Simple diagonal locally finite Lie algebras

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1 Introduction

A Lie algebra over a field F is called locally finite-dimensional (or locally finite, for brevity) if any finitely generated subalgebra is finite-dimensional. In this paper we continue the study of diagonal locally finite Lie algebras introduced in [4] (see also [5]). The aim is to consider in detail simple ones. The intensive investigation of simple locally finite Lie algebras was begun by Y. Bahturin and H. Strade in [2]. In particular, they have observed that the class of these algebras is very large, and their classification is a rather daunting task. So it is necessary to look for reasonable restrictions. In [4] the notion of *diagonality* of locally finite Lie algebras over an algebraically closed field of zero characteristic was introduced. It turns out that this notion is closely linked with the possibility of embedding a locally finite Lie algebra L into a locally finite associative algebra. Recall that in the finite-dimensional case the corresponding problem is solved by well-known Ado's theorem. So, if such an embedding exists, we shall call L an *Ado algebra*. Observe that L is an Ado algebra if and only if there exists an ideal X in the universal enveloping algebra $U(L)$ such that $X \cap L = 0$ and $U(L)/X$ is locally finite. For the case of simple locally finite Lie algebras the two notions coincide, i.e. L is an Ado algebra if and only if it is diagonal ([4, Corollary 5.11]). One of the main results of this paper is the description of simple diagonal Lie algebras (or equivalently, simple Ado algebras). To give precise statements of this and other results we need some notation.

As in [4], we assume that the ground field F is an algebraically closed field of zero characteristic. Let L be a locally finite Lie algebra over F. This means that every finite set of elements of L is contained in a finite-dimensional subalgebra. If the latter can always be chosen simple (resp., semisimple), then L is called locally simple (resp., semisimple). A set $\{L_i\}_{i\in I}$ of finite-dimensional subalgebras of L

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is called a *local system* of L if $L = \cup_{i \in I} L_i$ and for any pair $i, j \in I$ there exists $k \in I$ such that $L_i, L_j \subseteq L_k$. Set $i \leq j$ if $L_i \subseteq L_j$. Then I is a directed set, i.e. for any pair $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$. It is clear that L is the direct limit of the algebras L_i, that is, $L = \varinjlim L_i$. If L has countable dimension, then L_i can be chosen in such a way that $I = \mathbf{N}$ and $L_i \subset L_{i+1}$, $i \in \mathbf{N}$, where $\mathbf{N}$ is the set of positive integers. Let V be an L_i-module. By $\langle V \rangle$ we denote the set of composition factors of V (multiplicities are not taken into account).

Assume that L is simple. Then by [2, Theorem 3.2], all L_i can be chosen perfect (i.e. $[L_i, L_i] = L_i$). We shall call such local systems *perfect*. Let $S_i = S_i^1 \oplus \ldots \oplus S_i^{n_i}$ be a Levi subalgebra of L_i, $S_i^1, \ldots, S_i^{n_i}$ simple components of S_i, V_i^k the standard S_i^k-module. Since L_i is perfect, for every k there exists a unique irreducible L_i-module $\mathcal{V}_i^k$ such that the restriction $\mathcal{V}_i^k \downarrow S_i^k$ is isomorphic to V_i^k. An embedding $L_i \to L_j$ for $i < j$ is called *diagonal* if

$$\langle \mathcal{V}_j^l \downarrow L_i \rangle \subseteq \{\mathcal{V}_i^1, \ldots, \mathcal{V}_i^{n_i}, \mathcal{V}_i^{1^*}, \ldots, \mathcal{V}_i^{n_i^*}, T_i\}, \quad 1 \leq l \leq n_j,$$

where T_i is the trivial one-dimensional L_i-module, and $\mathcal{V}_i^{k^*}$ is the dual module to $\mathcal{V}_i^k$. We illustrate this definition by the following example. An embedding $\mathfrak{sl}(V) \to \mathfrak{sl}(W)$ is diagonal if and only if one can choose a basis of W such that

$$A \mapsto \operatorname{diag}(\underbrace{A, \ldots, A}_{l}, \underbrace{-A^t, \ldots, -A^t}_{r}, \underbrace{0, \ldots, 0}_{z})$$

for any matrix $A \in \mathfrak{sl}(V)$ where l, r, z do not depend on A, $z + (l + r) \dim V = \dim W$.

One of our main results is

Theorem 1.1 *A simple locally finite Lie algebra L is an Ado algebra (or equivalently, diagonal) if and only if there exists a perfect local system $\{L_i\}_{i \in I}$ of L such that all embeddings $L_i \to L_j$, $i < j$, are diagonal.*

It is worth mentioning that simple diagonal locally finite Lie algebras (constructed as direct limits of diagonal embeddings) appeared in a number of papers. For instance, Y. Bahturin and H. Strade have shown that there is an uncountable family of pairwise nonisomorphic ones [2] (see also [8]), and have given examples of ones that are not locally semisimple [3]. Embeddings similar to diagonal appear in [1] where highest weight modules for locally finite Lie algebras are studied. Very often a particular case of diagonal embeddings occurs. An embedding $L_i \to L_j$ of classical simple Lie algebras is called *natural* if the restriction of the standard L_j-module to L_i involves a unique nontrivial composition factor, and the latter is isomorphic to the standard L_i-module or dual to it. There are only three nonisomorphic simple Lie algebras of countable dimension that are limits of natural embeddings of simple finite-dimensional Lie algebras (over a fixed algebraically closed field of zero characteristic): $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$. They can be represented as the limits of the embeddings

$$\mathfrak{sl}_2 \to \mathfrak{sl}_3 \to \ldots \to \mathfrak{sl}_n \to \ldots$$
$$\mathfrak{so}_2 \to \mathfrak{so}_3 \to \ldots \to \mathfrak{so}_n \to \ldots$$
$$\mathfrak{sp}_2 \to \mathfrak{sp}_4 \to \ldots \to \mathfrak{sp}_{2n} \to \ldots$$

Let $L = \varinjlim L_i$. Denote by $\operatorname{Irr} L_i$ the set of irreducible L_i-modules (up to equivalence). Let Φ_i be a finite subset of $\operatorname{Irr} L_i$. We say that $\Phi = \{\Phi_i\}_{i\in I}$ is an *inductive system* for L if

$$\cup_{\varphi\in\Phi_j}\langle\varphi{\downarrow}L_i\rangle = \Phi_i$$

for any pair $i < j$. This notion was introduced by A.E. Zalesskii. Inductive systems play an important role in the theories of locally finite groups and Lie algebras (see, for instance, [12]). Notice that the lattice of inductive systems for a simple locally finite Lie algebra L is antiisomorphic to that of semiprimitive ideals of its universal enveloping algebra $U(L)$ with locally finite quotients (Corollary 2.4), so it does not depend on the choice of local system. In particular, L is an Ado algebra if and only if the lattice is nontrivial. Using the antiisomorphism above, we prove

Theorem 1.2 *Let L be a simple locally finite Lie algebra. Then semiprimitive ideals of $U(L)$ with locally finite quotients satisfy the ascending chain condition.*

Some estimates for inductive systems (Theorem 3.12) allow us to distinguish the class of "thin" simple locally finite Lie algebras (see Definition 3.11). In Section 5 all such algebras of countable dimension are classified. It turns out that there are only three: $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$ (Corollary 5.4). Lattices of inductive systems for these algebras were described by A.G. Zhilinskii [16]. Our experience hints that the lattices for other simple locally finite Lie algebras are much simpler.

Let V be a vector space. All linear transformations of V form a Lie algebra $\mathfrak{gl}(V)$. An element $x \in \mathfrak{gl}(V)$ is called *finitary* if $\dim xV < \infty$. The finitary transformations of V form an ideal $\mathfrak{fgl}(V)$ of $\mathfrak{gl}(V)$, and any subalgebra of $\mathfrak{fgl}(V)$ is called a *finitary Lie algebra*. Since $\mathfrak{fgl}(V)$ is locally finite, any finitary Lie algebra is also locally finite. One can easily check that finitary Lie algebras are diagonal. It is a remarkable fact that a simple locally finite Lie algebra is finitary if and only if it is thin (Theorem 5.6). As a result we have

Theorem 1.3 *There are only three nonisomorphic finitary simple Lie algebras of countable dimension: $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$.*

The author has classified all finitary simple Lie algebras and will expose this in a subsequent paper. The analogous problem in group theory has been recently solved by J.I. Hall [9]. He classified simple locally finite groups of finitary linear transformations.

Let G be an infinite locally finite group that is a direct limit of finite alternating groups. Using properties of inductive systems, A.E. Zalesskii [11] has shown that any proper (two-sided) ideal of the complex group algebra $\mathbf{C}G$ is contained in the augmentation ideal. The following problem arises naturally. Describe all simple locally finite groups whose group algebras do not satisfy the property above. Examples of such groups exist [13] (see also [10] for the case of group algebras over a field of positive characteristic). One can reformulate the above question for Lie algebras as follows. Describe all simple locally finite Lie algebras L such that there is a proper ideal X of $U(L)$ with locally finite quotient $U(L)/X$, not lying in the augmentation ideal $A(L)$. By $A(L)$ we mean the ideal of $U(L)$ generated by all $l \in L$. We prove (Theorem 6.3) that the only simple locally finite Lie algebras with the above property are those which are either finitary of type B or "diagonally dense".

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2 Notation and preliminary results

We use the notation of the paper [4].

By $\mathbf{N}$ we denote the set of positive integers. All Lie algebras in question are finite-dimensional or locally finite. In the latter case we assume in addition that they are infinite-dimensional. For finite-dimensional algebras we shall consider only finite-dimensional modules. $U(L)$ denotes the universal enveloping algebra of a Lie algebra L.

Let L be a finite-dimensional Lie algebra. Denote by $\operatorname{Irr} L$ the set of irreducible L-modules (up to equivalence). For an L-module V let $\langle V\rangle$ denote the set of composition factors of V (multiplicities are not taken into account). If $\Phi = \{V_i\}_{i\in I}$ is a set of L-modules, then $\langle\Phi\rangle = \cup_{i\in I}\langle V_i\rangle$. If S is a subalgebra of L (or more generally, a homomorphism $S \to L$ is given), then $V{\downarrow}S$ denotes the restriction of the L-module V to S. If $\Phi = \{V_i\}_{i\in I}$ is a set of L-modules, then $\Phi{\downarrow}S = \{V_i{\downarrow}S\}_{i\in I}$. Denote by $\operatorname{Rad} L$ the solvable radical of L. Let S be a Levi subalgebra of L, W an S-module. We can consider W as an L-module, setting $(\operatorname{Rad} L)W = 0$. Assume that L is perfect, i.e. $[L, L] = L$. Then $(\operatorname{Rad} L)V = 0$ for any irreducible L-module V. Therefore there is a natural bijection between irreducible L- and S-modules. Sometimes we shall identify corresponding modules.

Let $\mathfrak{h}$ be a classical simple finite-dimensional Lie algebra of rank m. Denote by $\omega_1, \dots, \omega_m$ its fundamental weights. (We label fundamental weights as in [7]). We define two linear functions δ and σ on the weights of $\mathfrak{h}$, by writing down their values on the fundamental weights. In the following list $\sigma(\omega_i) = p_i$, $\delta(\omega_i) = q_i$, $1 \le i \le m$, are abbreviated by $\sigma = (p_1, \dots, p_m)$, $\delta = (q_1, \dots, q_m)$. Put

$\sigma = (1, 1, \dots, 1, 1, 1)$, $\quad\delta = (1, 2, \dots, k, k, \dots, 2, 1)$ $\qquad (A_{2k},\ k \ge 1)$;
$\sigma = (1, 1, \dots, 1, 1, 1)$, $\quad\delta = (1, 2, \dots, k+1, \dots, 2, 1)$ $\qquad (A_{2k+1},\ k \ge 0)$;
$\sigma = (1, 1, \dots, 1, 1, 1)$, $\quad\delta = (1, 2, \dots, m-2, m-1, m)$ $\qquad (C_m,\ m \ge 2)$;

$\sigma = (1, 1, \ldots, 1, 1, \frac{1}{2}), \quad \delta = (1, 2, \ldots, m-2, m-1, [\frac{m}{2}]) \qquad (B_m,\ m \geq 3);$
$\sigma = (1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{2}), \quad \delta = (1, 2, \ldots, 2k-2, k-1, k) \qquad (D_{2k},\ k \geq 2);$
$\sigma = (1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{2}), \quad \delta = (1, 2, \ldots, 2k-1, k, k) \qquad (D_{2k+1},\ k \geq 2).$

To avoid repetition, we shall use the symbol χ below to denote both δ and σ. Let V be an $\mathfrak{h}$-module with the set of weights Λ. Put $\chi_{\mathfrak{h}}(V) = \sup\{\chi(\mu)\}_{\mu\in\Lambda}$. It is not difficult to show (see [4]) that if V is an irreducible $\mathfrak{h}$-module with highest weight λ, then $\chi_{\mathfrak{h}}(V) = \chi(\lambda)$. If $\Phi = \{V_i\}_{i\in I}$ is a set of $\mathfrak{h}$-modules, set $\chi_{\mathfrak{h}}(\Phi) = \sup\{\chi_{\mathfrak{h}}(V_i)\}_{i\in I}$. It is clear that $\chi_{\mathfrak{h}}(V) = \chi_{\mathfrak{h}}(\langle V\rangle)$. Let $\mathfrak{g}$ be another classical simple Lie algebra. Assume that $\mathfrak{h} \subset \mathfrak{g}$. Let W be a $\mathfrak{g}$-module. Set $\chi_{\mathfrak{h}}(W) = \chi_{\mathfrak{h}}(W{\downarrow}\mathfrak{h})$.

