of Proposition 3.10 does not hold.

Detail. Observe that $EX = 0$, and

$$\mathrm{CVaR}^{\Delta}_{\beta}(X) = -\frac{1}{\beta}\int_0^{\beta}\left(-\frac{1}{\sqrt{\alpha}}\right)d\alpha = \frac{2}{\sqrt{\beta}}, \qquad \beta \le \frac{1}{4},$$

and $\mathrm{CVaR}^{\Delta}_{\beta}(X) < 2/\sqrt{\beta}$ for $\beta > 1/4$. Thus, $\mathcal{D}_2(X) = 2$. Proposition 3.2 implies that the infimum in (3.1) is attained at some $Y^* = (Y_1^*, Y_2^*)$ with comonotone Y_1^* and Y_2^*, and we can assume $EY_1^* = EY_2^* = 0$. If $Y_1^* \ge -2$ a.s., then $\mathrm{CVaR}^{\Delta}_{\beta}(Y_2^*) \ge \mathrm{CVaR}^{\Delta}_{\beta}(X) - 2 = 2/\sqrt{\beta} - 2$, and thus, $\mathcal{D}_2(Y_2^*) \ge 2$, otherwise, $\mathcal{D}_1(Y_1^*) \ge 2$, and consequently, $\mathcal{D}_S(X) \ge 2$. On the other hand, $\mathcal{D}_1(Y_1) \le 2$ and $\mathcal{D}_2(Y_2) = 2$ for $Y = (Y_1, Y_2) \in \mathcal{C}(X)$ with $EY_1 = 0$ and $|Y_1| \le 2$, whence $\mathcal{D}_S(X) = 2$, and the infimum in (3.1) is attained at every such $Y = (Y_1, Y_2)$, in particular at $Y_1 = 0$, $Y_2 = X$. In this case, $\mathcal{D}_2(Y_2) = 2 > 0 = \mathcal{D}_1(Y_1)$, and the statement of Proposition 3.10 does not hold. □

3.4 General Formula for $\mathcal{D}_S$

This section develops a formula for $\mathcal{D}_S$. We first characterize the subdifferential set $\partial\mathcal{D}(Y)$.

Proposition 3.11 *Let $\mathcal{D}$ be a law-invariant deviation measure, and let $Y \in \mathcal{L}^2(\Omega)$ and $Q \in \partial\mathcal{D}(Y)$. Then for every $X \in \mathcal{L}^2(\Omega)$, the following inequality holds*

$$\mathcal{D}(X) \ge \int_0^1 q_X(\alpha) q_Q(\alpha) d\alpha = \int_0^1 g(\alpha) d(q_X(\alpha)), \quad \textit{where} \quad g(\alpha) = \int_0^{\alpha} (-q_Q(\beta)) d\beta, \tag{3.17}$$

and (3.17) *reduces to the equality for $X = Y$.*

Proof See Appendix D. □

Representation (3.5) establishes a correspondence between law-invariant deviation measures and g-envelopes $G \subset \mathcal{G}$, where $\mathcal{G}$ is the set of all positive concave functions $g : (0,1) \to (0,\infty)$. This correspondence is not a bijection, since (3.5) with different sets G can produce the same deviation measure. There is a one-to-one correspondence, however, between law-invariant deviation measures and maximal g-envelopes, defined as

$$G_M = \left\{ g(\alpha) \in \mathcal{G} \,\middle|\, \int_0^1 g(\alpha) \cdot d(q_X(\alpha)) \le \mathcal{D}(X)\ \forall X \in \mathcal{L}^2(\Omega) \right\}. \tag{3.18}$$

Maximality of G_M implies that $G \subseteq G_M$ for an arbitrary g-envelope G (Grechuk, Molyboha, and Zabarankin, 2009).

We say that $g_1 \in \mathcal{G}$ dominates $g_2 \in \mathcal{G}$ if $g_1(\alpha) \ge g_2(\alpha)$ for all α. A set $G \subset \mathcal{G}$ is called dominance-closed if $g_1 \in G$ implies $g_2 \in G$, whenever g_1 dominates g_2. A set $G \subset \mathcal{G}$ is the maximal g-envelope of some deviation measure if and only if it is convex, dominance-closed, and $G \cup \{0\}$ is closed with respect to pointwise convergence (Grechuk, Molyboha, and Zabarankin, 2009).

The next proposition provides a formula for the maximal g-envelope of $\mathcal{D}_S$ for the case $m = 2$.

Proposition 3.12 *Let S be a coalition of two investors with continuous law-invariant deviation measures $\mathcal{D}_1$ and $\mathcal{D}_2$ with the maximal g-envelopes G_1 and G_2, respectively. Then the maximal g-envelope of $\mathcal{D}_S$ is determined by*

$$G_S = \left\{ g(\alpha) = \min\{\lambda g_1(\alpha), (1-\lambda) g_2(\alpha)\} \,\middle|\, g_1 \in G_1,\ g_2 \in G_2,\ \lambda \in (0,1) \right\}. \tag{3.19}$$

Proof Let $\mathcal{D}'(X)$ be a deviation measure with the g-envelope $G_S = \{g_\lambda(\alpha) \big| g_1 \in G_1,\ g_2 \in G_2,\ \lambda \in (0,1)\}$, where $g_\lambda(\alpha) = \min\{\lambda g_1(\alpha), (1-\lambda)g_2(\alpha)\}$. For every two comonotone r.v.'s Y_1 and Y_2 with $Y_1 + Y_2 = X$, we have

$$\int_0^1 g_\lambda(\alpha)\cdot d(q_X(\alpha)) = \int_0^1 g_\lambda(\alpha)\cdot d(q_{Y_1}(\alpha)) + \int_0^1 g_\lambda(\alpha)\cdot d(q_{Y_2}(\alpha))$$
$$\le \lambda\mathcal{D}_1(Y_1) + (1-\lambda)\mathcal{D}_2(Y_2) \le \max\{\mathcal{D}_1(Y_1), \mathcal{D}_2(Y_2)\},$$

where the first inequality follows from (3.18). This implies

$$\sup_{g_1\in G_1, g_2\in G_2, \lambda\in(0,1)} \int_0^1 g_\lambda(\alpha)\cdot d(q_X(\alpha)) \le \inf_{Y\in\mathcal{C}(X)} \max\{\mathcal{D}_1(Y_1), \mathcal{D}_2(Y_2)\},$$

or $\mathcal{D}'(X) \le \mathcal{D}_S(X)$.

