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Enhanced residual–free bubble method for convection–diffusion problems

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SUMMARY

We analyse the performance of the *enhanced residual–free bubble* (RFBe) method for the solution of convection–dominated convection–diffusion problems in 2–D, and compare the present method with the standard residual–free bubble (RFB) method. The advantages of the RFBe method are two folded: it has better stability properties and it can be used to resolve boundary layers with high accuracy on globally coarse meshes.

KEY WORDS: convection–diffusion, residual–free bubbles

1. THE ENHANCED RESIDUAL–FREE BUBBLE METHOD

The *residual-free bubble method* (RFB) was introduced by Brezzi and Russo in [4] as a stable parameter–free finite element method for strongly convection–dominated convection–diffusion boundary value problems (for the a–priori error analysis of the method applied to general elliptic problems in divergence form, see Brezzi, Marini and Süli [3]).

The RFB finite element space includes all bubble functions on a given triangulation $\mathcal{T}$ of the computational domain Ω, i.e. all the function with compact support on every element of $\mathcal{T}$. The idea behind the RFBe method introduced in [5] is to combine this property with an enrichment of the finite element space on the skeleton of the triangulation to obtain a more stable method and one that is able to resolve the boundary–layer behaviour, if required.

We consider the following model boundary value problem for the convection-diffusion equation:

$$\begin{cases} Lu := -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u = f & \text{in } \Omega = (0,1)^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the convection field $\boldsymbol{a}$ is assumed continuously differentiable on $\overline{\Omega}$ and the diffusion parameter ε is a positive real number. The problem (1) admits a unique weak solution in $V = H_0^1(\Omega)$.

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Equation (1) models many physical phenomena. For example, we may think of u as representing the concentration of a pollutant in a fluid that is moving at velocity $\boldsymbol{a}$. In this case, the first–order term in equation (1) models the convection due to the flow velocity field $\boldsymbol{a}$, while the second-order term represents the diffusion of u. Equation (1) is also a fundamental model problem for the field of computational fluid dynamics since its stable and accurate solution is a crucial step in the treatment of the incompressible Navier-Stokes equations. Indeed, although fairly simple at first sight, the convection-diffusion equation does present some of the typical difficulties of the numerical solution of flow problems. In particular, in the convection-dominated regime, the solution of (1) exhibits a normal boundary layer on parts of $\partial\Omega$ where $\boldsymbol{a} \cdot \boldsymbol{n} > 0$, with $\boldsymbol{n}$ denoting the unit outward normal vector to $\partial\Omega$. This part of $\partial\Omega$ will be referred to as the outflow boundary for the reduced problem corresponding to $\varepsilon = 0$, or simply, as the outflow boundary. It is in the vicinity of the outflow boundary, in particular, that we aim to obtain increased accuracy over a standard Galerkin finite element method and over RFB.

The RFB method consists of seeking the weak solution of problem (1) onto the subspace of V given by

$$V_{\mathrm{RFB}} = V_h \oplus B_h,$$

where

$$V_h = \left\{ v_h \in H^1_0(\Omega) : w_{h|_T} \in \mathcal{P}_1(T) \ (\text{or } w_{h|_T} \circ F_T \in \mathcal{Q}_1(\widehat{T})) \quad \forall T \in \mathcal{T} \right\}, \tag{2}$$

$$B_h = \bigoplus_{T \in \mathcal{T}} H^1_0(T). \tag{3}$$

As regards the RFBe method, this is defined by enlarging the approximation space V_{RFB}, i.e. considering instead the *augmented* space

$$V_a = V_h \oplus B_h \oplus E_h,$$

where the *edge–bubbles* space E_h is defined as follows.

Thanks to the richness of the bubble space B_h, we suffices to fix the value of the functions belonging to E_h on the skeleton Σ of the triangulation. To this end, we introduce the following pieces of notation. We define as *boundary layer region* a neighbourhood of the outflow boundary of width $\kappa = \varepsilon \ln(1/\varepsilon)$ in the direction orthogonal to the boundary. Then, we denote by Ω_{bl} the union of all elements that intersect the boundary layer region and by Ω_{out} the union of all of the remaining elements. Moreover, let Γ_{bl} be the set of all edges that intersect the boundary layer region, and Γ_{out} the set of all the remaining edges.

We define an edge bubble only on edges which belong to Γ_{bl}. On every $\Gamma \in \Gamma_{\mathrm{bl}}$ we define the edge-bubble e on Γ as the solution of the one-dimensional boundary value problem

$$\begin{cases} L_\Gamma e_\Gamma = 1 & \text{in } \Gamma \\ e_\Gamma = 0 & \text{on } \partial\Gamma, \end{cases} \tag{4}$$

where we define the restriction of the differential operator L onto e_Γ as $L_\Gamma e_\Gamma = -\varepsilon e''_\Gamma + a_\Gamma e'_\Gamma$, where a_Γ is the projection of $\boldsymbol{a}$ along Γ (here $(\cdot)'$ represent derivative along the edge).