Definition 2.1 A locally finite Lie algebra is called *locally perfect* if no nontrivial quotient is locally solvable, or equivalently ([4, Lemma 2.4]), if it has a perfect local system.

It is known (see [2, Corollary 3.2 and Theorem 3.2], or [4, Proposition 2.8]) that any simple locally finite Lie algebra is semisimple (i.e. no nontrivial ideal is locally solvable) and locally perfect, so it has perfect local systems. As follows from the proof of [2, Theorem 3.2] one can construct perfect local systems in the following manner.

Remark 2.2 Let $\{L_i\}_{i\in I}$ be a local system of a locally perfect Lie algebra L. Denote by $L_i^{(\infty)}$ the smallest member of the derived series of L_i. Then $\{L_i^{(\infty)}\}_{i\in I}$ is a perfect local system of L.

In what follows, we shall assume, unless otherwise stated, that all local systems considered are perfect. So the notation $L = \varinjlim L_i$ means that $\{L_i\}_{i\in I}$ is a perfect local system of L.

Denote by $\mathfrak{IS}$, $\mathfrak{LF}$ the sets of inductive systems of a locally perfect Lie algebra L, and of ideals of its universal enveloping algebra $U(L)$ with locally finite quotients, respectively. Let $X \in \mathfrak{LF}$. Then the set

$$\Phi(X) = \{\langle U(L_i)/X \cap U(L_i)\rangle\}_{i\in I}$$

is an inductive system for L ([4, Lemma 3.8]). So we can define a map $f : \mathfrak{LF} \to \mathfrak{IS}$, setting $f(X) = \Phi(X)$ where $X \in \mathfrak{LF}$. Denote by $\mathfrak{LF}(\Phi)$ the inverse image of the inductive system Φ.

Theorem 2.3 ([4, Theorem 3.9]) *Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $f : \mathfrak{LF} \to \mathfrak{IS}$ the above map. Then for any inductive system Φ the set $\mathfrak{LF}(\Phi)$ is nonempty and has a smallest element $N(\Phi)$ and a largest element $M(\Phi)$ so that $N(\Phi) \subseteq X \subseteq M(\Phi)$ for any $X \in \mathfrak{LF}(\Phi)$. The algebra $U(L)/M(\Phi)$ is semisimple, the algebra $M(\Phi)/N(\Phi)$ is locally nilpotent. Moreover, the map f produces a 1-1 correspondence between semiprimitive ideals in $\mathfrak{LF}$ and inductive systems for L (the inverse map is given by $\Phi \mapsto M(\Phi)$).*

Let $\Phi = \{\Phi_i\}_{i\in I}$ be the empty inductive system, i.e. $\Phi_i = \emptyset$, $i \in I$. It is convenient to assume that $\mathfrak{LF}(\Phi) = \{U(L)\}$. Let $\Phi = \{\Phi_i\}_{i\in I}$ be the trivial inductive system, i.e. $\Phi_i = T_i$, $i \in I$, where T_i is the trivial one-dimensional L_i-module. The corresponding semiprimitive ideal $A(L) = M(\Phi)$ of $U(L)$ is called the augmentation ideal. This is the ideal of $U(L)$ of codimension one generated by all $l \in L$. One can derive from Theorem 2.3 that a simple locally finite Lie algebra L is an Ado algebra if and only if it has a nontrivial inductive system. Moreover, nontrivial inductive systems give a parameterization of semisimple locally finite enveloping algebras of L.

Let $\Phi^1 = \{\Phi_i^1\}_{i\in I}$, $\Phi^2 = \{\Phi_i^2\}_{i\in I}$ be inductive systems for a locally perfect Lie algebra $L = \varinjlim L_i$. We say that $\Phi^1 \subseteq \Phi^2$ if $\Phi_i^1 \subseteq \Phi_i^2$ for any $i \in I$. Clearly, $\mathfrak{IS}$ is a lattice with respect to the operations $\Phi^1 \cup \Phi^2 = \{\Phi_i^1 \cup \Phi_i^2\}_{i\in I}$ and $\Phi^1 \wedge \Phi^2 = \Phi$ where Φ is the largest inductive system contained in Φ^1 and Φ^2. Let $X_1 \subseteq X_2$ be ideals in $\mathfrak{LF}$. Then it is not difficult to see that $\Phi(X_2) \subseteq \Phi(X_1)$. We now show that for semiprimitive ideals the converse statement also holds. That is, $\Phi^1 \subseteq \Phi^2$ implies $M(\Phi^2) \supseteq M(\Phi^1)$ where $M(\Phi^1)$ and $M(\Phi^2)$ are the corresponding semiprimitive ideals of $U(L)$. First of all note that for any $X_1, X_2 \in \mathfrak{LF}$ we have $\Phi(X_1 \cap X_2) = \Phi(X_1) \cup \Phi(X_2)$. Indeed, this follows from the equality

$$\langle U(L_i)/(X_1 \cap X_2 \cap U(L_i))\rangle = \langle U(L_i)/(X_1 \cap U(L_i))\rangle \cup \langle U(L_i)/(X_2 \cap U(L_i))\rangle, \quad i \in I.$$

In particular,

$$\Phi(M(\Phi^1) \cap M(\Phi^2)) = \Phi_1 \cup \Phi_2 = \Phi_2.$$

The ideal $M(\Phi^1) \cap M(\Phi^2)$ is semiprimitive (as the intersection of semiprimitive ideals). Therefore by Theorem 2.3, $M(\Phi^1) \cap M(\Phi^2) = M(\Phi^2)$, i.e. $M(\Phi^2) \supseteq M(\Phi^1)$. So we have

Corollary 2.4 *Let L be a locally perfect Lie algebra. Then the lattices of inductive systems for L and semiprimitive ideals of $U(L)$ with locally finite quotients are antiisomorphic.*

Definition 2.5 A subalgebra S of a locally finite Lie algebra $L = \varinjlim L_i$ is called a *Levi subalgebra* associated with the local system $\{L_i\}_{i\in I}$, if $S = \varinjlim S_i$ where S_i is a Levi subalgebra of L_i and $S_i \subseteq S_j$ for any $i \leq j$.

Observe that a Levi subalgebra is locally semisimple.

Given a perfect local system $\{L_i\}_{i\in I}$ for a locally perfect Lie algebra L of countable dimension, one can construct a Levi subalgebra S of L associated with $\{L_i\}_{i\in I}$ [4, Lemma 4.3]. Recall that there is a natural bijection between the irreducible modules of a finite-dimensional perfect Lie algebra and those of its Levi subalgebra. So the description of inductive systems for L is equivalent to that for S. It is not clear whether all locally perfect Lie algebras have Levi subalgebras. To handle the general case we introduced in [4] the notions of abstract Levi subalgebras and inductive systems for them. However, to simplify arguments, we assume here that all locally perfect Lie algebras in question have Levi subalgebras. This is in fact a purely technical assumption (for details see [4]). Note also that recently A.E.Zalesskii [14] introduced the notion of an *external Levi subalgebra* for a locally finite Lie algebra of countable dimension. This notion is useful for proving that all Levi subalgebras of a locally finite Lie algebra of countable dimension are isomorphic.

Now we are going to introduce the notion of a *Bratteli diagram.* This notion appeared in the investigation of locally finite associative algebras. In [4] we adapted it for locally finite Lie algebras. Let L be a locally finite Lie algebra. Assume that L has countable dimension. Then L can be expressed in the form $L = \varinjlim L_i$ where $L_i \subset L_{i+1}$, $i \in \mathbf{N}$. Let $S = \varinjlim S_i$ be a Levi subalgebra of L associated with the local system $\{L_i\}_{i\in\mathbf{N}}$. Recall that S_i is a Levi subalgebra of L_i. Let $S_i = S_i^1 \oplus \ldots \oplus S_i^{n_i}$ where S_i^k are the simple components of S_i. Recall that any irreducible S_i-module M can be written in the (canonical) form $M = M_1 \otimes \ldots \otimes M_{n_i}$ where M_k is an irreducible S_i-module such that $M_k{\downarrow}S_i^k$ is irreducible and $\langle M_k{\downarrow}S_i^l\rangle = \{T_i^l\}$ for $l \neq k$ with T_i^l the trivial one-dimensional S_i^l-module. Denote by V_i^k the standard module for S_i^k classical and the module of the minimal dimension for S_i^k exceptional. We shall identify this module with the corresponding irreducible S_i-module (such that all S_i^l act trivially for $l \neq k$). A Bratteli diagram of L associated with the local system $\{L_i\}_{i\in\mathbf{N}}$ is an $\mathbf{N}$-graded graph $\mathfrak{B}$ defined as follows. The nodes at level i of $\mathfrak{B}$ are in one-to-one correspondence with the simple components of S_i, and they are labelled by the corresponding components. Two nodes S_i^k and S_{i+1}^l at neighboring levels are joined by an edge (denoted by (S_i^k, S_{i+1}^l)) if $\langle V_{i+1}^l{\downarrow}S_i^k\rangle \neq \{T_i^k\}$. By path we mean an increasing (finite or infinite) sequence of nodes $\Gamma = (S_i^k, S_{i+1}^l, S_{i+2}^m, \ldots)$ such that all the neighboring nodes are joined by edges. Denote by $\mathcal{P}(S_i^k)$ the set of paths beginning in S_i^k, by $\mathcal{P}(S_i^k, S_j^l)$ the set of paths beginning in S_i^k and ending in S_j^l. A node S_j^l is called S_i^k-accessible if $\mathcal{P}(S_i^k, S_j^l) \neq \emptyset$. A node S_j^l is called critical if $\operatorname{rk} S_j^l > 12$ and there exists $M \in \langle V_j^l{\downarrow}S_{j-1}\rangle$ such that $\langle M{\downarrow}S_{j-1}^p\rangle \neq \{T_{j-1}^p\}$ and $\langle M{\downarrow}S_{j-1}^q\rangle \neq \{T_{j-1}^q\}$ for two distinct nodes at level $j-1$. Let $\Gamma \in \mathcal{P}(S_i^k)$. A node $S_j^l \in \Gamma$ is called Γ-critical of degree g if $\operatorname{rk} S_j^l > 12$ and there exists $M \in \langle V_j^l{\downarrow}S_{j-1}\rangle$ such that $\langle M{\downarrow}S_{j-1}^{p_i}\rangle \neq \{T_{j-1}^{p_i}\}$ for $g \geq 2$ distinct S_i^k-accessible nodes $S_{j-1}^{p_i}$, $i = 1, \ldots, g$, with $S_{j-1}^{p_1} \in \Gamma$, and such g is maximal. An edge (S_i^k, S_{i+1}^l) is called nonstandard if $\langle V_{i+1}^l{\downarrow}S_i^k\rangle \not\subseteq \{T_i^k, V_i^k, V_i^{k^*}\}$ and $\operatorname{rk} S_{i+1}^l > 12$, and standard otherwise. Let $j > i$, V be an S_j-module. Set $\delta_i^k(V) = \delta_{S_i^k}(V{\downarrow}S_i^k)$, $\sigma_i^k(V) = \sigma_{S_i^k}(V{\downarrow}S_i^k)$ By δ-rank and σ-rank of the edge (S_i^k, S_{i+1}^l) we mean the numbers $\delta_i^k(V_{i+1}^l)$ and $\sigma_i^k(V_{i+1}^l)$ respectively. Denote by $\tau(S_i^k, S_j^l)$ the number of nontrivial composition factors of $V_j^l{\downarrow}S_i^k$.

For the case of a Lie algebra $L = \varinjlim L_i$ of uncountable dimension the picture is as follows. For every ascending chain of indices $C : i_1 < i_2 < \ldots$ we construct its own Bratteli diagram $\mathfrak{B}_C$ (the Bratteli diagram of the Lie algebra $L_C = \cup_{j\in\mathbf{N}} L_{i_j}$). By the Bratteli diagram $\mathfrak{B}$ of L we mean the set of all $\mathfrak{B}_C$. We say that $\mathfrak{B}$ satisfies a property Π if all $\mathfrak{B}_C$ satisfy Π.

As has been shown in [4], a Bratteli diagram associated with the local system $\{L_i\}_{i\in I}$ does not depend on the choice of a Levi subalgebra associated with $\{L_i\}_{i\in I}$. Therefore the choice of $\{L_i\}_{i\in I}$ uniquely determines the corresponding Bratteli diagram.

3 Local systems

In this section we construct nice local systems for simple diagonal locally finite Lie algebras.

Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $\mathfrak{B}$ the corresponding Bratteli diagram. Let S_i^k be a node of $\mathfrak{B}$. Denote by $\mathfrak{B}_i^k$ the subdiagram of $\mathfrak{B}$ defined as follows. $\mathfrak{B}_i^k$ consists of all S_i^k-accessible nodes of $\mathfrak{B}$ and the edges joining them. Let $\{S_j^{p_1}, \dots, S_j^{p_m}\}$ be the set of all S_i^k-accessible nodes at level $j \geq i$. Set $S'_j = S_j^{p_1} \oplus \dots \oplus S_j^{p_m}$, $L'_j = (S'_j \oplus R_j)^{(\infty)}$. Observe that L'_j is a perfect ideal of L_j, S'_j is a Levi subalgebra of L'_j, and $S'_r \subseteq S'_t$ if $i \leq r \leq t$. We now show that $L'_r \subseteq L'_t$. Since R_r is solvable, there exists n such that $R_r^{(n)} = 0$. It follows that

$$L'_r \subseteq (S'_j \oplus R_j)^{(n)} \subseteq \mathrm{id}_{L_r}(S'_r) \subseteq \mathrm{id}_{L_t}(S'_t) \subseteq S'_t \oplus R_t,$$

where $\mathrm{id}_{L_r}(S'_r)$ is the ideal of L_r generated by S'_r. Since L'_r is perfect, $L'_r \subseteq (S'_t \oplus R_t)^{(\infty)} = L'_t$. Therefore $\mathfrak{C}_i^k = \{L'_j\}_{j\geq i}$ is a local system of $N_i^k = \varinjlim L'_j$. The corresponding Bratteli diagram of N_i^k has the "minimal" node S_i^k, and all other nodes are S_i^k-accessible. We shall call such Bratteli diagrams and local systems *conical*. By the *rank* of a conical local system we mean the rank of its "minimal" node. It is clear that N_i^k is an ideal of L, and $\mathfrak{B}_i^k$ is a Bratteli diagram of N_i^k. Assume that L is simple. Then $N_i^k = L$. Therefore $\mathfrak{C}_i^k$ is a local system for L.

Theorem 3.1 *Let $L = \varinjlim L_i$ be a semisimple locally perfect Lie algebra, $\mathfrak{B}$ the corresponding Bratteli diagram of L. Then L is simple if and only if $\mathfrak{C}_i^k$ is a local system of L for any node S_i^k of $\mathfrak{B}$.*

Proof. In view of the above remark it remains to prove the "if" part. Let N be a nonzero ideal of L. Since L is semisimple, N is not locally solvable. Therefore there exists $i \in I$ such that the ideal $N \cap L_i$ of L_i is not solvable. Hence $N \cap S_i \neq 0$, so $S_i^k \subset N$ for some $k = 1, \dots, n_i$. It is clear that N contains all S_i^k-accessible nodes of $\mathfrak{B}$, i.e. $S'_j \subset N$ for any $j \geq i$. Since $\mathfrak{C}_i^k = \{L'_j\}_{j\geq i}$ is a local system of L, and S'_j is a Levi subalgebra of L'_j, we conclude that L/N is locally solvable. Since L is locally perfect, $N = L$. Therefore L is simple. This proves the theorem.

Remark that a result similar to Theorem 3.1 (the "only if" part) has been obtained earlier by Yu. Bahturin and H. Strade [2, Proposition 3.1]. Their paper contains some other information about local systems of simple Lie algebras.

We shall use also another simplicity criterion.

Theorem 3.2 *Let $\mathfrak{B}$ be a Bratteli diagram of a semisimple locally perfect Lie algebra L. Then L is simple if and only if for any nodes S_i^k, S_j^l of $\mathfrak{B}$ there exists $p > i, j$ such that for any $q \geq p$ the sets of S_i^k- and S_j^l-accessible nodes at level q coincide.*

Proof. Assume that L is simple. Denote by $A_1(t)$, $A_2(t)$ the sets of S_i^k- and S_j^l-accessible nodes at level t, respectively. By Theorem 3.1, $\mathfrak{C}_i^k$ is a local system of L. Therefore there exists $t \geq i, j$ such that $S_j^l \subseteq L'_t = (S_t^{p_1} \oplus \ldots \oplus S_t^{p_m} \oplus R_t)^{(\infty)}$ where $\{S_t^{p_1}, \ldots, S_t^{p_m}\} = A_1(t)$, i.e. $A_2(t) \subset A_1(t)$. Moreover, it is clear that $A_2(r) \subset A_1(r)$ for any $r \geq t$. Since $\mathfrak{C}_j^l$ is another local system of L, there exists $p > t$ such that $A_1(p) \subset A_2(p)$. Hence $A_1(p) = A_2(p)$. It is obvious that $A_1(q) = A_2(q)$ for any $q \geq p$.

Conversely. By the arguments preceding Theorem 3.1, $\mathfrak{C}_i^k = \{L'_j\}_{j \geq i}$ is a local system of an ideal $N_i^k = \varinjlim L'_j$ of L. It follows from the assumptions that for any node S_j^l of $\mathfrak{B}$ there exists $p > i$ such that $S_j^l \subset L'_p$, so $S_j^l \subset N_i^k$. Therefore the quotient $L_j/(N_i^k \cap L_j)$ is solvable for any $j \in I$, so L/N_i^k is locally solvable. Since L is locally perfect, we conclude that $L = N_i^k$. Hence $\mathfrak{C}_i^k$ is a local system of L for any node S_i^k of $\mathfrak{B}$. Therefore by Theorem 3.1, L is simple.

Let $S = \varinjlim S_i$ be a Levi subalgebra of $L = \varinjlim L_i$. Since the corresponding Bratteli diagrams of these algebras coincide, we obtain the following useful

Corollary 3.3 *Let S be a Levi subalgebra of a semisimple locally perfect Lie algebra L. Then L is simple if and only if S is.*

The following is a generalization of Theorem 3.2.

Corollary 3.4 *Let $\mathfrak{B}$ be a Bratteli diagram of a semisimple locally perfect Lie algebra L. Then L is simple if and only if for any $n \geq 2$ nodes $S_{i_1}^{k_1}, \ldots, S_{i_n}^{k_n}$ of $\mathfrak{B}$ there exists $p > i_1, \ldots, i_n$ such that for any $q \geq p$ the sets of $S_{i_1}^{k_1}$-,…,$S_{i_n}^{k_n}$-accessible nodes at level q coincide.*

Proof. It suffices to prove the "only if" part. By Theorem 3.2, for any $1 \leq l, r \leq n$ there exists $p(l,r) > \max(i_l, i_r)$ such that for any $q \geq p(l,r)$ the sets of $S_{i_l}^{k_l}$- and $S_{i_r}^{k_r}$-accessible nodes at level q coincide. It is clear that the number $p = \sup\{p(l,r) \mid 1 \leq l, r \leq n\}$ has the required property.

If L has countable dimension, then we obtain the following

Corollary 3.5 *Let L be a simple locally finite Lie algebra of countable dimension. Then there exists a local system of subalgebras of L indexed by positive integers such that any two nodes at neighboring levels of the corresponding Bratteli diagram are joined by an edge.*

Proof. Let $\{L_i\}_{i\in\mathbf{N}}$ be a local system of L. One can assume that the corresponding Bratteli diagram $\mathfrak{B}$ is conical with minimal node S_1^1. Choose a sequence of indices $p_1 < p_2 < p_3 < \ldots$ as follows. Set $p_1 = 1$. Assume that $p_1, \ldots, p_k$ have been found. Let $S_{p_k}^1, \ldots, S_{p_k}^n$ be nodes of $\mathfrak{B}$ at level p_k. By Theorem 3.4 there exists $t > p_k$ such that the sets of $S_{p_k}^1$-,…,$S_{p_k}^n$-accessible nodes at level t coincide. It is clear that these sets coincide with the set of all nodes at level t, since all these nodes are S_1^1-accessible. Put $p_{k+1} = t$. It is not difficult to check that the Bratteli diagram associated with the local system $\{L_{p_j}\}_{j\in\mathbf{N}}$ satisfies the conditions of the theorem.

Definition 3.6 Let $i < j$. An embedding $L_i \subset L_j$ is called *diagonal* if

$$\langle V_j^l{\downarrow}S_i\rangle \subseteq \{V_i^1, \ldots, V_i^{n_i}, V_i^{1^*}, \ldots, V_i^{n_i^*}, T_i\}, \quad 1 \le l \le n_j,$$

where T_i is the trivial S_i-module.

One can easily check that this definition does not depend on the choice of Levi subalgebras in L_i and L_j.

Definition 3.7 A local system $\{L_i\}_{i\in I}$ and the corresponding Bratteli diagram $\mathfrak{B}$ of a locally perfect Lie algebra L are called *pure diagonal* if all embeddings $L_i \subset L_j$, $i < j$, are diagonal.

Theorem 3.8 *Let L be a simple diagonal locally finite Lie algebra, $n \in \mathbf{N}$. Then it has a conical pure diagonal local system of rank greater than n.*

Proof. One can derive from [2, Corollary 3.3] that there exists a local system $\{L_i\}_{i\in I}$ of L such that all nodes of the corresponding Bratteli diagram $\mathfrak{B}$ have ranks greater than $\max(12, n)$. By Theorem 3.1, we can assume that $\{L_i\}_{i\in I} = \mathfrak{C}_1^1$, i.e. all nodes of $\mathfrak{B}$ are S_1^1-accessible. Since L is diagonal, by [4, Corollary 8.5], $\mathfrak{B}$ is diagonal, i.e. for any path $\Gamma \in \mathcal{P}(S_1^1)$ the number of nonstandard edges and Γ-critical nodes of Γ is globally bounded. Take a node S_i^k such that this number is maximal for some $\Gamma \in \mathcal{P}(S_1^1, S_i^k)$. It is clear that any $\Gamma' \in \mathcal{P}(S_i^k)$ does not contain nonstandard edges and Γ'-critical nodes. Therefore $\mathfrak{B}_i^k$ and $\mathfrak{C}_i^k$ are pure diagonal. Hence by Theorem 3.1, $\mathfrak{C}_i^k$ is a pure diagonal local system of L. Observe that $\mathfrak{C}_i^k$ is conical of rank greater than n.

We collect information about simple diagonal locally finite Lie algebras in the following

Corollary 3.9 *Let L be a simple locally finite Lie algebra. Then the following conditions are equivalent.*

(1) L is diagonal.

(2) There exists a pure diagonal local system for L.

(3) There exists a nontrivial inductive system for L.

(4) There exists a proper ideal X of $U(L)$ such that $X \neq A(L)$ and $U(L)/X$ is locally finite.

(5) L is an Ado algebra.

(6) For any $x \in L$ there exists a polynomial f_x such that $f_x(\operatorname{ad} x) = 0$ (in $\operatorname{End} L$).

Proof. The equivalences (1)⇔(3)⇔(4)⇔(5)⇔(6) follow from [4, Theorem 5.7, Corollary 5.11 and Corollary 9.2]. The implication (1)⇒(2) was proved in Theorem 3.8. Finally, let $\{L_i\}_{i\in I}$ be a pure diagonal local system for L. Set

$$\Phi_i = \{V_i^1, \dots, V_i^{n_i}, V_i^{1^*}, \dots, V_i^{n_i^*}, T_i\}.$$

It is not difficult to check that $\Phi = \{\Phi_i\}_{i\in I}$ is an inductive system for L. Therefore (2)⇒(3).

It remains to observe that Theorem 1.1 is a particular case of Corollary 3.9.

The following lemma will be used in Section 5.

Lemma 3.10 *Let $\{L_i\}_{i\in I}$ be a pure diagonal local system, $\mathfrak{B}$ the corresponding Bratteli diagram. Let S_i^k, S_j^l be nodes of $\mathfrak{B}$, $i < t < j$. Then*

$$\tau(S_i^k, S_j^l) = \tau(S_i^k, S_t^1)\tau(S_t^1, S_j^l) + \ldots + \tau(S_i^k, S_t^{n_t})\tau(S_t^{n_t}, S_j^l).$$

Proof. Since the embedding $L_t \subset L_j$ is diagonal, we have

$$V_j^l{\downarrow}S_t = W_1^1 \oplus \ldots \oplus W_{a_1}^1 \oplus \ldots \oplus W_1^{n_t} \oplus \ldots \oplus W_{a_{n_t}}^{n_t} \oplus T_t \oplus \ldots \oplus T_t$$

where $a_p = \tau(S_t^p, S_j^l)$, $1 \le p \le n_t$, and W_q^p denotes one of the modules V_t^p, $(V_t^p)^*$. It remains to note that $V_j^l{\downarrow}S_i^k = V_j^l{\downarrow}S_t{\downarrow}S_i^k$.

Definition 3.11 A Bratteli diagram $\mathfrak{B}$ of a simple locally finite Lie algebra L is called *thin* if it is pure diagonal and for any node S_i^k there exists $m \in \mathbf{N}$ such that $\tau(S_i^k, S_j^l) \le m$ for any node S_j^l with $j > i$. L is called *thin* if it has a thin Bratteli diagram, and *thick* otherwise.