Now we prove the reverse inequality. Let $X \neq C$ (otherwise the statement is trivial), and let $Y = (Y_1, Y_2) \in \mathcal{C}(X)$ be an optimal risk sharing of X. Proposition 3.10 implies that $\mathcal{D}_1(Y_1) = \mathcal{D}_2(Y_2)$, and also that $Z_1 = kZ_2$ for some $Z_1 \in \partial\mathcal{D}_1(Y_1)$, $Z_2 \in \partial\mathcal{D}_2(Y_2)$, and $k > 0$. Then $g_1(\alpha) = kg_2(\alpha)$ for $g_i(\alpha) = \int_0^\alpha(-q_{Z_i}(\beta))d\beta$, $i = 1,2$. Proposition 3.11 implies $g_i(\alpha) \in G_i$, $i = 1,2$, and thus, $g_S(\alpha) \in G_S$, where $g_S(\alpha) = \min\{\frac{1}{k+1}g_1(\alpha), \frac{k}{k+1}g_2(\alpha)\} = \frac{1}{k+1}g_1(\alpha) = \frac{k}{k+1}g_2(\alpha)$. Consequently,

$$\mathcal{D}'(X) \ge \int_0^1 g_S(\alpha)\cdot d(q_X(\alpha)) = \frac{1}{k+1}\int_0^1 g_1(\alpha)d(q_{Y_1}(\alpha)) + \frac{k}{k+1}\int_0^1 g_2(\alpha)d(q_{Y_2}(\alpha))$$
$$= \frac{1}{k+1}\mathcal{D}_1(Y_1) + \frac{k}{k+1}\mathcal{D}_2(Y_2) = \mathcal{D}_S(X),$$

where the first equality uses comonotonicity of Y_1 and Y_2, and the second one follows from Proposition 3.11. Hence, $\mathcal{D}'(X) = \mathcal{D}_S(X)$.

Finally, we prove that G_S is the maximal g-envelope of $\mathcal{D}_S(X)$. The facts that G_S is dominance-closed and that $G_S \cup \{0\}$ is closed with respect to pointwise convergence are immediate corollaries from the corresponding properties of G_i, $i = 1,2$. Next we prove that G_S is convex. Indeed, let $f(\alpha) = kg(\alpha) + (1-k)h(\alpha)$ for $k \in (0,1)$, and $g, h \in G_S$. Then $g(\alpha) = \min\{\lambda_g g_1(\alpha), (1-\lambda_g)g_2(\alpha)\}$, and $h(\alpha) = \min\{\lambda_h h_1(\alpha), (1-\lambda_h)h_2(\alpha)\}$ for some $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$, and $\lambda_g, \lambda_h \in (0,1)$. Since G_1 and G_2 are convex, $f^*(\alpha) = \min\{\lambda(k_1 g_1(\alpha) + (1-k_1)h_1(\alpha)), (1-\lambda)(k_2 g_2(\alpha) + (1-k_2)h_2(\alpha))\} \in G_S$ for any $k_1, k_2, \lambda \in (0,1)$. But $f^*(\alpha) = \min\{k\lambda_g g_1(\alpha) + (1-k)\lambda_h h_1(\alpha), k(1-\lambda_g)g_2(\alpha) + (1-k)(1-\lambda_h)h_2(\alpha)\} \ge kg(\alpha) + (1-k)h(\alpha) = f(\alpha)$ for $\lambda = k\lambda_g + (1-k)\lambda_h$, $k_1 = k\lambda_g/\lambda$, and $k_2 = k(1-\lambda_g)/\lambda$. Since G_S is dominance-closed, this implies $f(\alpha) \in G_S$, i.e. G_S is convex, and thus, G_S is the maximal g-envelope of $\mathcal{D}_S(X)$. □

A generalization of Proposition 3.12 to the case of m investors is straightforward.

Proposition 3.13 *Let S be a coalition of m investors with continuous law-invariant deviation measures $\mathcal{D}_i$, $i = 1,\dots,m$, with the maximal g-envelopes G_i, $i = 1,\dots,m$, respectively. Then the maximal g-envelope of $\mathcal{D}_S$ is given by*

(3.20) $$G_S = \left\{g(\alpha) = \min_{i=1,\dots,m}\{\lambda_i g_i(\alpha)\} \Big| g_i \in G_i,\ \lambda_i \in (0,1),\ i = 1,\dots,m,\ \textstyle\sum_{i=1}^m \lambda_i = 1\right\}.$$

Proof We prove the formula (3.20) by induction on m. The case $m = 2$ follows from Proposition 3.12. Assume that (3.20) holds for $m-1$. Let S' be a coalition of investors $1,\dots,m-1$. Then Proposition 3.5 implies

$$\mathcal{D}_S(X) = \inf_{(Z,Y_m):\ Z+Y_m=X} \max\{\mathcal{D}_{S'}(Z), \mathcal{D}_m(Y_m)\}.$$

According to Proposition 3.12, G_S is given by

$$G_S = \left\{ g(\alpha) = \min\{\lambda_S g_S(\alpha), (1-\lambda_S) g_m(\alpha)\} \middle| g_S \in G'_S,\ g_m \in G_m,\ \lambda_S \in (0,1) \right\}, \tag{3.21}$$

where G'_S is the maximal g-envelope of $\mathcal{D}_{S'}$, which, by the induction assumption, is determined by

$$G'_S = \left\{ g_S(\alpha) = \min_{i=1,\ldots,m-1} \{\lambda_i g_i(\alpha)\} \middle| g_i \in G_i, \quad \lambda_i \in (0,1),\ \ i=1,\ldots,m-1, \quad \textstyle\sum_{i=1}^{m-1} \lambda_i = 1 \right\}. \tag{3.22}$$

Combining (3.22) with (3.21), we obtain (3.20). □

4 Cooperative Portfolio

4.1 Pareto-Optimal Solutions

This section returns to finding an optimal cooperative portfolio. Proposition 2.1 implies that for every Pareto-optimal division $Y = (Y_1, \ldots, Y_m)$ with $\mathcal{D}_i(Y_i) = d_i$, $i = 1, \ldots, m$, the r.v. $X_S = \sum_{i=1}^m Y_i$ solves (2.9). Proposition 3.4 states that (2.9) is equivalent to (1.2) with the deviation measure

$$\mathcal{D}_S(X) = \inf_{Y \in \mathcal{A}(X)} \max\{\mathcal{D}_1(Y_1)/d_1, \ldots, \mathcal{D}_m(Y_m)/d_m\}, \tag{4.1}$$

where $\mathcal{A}(X) = \{Y : X = \sum_{i=1}^m Y_i\}$. $\mathcal{D}_S(X)$ can be evaluated by (3.5) with the g-envelope given by (3.20).

Next we prove that the statement converse to Proposition 2.1 is also true.

Proposition 4.1 *Let $d = (d_1, \ldots, d_m)$, where $d_i > 0$, $i = 1, \ldots, m$. If $Y = (Y_1, \ldots, Y_m)$ is a feasible division with $\mathcal{D}_i(Y_i) = d_i$, $i = 1, \ldots, m$, and the r.v. $X_S = \sum_{i=1}^m Y_i$ solves (2.9), then Y is a Pareto-optimal division. Moreover, for every $y = (y_1, \ldots, y_m)$ with $\sum_{i=1}^m y_i = EX_S$, there exists a Pareto-optimal division $Z = (Z_1, \ldots, Z_m)$ with $\mathcal{D}_i(Z_i) = d_i$, $i = 1, \ldots, m$, and $EZ_i = y_i$, $i = 1, \ldots, m$.*

Proof Suppose Y is not a Pareto-optimal division. Then $\mathcal{D}_i(Y_i^*) \le \mathcal{D}_i(Y_i) = d_i$ and $EY_i^* \ge EY_i$, $i = 1, \ldots, m$, (with at least one inequality being strict) for some Pareto-optimal division $Y^* = (Y_1^*, \ldots, Y_m^*)$. By definition, $X^* \in \mathcal{P}_d$ for $X^* = \sum_{i=1}^m Y_i^*$, which implies $EX^* \le EX_S$, and thus, $EX^* = EX_S$ and $EY_i^* = EY_i$, $i = 1, \ldots, m$. Consequently, X^* solves (2.9), and hence, X^* solves (1.2) with $\mathcal{D}_S$ given by (4.1). Thus, $\mathcal{D}_S(X^*) = 1$ and Y^* is an optimal risk sharing of X^*, and then Proposition 3.10 implies $\mathcal{D}_1(Y_1^*)/d_1 = \ldots = \mathcal{D}_m(Y_m^*)/d_m = 1$, which contradicts the assumption that $Y^* \succ Y$.