This completely defines the space of edge-bubbles E_h and hence the RFBe method for the solution of (1).

The a–priori analysis of the method in the preasymptotic regime is carried out in [5]. The error bounds obtained for the RFB and RFBe methods are displayed in the following theorem.

Theorem 1.1. *Let $u \in H^1_0(\Omega)$ be the solution of the boundary-value problem (1) assuming that $\varepsilon \in \mathbb{R}^+$, $f \in W^2_\infty(\Omega)$ and $\boldsymbol{a} = (a_1, a_2) \in [\mathcal{C}^1(\overline{\Omega})]^2$, with $\operatorname{div} \boldsymbol{a} \leq 0$ and $a_1, a_2 \geq c_a > 0$. Moreover, let $\mathcal{T}$ be a quasi–uniform axiparallel rectangular mesh. Then, as long as $h \geq C\,\kappa$ and $\varepsilon \leq 1/\mathrm{e}$, the RFB solution $u_{RFB} \in V_{RFB} = V_h \oplus B_h$ satisfies*

$$\varepsilon^{1/2}|u-u_{RFB}|_{1,\Omega}+h^{-1/2}||\boldsymbol{a}\cdot\nabla(u-u_{RFB})||_{-1,\Omega} \leq C_1\left(\varepsilon^{1/2}h^{-1/2}+\min\left(\varepsilon^{1/4},h^{1/2}\right)\right)+C_2. \quad (5)$$

Let $u_a \in V_a = V_h \oplus B_h \oplus E_h$ be the RFBe solution. Then, under the same hypotheses as above,

$$\varepsilon^{1/2}|u-u_a|_{1,\Omega}+h^{-1/2}||\boldsymbol{a}\cdot\nabla(u-u_a)||_{-1,\Omega} \leq C_1\left(\varepsilon^{1/2}h^{-1/2}+\min\left(\varepsilon^{1/4},h^{1/2}\right)\right)+C_3 h. \quad (6)$$

The constants C_1, C_2 and C_3 are independent of the mesh size h and of ε, but may depend on the vector field $\boldsymbol{a}$.

We remark once again that the error bounds (5) and (6) are valid only when $h \geq C\kappa$ (for a quantification of the constant C see, again, [5]). This is the regime of interest, since only when the mesh does not resolve the normal layers typical of convection–dominated boundary value problems is there need for a stabilised finite element method. Below, the a–priori error bounds are tested by considering a problem with known exact solution to compare with.

2. EXAMPLES

Example 1. We consider the boundary value problem

$$\begin{cases} -\varepsilon\Delta u + u_x + u_y = f & \text{in } \Omega = (0,1)^2 \\ u = 0 & \text{in } \partial\Omega. \end{cases} \quad (7)$$

with f defined in such a way that the exact solution is given by

$$u(x,y) = 2\sin x(1-\mathrm{e}^{-(1-x)/\varepsilon})y^2(1-\mathrm{e}^{-(1-y)/\varepsilon}).$$

To confirm the a-priori bounds on the energy-norm error, we solve this model problem on a sequence of uniform meshes using both the RFB and the RFBe methods for $\varepsilon = 10^{-2}$.

In order to implement the method, this has to be fully discretised as described in [5] (see also [2]). In particular, a finite number of *bubble functions* belonging to B_h has to be computed. Similarly the edge–bubbles belonging to E_h are computed by solving on the appropriate edge the problem (4) and using the obtained function as boundary value for the computation of the edge–bubble defined inside the element. We use finite elements for such elemental computations, with a subgrid of Shishkin; the details of this are described below.

Given the one–dimensional boundary value problem

$$\begin{cases} -\varepsilon v_h'' + a v_h' = f & \text{in } I_h = (0,h), \\ v_h(0) = 0, \quad v_h(h) = 0, \end{cases}$$

we scale it back to the unit interval:

$$\begin{cases} -\varepsilon^* v'' + a v' = hf & \text{in } I = (0,1), \\ v(0) = 0, \quad v(h) = 0, \end{cases} \quad (8)$$

with $\varepsilon^* = \varepsilon/h$. Next we perform a finite element approximation of v on a Shishkin mesh on I.