Theorem 3.12 *Let L be a simple diagonal locally finite Lie algebra, $\{L_i\}_{i\in I}$ a conical pure diagonal local system of L of rank greater than 12. Denote by $\mathfrak{B}$ the corresponding Bratteli diagram. Let $\Phi = \{\Phi_i\}_{i\in I}$ be an inductive system for L. Then there exist $c_1, c_2 \in \mathbf{N}$ such that for any $i \in I$*

(i) for any $M \in \Phi_i$ the number of nodes S_i^k at level i such that $\langle M{\downarrow}S_i^k\rangle \neq \{T_i^k\}$ is at most c_1,

(ii) $\sigma_i^k(\Phi_i) \le c_2,\ 1 \le k \le n_i$*;*

(iii) $\delta_i^k(\Phi_i) \le c_2,\ 1 \le k \le n_i$*, if* L *is thick.*

Proof. Let S_1^1 be the minimal node of $\mathfrak{B}$. By [4, Theorem 9.6] all nodes of $\mathfrak{B}$ are Φ-regular. In particular, S_1^1 is. This means that there exist $d_1, d_2, d_3 \in \mathbf{N}$ such that for any $i \in I$ one has

(1) for any $M \in \Phi_i{\downarrow}S_i$ the number of S_1^1-accessible nodes S_i^k at level i such that $\langle M{\downarrow}S_i^k\rangle \neq \langle T_i^k\rangle$ is at most d_1,

(2) $\sigma_i^k(\Phi_i) \le d_2$ for any S_1^1-accessible node S_i^k,

(3) if $\tau(S_1^1, S_i^k) > d_3$, then $\delta_i^k(\Phi_i) \le d_2$.

Set $c_1 = d_1$, $c_2 = d_1 d_2 \ge d_2$. Since all nodes of $\mathfrak{B}$ are S_1^1-accessible, *(i)* and *(ii)* hold. It remains to prove part *(iii)*. Assume that L is thick. We have to show that $\delta_i^k(\Phi_i) \le c_2$ for any node S_i^k. By the definition of thick algebras, there exists a node S_j^l such that $\tau(S_1^1, S_j^l) > d_3$, so $\delta_j^l(\Phi_j) \le d_2$. Assume that S_i^k is S_j^l-accessible. Then

$$\tau(S_1^1, S_i^k) \ge \tau(S_1^1, S_j^l)\tau(S_j^l, S_i^k) \ge \tau(S_1^1, S_j^l) > d_3.$$

Therefore $\delta_i^k(\Phi_i) \le d_2 \le c_2$. Assume now that S_i^k is not S_j^l-accessible. By Theorem 3.1, $\mathfrak{C}_j^l$ is a local system of L, so there exists $r \in I$ such that

$$S_i^k \subset L'_r = (S_r^{p_1} \oplus \ldots \oplus S_r^{p_m} \oplus R_r)^{(\infty)}$$

where $L'_r \in \mathfrak{C}_j^l$ (see the arguments preceding Theorem 3.1). Since the nodes $S_r^{p_1}, \ldots, S_r^{p_m}$ are S_j^l-accessible, as has been shown above, $\delta_r^{p_t}(\Phi_r) \le d_2$, $1 \le t \le m$. Since the embedding $S_i^k \subset L'_r$ is diagonal, by [4, Proposition 6.12 (a)], $\delta_i^k(\Phi_r{\downarrow}S_r^{p_t}) \le \delta_r^{p_t}(\Phi_r{\downarrow}S_r^{p_t}) \le d_2$, $1 \le t \le m$. Therefore by [4, Lemma 7.2],

$$\delta_i^k(\Phi_i) = \delta_i^k(\Phi_r{\downarrow}S_i) = \delta_i^k(\Phi_r) = \delta_i^k(\Phi_r{\downarrow}L'_r) \le$$

$$d_1 \max\{\delta_i^k(\Phi_r{\downarrow}S_r^{p_t}) \mid 1 \le t \le m\} \le d_1 d_2 = c_2,$$

as required. So the theorem follows.

4 Natural embeddings

This section contains some auxiliary lemmas needed for the classification of finitary simple Lie algebras. All Lie algebras considered in this section are finite-dimensional.

Let $S_1 \subset S_2 \subset S_3$ be classical simple Lie algebras of the same type. Let $\operatorname{rk} S_3 = n$, $\operatorname{rk} S_2 = n - l + 1$, $\operatorname{rk} S_1 = n - m + 1$ where $l < m < n$. Suppose that all embeddings $S_1 \subset S_2 \subset S_3$ are natural. Recall that an embedding of classical simple Lie algebras $\mathfrak{h} \to \mathfrak{g}$ is called natural if the restriction of the standard $\mathfrak{g}$-module to $\mathfrak{h}$ contains a unique nontrivial composition factor, and this factor is isomorphic to the standard $\mathfrak{h}$-module or dual to it. Then we can choose Cartan subalgebras $H_1 \subset H_2 \subset H_3$ of S_1, S_2, S_3, respectively, and simple roots $\alpha_1, \ldots, \alpha_l, \ldots, \alpha_m, \ldots, \alpha_n$ of S_3 (we label simple roots as in [7]) such that $\alpha_l, \ldots, \alpha_m, \ldots, \alpha_n$ are the simple roots of S_2, and $\alpha_m, \ldots, \alpha_n$ are those of S_1. Let ε_i be weights of the standard S_3-module (as in [7]). Let $i \neq j$, $1 \leq i, j \leq n$ ($1 \leq i, j \leq n+1$ for S_3 of type A_n). Then the roots of S_3 have the form: $\varepsilon_i - \varepsilon_j$ (A_n); $\pm\varepsilon_i$, $\pm\varepsilon_i \pm \varepsilon_j$ (B_n); $\pm 2\varepsilon_i$, $\pm\varepsilon_i \pm \varepsilon_j$ (C_n); or $\pm\varepsilon_i \pm \varepsilon_j$ (D_n). Denote by R_1, R_2, R_3 the sets of roots of S_1, S_2, S_3, respectively; by $\omega_1, \ldots, \omega_n$ the fundamental weights of S_3. Let $\{x_\alpha, h_\alpha \mid \alpha \in R_3\}$ be the standard basis of S_3. Recall that $h_\alpha \in H_3$, $[x_{-\alpha}, x_\alpha] = h_\alpha$, $[h_\alpha, x_\beta] = \langle \beta, \alpha \rangle x_\beta$ with $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ where $(\ ,\ )$ is the standard nondegenerate symmetric bilinear form on the weight system associated with the Killing form.

Lemma 4.1 *Any nontrivial irreducible submodule of the S_1-module S_3 (with respect to the adjoint action) which is not contained in S_1 is either standard or dual to the standard module.*

Proof. This follows from the branching rules for natural embeddings [15].

Let V and W be S_1-submodules of S_3 such that V is standard, W is dual to standard (for type A). Denote by v and w their highest weight vectors. Set $\Lambda = \{\alpha \in R_3 \mid \alpha = \pm\varepsilon_i + \varepsilon_m,\ i < m\}$, $\hat{\Lambda} = \{\alpha \in R_3 \mid \alpha = \varepsilon_i - \varepsilon_{n+1},\ i < m\}$ (for type A).

Lemma 4.2 $v = \sum_{\alpha \in \Lambda} a_\alpha x_\alpha$*,* $w = \sum_{\alpha \in \hat{\Lambda}} \hat{a}_\alpha x_\alpha$ *where* $a_\alpha, \hat{a}_\alpha \in F$*.*

Proof. Let $v = h + \sum_{\alpha \in R_3} a_\alpha x_\alpha$ where $h \in H_3$. Since V is a standard S_1-module, $[h_{\alpha_m}, v] = v$, $[h_{\alpha_{m+1}}, v] = \ldots = [h_{\alpha_n}, v] = 0$. Therefore $h = 0$, and if $a_\alpha \neq 0$, then $\langle \alpha, \alpha_m \rangle = 1$, $\langle \alpha, \alpha_{m+1} \rangle = \ldots = \langle \alpha, \alpha_n \rangle = 0$. Observe that

$$\langle \varepsilon_i, \alpha_j \rangle = \langle \omega_i - \omega_{i-1}, \alpha_j \rangle = \delta_{ij} - \delta_{i-1,j},$$

for any i, j except in the following cases

$$\langle \varepsilon_{n+1}, \alpha_j \rangle = \langle -\omega_n, \alpha_j \rangle = -\delta_{nj} \quad (A_n);$$

$$\langle \varepsilon_n, \alpha_j \rangle = \langle 2\omega_n - \omega_{n-1}, \alpha_j \rangle = 2\delta_{nj} - \delta_{n-1,j} \quad (B_n);$$
$$\langle \varepsilon_{n-1}, \alpha_j \rangle = \langle \omega_n + \omega_{n-1} - \omega_{n-2}, \alpha_j \rangle = \delta_{nj} + \delta_{n-1,j} - \delta_{n-2,j} \quad (D_n);$$
$$\langle \varepsilon_n, \alpha_j \rangle = \langle \omega_n - \omega_{n-1}, \alpha_j \rangle = \delta_{nj} - \delta_{n-1,j} \quad (D_n).$$

Recall that α has the form $\pm\varepsilon_i$, $\pm 2\varepsilon_i$, or $\pm\varepsilon_i \pm \varepsilon_j$, $i > j$. If $m < i \leq n$, then $\langle \varepsilon_i, \alpha_i \rangle \neq 0$ and $\langle \varepsilon_j, \alpha_i \rangle = 0$, so $\langle \alpha, \alpha_i \rangle \neq 0$, which contradicts the hypothesis. If

$\alpha = \pm\varepsilon_{n+1} \mp \varepsilon_j$, then $\langle \alpha, \alpha_n \rangle = \mp 1 \mp \delta_{nj} \neq 0$. Therefore $i \leq m$. It is not difficult to check that α has the form $\alpha = \pm\varepsilon_i + \varepsilon_m$, $i < m$, or $\alpha = \varepsilon_m = \alpha_m + \ldots + \alpha_n$ (for type B_n). Since v is the highest weight vector of V, $[x_{\alpha_n}, v] = 0$. Since $\alpha_n + \varepsilon_m = \varepsilon_n + \varepsilon_m$ is a root of S_3 (type B_n), $[x_{\alpha_n}, x_{\varepsilon_m}] \neq 0$, so $a_{\varepsilon_m} = 0$. Now consider $w = h + \sum_{\alpha \in R_3} \hat{a}_\alpha x_\alpha$ (type A). Then arguments similar to those above show that $h = 0$, and if $\hat{a}_\alpha \neq 0$, then $\langle \alpha, \alpha_m \rangle = \ldots = \langle \alpha, \alpha_{n-1} \rangle = 0$, $\langle \alpha, \alpha_n \rangle = 1$. Since any root of S_3 has the form $\varepsilon_i - \varepsilon_j$, we conclude $\alpha = \varepsilon_i - \varepsilon_{n+1}$, $i < m$. So the lemma follows.

Observe that

$$-\varepsilon_i + \varepsilon_m = -\alpha_i - \ldots - \alpha_{m-1},$$

$$\varepsilon_i + \varepsilon_m = \alpha_i + \ldots + \alpha_{m-1} + 2\alpha_m + \ldots,$$

$$\varepsilon_i - \varepsilon_{n+1} = \alpha_i + \ldots + \alpha_n.$$

Set

$$\Lambda_1 = \{\alpha \in R_3 \mid \alpha = \pm\varepsilon_i + \varepsilon_m,\ l \leq i < m\} \subset R_2;$$

$$\hat{\Lambda}_1 = \{\alpha \in R_3 \mid \alpha = \varepsilon_i - \varepsilon_{n+1},\ l \leq i < m\} \subset R_2;$$

$$\Lambda_2 = \{\alpha \in R_3 \mid \alpha = \pm\varepsilon_i + \varepsilon_m,\ i < l\};$$

$$\Lambda_- = \{\alpha \in R_3 \mid \alpha = \pm\varepsilon_i \pm \varepsilon_j,\ i < l \leq j < m\}.$$

Notice that $\Lambda_1 \cup \Lambda_2 = \Lambda$, $\hat{\Lambda}_1 \subset \hat{\Lambda}$. Put $\bar{R}_3 = R_3 \cup \{0\}$.

Lemma 4.3 *(1) $\alpha + \beta \notin \bar{R}_3$ for any $\alpha \in R_1$, $\beta \in \Lambda_-$;*

(2) $\langle \beta, \alpha \rangle = 0$ for any $\alpha \in R_1$, $\beta \in \Lambda_-$;

(3) $\alpha + \beta \notin \bar{R}_3$ for any $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$;

(4) $\langle \beta, \alpha \rangle = 1$ for any $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$;

(5) $-\alpha + \beta + \gamma \notin \bar{R}_3$ for any $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$, $\gamma \in \hat{\Lambda}_1$ (for type A);

(6) if $-\alpha + \beta + \gamma \in \bar{R}_3$ where $\alpha, \beta \in \Lambda_1$, $\gamma \in \Lambda_2$, then $\alpha = \beta$;

(7) if $-\alpha + \beta + \gamma \in \bar{R}_3$ where $\alpha \in \Lambda_1$, $\beta, \gamma \in \Lambda_2$, then $-\alpha + \beta + \gamma \in R_2$.

Proof. (1), (2). By definition, $\alpha = \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm 2\varepsilon_i$, $\beta = \pm\varepsilon_p \pm \varepsilon_q$ where $p < l \leq q < m \leq i < j$. Therefore $\alpha + \beta \notin \bar{R}_3$. Since α is a sum of α_i, $m \leq i \leq n$, and $\langle \varepsilon_k, \alpha_i \rangle = 0$, $k < m \leq i$, we conclude that $\langle \beta, \alpha \rangle = 0$.