Now we prove the second part of the proposition. Let $y = (y_1, \ldots, y_m)$ be such that $\sum_{i=1}^m y_i = EX_S$. Then $\mathcal{D}_i(Z_i) = \mathcal{D}_i(Y_i) \le d_i$ and $EZ_i = y_i$, $i = 1, \ldots, m$, for $Z = (Z_1, \ldots, Z_m)$ with $Z_i = Y_i - EY_i + y_i$, $i = 1, \ldots, m$. Since $\sum_{i=1}^m EZ_i = EX_S$, Z is a Pareto-optimal division. □

Next we restate the necessary and sufficient optimality conditions for a solution to the problem (1.2), obtained by Rockafellar, Uryasev, and Zabarankin (2006c). We assume that

(F1) $EX \neq 0$ for some $X \in \mathcal{F}$,

(F2) $C \notin \mathcal{F}$ for constants $C \neq 0$.

Conditions (F1) and (F2) imply that for an optimal solution, the constraint in (1.2) is active. Consequently, an r.v. X^* with $EX^* = \mu_0 > 0$ solves (1.2) if and only if it solves the following problem

$$\min_{X \in \mathcal{F}} \mathcal{D}_S(X) \qquad \text{subject to} \quad EX \ge \mu_0. \tag{4.2}$$

Theorem 4 in Rockafellar, Uryasev, and Zabarankin (2006c) states that an r.v. X^* solves (4.2) if and only if $X^* \in \mathcal{F}$, $EX^* \ge \mu_0$, and

$$E[(\bar{r}_i - r_i)(1-Q)] = \lambda(\bar{r}_i - r_0), \quad i = 1, \ldots, n, \tag{4.3}$$

for some $Q \in \partial\mathcal{D}_S(X^*)$ and for the Lagrange multiplier λ to be $\lambda = \mathcal{D}_S(X^*)/\mu_0$. In this case, $\mathcal{D}_S(X^*) = 1$, and $\mu_0 = EX^*$, whence $\lambda = 1/EX^*$. Since $EQ = 0$ for every $Q \in \partial\mathcal{D}_S(X^*)$ (see Rockafellar, Uryasev, and Zabarankin, 2006c), the condition (4.3) can be rewritten as $E[r_i Q] = \lambda\bar{r}_i$, $i = 1,\dots,n$, or equivalently, $E[ZQ] = \lambda EZ$ for every $Z \in \mathcal{F}$. This discussion is summarized in the next proposition.

Proposition 4.2 *Let $\mathcal{D}_S$ be a continuous deviation measure. An r.v. X^* solves problem* (1.2) *if and only if $X^* \in \mathcal{F}$, $EX^* > 0$, and there exists an r.v. $Q \in \partial\mathcal{D}_S(X^*)$ such that $EZ = E[ZQ]EX^*$ for every $Z \in \mathcal{F}$.*

Proof If X^* solves (1.2), then it also solves (4.2) with $\mu_0 = EX^* > 0$, and thus, (4.3) holds with $\lambda = 1/EX^*$, which implies $EZ = E[ZQ]EX^*$ for all $Z \in \mathcal{F}$; and vice versa, the last condition together with $X^* \in \mathcal{F}$ and $EX^* > 0$ implies (4.3) with $\mu_0 = EX^* > 0$ and $\lambda = 1/EX^* > 0$. □

We summarize key steps in constructing a Pareto-optimal set for the cooperative portfolio.

- For every fixed $d = (d_1,\dots,d_m)$, determine the deviation measure $\mathcal{D}_S$ of the coalition $S = \{1,\dots,m\}$ with $\mathcal{D}_i$, $i = 1,\dots,m$, by (4.1).
- Find optimal portfolio X^* in (1.2) from the optimality conditions in Proposition 4.2.
- Find optimal risk sharing $Y = (Y_1,\dots,Y_m) \in \mathcal{C}(X)$ using (3.13). Then $(Y_1 - EY_1 + y_1,\dots,Y_m - EY_m + y_m)$ is a Pareto-optimal solution for any $y = (y_1,\dots,y_m) \in \mathbb{R}^m$ with $\sum_{i=1}^m y_i = EX^*$.

Pairwise comonotonicity of $Y_1,\dots,Y_m$, established in Proposition 3.1, implies that $\overline{Y_i} = Y_i - EY_i$, $i = 1,\dots,m$, is a function of X, i.e., $\overline{Y_i} = f_i(X)$, $i = 1,\dots,m$, and thus, investor i's gain is $f_i(X) + y_i$. Consequently, if y_i is chosen independently of a particular realization $X(\omega)$ of X, the coalition can divide X prior to observing $X(\omega)$.

Next, we address the question of selecting a division from the Pareto-optimal set.

Let X_i^* solve the individual portfolio selection problem (2.4), and let $d_i = \mathcal{D}(X_i^*)$, $i = 1,\dots,m$. Suppose that all the investors in the coalition agree on functions $U_i(EX_i, \mathcal{D}(X_i))$, $i = 1,\dots,m$, which are increasing in the first argument and are decreasing in the second. A cooperative game is then formulated as m-parameter optimization problem

$$\text{(4.4)} \qquad \max_{X_S \in \mathcal{F}, Y} U_i(EY_i, \mathcal{D}(Y_i)) \quad \text{subject to} \quad \sum_{i=1}^n Y_i = X_S, \quad EY_i \geq \pi_i, \quad \mathcal{D}(Y_i) \leq d_i \quad i = 1,\dots,n.$$

There are two special cases of the cooperative game (4.4): (i) expected gain maximization, in which case $U_i(EY_i, \mathcal{D}(Y_i)) = EY_i$; and (ii) deviation minimization, i.e., $U_i(EY_i, \mathcal{D}(Y_i)) = -\mathcal{D}(Y_i)$.

4.2 Cooperative Game: Expected Gain Maximization

Let $U_i(EY_i, \mathcal{D}(Y_i)) \equiv EY_i$. We temporarily omit the condition $EY_i \geq \pi_i$. In this case, the cooperative game (4.4) reduces to

$$\text{(4.5)} \qquad \max_{X_S \in \mathcal{F}, Y} \{EY_1,\dots,EY_m\} \quad \text{subject to} \quad \sum_{i=1}^m Y_i = X_S, \quad \mathcal{D}(Y_i) \leq d_i, \quad i = 1,\dots,m.$$

Since in (4.5), $d = (d_1,\dots,d_m)$ is fixed, without loss of generality, we may assume that $d_1 = \dots = d_m = 1$ (otherwise, a deviation measure $\mathcal{D}_i(X)$ can be replaced by $\mathcal{D}_i'(X) = \mathcal{D}_i(X)/d_i$).