A Shishkin piecewise equidistant mesh consisting of N subdivisions (with N even) is defined as follows. Given the *turning point* $\lambda^* = c_s(\varepsilon^*/c_a)\ln N$ of the Shishkin mesh, where c_s is a constant independent of ε^* and N, the mesh is taken to be uniform with $N/2$ subdivisions on the two subintervals $(0, 1-\lambda^*)$ and $(1-\lambda^*, 1)$. Thus, the mesh on $[0, 1]$ is piecewise uniform. For the continuous piecewise linear finite element approximation v^I of the solution v of (8), the interpolation error over such mesh satisfies (see [1]),

$$\varepsilon^*|v - v^I|^2_{1,I} + ||v - v^I||^2_{0,I} \leq CN^{-2}\ln^2 N,$$

with the constant C independent of ε and N.

Scaling back to the interval I_h we obtain a Shishkin mesh with turning point $\lambda = c(\varepsilon/c_a)\ln N$ and the scaled error bound

$$\varepsilon|v - v^I|^2_{1,I_h} + h^{-1}||v - v^I||^2_{0,I_h} \leq CN^{-2}\ln^2 N.$$

Shishkin meshes on rectangles are constructed by taking a tensor–product of 1-D meshes, and then similar approximation results apply.

In the present example, the subgrid mesh is axiparallel and of Shishkin type with turning point $\lambda = c_s \frac{\varepsilon}{c_a}\ln N$ and the value of the Shishkin parameter $c_s = 1$.

The convergence in terms of the mesh parameter h is shown in the log-log plots of Figure 1; see the left–hand plots for the $\varepsilon^{1/2}$–weighted H^1–seminorm (in short, H^1_ε) errors, and those on the left for the error in the L^2–norm. To study the stabilisation properties of the new method, we also plot the error in the norms restricted to the outside region Ω_{out} (see the plots below in Figure 1), which are defined by

$$|w|_{H^1_{\varepsilon,\text{out}}} = \left(\frac{\varepsilon}{|\Omega_{\text{out}}|}\right)^{1/2}|w|_{H^1(\Omega)}; \quad ||w||_{L^2_{\text{out}}} = \left(\frac{1}{|\Omega_{\text{out}}|}\right)^{1/2}||w||_{L^2(\Omega)}.$$

The error reduction rates predicted by the a–priori analysis are confirmed by the top–left plot in Figure 1: indeed, the RFBe(64) solution initially converges to the reference solution with rate 1. As we keep refining, the slope of the error curve changes sign until the error curve joins the corresponding error curve for the RFB method.

The same is not true for the RFBe(4) method: since the error is concentrated in the boundary–layer region, the poor evaluation of the edge bubbles dominates the overall computational error. Hence it is the subgrid discretisation error that dominates. Indeed, solving repeatedly on a fixed uniform 8×8 mesh on Ω, but using different smaller mesh sizes for the subgrid mesh, we observe the characteristic $N^{-1}\log N$ convergence rate on Shishkin meshes described above, see Figure 2.

The bottom plots in Figure 1, reporting the error in Ω_{out}, show that the new method RFBe has better stability properties than RFB. Notice that, this time, there is no important difference between RFBe(4) and RFBe(64). We conclude that the stabilisation effect due to the introduction of the edge–bubbles is quite robust with respect to the accuracy of the computation of the edge–bubbles.

Example 2. To exemplify the stabilisation effect of the edge–bubbles we solve problem (7), this time with $f = 0$ and boundary conditions $u = 1$ for $x = 0$ and $y = 0$, homogeneous Dirichlet boundary conditions elsewhere. As we can see by comparing the solution profiles in Figure 3 (here only the bilinear part of the solution (belonging to V_h) is plotted), the edge bubbles have the effect of reducing the over– and under–shoots typical of RFB (and of most stabilised finite element methods) near the boundary layer.