(3), (4). By definition, $\alpha = \pm\varepsilon_i + \varepsilon_m$, $\beta = \pm\varepsilon_j + \varepsilon_m$ where $j < l \leq i < m$. Therefore $\alpha + \beta = \pm\varepsilon_i \pm \varepsilon_j + 2\varepsilon_m \notin \bar{R}_3$. Observe that the roots α, β, $\alpha_1, \ldots, \alpha_m$ have the same length (they are long for the system B, and short for C). If $\alpha = -\varepsilon_i + \varepsilon_m = -\alpha_i - \ldots - \alpha_{m-1}$, then

$$\langle \beta, \alpha \rangle = -\langle \beta, \alpha_i \rangle - \ldots - \langle \beta, \alpha_{m-1} \rangle = -\langle \varepsilon_m, \alpha_{m-1} \rangle = 1.$$

If $\alpha = \varepsilon_i + \varepsilon_m = \alpha_i + \ldots + \alpha_{m-1} + 2\alpha_m + \ldots$, then

$$\langle \beta, \alpha \rangle = \frac{2((\beta, \alpha_i) + \ldots + (\beta, \alpha_{m-1}) + 2(\beta, \alpha_m) + \ldots)}{(\alpha, \alpha)} =$$

$$\frac{2(\beta, \alpha_{m-1})}{(\alpha, \alpha)} + \frac{4(\beta, \alpha_m)}{(\alpha, \alpha)} = \langle \beta, \alpha_{m-1} \rangle + 2\langle \beta, \alpha_m \rangle =$$

$$\langle \varepsilon_m, \alpha_{m-1} \rangle + 2\langle \varepsilon_m, \alpha_m \rangle = -1 + 2 = 1,$$

as required.

(5). Since S_3 is of type A, we have $\alpha = -\varepsilon_i + \varepsilon_m$, $\beta = -\varepsilon_j + \varepsilon_m$, $\gamma = \varepsilon_k - \varepsilon_{n+1}$ where $j < l \leq i, k < m$. Therefore

$$-\alpha + \beta + \gamma = \varepsilon_i + \varepsilon_k - \varepsilon_j - \varepsilon_{n+1} \notin \bar{R}_3.$$

(6). By definition, $\alpha = \pm\varepsilon_i + \varepsilon_m$, $\beta = \pm\varepsilon_k + \varepsilon_m$, $\gamma = \pm\varepsilon_j + \varepsilon_m$ where $j < l \leq i, k < m$. Therefore

$$-\alpha + \beta + \gamma = \mp\varepsilon_i \pm \varepsilon_k \pm \varepsilon_j + \varepsilon_m.$$

It is clear that $-\alpha + \beta + \gamma \in \bar{R}_3$ only for the case $\alpha = \beta$.

(7). By definition, $\alpha = \pm\varepsilon_i + \varepsilon_m$, $\beta = \pm\varepsilon_j + \varepsilon_m$, $\gamma = \pm\varepsilon_k + \varepsilon_m$ where $j, k < l \leq i < m$. Therefore

$$-\alpha + \beta + \gamma = \mp\varepsilon_i \pm \varepsilon_j \pm \varepsilon_k + \varepsilon_m.$$

It is clear that $-\alpha + \beta + \gamma \in \bar{R}_3$ only for the case $\pm\varepsilon_j \pm \varepsilon_k = 0$. In this case $-\alpha + \beta + \gamma \in R_2$.

The following fact is well known (see, for example, [6]).

Lemma 4.4 *The solvable radical* $\operatorname{Rad} L$ *of a perfect Lie algebra* L *coincides with the nilpotent radical of* L*, i.e. annihilates any irreducible* L*-module. In particular,* $\operatorname{Rad} L$ *is nilpotent.*

Lemma 4.5 *Let* $L = S \oplus R$ *be perfect where* S *is a Levi subalgebra of* L*,* $R = \operatorname{Rad} L$*. Then there exists an* S*-submodule* M *of* R *(in fact* $M \cong R/[R,R]$*) with respect to the adjoint action, such that* $SM = M$*, and* M *generates* R *(as an algebra).*

Proof. Observe that $[R, R]$ is an S-submodule of R. In view of complete reducibility we have $R = M \oplus [R, R]$ where M is an S-submodule of R. Since L is perfect, $[S, R] + [R, R] = R = M + [R, R]$. Hence $SM = M$. By Lemma 4.4, R is nilpotent. Therefore M generates R (this is a well known fact for nilpotent Lie algebras, see, for instance, [6, ch. I, §4, exercise 4]).

Recall that an automorphism $e^{\operatorname{ad} x} = 1 + \operatorname{ad} x + (\operatorname{ad} x)^2/2 + \ldots$ of a Lie algebra L, with x belonging to the nilpotent radical of L, is called special.

Lemma 4.6 *Let $L_1 \subset L_2$ be perfect subalgebras of a simple Lie algebra S_3 such that $L_1 \cap \operatorname{Rad} L_2 = 0$. Let also $S_1 \subset S_2$ be Levi subalgebras of L_1, L_2, respectively. Suppose that S_1, S_2, S_3 are classical simple Lie algebras of the same type, and the embeddings $S_1 \subset S_2 \subset S_3$ are natural. Then there exists a Levi subalgebra S_2' of L_2 such that $L_1 \subset S_2'$.*

Proof. We adopt the notation above (for the algebras $S_1 \subset S_2 \subset S_3$). If $\operatorname{Rad} L_1 = 0$ or $\operatorname{Rad} L_2 = 0$, then $L_1 = S_1 \subset S_2$ or $L_1 \subset L_2 = S_2$, respectively, and we are done. So one can assume that $\operatorname{Rad} L_1 \neq 0$, $\operatorname{Rad} L_2 \neq 0$. By Lemma 4.5, there is a submodule M of $\operatorname{Rad} L_1$ such that $S_1 M = M$ and M generates $\operatorname{Rad} L_1$ (as an algebra). Since $M \subset S_3$, by Lemma 4.1, every composition factor of M is either a standard S_1-module or dual to one. It is clear that $N = M \cap S_2$ is a submodule of M. In view of complete reducibility, there is a submodule Q of M such that $M = N \oplus Q$. Let k be the length of Q (the number of composition factors). Proceed by induction on k. If $k = 0$, then $M = N \subset S_2$, forcing $\operatorname{Rad} L_1 \subset S_2$, as desired. Assume that $k > 0$. Let V be an irreducible submodule of Q. One can assume that V is standard. Indeed, if V is dual to standard (type A), then we can take another base: $\alpha_i' = -\alpha_i$, $1 \le i \le n$. One checks that V becomes standard in this situation. Recall that $V \subset \operatorname{Rad} L_1 \subset S_2 \oplus \operatorname{Rad} L_2$, $V \cap S_2 = V \cap \operatorname{Rad} L_2 = 0$. Let V_s and V_r be the projections of V to S_2 and $\operatorname{Rad} L_2$, respectively. It is clear that V_s and V_r are S_1-modules isomorphic to V. Moreover, $v = v_s + v_r$ where v, v_s, v_r are the highest weight vectors of V, V_s, V_r, respectively. Observe that $V_s \cap S_1 = 0$. Indeed, if $x = x_s + x_r \in V$ and $x_s \in S_1$ then $x - x_s = x_r \in (L_1 \cap \operatorname{Rad} L_2)$. Since $L_1 \cap \operatorname{Rad} L_2 = 0$, $x = x_s \in S_1$, so $x = 0$. By Lemma 4.2, $v = \sum_{\alpha \in \Lambda} a_\alpha x_\alpha$. It is clear that $v_s = \sum_{\alpha \in \Lambda_1} a_\alpha x_\alpha$, $v_r = \sum_{\alpha \in \Lambda_2} a_\alpha x_\alpha$.

Let $N = N^+ \oplus N^-$ where N^+ is the sum of standard S_1-modules, and N^- is the sum of modules dual to standard (for type A). Let P^+, P^- be the subspaces of primitive vectors of N^+, N^-, respectively. By Lemma 4.2, any $u \in P^+$ has the form $u = \sum_{\alpha \in \Lambda_1} p_\alpha(u) x_\alpha$ where $p_\alpha(u) \in F$. Thus we can identify P^+ with the corresponding subspace of F^{Λ_1} via the map $u \mapsto (p_\alpha(u))_{\alpha \in \Lambda_1}$. Recall that $v_s = \sum_{\alpha \in \Lambda_1} a_\alpha x_\alpha$, so $v_s \mapsto (a_\alpha)_{\alpha \in \Lambda_1}$. Observe that $v_s \notin M$. Indeed, otherwise $v_r = v - v_s \in M$. This forces $L_1 \cap \operatorname{Rad} L_2 \neq 0$ which contradicts the hypothesis. Therefore $v_s \notin P^+$. It is clear that there exists a linear function $b = (b_\alpha)_{\alpha \in \Lambda_1}$ on F^{Λ_1} such that $b(P^+) = 0$ (i.e. $\sum_{\alpha \in \Lambda_1} b_\alpha p_\alpha(u) = 0$ for any $u \in P^+$) and $b(v_s) = \sum_{\alpha \in \Lambda_1} b_\alpha a_\alpha = 1$. Set $v^* = \sum_{\alpha \in \Lambda_1} b_\alpha x_{-\alpha}$. Observe that $v^* \in S_2$. Put

$$x = [v_r, v^*] = [\sum_{\alpha \in \Lambda_2} a_\alpha x_\alpha, \sum_{\alpha \in \Lambda_1} b_\alpha x_{-\alpha}] = \sum_{\alpha \in R_3} c_\alpha x_\alpha.$$

Observe that if $c_\alpha \neq 0$, then α has the form $\alpha = \pm\varepsilon_i \pm \varepsilon_j$, $i < l \le j < m$, i.e. $\alpha \in \Lambda_-$. Therefore $x = \sum_{\alpha \in \Lambda_-} c_\alpha x_\alpha$. Since $v_r \in \operatorname{Rad} L_2$ and $v^* \in S_2$, we have $x = [v_r, v^*] \in \operatorname{Rad} L_2$. By Lemma 4.4, $\operatorname{Rad} L_2$ is nilpotent. Therefore

$$\theta = e^{\operatorname{ad} x} = 1 + \operatorname{ad} x + (\operatorname{ad} x)^2/2 + \dots$$

is a (special) automorphism of L_2. By lemma 4.3 (2), for any $\alpha \in R_1$ and any $\beta \in \Lambda_-$ we have $[h_\alpha, x_\beta] = \langle \beta, \alpha \rangle x_\beta = 0$, so $[h_\alpha, x] = 0$. By Lemma 4.3 (1), $\alpha + \beta \notin \bar{R}_3$ for any $\alpha \in R_1$, $\beta \in \Lambda_-$, so $[x, x_\alpha] = 0$ for any $\alpha \in R_1$. Thus $[x, S_1] = 0$. Therefore $S_1^\theta = S_1$. Let $y = \sum_{\gamma \in \hat{\Lambda}_1} y_\gamma x_\gamma \in P^-$, $y_\gamma \in F$. Then by Lemma 4.3 (5)

$$[x, y] = [[v_r, v^*], y] = \sum_{\substack{\alpha \in \Lambda_1, \beta \in \Lambda_2 \\ \gamma \in \hat{\Lambda}_1}} a_\beta b_\alpha y_\gamma [[x_\beta, x_{-\alpha}], x_\gamma] = 0.$$

Therefore $[x, P^-]$=0, so $(P^-)^\theta = P^-$ and $(N^-)^\theta = N^-$. Let now $u = \sum_{\alpha \in \Lambda_1} d_\alpha x_\alpha$, $d_\alpha \in F$. By Lemma 4.3 (3), $[x_\alpha, x_\beta] = 0$ for any $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$. Therefore $[v_r, u] = 0$. Thus we have

$$[x, u] = [[v_r, v^*], u] = [[v_r, u], v^*] + [v_r, [v^*, u]] = -[[v^*, u], v_r] =$$

$$- \sum_{\substack{\alpha, \beta \in \Lambda_1 \\ \gamma \in \Lambda_2}} b_\alpha d_\beta a_\gamma [[x_{-\alpha}, x_\beta], x_\gamma].$$

By Lemma 4.3 (6), if $[[x_{-\alpha}, x_\beta], x_\gamma] \neq 0$, then $\alpha = \beta$. Hence by Lemma 4.3 (4),

$$[x, u] = - \sum_{\substack{\alpha \in \Lambda_1 \\ \gamma \in \Lambda_2}} b_\alpha d_\alpha a_\gamma [h_\alpha, x_\gamma] = - \sum_{\substack{\alpha \in \Lambda_1 \\ \gamma \in \Lambda_2}} b_\alpha d_\alpha a_\gamma \langle \gamma, \alpha \rangle x_\gamma =$$

$$- \sum_{\substack{\alpha \in \Lambda_1 \\ \gamma \in \Lambda_2}} b_\alpha d_\alpha a_\gamma x_\gamma = -(\sum_{\alpha \in \Lambda_1} b_\alpha d_\alpha)(\sum_{\gamma \in \Lambda_2} a_\gamma x_\gamma) = -q v_r$$

where $q = \sum_{\alpha \in \Lambda_1} b_\alpha d_\alpha \in F$. If $u \in P^+$, then by the choice of b_α, $q = 0$, so $[x, u] = 0$. Therefore $[x, P^+] = 0$ and $(P^+)^\theta = P^+$. Hence $(N^+)^\theta = N^+$. Since $(N^-)^\theta = N^-$, we conclude that $N^\theta = N$. If $u = v_s$ (i.e. $d_\alpha = a_\alpha$, $\alpha \in \Lambda_1$), then by the choice of b_α, $q = 1$, so $[x, v_s] = -v_r$. Observe that $[x, [x, v]] = -[x, v_r] = -[[v_r, v^*], v_r] = 0$. Indeed, by Lemma 4.3 (7), if $e_\alpha \neq 0$ in the decomposition $[x, v_r] = \sum_{\alpha \in R_3} e_\alpha x_\alpha$, then $\alpha \in R_2$. Therefore $[x, v_r] \in S_2$. On the other hand $[x, v_r] \in \operatorname{Rad} L_2$. This forces $[x, v_r] = 0$. Consequently,

$$v^\theta = v + [x, v] + \frac{[x, [x, v]]}{2} + \ldots = v - v_r + 0 + \ldots = v_s \in S_2.$$

This forces $V^\theta \subset S_2$. Summarizing, $S_1^\theta \subset S_2$ and $(N + V)^\theta \subset S_2$, or equivalently, $S_1 \subset S_2^{\theta^{-1}}$ and $N + V \subset S_2^{\theta^{-1}}$. It is clear that $S_2^{\theta^{-1}}$ is a Levi subalgebra of L_2. Set $N' = M \cap S_2^{\theta^{-1}}$. Since $N' \supset N + V$, we have $\dim N' > \dim N$. Therefore the length of the module $Q' = M/N'$ less than k. By inductive hypothesis (for the Levi subalgebras $S_1 \subset S_2^{\theta^{-1}}$ of the Lie algebras $L_1 \subset L_2$), there exists a Levi subalgebra S_2' of L_2 such that $L_1 \subset S_2'$, as required.