Let X^* be the value of the cooperative portfolio that solves (1.2), and let $Y = (Y_1,\dots,Y_m) \in \mathcal{C}(X^*)$ be any optimal risk sharing. Then Proposition 4.1 implies that $(Y_1 - EY_1 + y_1,\dots,Y_m - EY_m + y_m)$ is a Pareto-optimal solution for any $y = (y_1,\dots,y_m) \in \mathbb{R}^m$ with $\sum_{i=1}^m y_i = EX^*$. Here, y_i is a premium for investor i for accepting uncertain $\overline{Y_i} = Y_i - EY_i$. This section addresses selecting a division from the Pareto-optimal set, i.e. assigning $y_1,\dots,y_m$ "fairly."

First, assume that there exists a Pareto-optimal division $Y^* = (Y_1^*,\dots,Y_m^*) \in \mathcal{C}(X^*)$ such that $Y_i^* \in \mathcal{F}$, $i = 1,\dots,m$. In this case, investor i can form an individual portfolio with the net value Y_i^*, and the investor's premium for accepting uncertain $\overline{Y_i}$ is $y_i = EY_i^*$. Proposition 4.2 states that in this case, $y_i = EY_i^* = E[QY_i^*]EX^* =$

$E[Q(Y_i^* - EY_i^*)]EX^* = E[Q(Y_i - EY_i)]EX^* = E[Q\overline{Y_i}]EX^*$. In general, however, $Y_i \notin \mathcal{F}$, i.e. investor's share of the cooperative portfolio cannot be replicated by the investor if he/she decides to invest alone, whereas we suggest investor i to be offered the premium for taking the risk based on the same formula

$$y_i = E[Q\overline{Y_i}]EX^*, \quad \overline{Y_i} = Y_i - EY_i \quad i = 1,\dots,m, \tag{4.6}$$

where $Q \in \partial\mathcal{D}_S(X^*)$ is described in Proposition 4.2. In other words, investor i is offered the same premium for accepting uncertain $\overline{Y_i}$ which he/she would obtain as if the instrument with the return Y_i were available on the market. Obviously, $\sum_{i=1}^m y_i = EX^* \sum_{i=1}^m E[QY_i] = EX^* E[QX^*] = EX^* \mathcal{D}_S(X^*) = EX^*$, i.e. the division (4.6) is feasible. Also, $y_1,\dots,y_m$ do not depend on a particular realization of X, and consequently, the investors may accept the division (4.6) at the moment of forming a cooperative portfolio.

In fact, the problem (4.5) is a cooperative game with transferable payoffs with the set of players $M = \{1,2,\dots,m\}$, with associated deviation measures $\mathcal{D}_1,\dots,\mathcal{D}_m$, and with the *value* $v(S)$ of every coalition $S \subset M$ given by

$$v(S) = \max_{X\in\mathcal{F}} EX \quad \text{subject to} \quad \mathcal{D}_S(X) \le 1. \tag{4.7}$$

The *core* of a cooperative game is the set of vectors $y = (y_1,\dots,y_m) \in \mathbb{R}^m$ with $\sum_{i=1}^m y_i = v(M)$ such that $\sum_{i\in S} y_i \ge v(S)$ for every $S \subset M$, i.e., it is the set of feasible payoffs under which no coalition has a value greater than the sum of its members' payoffs (Osborne and Rubinstein, 1994). The next proposition shows that the devision (4.6) of the cooperative portfolio expected gain belongs to the core of the game (4.7).

Proposition 4.3 *Let M be a coalition of m investors with continuous law-invariant deviation measures $\mathcal{D}_i$, $i = 1,\dots,m$, and let X^* solve* (1.2) *with $\mathcal{D}_M$. Also, let $Y = (Y_1,\dots,Y_m)$ minimize (3.2), and let $Q \in \partial\mathcal{D}_M(X^*)$ be defined as in Proposition 4.2. Then $y = (y_1,\dots,y_m)$ given by (4.6) belongs to the core of the game (4.7).*

Proof Proposition 3.11 implies that the function $g_Q(\alpha) = \int_0^\alpha (-q_Q(\beta))d\beta$ belongs to the maximal g-envelope G_M of $\mathcal{D}_M$. By Proposition 3.13, $g_Q(\alpha) = \min_{i\in M}\{\lambda_i g_i(\alpha)\}$ for some $\lambda_i \in (0,1)$, $i = 1,\dots,m$ with $\sum_{i=1}^m \lambda_i = 1$, where g_i, $i = 1,\dots,m$, are elements of the maximal g-envelopes of $\mathcal{D}_i$, $i = 1,\dots,m$. Consequently,

$$\frac{y_i}{EX^*} = E[QY_i] \le \int_0^1 g_Q(\alpha)d(q_{Y_i}(\alpha)) \le \lambda_i \int_0^1 g_i(\alpha)d(q_{Y_i}(\alpha)) \le \lambda_i \mathcal{D}_i(Y_i) = \lambda_i, \quad i = 1,\dots,m, \tag{4.8}$$

where the first inequality is due to Hardy-Littlewood (see Föllmer and Schied, 2004, Theorem A.24), and the last equality follows from Proposition 3.10. Since $\sum_{i=1}^m y_i/EX^* = \sum_{i=1}^m \lambda_i = 1$, (4.8) reduces to $y_i/EX^* = \lambda_i$, $i = 1,\dots,m$.

We need to prove that $y_S = \sum_{i\in S} y_i \ge v(S)$ for any coalition $S \subset M$. Let $Z \in \mathcal{F}$ solve (4.7). Proposition 3.13 implies that $g_S(\alpha) = \min_{i\in S}\{\frac{y_i}{y_S} g_i(\alpha)\} = \frac{EX^*}{y_S}\min_{i\in S}\{\lambda_i g_i(\alpha)\}$ belongs to the maximal g-envelope G_S of $\mathcal{D}_S$, and thus,

$$1 \ge \mathcal{D}_S(Z) \ge \int_0^1 g_S(\alpha)d(q_Z(\alpha)) \ge \frac{EX^*}{y_S}\int_0^1 g_Q(\alpha)d(q_Z(\alpha)) \ge \frac{EX^*}{y_S}E[QZ] = \frac{EZ}{y_S} = \frac{v(S)}{y_S},$$

where the equality $EX^*E[QZ]/y_S = EZ/y_S$ follows from Proposition 4.2. This finishes the proof. □

As a corollary from Proposition 4.3, the core of the game (4.7) as well as the core of any its subgame is always non-empty. Such games are called *totally balanced*. Also, Proposition 4.3 implies that the constraints $EY_i \ge \pi_i$, included into (4.4) and omitted in (4.5), are satisfied automatically.

The cooperative game theory offers several concepts for choosing a solution from a Pareto-optimal set, including nucleolus, Shapley value, Nash bargaining solution with disagreement point $(v(1),\dots,v(m))$, etc. (see Osborne and Rubinstein, 1994). However, according to Osborne and Rubinstein (1994), "there are few persuasive applications of these concepts." In addition, some of those solutions, e.g., Shapley value, are not even guaranteed to belong to the core of the game (4.7).