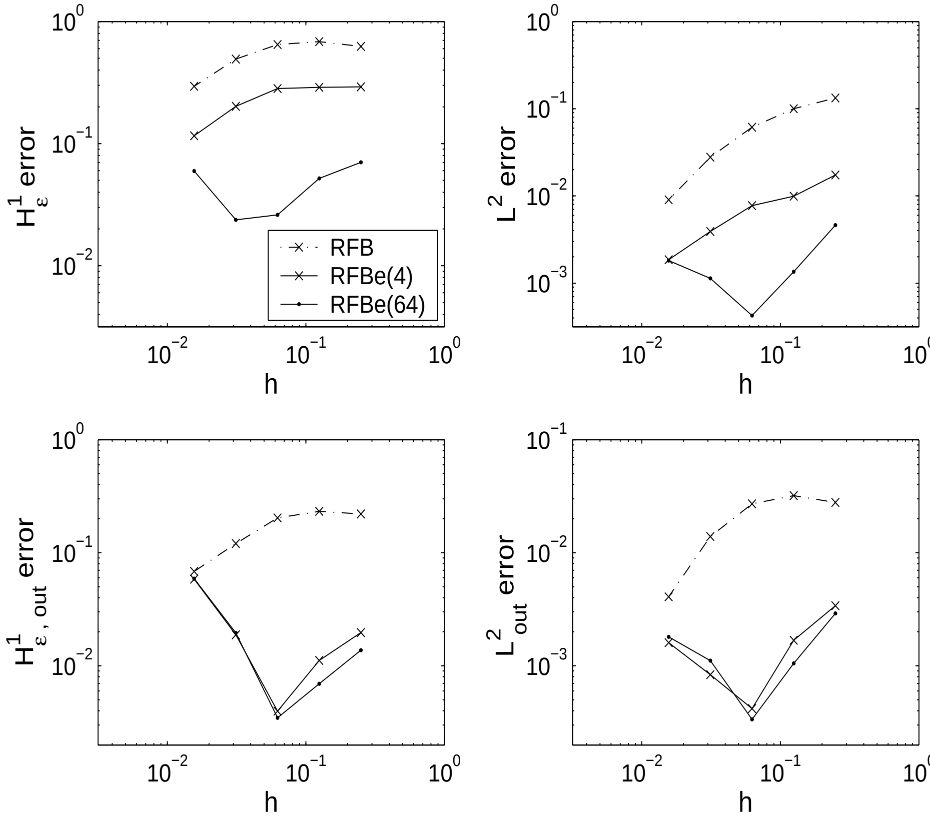

Figure 1. Example 1. $\sqrt{\varepsilon}$-weighted energy norm and L_2 norm error as a function of h; $\varepsilon = 10^{-2}$.

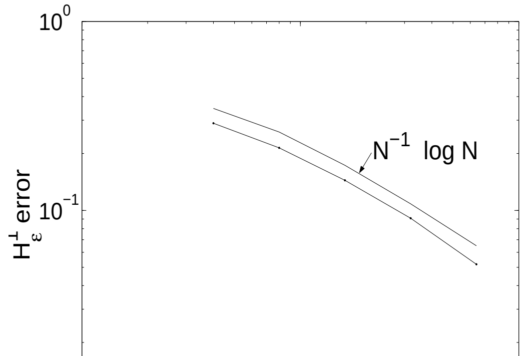

Figure 2. Example 1. $\sqrt{\varepsilon}$-weighted energy norm error on a uniform 8×8 mesh as a function of the subgrid discretisation parameter N with $\varepsilon = 10^{-2}$.

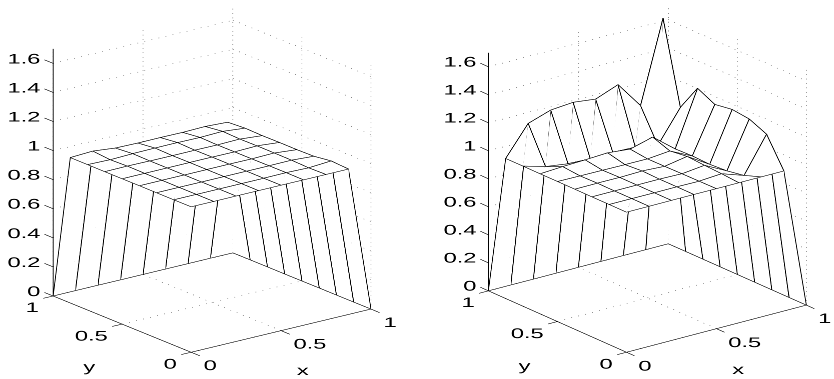

Figure 3. Example 2. Solution profiles ($\varepsilon = 10^{-2}$): RFBe(4) (left) and RFB (right).

3. CONCLUSIONS

We have shown how a small number of edge-bubbles can be introduced to improve the resolution of boundary layers of the RFB method in the context of convection–dominated convection-diffusion boundary value problems. The resulting scheme has better accuracy and stability properties than RFB in the regime $\varepsilon < h$.

Indeed, although our actions were local (i.e. confined to the boundary layer), the new scheme obtains increased resolution globally, indicating that the introduction of the edge-bubbles has a stabilising effect. Moreover, the method is sensitive to the accuracy of the evaluation of the edge–bubbles only inasmuch as accuracy within the layer is concerned. In other words, the method can be seen either as a way of performing local mesh refinement near the layer, or as a better stabilised and computationally competitive alternative to the classical RFB method.

The RFBe method can be generalised to the solution of convection–diffusion equations with a symmetric tensor diffusion coefficient, see [5]. the adaptive detection of layers where edge–bubbles are required is another area of our research. Our results in this direction will be reported elsewhere.

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