5 Finitary simple Lie algebras

The aim of this section is to classify finitary simple Lie algebras (of countable dimension). It turns out that this is equivalent to the classification of the thin simple locally finite Lie algebras introduced in Section 3. We shall show that there are only three nonisomorphic such algebras with countable dimension (over a given algebraically closed field of zero characteristic). They are $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$. One can construct these algebras as the limits of the natural embeddings of the corresponding finite-dimensional simple Lie algebras. The case of uncountable dimension is somewhat more complicated. However we have the following general result.

Theorem 5.1 *Let L be a thin simple locally finite Lie algebra. Then there exists a conical local system of L consisting of classical simple Lie algebras of the same type A, B, or C, such that all the corresponding embeddings of each into another are natural.*

Proof. Let $\{L_i\}_{i\in I}$ be a local system of L such that the corresponding Bratteli diagram $\mathfrak{B}$ is thin (see Definition 3.11). We proceed by steps.

Step 1. One can assume that $\mathfrak{B}$ is conical with the minimal node S_1^1 of rank greater than 8 (i.e all nodes of $\mathfrak{B}$ are classical Lie algebras).

This follows from Theorem 3.1.

Step 2. One can assume that $\tau(S_j^l, S_r^p) \leq 1$ for any nodes S_j^l, S_r^p of $\mathfrak{B}$.

Take a node S_i^k such that the value $\tau(S_1^1, S_i^k)$ is maximal (this value is bounded by definition). By Theorem 3.1, $\mathfrak{C}_i^k = \{L'_j\}_{j\geq i}$ is a local system of L. We shall show that this local system satisfies the required property. Observe that the corresponding Bratteli diagram $\mathfrak{B}_i^k$ is thin. Moreover, by Lemma 3.10, for any node S_j^l of $\mathfrak{B}_i^k$ we have $\tau(S_i^k, S_j^l) = 1$. By Lemma 3.10, we have also $\tau(S_i^k, S_j^l)\tau(S_j^l, S_r^p) \leq \tau(S_i^k, S_r^p) = 1$ where $j < r$. Therefore $\tau(S_j^l, S_r^p) \leq 1$ for any nodes of $\mathfrak{B}_i^k$ with $i \leq j < r$, so $\mathfrak{B}_i^k$ has the required property.

Step 3. For any $i \in I$ there is only one node (S_i^1) at level i of $\mathfrak{B}$, i.e. $S_i = S_i^1$.

Indeed, assume that there are at least two nodes S_i^1 and S_i^2 at level i. Then by Theorem 3.2, there exists an S_i^1- and S_i^2-accessible node S_j^l of $\mathfrak{B}$ with $j > i$. Since all nodes of $\mathfrak{B}$ are S_1^1-accessible, we have

$$\tau(S_1^1, S_j^l) \geq \tau(S_1^1, S_i^1)\tau(S_i^1, S_j^l) + \tau(S_1^1, S_i^2)\tau(S_i^2, S_j^l) \geq 1 + 1 = 2 > 1.$$

This contradicts the claim above.

Step 4. One can assume that all nodes of $\mathfrak{B}$ are classical Lie algebras of the same type.

Assume that there is a node S_i^1 of $\mathfrak{B}$ of type A. Since there are no natural embeddings of type $A \to B$, $A \to C$ and $A \to D$, all S_i^1-accessible nodes of $\mathfrak{B}$ have type A. Therefore the local system $\mathfrak{C}_i^1$ has the required property.

If all nodes of $\mathfrak{B}$ are of type C, then we are done. Thus one can assume that this is false. Since there are no natural embeddings between orthogonal and symplectic Lie algebras, one can assume that all nodes are of types B and D. If there exists a node S_i^1 such that all S_i^1-accessible nodes are of type B, then the local system $\mathfrak{C}_i^1$ has the required property. Otherwise, denote by J the set of indices $j \in I$ such that S_j^1 has type D. Then $\{L_j\}_{j \in J}$ is a local system of L. Indeed, it suffices to show that for any node S_i^1 of type B there exists $j \in J$ such that $L_i \subset L_j$, i.e. $i < j$. But this follows immediately from our assumption.

Step 5. One can assume that all L_i are simple, i.e. $L_i = S_i = S_i^1$ for any $i \in I$.

Since L is simple, by [2, Lemma 3.3], for any $i \in I$ there exists $j = j(i) > i$ such that $L_i \cap \operatorname{Rad} L_j = 0$. For the same reason, there exists $r > j$ such that $L_j \cap \operatorname{Rad} L_r = 0$. Thus we have a natural embedding of L_j into $S_r \cong L_r / \operatorname{Rad} L_r$. So we obtain the chain of natural embeddings $S_i \subset S_j \subset S_r$. Therefore by Lemma 4.6, there exists a Levi subalgebra P_j of L_j such that $L_i \subset P_j$. Set $J = \{j(i) \mid i \in I\}$. It is clear that $\{P_j\}_{j \in J}$ is a local system of L consisting of simple subalgebras and satisfying all properties above.

Step 6. One can assume that all L_i are of type either A, or B, or C.

Indeed, assume that all L_i have type D. Since any natural embedding of type $D_m \to D_n$ can be represented as the composition of the natural embeddings $D_m \to B_m \to D_n$, we can choose a local system of simple subalgebras of type B such that all the corresponding embeddings are natural. This proves *Step 6* and with it the theorem.

By Theorem 5.1, one can distribute thin Lie algebras by types (A, B, or C). To show that algebras of different types are nonisomorphic we need the following

Lemma 5.2 *Let $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{s}$ be classical simple Lie algebras,* $\operatorname{rk} \mathfrak{h} > 10$. *Assume that the embedding $\mathfrak{h} \to \mathfrak{s}$ is diagonal. Then the embeddings $\mathfrak{h} \to \mathfrak{g}$ and $\mathfrak{g} \to \mathfrak{s}$ are also diagonal.*

Proof. Denote by $V_{\mathfrak{g}}$ and $V_{\mathfrak{s}}$ the standard modules of $\mathfrak{g}$ and $\mathfrak{s}$. By assumption, we have $\delta_{\mathfrak{h}}(V_{\mathfrak{s}}) = 1$. Assume that the embedding $\mathfrak{g} \to \mathfrak{s}$ is nondiagonal. Then by [4, Lemma 6.7],

$$1 = \delta_{\mathfrak{h}}(V_{\mathfrak{s}}) = \delta_{\mathfrak{h}}(V_{\mathfrak{s}} \downarrow \mathfrak{g}) > \delta_{\mathfrak{h}}(V_{\mathfrak{g}}) \geq 1,$$

so we have a contradiction. Therefore $\mathfrak{g} \to \mathfrak{s}$ is diagonal. It immediately follows that $\mathfrak{h} \to \mathfrak{g}$ is also diagonal.

Proposition 5.3 *Thin simple locally finite Lie algebras of different types are nonisomorphic.*

Proof. Suppose that L is thin, and let $\{L_i\}_{i \in I}$ and $\{M_j\}_{j \in J}$ be local systems of L as in Theorem 5.1. Take any $i \in I$ such that $\operatorname{rk} L_i > 10$. Choose $j \in J$ and $i' \in I$ such that $L_i \subset M_j \subset L_{i'}$. Since the embedding $L_i \to L_{i'}$ is diagonal, by

Lemma 5.2, the embeddings $L_i \to M_j$ and $M_j \to L_{i'}$ are diagonal. Moreover, these embeddings are natural since $L_i \to L_{i'}$ is. Therefore L_i, M_j, and $L_{i'}$ have the same type, as required.

From Theorem 5.1 we obtain the following

Corollary 5.4 *Any thin simple locally finite Lie algebra of countable dimension is isomorphic to* $\mathfrak{sl}_\infty$*,* $\mathfrak{so}_\infty$ *or* $\mathfrak{sp}_\infty$*.*

Proof. Using Theorem 5.1, one can choose an ascending chain of simple subalgebras of the same type (A, B, or C) naturally embedded each into another and forming a local system. Clearly, their inductive limit is either $\mathfrak{sl}_\infty$, or $\mathfrak{so}_\infty$, or $\mathfrak{sp}_\infty$.

Inductive systems for the algebras $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$ were described by A. Zhilinskii [16]. His results can easily be generalized to all thin simple locally finite Lie algebras. In particular, one can make the following

Remark 5.5 Let $L = \varinjlim L_i$ be a thin simple locally finite Lie algebra, $\{L_i\}_{i \in I}$ as in Theorem 5.1. Set

$$\Psi_i^d = \{V \in \operatorname{Irr} L_i \mid \sigma_{L_i}(V) \in \mathbf{N} \cup \{0\}, \quad \sigma_{L_i}(V) \le d\},$$

$$\tilde{\Psi}_i^d = \{V \in \operatorname{Irr} L_i \mid \sigma_{L_i}(V) \notin \mathbf{N} \cup \{0\}, \quad \sigma_{L_i}(V) \le d\} \quad \text{(for type } B).$$

It follows from [16] that $\Psi^d = \{\Psi_i^d\}_{i \in I}$ and $\tilde{\Psi}^d = \{\tilde{\Psi}_i^d\}_{i \in I}$ are inductive systems for L for any $d \in \mathbf{N} \cup \{0\}$. Furthermore, let $\Phi = \{\Phi_i\}_{i \in I}$ be a nonempty join-irreducible inductive system for L. Then there exist $c, d \in \mathbf{N} \cup \{0\}$ and an inductive system $\Psi = \{\Psi_i\}_{i \in I} \supseteq \Psi^0$ with $\delta_{L_i}(\Psi_i) \le c$ for any $i \in I$, such that $\Phi = \Psi \otimes \Psi^d$ or $\Psi \otimes \tilde{\Psi}^d$, i.e. $\Phi_i = \{V \otimes W \mid V \in \Psi_i, \quad W \in \Psi_i^d \text{ (or } \tilde{\Psi}_i^d)\}$.

Fix $d \in \mathbf{N}$. Observe that for any c there exists $i \in I$ such that $\delta_{L_i}(\Psi_i^d) > c$ (cf. Theorem 3.12).

Let L be a thin simple locally finite Lie algebra, $\{L_i\}_{i \in I}$ be a local system of L as in Theorem 5.1 with the minimal node L_1. Denote by V_i the standard L_i-module. Since for any pair $i < j$ the embedding $L_i \subset L_j$ is natural, one can identify V_i with the linear subspace of V_j as follows. For types B and C we take a nonzero homomorphism of V_i onto the nontrivial irreducible L_i-submodule of V_j (isomorphic to V_i). For type A we identify every L_i with $\mathfrak{sl}(V_i)$. So for any pair $i < j$ the corresponding embedding $\mathfrak{sl}(V_i) \to \mathfrak{sl}(V_j)$ produces a homomorphism $V_i \to V_j$. Observe that these embeddings are unique up to scalar multiplication. Summarizing, for any pair $i < j$ there exists an embedding of linear spaces ρ_{ij} : $V_i \to V_j$ compatible with the embedding $L_i \to L_j$. Construct the set of scalars $\{k_{ij}\}_{i<j}$ in the following manner. Put $k_{1j} = 1$ for all j; the k_{ij} with $1 < i < j$ are determined from the equality $k_{ij}\rho_{ij} \circ \rho_{1i} = \rho_{1j}$. Define the homomorphism $\bar{\rho}_{ij} : V_i \to V_j$ by $\bar{\rho}_{ij} = k_{ij}\rho_{ij}$. One can easily check that $\bar{\rho}_{jk} \circ \bar{\rho}_{ij} = \bar{\rho}_{ik}$ for any

$i < j < k$. Therefore one can construct a linear space $V = \varinjlim V_i$. Obviously, L is a subalgebra of $\mathfrak{fgl}(V)$, so L is *finitary* (see Introduction). We now show that the converse conclusion also holds.