As an example of the division (4.6), suppose there are only two assets on the market: risk-free asset with the rate of return r_0 and risky asset with the rate of return $r_0 + r$ with $Er > 0$. In this case, an investor with capital C forms a portfolio $(C - x, x)$, i.e. invest x amount of money into the risky asset and the rest into the risk-free asset. It is assumed that borrowing ($x > C$) and opening short positions ($x < 0$) are allowed, i.e., x is unconstrained. The uncertain gain of the portfolio is $X = x(1 + r_0 + r) + (C - x)(1 + r_0) - C(1 + r_0) = xr$. If investor i maximizes his/her portfolio expected gain EX subject to $\mathcal{D}_i(X) \le 1$, his/her optimal portfolio is, obviously, $x_i^* = 1/\mathcal{D}_i(r)$. Similarly, the optimal cooperative portfolio is $x_S^* = 1/\mathcal{D}_S(r)$. Investors $i = 1, \ldots, m$ form the cooperative portfolio only if a division $(Y_1, \ldots, Y_m)$ of $X^* = r/\mathcal{D}_S(r)$ satisfies $EY_i \ge Er/\mathcal{D}_i(r)$ and $\mathcal{D}_i(Y_i) \le 1$, $i = 1, \ldots, m$. Obviously, if $x_S^* > \sum_{i=1}^m x_i^*$ then forming the cooperative portfolio is strictly preferable for all the investors.

Example 4.1 *Let S be a coalition of two investors with deviation measures $\mathcal{D}_1(X) = \mathrm{CVaR}^{\Delta}_{\beta_1}(X)$ and $\mathcal{D}_2(X) = \mathrm{CVaR}^{\Delta}_{\beta_2}(X)$, where $\beta_1 < \beta_2$. Then forming a cooperative portfolio on the market with the only two assets (risk free with r_0 and risky with continuous $r_0 + r$ and $Er > 0$) is strictly preferable for the both investors.*

Detail. The deviation measure $\mathcal{D}_S$ of S is given by (3.7). For every continuous r.v. X, the function $f(\beta) = \frac{\beta}{1-\beta}\mathrm{CVaR}^{\Delta}_{\beta}(X) = \mathrm{CVaR}^{\Delta}_{1-\beta}(-X)$ is strictly increasing, and $f(\beta) < f(\beta_2)$ is equivalent to

$$1 \Big/ \left(\frac{\beta(1-\beta_2)}{\beta - 2\beta\beta_2 + \beta_2} \mathrm{CVaR}^{\Delta}_{\beta}(X) \right) > 1/\mathrm{CVaR}^{\Delta}_{\beta}(X) + 1/\mathrm{CVaR}^{\Delta}_{\beta_2}(X).$$

Since $\mathrm{CVaR}^{\Delta}_{\beta}(X)$ is also a strictly increasing function of β, for $\beta_1 < \beta_2$ this implies $1/\mathcal{D}_S(X) > 1/\mathrm{CVaR}^{\Delta}_{\beta_1}(X) + 1/\mathrm{CVaR}^{\Delta}_{\beta_2}(X)$. Thus, $x_S^* = 1/\mathcal{D}_S(r) > 1/\mathcal{D}_1(r) + 1/\mathcal{D}_2(r) = x_1^* + x_2^*$. □

Example 4.2 *In Example 4.1, let r be uniformly distributed on $(-1, 2)$, and let $\beta_1 = 0.2$ and $\beta_2 = 0.6$. Investing alone, the investors obtain expected gains $Er \cdot x_1^* = 5/12$ and $Er \cdot x_2^* = 5/6$, while the cooperative portfolio expected gain is $Er \cdot x_S^* = (5 + 2\sqrt{6})/6 \approx 1.65 > 1.25 = Er \cdot (x_1^* + x_2^*)$. According to (4.6), $Er \cdot x_S^*$ is divided as $y_1 = (2 + \sqrt{6})/6 \approx 0.74 > 5/12$ and $y_2 = (3 + \sqrt{6})/6 \approx 0.91 > 5/6$.*

Detail. The quantile function of r is $q_r(\alpha) = 3\alpha - 1$, and thus, $\mathrm{CVaR}^{\Delta}_{\beta}(r) = \frac{3}{2}(1 - \beta)$, which implies $Er \cdot x_1^* = \frac{2}{3(1-\beta_1)} Er = 5/12$ and $Er \cdot x_2^* = \frac{2}{3(1-\beta_2)} Er = 5/6$. In this case, $\mathcal{D}_S$, given by (3.7), reduces to

$$\mathcal{D}_S(X) = \sup_{\beta \in [0.2, 0.6]} \left(\frac{0.4\beta}{-0.2\beta + 0.6} \cdot \frac{3}{2}(1 - \beta) \right).$$

The supremum is attained at $\beta = 3 - \sqrt{6}$, and thus, $Er \cdot x_S^* = Er/\mathcal{D}_S(r) = (5 + 2\sqrt{6})/6$. The devision (4.6) implies that the investors obtain $y_1 = (\sqrt{6} - 2)Er \cdot x_S^* = (2 + \sqrt{6})/6$ and $y_2 = (3 - \sqrt{6})Er \cdot x_S^* = (3 + \sqrt{6})/6$, respectively. □

Remark 4.1 *If preferences of all the investors on the market are represented by the mean-deviation model with general deviation measures, and if all the investors form a single "market" coalition, a cooperative portfolio that solves* (1.2) *consists of only two instruments:* master fund *(market portfolio) and risk-free asset (Rockafellar, Uryasev, and Zabarankin, 2006b). Since the shares of the cooperative portfolio value that the investors obtain after division are pairwise comonotone, the portfolio of every investor consists of the risk-free asset and some derivative of the master fund. This fact can be viewed as a generalization of the one-fund theorem (Rockafellar, Uryasev, and Zabarankin, 2006b). However, this interpretation depends on conditions of the market equilibrium for the investors using a diversity of deviation measures (Rockafellar, Uryasev, and Zabarankin, 2007).*

4.3 Cooperative Game: Deviation Minimization

This section investigates the cooperative game (4.4) with $U_i(EY_i, \mathcal{D}(Y_i)) = -\mathcal{D}(Y_i)$. In this case, (4.4) reduces to

$$\text{(4.9)} \qquad \min_{X_S\in\mathcal{F},Y} \{\mathcal{D}(Y_1),\ldots,\mathcal{D}(Y_m)\} \quad \text{subject to} \quad \sum_{i=1}^m Y_i = X_S, \quad EY_i \geq \pi_i, \quad \mathcal{D}(Y_i) \leq d_i \quad i = 1,\ldots,m,$$

which can be considered as a cooperative game *without* transferable payoffs. A Pareto-optimal solution to (4.9) is chosen as follows. Let $\mathcal{D}_S(X)$ be determined by (4.1), and let X_S^* solve the problem

$$\text{(4.10)} \qquad \min_{X\in\mathcal{F}} \mathcal{D}_S(X) \quad \text{subject to} \quad EX \geq \sum_{i=1}^m \pi_i,$$

with $d_S = \mathcal{D}_S(X_S^*)$. If X_i is the value of an optimal individual portfolio in (2.4), then $EX_i = \pi_i$ and $\mathcal{D}_i(X_i) = d_i$. Consequently, $E[\sum_{i=1}^m X_i] = \sum_{i=1}^m \pi_i$, and $\mathcal{D}_S(\sum_{i=1}^m X_i) \leq 1$, whence $d_S \leq 1$.