Theorem 5.6 *A simple locally finite Lie algebra is thin if and only if it is finitary.*

The following lemma precedes the proof of the theorem.

Lemma 5.7 *Let $\mathfrak{h}$ be a classical finite-dimensional Lie algebra, V an $\mathfrak{h}$-module, $\operatorname{rk}\mathfrak{h} = m$, $\delta_{\mathfrak{h}}(V) = d > 0$. Denote by $\wedge V$ the module $V + \wedge^2 V + \ldots + \wedge^l V$ where $l = \dim V$. Assume that $n = [m/2] - d - 1 > 0$. Then $\delta_{\mathfrak{h}}(\wedge V) \geq nd$.*

Proof. Denote by $\omega_1, \ldots, \omega_m$ the fundamental weights of $\mathfrak{h}$, by $\alpha_1, \ldots, \alpha_m$ its simple roots. Let W be an irreducible submodule of V such that $\delta_{\mathfrak{h}}(W) = \delta_{\mathfrak{h}}(V)$. Let $\lambda = a_1\omega_1 + \ldots + a_m\omega_m$ be its highest weight. We have $\delta(\lambda) = a_1 + 2a_2 + \ldots = d$. Since $m > 2d+2$, there exists $k \leq d$ such that $a_k \neq 0$, $a_{k+1} = \ldots = a_{k+n} = 0$. (It can occur for type A that $a_1 = \ldots = a_d = 0$. Then there exists $m-d+1 \leq k \leq m$ such that $a_k \neq 0$, $a_{k-1} = \ldots = a_{k-n} = 0$, and the proof is analogous.) Put $\beta_j = \alpha_k + \ldots + \alpha_{k+j}$, $1 \leq j \leq n$. Observe that the β_j are roots of $\mathfrak{h}$, and

$$\langle \lambda, \beta_j \rangle = \frac{a_k(\omega_k, \alpha_k)}{(\beta_j, \beta_j)} = a_k \frac{(\alpha_k, \alpha_k)}{(\beta_j, \beta_j)} > 0.$$

Therefore $\mu_j = \lambda - \beta_j$ is a weight of W. Since $k + n \leq d + n < m/2$, we have $\delta(\beta_j) = \delta(\alpha_k) + \ldots + \delta(\alpha_{k+j}) = 0$, so $\delta(\mu_j) = \delta(\lambda) = d$. Let w_j be a vector of weight μ_j in W. Then the vector $w_1 \wedge \ldots \wedge w_n$ of $\wedge V$ has weight $\lambda' = \mu_1 + \ldots + \mu_n$. Therefore

$$\delta_{\mathfrak{h}}(\wedge V) \geq \delta(\lambda') = \delta(\mu_1) + \ldots + \delta(\mu_n) = nd,$$

as required.

Proof of the theorem. It remains to prove the "if" part. Assume that L is a simple infinite-dimensional subalgebra of $\mathfrak{fgl}(V)$. Then V is a faithful L-module. Let $\{L_i\}_{i \in I}$ be a perfect local system for L. Fix $i \in I$. Let $\{e_1, \ldots, e_q\}$ be a basis of L_i. By definition, there exist finite-dimensional subspaces $W_i^1, \ldots, W_i^q$ of V such that $e_k V \subseteq W_i^k$, $1 \leq k \leq q$. Put $W_i = W_i^1 + \ldots + W_i^q$. Clearly, $L_i V \subseteq W_i$. Since $L_i(V/W_i) = 0$, any nontrivial irreducible L_i-submodule of V lies in W_i. Since W_i is finite-dimensional, the set $\Phi_i = \langle V{\downarrow}L_i \rangle = \langle W{\downarrow}L_i \rangle \cup \{T_i\}$ is finite. Observe that for any pair $i < j$ we have

$$\langle \Phi_j{\downarrow}L_i \rangle = \langle \langle V{\downarrow}L_j \rangle{\downarrow}L_i \rangle = \langle V{\downarrow}L_i \rangle = \Phi_i.$$

Hence $\Phi = \{\Phi_i\}_{i \in I}$ is an inductive system for L. Obviously, it is nontrivial. Therefore by Corollary 3.9, L is diagonal. By Theorem 3.8, one can assume that $\{L_i\}_{i \in I}$ is a conical pure diagonal local system of rank greater than 12. Denote by $\wedge V$ the L-module $V + \wedge^2 V + \wedge^3 V + \dots$. Observe that all nontrivial composition factors of the L_i-module $\wedge V$ are isomorphic to those of $\wedge W_i$. Therefore $\Phi_i^\wedge = \langle \wedge V{\downarrow}L_i \rangle = \langle \wedge W_i \rangle \cup \{T_i\}$. Since $\langle \wedge W_i \rangle$ is finite-dimensional, $\Phi_i^\wedge$ is finite. Consequently, $\Phi^\wedge = \{\Phi_i^\wedge\}_{i \in I}$ is a nontrivial inductive system for L. In view of Theorem 3.12, it suffices to show that for any $c \in \mathbf{N}$ there exists a node S_i^k such that $\delta_i^k(\Phi_i^\wedge) = \delta_i^k(\wedge W_i) > c$. Take a node S_i^k such that $m = \operatorname{rk} S_i^k \geq 4(c+1)$ (this is possible thanks to [2, Corollary 3.3]). Since L is simple, Φ is nondegenerate (see the remark above Theorem 5.7 in [4]), i.e. $\langle \Phi_i{\downarrow}S_i^k \rangle \neq \{T_i^k\}$. Therefore $d = \delta_i^k(W_i) = \delta_i^k(\Phi_i) \geq 1$. If $d > c$, then we are done ($\delta_i^k(\wedge W_i) \geq \delta_i^k(W) = d > c$). Assume that $d \leq c$. Then $n = [m/2] - d - 1 \geq 2(c+1) - c - 1 = c + 1 > 0$. Therefore by Lemma 5.7, $\delta_i^k(\wedge W_i) \geq nd \geq c + 1 > c$, as required.

We conclude this section by a fact about properties of Levi subalgebras of locally finite Lie algebras. As is shown above, they play an important role in the theory of locally finite Lie algebras. The definition yields the fact that Levi subalgebras are locally semisimple. The question whether they are maximal locally semisimple subalgebras naturally arises. The following theorem gives a negative answer.

Theorem 5.8 *The algebra $\mathfrak{sl}_\infty$ has a proper Levi subalgebra.*

Proof. Denote by L_n the Lie algebra of $((n+1) \times (n+1))$-matrices with zero traces and zero lowest rows, by R_n its radical, and by S_n the Levi subalgebra consisting of matrices with zero right column. We have $L_n = S_n \oplus R_n$,

$$L_n = \left\{ \begin{pmatrix} M_n & X_n \\ 0 & 0 \end{pmatrix} \right\}, \quad S_n = \left\{ \begin{pmatrix} M_n & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad R_n = \left\{ \begin{pmatrix} 0 & X_n \\ 0 & 0 \end{pmatrix} \right\}$$

where M_n is an $(n \times n)$-matrix with zero trace, X_n is an $(n \times 1)$-matrix. Observe that L_n is perfect, $S_n \cong \mathfrak{sl}_n$. Construct the embedding $L_n \to L_{n+1}$, setting

$$\begin{pmatrix} M_n & X_n \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} M_n & X_n & X_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that $S_n \to S_{n+1}$. Therefore $S = \varinjlim S_n$ is a Levi subalgebra of $L = \varinjlim L_n$. Denote by r the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_1 \subset L$. Observe that $r \notin S_n$ for any n. Indeed, the image of r in L_n has the form

$$\begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and does not belong to S_n. Therefore S is a proper Levi subalgebra of L. Observe that $R_n \cap R_{n+1} = 0$ for any n, so L is semisimple ($L_n \cap \operatorname{Rad} L \subseteq \operatorname{Rad} L_n$ for any

n). Since S is simple, by Corollary 3.3, L is simple. Finally, Theorem 5.4 implies $L \cong \mathfrak{sl}_\infty$.

Observe that $S \cong L \cong \mathfrak{sl}_\infty$. A.E.Zalesskii [14] proved that all Levi subalgebras of a locally finite Lie algebra of countable dimension are isomorphic.

6 Inductive systems and ideals

In this section we are going to obtain some results about structures of the lattices of inductive systems for simple locally finite Lie algebras. In view of Corollary 2.4 this will allow us to establish some properties of ideals in universal enveloping algebras.

Theorem 6.1 *Inductive systems for simple locally finite Lie algebras satisfy the descending chain condition.*

Proof. In view of Corollary 3.9, one need consider only diagonal Lie algebras. Thus, let $L = \varinjlim L_i$ be a simple diagonal locally finite Lie algebra. Suppose that

$$\Phi^1 \supset \Phi^2 \supset \ldots \supset \Phi^n \supset \ldots$$

is an infinite descending chain of inductive systems for L. By the definition, for any $n \in \mathbf{N}$ there exists $i_n \in I$ such that $\Phi^n_{i_n} \neq \Phi^{n+1}_{i_n}$. Construct a sequence of indices $j_1, j_2, \ldots, j_n, \ldots$ as follows. Set $j_1 = i_1$. If j_{n-1} has been chosen, take arbitrary $j_n > j_{n-1}, i_n$. Observe that $j_n > i_1, \ldots, i_n$. By the definition of inductive systems,

$$\langle \Phi^m_{j_n} \!\downarrow\! L_{i_m} \rangle = \Phi^m_{i_m} \neq \Phi^{m+1}_{i_m} = \langle \Phi^{m+1}_{j_n} \!\downarrow\! L_{i_m} \rangle$$

for any $1 \leq m \leq n$. Therefore $\Phi^m_{j_n} \neq \Phi^{m+1}_{j_n}$ for any $1 \leq m \leq n$. Consequently, $\Phi^1_{j_n} \supset \ldots \supset \Phi^{n+1}_{j_n}$. Suppose that L is thin. One can assume that the local system $\{L_i\}_{i \in I}$ is as in Theorem 5.1. Therefore $L' = \cup_{n \in \mathbf{N}} L_{j_n}$ is a thin simple locally finite Lie algebra of countable dimension. By Corollary 5.4, L' is isomorphic to either $\mathfrak{sl}_\infty$, or $\mathfrak{so}_\infty$, or $\mathfrak{sp}_\infty$. A. Zhilinskii has proved [16, Corollary 2.4.1] that inductive systems for them satisfy DCC. Observe that $\Psi^k = \{\Phi^k_{j_n}\}_{n \in \mathbf{N}}$, $k = 1, 2, \ldots$, are inductive systems for L', and they form a descending chain. So we have a contradiction. Assume now that L is thick. By Theorem 3.12, there exist $c_1, c_2 \in \mathbf{N}$ such that for any $i \in I$ we have: (i) for any $M \in \Phi_i$ the number of nodes S^k_i at level i such that $\langle M \!\downarrow\! S^k_i \rangle \neq \{T^k_i\}$ is at most c_1; (ii) $\delta^k_i(\Phi_i) \leq c_2$, $1 \leq k \leq n_i$. Clearly, these estimates hold for all inductive systems Φ^n, $n = 1, 2, \ldots$. It follows from the definition of the function δ, that there exists $p \in \mathbf{N}$ such that for any classical simple finite-dimensional Lie algebra $\mathfrak{h}$ the number of nonisomorphic irreducible $\mathfrak{h}$-modules V with $\delta_{\mathfrak{h}}(V) \leq c_2$ does not exceed p. Therefore there exists $q \in \mathbf{N}$ such that for any semisimple finite-dimensional Lie algebra $\mathfrak{g} =$

$\mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r$ with classical simple components $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$, $r \le c_1$, the number of nonisomorphic irreducible $\mathfrak{g}$-modules V with $\delta_{\mathfrak{h}_k}(V) \le c_2$, $1 \le k \le r$, does not exceed q. Therefore for any $j \in I$ the length of the chain of the strict embeddings $\Phi_j^1 \supset \Phi_j^2 \supset \ldots$ does not exceed q. This contradicts the remark above.

Taking into account Corollary 2.4, we immediately obtain Theorem 1.2.

Now we introduce another important class of simple locally finite Lie algebras.

Definition 6.2 Let L be a simple diagonal locally finite Lie algebra, $\{L_i\}_{i\in I}$ a pure diagonal local system of L, and $\mathfrak{B}$ the corresponding Bratteli diagram. $\{L_i\}_{i\in I}$ and $\mathfrak{B}$ are called *diagonally dense* if for any pair $i < j$ and any $l = 1, \ldots, n_j$ the set $\langle V_j^l{\downarrow}S_i\rangle$ does not contain the trivial (one-dimensional) S_i-module T_i, i.e.

$$\langle V_j^l{\downarrow}S_i\rangle \subset \{V_i^1, \ldots V_i^{n_i}, V_i^{1^*}, \ldots, V_i^{n_i^*}\} = \Omega_i.$$

L is called *diagonally dense* if it has a diagonally dense local system.