Let $Y_S = (Y_1^*,\ldots,Y_m^*) \in \mathcal{A}(X_S^*)$ be a minimizer in (4.1). Then Proposition 4.1 implies that $Y^* = (Y_1^* - EY_1^* + \pi_1,\ldots,Y_m^* - EY_m^* + \pi_m)$ is a Pareto-optimal division in (4.9). On the other hand, Proposition 3.10 implies that $\mathcal{D}_1(Y_1^*)/d_1 = \ldots = \mathcal{D}_m(Y_m^*)/d_m = d_S$, i.e., compared to the individual investment, the coalition offers every investor the same relative decrease in deviation. This is the reason for the devision Y^* to be preferable among all Pareto-optimal divisions in (4.9).

This approach is demonstrated in the following example.

Example 4.3 *In Example 4.1, $\pi_i = Er/\mathcal{D}_i(r)$, $i = 1,2$. A solution to (4.10) is the portfolio $x_S = (\pi_1+\pi_2)/Er = 1/\mathcal{D}_1(r)+1/\mathcal{D}_2(r)$ with $d_S = \mathcal{D}_S(x_S r) = \mathcal{D}_S(r)(1/\mathcal{D}_1(r)+1/\mathcal{D}_2(r))$. Under the conditions of Example 4.1, $d_S < 1$, i.e. the coalition reduces investors' deviations while offering the same expected gain.*

5 Conclusions

This work has formulated and solved the cooperative games with m investors using different law-invariant deviation measures as numerical representations for their attitudes towards risk. Each investor either selects his/her individual portfolio by minimizing a deviation measure $\mathcal{D}$ of the portfolio gain subject to a constraint on the expected gain or joins a coalition and obtains a share of a cooperative portfolio. The cooperative games reduce to two stages: (i) formulating a cooperative portfolio selection problem, and (ii) dividing the cooperative portfolio gain among investors in the coalition. It has been shown that the coalition behaves as a single imaginary investor with a certain deviation measure $\mathcal{D}_S$ representing "integral" risk preferences of the coalition. This result has the following implications. From the mathematical point of view, finding Pareto-optimal solutions of the multi-criteria cooperative portfolio selection problem is reduced to a simpler problem, which can be readily solved in the framework of the mean-deviation analysis (Rockafellar, Uryasev, and Zabarankin, 2006c,b, 2007). From the finance point of view, the mean-deviation model implies that the risk preferences of a coalition of the investors in a cooperative game can be represented by a single deviation measure. This result is not known to hold for other models of risk, e.g. the expected utility theory.

Also, the approach to dividing the optimal cooperative portfolio gain has been developed, and the components of an optimal division, i.e., investors' shares, have been shown to be pairwise comonotone and to have equal corresponding deviations. As an illustration, two cooperative games have been solved: "cooperative expected gain maximization" and "cooperative deviation minimization." For the former, the suggested division of the expected gain has been proved to be in the core of the game.

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A Proof of Proposition 3.3

The functional $\mathcal{D}_S(X)$ is obviously law-invariant, and we need to prove that $\mathcal{D}_S(X)$ is a deviation measure. Properties D1 and D2 follow from Proposition 3.2. Subadditivity D3, i.e. $\mathcal{D}_S(X_1+X_2) \le \mathcal{D}_S(X_1)+\mathcal{D}_S(X_2)$ for any X_1 and X_2, is shown as follows. Proposition 3.2 implies that there exist $Y^{(1)} \in \mathcal{A}(X_1)$ and $Y^{(2)} \in \mathcal{A}(X_2)$ such that $\mathcal{D}_S(X_j) = \max\{\mathcal{D}_1(Y_1^{(j)}),\ldots,\mathcal{D}_m(Y_m^{(j)}\}$, $j=1,2$. This implies

$$
\begin{aligned}
\mathcal{D}_S(X_1+X_2) &= \inf_{Y\in\mathcal{A}(X_1+X_2)} \max\{\mathcal{D}_1(Y_1),\ldots,\mathcal{D}_m(Y_m)\}\\
&\le \max\{\mathcal{D}_1(Y_1^{(1)}+Y_1^{(2)}),\ldots,\mathcal{D}_m(Y_m^{(1)}+Y_m^{(2)})\}\\
&\le \max\{\mathcal{D}_1(Y_1^{(1)})+\mathcal{D}_1(Y_1^{(2)}),\ldots,\mathcal{D}_m(Y_m^{(1)})+\mathcal{D}_m(Y_m^{(2)})\}\\
&\le \max\{\mathcal{D}_1(Y_1^{(1)}),\ldots,\mathcal{D}_m(Y_m^{(1)})\}+\max\{\mathcal{D}_1(Y_1^{(2)}),\ldots,\mathcal{D}_m(Y_m^{(2)})\}\\
&= \mathcal{D}_S(X_1)+\mathcal{D}_S(X_2).
\end{aligned}
$$

It is left to prove lower semicontinuity D4. Let a sequence $\{X_n\}_{n=1}^{\infty}$ converge to X in $\mathcal{L}^2(\Omega)$, and let $\mathcal{D}_S(X_n) \le C$ for all n and a constant $C<\infty$. Then $\{X_n\}_{n=1}^{\infty}$ is bounded, i.e., there exists $A<\infty$ such that $\|X_n\|_2 \le A$ for all n. Proposition 3.2 implies that for every n there exists $Y^{(n)} \in \mathcal{C}(X_n)$ such that $\mathcal{D}_S(X_n) = \max\{\mathcal{D}_1(Y_1^{(n)}),\ldots,\mathcal{D}_m(Y_m^{(n)})\} \le C$, and (2.1) implies that without loss of generality, we can assume $EY_1^{(n)} = \ldots = EY_m^{(n)} = EX_n = 0$. Then $Y^{(n)} \in \mathcal{C}(X_n)$, $i=1,\ldots,m$, implies $X_n \succcurlyeq_{cx} Y_i^{(n)}$, whence $\sigma(Y_i^{(n)}) \le \sigma(X_n)$, and consequently, $\|Y_i^{(n)}\|_2 \le \|X\|_2 \le A$. Thus, the sequence $Y_i^{(1)}, Y_i^{(2)},\ldots,Y_i^{(n)},\ldots$ belongs to the set $\{Z\in\mathcal{L}^2(\Omega) : \|Z\| \le A\}$, which is bounded, convex, and closed, and hence, is weakly compact. We conclude that there exists a sequence $n_1<n_2<\ldots<n_k<\ldots$ such that the sequences $\{Y_i^{(n_k)}\}_{k=1}^{\infty}$, $i=1,\ldots,m$, converge weakly to some limits Y_i, $i=1,\ldots,m$. It follows from $\sum_{i=1}^m Y_i^{n_k} = X_{(n_k)}$ and $X_{(n_k)} \to X$ that $\sum_{i=1}^m Y_i = X$, and thus, $Y=(Y_1,\ldots,Y_m) \in \mathcal{A}(X)$. Since all $\mathcal{D}_i$, $i=1,\ldots,m$, are convex and lower semicontinuous, they are weakly lower semicontinuous, and thus, $\mathcal{D}_i(Y_i) \le \limsup_k \mathcal{D}_i(Y_i^{(n_k)}) \le C$, $i=1,\ldots,m$, which implies $\mathcal{D}_S(X) \le C$.

B Proof of Proposition 3.7

Let us prove the "only if" part of (a). Assume X to be a nonconstant r.v. (otherwise, the statement is obvious) with $EX=0$. First we prove that for any $\lambda\in\Lambda$, a set $\mathcal{Z}_\lambda$ of r.v.'s $Z\in\mathcal{C}(X)$, satisfying (ii), is nonempty. Let $A_i = \{\alpha : g_\lambda(\alpha) = \lambda_i g_i(\alpha)\}$, $i=1,\ldots,m$ and $B_i = A_i \backslash (A_1\cup A_2\cup\ldots\cup A_{i-1})$. Then the sets $B_1,\ldots,B_m$ are pairwise disjoint, and $\bigcup_{i=1}^m B_i = (0,1)$. Let $Z_1,\ldots,Z_m$ be pairwise comonotone r.v.'s with $EZ_1=\ldots=EZ_m=0$ and $\mu_{Z_i}(I)$ such that $\mu_{Z_i}(I)=\mu_X(I)$ for $I\subset B_i$, and $\mu_{Z_i}(I)=0$ for $I\cap B_i=\emptyset$. Then $\mu_{Z_i}(\{\alpha : \lambda_i g_i(\alpha) > g_\lambda(\alpha)\}) = \mu_{Z_i}((0,1)\backslash A_i) = 0$, $i=1,\ldots,m$, and also $d(q_X(\alpha)) = \sum_{i=1}^m d(q_{Z_i}(\alpha))$. Since $EX=EZ_1=\ldots=EZ_m=0$, this implies $q_X(\alpha)=\sum_{i=1}^m q_{Z_i}(\alpha)$, which together with the comonotonicity of Z_i implies $X=\sum_{i=1}^m Z_i$, i.e. $\mathcal{Z}_\lambda$ is nonempty.

Let $A\subset\mathbb{R}^m$ be a feasible set in (3.2), i.e. $(a_1,\ldots,a_m)\in A$ if and only if $a_i \ge \mathcal{D}_i(Y_i)$, $i=1,\ldots,m$, for some $Y\in\mathcal{C}(X)$. For any optimal risk sharing $Y^*\in\mathcal{C}(X)$, the vector $y^* = (\mathcal{D}_1(Y_1^*),\ldots,\mathcal{D}_m(Y_m^*))$ belongs to the boundary of the convex set A, and thus, there exists a tangent plane to A passing through Y^*, i.e., $\sum_{i=1}^m \lambda_i a_i \ge \sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i^*)$ for any $(a_1,\ldots,a_m)\in A$ and some $\lambda_1,\ldots,\lambda_m$ with $\sum_{i=1}^m \lambda_i^2 > 0$. Assume that $\lambda_1 \le 0$. Then, since $Y=(X,0,\ldots,0)\in\mathcal{C}(X)$, we have $a=(\mathcal{D}_1(X),0,\ldots,0)\in A$, and thus, $0 \ge \lambda_1\mathcal{D}_1(X) = \sum_{i=1}^m \lambda_i a_i \ge \sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i^*)$, whence $0 \ge \mathcal{D}_S(X)$, which contradicts the assumption that X is nonconstant. We conclude that $\lambda_1>0$, and by similar reasoning, obtain $\lambda_i>0$, $i=2,\ldots,m$. Thus, we can assume that $\lambda=(\lambda_1,\ldots,\lambda_m)\in\Lambda$, and for any $Z\in\mathcal{Z}_\lambda\subset\mathcal{C}(X)$ can write

$$
\text{(B.1)}\qquad \sum_{i=1}^m \lambda_i\mathcal{D}_i(Y_i^*) \le \sum_{i=1}^m \lambda_i\mathcal{D}_i(Z_i) = \sum_{i=1}^m \lambda_i \int_{B_i} g_i(\alpha)d(q_{Z_i}(\alpha)) = \int_0^1 g_\lambda(\alpha)d(q_X(\alpha)).
$$

On the other hand,

$$\sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i^*) = \sum_{i=1}^m \lambda_i \int_0^1 g_i(\alpha) d(q_{Y_i^*}(\alpha)) \geq \sum_{i=1}^m \int_0^1 g_\lambda(\alpha) d(q_{Y_i^*}(\alpha)) = \int_0^1 g_\lambda(\alpha) d(q_X(\alpha)),$$

which along with the previous inequality results in $\sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i^*) = \int_0^1 g_\lambda(\alpha) d(q_X(\alpha))$. Consequently,

$$\sum_{i=1}^m \int_0^1 (\lambda_i g_i(\alpha) - g_\lambda(\alpha)) d(q_{Y_i^*}(\alpha)) = 0. \tag{B.2}$$

Since $\lambda_i g_i(\alpha) - g_\lambda(\alpha) \geq 0$ for all i and α, all the integrals in (B.2) should vanish, and thus, Y^* satisfies (ii). Since Y^* also satisfies (i), by Proposition 3.6, this proves the "only if" part of (a).

Now, we prove (b). Let $\mathcal{D}'(X)$ be a deviation measure given by (3.5) with the g-envelope $G = \{g_\lambda(\alpha) | \lambda \in \Lambda\}$. Then, if $\mathcal{D}_S(X) < \infty$, the infimum in (3.2) is attained at some $Y^* \in \mathcal{C}(X)$ satisfying (i). Thus, (B.1) implies $\mathcal{D}_S(X) = \sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i^*) \leq \int_0^1 g_\lambda(\alpha) d(q_X(\alpha))$ for some $\lambda \in \Lambda$, and consequently, $\mathcal{D}_S(X) \leq \mathcal{D}'(X)$. If $\mathcal{D}_S(X) = \infty$, then $\mathcal{D}_i(Z_i) = \infty$ for any $\lambda \in \Lambda$, any $Z \in \mathcal{Z}_\lambda$, and some i, and (B.1) implies $\mathcal{D}'(X) \geq \int_0^1 g_\lambda(\alpha) d(q_X(\alpha)) = \sum_{i=1}^m \lambda_i \mathcal{D}_i(Z_i) = \infty = \mathcal{D}_S(X)$.

On the other hand, for every $\lambda \in \Lambda$ and every $Y \in \mathcal{C}(X)$, we can write

$$\int_0^1 g_\lambda(\alpha) \cdot d(q_X(\alpha)) = \sum_{i=1}^m \int_0^1 g_\lambda(\alpha) \cdot d(q_{Y_i}(\alpha)) \leq \sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i) \leq \max\{\mathcal{D}_1(Y_1), \ldots, \mathcal{D}_m(Y_m)\},$$

which implies $\mathcal{D}'(X) \leq \mathcal{D}_S(X)$ and proves (b).