One can easily check that if $\{L_i\}_{i\in I}$ is a diagonally dense local system of L, then $\Omega = \{\Omega_i\}_{i\in I}$ is an inductive system for L. Indeed, it suffices to show that for any $k \in \{1, \ldots, n_i\}$ there exists $l \in \{1, \ldots, n_j\}$ such that $V_i^k \in \langle V_j^l{\downarrow}S_i\rangle$ (this implies $V_i^{k^*} \in \langle V_j^{l^*}{\downarrow}S_i\rangle$) or $V_i^{k^*} \in \langle V_j^l{\downarrow}S_i\rangle$ (this implies $V_i^k \in \langle V_j^{l^*}{\downarrow}S_i\rangle$. But this follows from injectivity of embedding $S_i^k \to S_j$. Denote by Φ^0 the trivial inductive system for L ($\Phi^0 = \{T_i\}_{i\in I}$). Observe that $\Phi^0 \not\subseteq \Omega$. Recall that $A(L) = M(\Phi^0)$ is the augmentation ideal of $U(L)$. By Corollary 2.4, $M(\Omega) \not\subseteq A(L)$ where $M(\Omega)$ is the semiprimitive ideal of $U(L)$ corresponding to Ω. It turns out that this property almost completely characterizes diagonally dense Lie algebras. More exactly, we have the following

Theorem 6.3 *Let L be a simple locally finite Lie algebra. Then the following conditions are equivalent.*

(1) L is either diagonally dense or finitary of type B.

(2) There is a nonempty inductive system Φ for L such that $\Phi^0 \not\subseteq \Phi$.

(3) There is a proper ideal M of $U(L)$ such that $U(L)/M$ is locally finite and $M \not\subseteq A(L)$.

(4) There is a simple locally finite associative enveloping algebra for L (a quotient of $U(L)$ containing L).

Denote by $\zeta(S_i, S_j^l)$ the number of trivial (one-dimensional) composition factors of the restriction $V_j^l{\downarrow}S_i$. The following lemmas precede the proof of the theorem.

Lemma 6.4 *Let $\mathfrak{B}$ be a pure diagonal Bratteli diagram, S_t^p be an S_j^l-accessible node of $\mathfrak{B}$, $i < j < t$. Then $\zeta(S_i, S_t^p) \ge \zeta(S_i, S_j^l) + \zeta(S_j, S_t^p)$.*

Proof. Observe that either $V_j^l \in \langle V_t^p{\downarrow}S_j \rangle$ or $V_j^{l*} \in \langle V_t^p{\downarrow}S_j \rangle$. This implies the required inequality.

Lemma 6.5 *Let $\mathfrak{B}$ be a pure diagonal Bratteli diagram of a simple diagonal locally finite Lie algebra L. Assume that L is not diagonally dense. Fix $i \in I$. Then the values $\zeta(S_i, S_j^l)$ where S_j^l runs over S_i-accessible (i.e. S_i^k-accessible for some k) nodes of $\mathfrak{B}$ are unbounded.*

Proof. Assume that this is false. Take a node S_j^l such that $\zeta(S_i, S_j^l)$ is maximal. By Definition 6.2, there exists a node S_t^p with $t > j$ such that $\zeta(S_j, S_t^p) \geq 1$. Since L is simple, by Theorem 3.2, there exists $r > t$ and $1 \leq q \leq n_r$ such that S_r^q is S_j^l- and S_t^p-accessible. By Lemma 6.4,

$$\zeta(S_j, S_r^q) \geq \zeta(S_j, S_t^p) + \zeta(S_t, S_r^q) \geq 1.$$

Since S_r^q is S_j^l-accessible, applying Lemma 6.4 again, we have

$$\zeta(S_i, S_r^q) \geq \zeta(S_i, S_j^l) + \zeta(S_j, S_r^q) \geq \zeta(S_i, S_j^l) + 1,$$

contradicting the maximality of $\zeta(S_i, S_j^l)$.

Lemma 6.6 *Let $\mathfrak{B}$ be a pure diagonal Bratteli diagram of a simple diagonal locally finite Lie algebra L. Assume that L is not diagonally dense. Fix $i \in I$ and $n \in \mathbf{N}$. Then there exists $j \in I$ such that for any S_i-accessible node S_j^l at level j we have $\operatorname{rk} S_j^l > n$ and $\zeta(S_i, S_j^l) > n$.*

Proof. One can derive from [2, Corollary 3.3] that there exists a node S_t^p of rank greater than n. On the other hand, by Lemma 6.5, there exists an S_i-accessible node S_u^m such that $\zeta(S_i, S_u^m) > n$. By Corollary 3.4, there exists $j > i, t, u$ such that the sets of S_i^1-,$\ldots$,$S_i^{n_i}$-, S_t^p- and S_u^m-accessible nodes at level j coincide. In particular, all S_i-accessible nodes at level j have ranks greater than n. Let S_j^l be S_i-accessible. Then by Lemma 6.4,

$$\zeta(S_i, S_j^l) \geq \zeta(S_i, S_u^m) + \zeta(S_u, S_j^l) > n,$$

as required.

Lemma 6.7 *Let $\mathfrak{h} \to \mathfrak{g}$ be an embedding of a semisimple Lie algebra $\mathfrak{h}$ in a simple classical Lie algebra $\mathfrak{g}$ of rank n. Set $k = \min([\zeta(\mathfrak{h}, \mathfrak{g})/2], [(n-3)/2])$. Assume that V is an irreducible $\mathfrak{g}$-module with $\delta_{\mathfrak{g}}(V) \leq k$. Then $T_{\mathfrak{h}} \in \langle V{\downarrow}\mathfrak{h} \rangle$ where $T_{\mathfrak{h}}$ is the trivial $\mathfrak{h}$-module.*

Proof. Denote by V_1 the standard $\mathfrak{g}$-module, and by Z the maximal subspace of V_1 annihilated by $\mathfrak{h}$. Observe that $\dim Z = \zeta(\mathfrak{h}, \mathfrak{g})$. Consider two cases.
Case 1. Let $\mathfrak{g}$ be of type B_n, C_n, or D_n. Then $\mathfrak{g}$ preserves some symmetric (or skew-symmetric form on V_1. Let $Z' \subseteq Z$ be a completely isotropic subspace of

maximal dimension. Observe that $l = \dim Z' \geq [\zeta(\mathfrak{h}, \mathfrak{g})/2] \geq k$. Let $e_1, \ldots, e_l$ be a basis of Z'. Obviously, one can add some vectors to obtain a Witt basis of V_1. Fix the corresponding Cartan subalgebra of $\mathfrak{g}$ (consisting of diagonal matrices). Choose a base of the root system such that e_i has weight ε_i ($1 \leq i \leq k$). In particular, e_1 is a highest weight vector of V_1. Let $\omega_1, \ldots, \omega_n$ be the fundamental weights of $\mathfrak{g}$, V_i the irreducible $\mathfrak{g}$-module with highest weight ω_i ($1 \leq i \leq k$). Then V_i can be realized as the submodule of $\wedge^i V_1$ generated by the vector $v_i = e_1 \wedge \ldots \wedge e_i$. Furthermore, v_i is a highest weight vector of V_i. Observe that $\mathfrak{h} v_i = 0$, $1 \leq i \leq k$. Let $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ be the highest weight of V. Since $\delta_{\mathfrak{g}}(V) = \delta(\lambda) \leq k \leq [(n-3)/2]$, we have $a_j = 0$ for any $j > k$. Therefore V can be realized as the submodule of

$$\underbrace{V_1 \otimes \ldots \otimes V_1}_{a_1} \otimes \ldots \otimes \underbrace{V_k \otimes \ldots \otimes V_k}_{a_k}$$

generated by the vector

$$v = \underbrace{v_1 \otimes \ldots \otimes v_1}_{a_1} \otimes \ldots \otimes \underbrace{v_k \otimes \ldots \otimes v_k}_{a_k}.$$

Since $\mathfrak{h} v_i = 0$ for any $1 \leq i \leq k$, we have $\mathfrak{h} v = 0$. Therefore $T_{\mathfrak{h}} \in \langle V {\downarrow} \mathfrak{h} \rangle$, as required.

Case 2. Let $\mathfrak{g}$ be of type A_n. Since $\dim Z = \zeta(\mathfrak{h}, \mathfrak{g}) \geq 2k$, there exists a basis $e_1, \ldots, e_{n+1}$ of V_1 such that $e_1, \ldots, e_k, e_{n-k+2}, \ldots, e_{n+1} \in Z$. Obviously, one can assume that e_i has weight ε_i ($1 \leq i \leq n+1$). Let $e'_1, \ldots, e'_{n+1}$ be the dual basis in $V_1^* \cong V_n$. One can check that e'_{n+1} is a highest weight vector of V_1^* (of weight ω_n), and $\mathfrak{h} e'_{n-k+2} = \ldots = \mathfrak{h} e'_{n+1} = 0$. Denote by V_i (resp., W_i) the irreducible $\mathfrak{g}$-module with highest weight ω_i (resp., ω_{n-i+1}), $1 \leq i \leq k$. Recall that $V_i \cong \wedge^i V_1$, $W_i \cong \wedge^i V_1^*$; and $v_i = e_1 \wedge \ldots \wedge e_i$, $w_i = e'_{n+1} \wedge \ldots \wedge e'_{n-i+2}$ are their highest weight vectors. One has $\mathfrak{h} v_i = \mathfrak{h} w_i = 0$, $1 \leq i \leq k$. Let $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ be the highest weight of V. Since $\delta_{\mathfrak{g}}(V) = \delta(\lambda) \leq k \leq [(n-3)/2]$, we have $a_j = 0$ for any $k < j < n-k+1$. Therefore V can be realized as the submodule of

$$\underbrace{V_1 \otimes \ldots \otimes V_1}_{a_1} \otimes \ldots \otimes \underbrace{V_k \otimes \ldots \otimes V_k}_{a_k} \otimes \underbrace{W_k \otimes \ldots \otimes W_k}_{a_{n-k+1}} \otimes \ldots \otimes \underbrace{W_1 \otimes \ldots \otimes W_1}_{a_n}$$

generated by the vector

$$\underbrace{v_1 \otimes \ldots \otimes v_1}_{a_1} \otimes \ldots \otimes \underbrace{v_k \otimes \ldots \otimes v_k}_{a_k} \otimes \underbrace{w_k \otimes \ldots \otimes w_k}_{a_{n-k+1}} \otimes \ldots \otimes \underbrace{w_1 \otimes \ldots \otimes w_1}_{a_n}$$

Observe that $\mathfrak{h} v = 0$. Therefore $T_{\mathfrak{h}} \in \langle V {\downarrow} \mathfrak{h} \rangle$, as required.

Proof of Theorem 6.3. (1)$\Rightarrow$(2). If L is diagonally dense, then the inductive system $\Omega = \{\Omega_i\}_{i \in I}$ has the required property (see the remark above). If L is finitary (or equivalently, thin) of type B, then one can take the inductive system

$\tilde{\Psi}^1$ (see Remark 5.5). Observe that $\tilde{\Psi}^1_i = \{\Pi_i\}$ where Π_i is the spinor module for L_i.

(2)⇒(1). Let L be neither diagonally dense, nor finitary of type B, Φ a nonempty inductive system for L. We have to prove that $\Phi^0 \subseteq \Phi$. If L is finitary (thin) of type A or C, then by Remark 5.5, $\Phi^0 \subseteq \Phi$. Assume now that L is non-finitary (thick). Then by Theorem 3.12, there exists $c \in \mathbf{N}$ such that $\delta(\Phi_i) < c$ for any $i \in I$. Fix $i \in I$. We have to show that $T_i \in \Phi_i$. By Lemma 6.6, there exists $j \in I$ such that for any S_i-accessible node S^l_j at the level j we have $\operatorname{rk} S^l_j > 2c$ and $\zeta(S_i, S^l_j) > 2c$. Let $M = M_1 \otimes \ldots \otimes M_{n_j}$ be a module from Φ_j represented in canonical form. By the remark above, $\delta^l_j(M_l) < c$, $1 \leq l \leq n_j$. Therefore by Lemma 6.7, $T_i \in \langle M_l \downarrow S_i \rangle$ for any $l = 1, \ldots, n_j$. Hence every M_l contains a nonzero vector v_l annihilated by S_i. Observe that $v = v_1 \otimes \ldots \otimes v_{n_j} \in M$ is annihilated by S_i. So $T_i \in \langle M \downarrow S_i \rangle \subseteq \langle \Phi_j \downarrow S_i \rangle = \Phi_i$, as required.

(2)⇔(3). This follows from Theorem 2.3 and Corollary 2.4.

(3)⇒(4). Let M be as in (3). Denote by $\bar{M}$ a maximal proper ideal of $U(L)$ containing M. Observe that $U(L)/\bar{M}$ is simple, and $\bar{M} \cap L$ is an ideal of L. If $\bar{M} \cap L = L$, then $\bar{M} = A(L)$, so $M \subseteq A(L)$, a contradiction. Since L is simple, we have $\bar{M} \cap L = 0$. Therefore we have an injective homomorphism $L \to U(L)/\bar{M}$, as required.

(4)⇒(3). Let A be a simple locally finite associative enveloping algebra for L, M the kernel of the natural epimorphism $U(L) \to A$. Then M is a proper maximal ideal of $U(L)$ different from $A(L)$. Therefore $M \not\subseteq A(L)$.

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