Finally, if for $Y \in \mathcal{C}(X)$, conditions (i) and (ii) hold true, then

$$\mathcal{D}_S(X) \leq \max_i \mathcal{D}_i(Y_i) = \sum_{i=1}^m \lambda_i \mathcal{D}_i(Y_i) = \int_0^1 g_\lambda(\alpha) d(q_X(\alpha)) \leq \mathcal{D}'(X) = \mathcal{D}_S(X),$$

which proves the "if" part of (a).

C Proof of Proposition 3.10

A vector $Y = (Y_1, \ldots, Y_m) \in \mathcal{C}(X)$ minimizes (3.1) if and only if

$$0 \in \partial_Y (\max\{\mathcal{D}_1(Y_1), \ldots, \mathcal{D}_m(Y_m)\}) = \operatorname{conv}\left\{ \bigcup_{j:\mathcal{D}_j(Y_j)=\mathcal{D}_S(X)} \partial_Y \mathcal{D}_j(Y_j) \right\}, \tag{C.1}$$

where $\partial_Y (\max\{\mathcal{D}_1(Y_1), \ldots, \mathcal{D}_m(Y_m)\})$ is taken with respect to Y_1, …, Y_{m-1} since $Y_m = X - \sum_{j=1}^{m-1} Y_j$. Then $\partial_Y \mathcal{D}_j(Y_j) = (0, \ldots, 0, \partial_{Y_j} \mathcal{D}_j(Y_j), 0, \ldots, 0)$ for $j = 1, \ldots, m-1$, and $\partial_Y \mathcal{D}_m(Y_m) = (-\partial_{Y_m} \mathcal{D}_m(Y_m), \ldots, -\partial_{Y_m} \mathcal{D}_m(Y_m))$ for $j = m$.

The "if" part of the proposition is obvious. Under (i), the union in (C.1) is taken over all $j = 1, \ldots, m$, and thus, (ii) implies (C.1). Let us prove the "only if" part, i.e., that (C.1) implies (i) and (ii). Assume that $\mathcal{D}_j(Y_j) < \mathcal{D}_S(X)$ for some $j \in \{1, \ldots, m\}$. In this case, if $\mathcal{D}_m(Y_m) = \mathcal{D}_S(X)$, then (C.1) implies $0 \in \partial_{Y_m} \mathcal{D}_m(Y_m)$. If $\mathcal{D}_m(Y_m) < \mathcal{D}_S(X)$, then $\mathcal{D}_i(Y_i) = \mathcal{D}_S(X)$ for some $i \in \{1, \ldots, m-1\}$, and (C.1) implies $0 \in \partial_{Y_i} \mathcal{D}_i(Y_i)$. In any case, (3.12) implies $\mathcal{D}_i(X') \geq \mathcal{D}_i(Y_i)$ for all X', whence $0 = \mathcal{D}_i(Y_i) = \mathcal{D}_S(X)$ for all i, which contradicts the assumption that $\mathcal{D}_j(Y_j) < \mathcal{D}_S(X)$ for some $j \in \{1, \ldots, m\}$. This proves (i).

Under (i), (C.1) implies that $\sum_{j=1}^m \lambda_j = 1$ for some $\lambda_j \geq 0$, and $\lambda_j Z_j + \lambda_m(-Z_m) = 0$, $j = 1, \ldots, m-1$, for some $Z_j \in \partial_{Y_j} \mathcal{D}_j(Y_j)$, $j = 1, \ldots, m$. Since X is nonconstant, $\mathcal{D}(Y_m) = \mathcal{D}_S(X) > 0$, and thus, Y_m is nonconstant, whence $0 \notin \partial_{Y_m} \mathcal{D}_m(Y_m)$, or $Z_m \neq 0$, which implies $\lambda_j > 0$, $j = 1, \ldots, m-1$. Consequently, condition (ii) holds, say, for $Z_0 = Z_m$.

D Proof of Proposition 3.11

First, we prove the inequality in (3.17). Rockafellar, Uryasev, and Zabarankin (2006a) showed that there is one-to-one correspondence between deviation measures $\mathcal{D}$ and risk envelopes $\mathcal{Q}$ defined by

$$\mathcal{Q} = \left\{ Q \in \mathcal{L}^2(\Omega) \;\middle|\; E[X(1-Q)] \leq \mathcal{D}(X) \quad \text{for all } X \in \mathcal{L}^2(\Omega) \right\}, \qquad \mathcal{D}(X) = \sup_{1-Q\in\mathcal{Q}} E[XQ],$$

and $Q \in \partial\mathcal{D}(Y)$ if and only if $1-Q \in \mathcal{Q}$ and $\mathcal{D}(Y) = E[YQ]$ (see Rockafellar, Uryasev, and Zabarankin, 2006c, Proposition 1). On the other hand, as shown by Grechuk, Molyboha, and Zabarankin (2009),

$$\mathcal{D}(X) = \sup_{1-Q\in\mathcal{Q}} \int_0^1 q_Q(\alpha) q_X(\alpha) d\alpha.$$

This implies $\mathcal{D}(X) \geq \int_0^1 q_X(\alpha) q_Q(\alpha) d\alpha$. Also, by the Hardy-Littlewood inequality (see Föllmer and Schied, 2004, Theorem A.24), $\mathcal{D}(Y) = E[YQ] \leq \int_0^1 q_Q(\alpha) q_Y(\alpha)$, and thus, $\mathcal{D}(X) = \int_0^1 q_X(\alpha) q_Q(\alpha) d\alpha$ for $X = Y$.

Next, we prove that $\int_0^1 q_X(\alpha) q_Q(\alpha) d\alpha = \int_0^1 g(\alpha) d(q_X(\alpha))$. Since $EQ = 0$, which follows from the properties of the risk envelope (Rockafellar, Uryasev, and Zabarankin, 2006a), $g(\alpha)$ in (3.17) is a nonnegative concave function. Integrating $\int_0^1 g(\alpha) \cdot d(q_X(\alpha))$ by parts, we obtain

$$\int_0^1 g(\alpha) \cdot d(q_X(\alpha)) = \lim_{\alpha\to 1} g(\alpha) q_X(\alpha) - \lim_{\alpha\to 0} g(\alpha) q_X(\alpha) + \int_0^1 q_Q(\alpha) q_X(\alpha) d\alpha,$$

in which both limits vanish. Indeed, if $q_X(\alpha) \to -\infty$ for $\alpha \to 0$ then for sufficiently small α, the function $|q_X(\alpha)|$ monotonously decreases on $(0,\alpha)$, and

$$|q_X(\alpha)||g(\alpha)| = \int_0^\alpha |q_X(\alpha)|(-q_Q(\beta)) d\beta \leq \int_0^\alpha |q_X(\beta)||q_Q(\beta)| d\beta.$$

The integral $\int_0^1 |q_X(\beta)||q_Q(\beta)| d\beta$, being the inner product of two functions from $\mathcal{L}^2(0,1)$, is finite. Consequently, the fact $\int_0^\alpha \big|q_X(\beta)||q_Q(\beta)|\big| d\beta \to 0$ as $\alpha \to 0$ can be shown by Lebesgue's dominated convergence theorem and implies that $g(\alpha) q_X(\alpha) \to 0$ as $\alpha \to 0$. Similarly, $g(\alpha) q_X(\alpha) \to 0$ as $\alpha \to 1$, and thus, (3.17) holds.

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