# Infinite dimensional irreducible Lie algebras containing transformations of finite rank

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**Abstract**

Let $\mathbb{F}$ be a field of characteristic zero and let $V$ be an infinite dimensional vector space over $\mathbb{F}$. A linear transformation $x$ of $V$ is called *finitary* if $\dim xV < \infty$. The aim of this paper is to describe irreducible Lie subalgebras of $\mathfrak{gl}(V)$ containing nonzero finitary transformations. It turns out that any such algebra is a semidirect product of a finite dimensional Lie algebra and a "dense" Lie subalgebra of $\mathfrak{gl}(W)$ for some vector space $W$.

## 1 Introduction

Let us recall two classical theorems, which seem to be of the same nature. The first one is Nathan Jacobson's description of the structure of irreducible associative rings of linear transformations containing finitary transformations ([4], [5, Ch. IX], [6, §4.9]). It turns out that the finitary transformations in such rings form a dense simple ideal of explicit structure (see Jacobson's theorem below). The second result is Helmut Wielandt's [7] characterization of infinite permutation groups which contain finitary (i.e. fixing all but finitely many elements) permutations:

**Theorem 1.1 (Wielandt)** *Let $G$ be a primitive permutation group on an infinite set $\Omega$ containing a nonidentity finitary permutation. Then $G$ contains the (finitary) alternating group* $\mathrm{Alt}(\Omega)$.

In this paper we extend Jacobson-Wielandt philosophy to Lie algebras. We need some notation to state our results. Let $\mathbb{F}$ be a field and let $V$ be a vector space over $\mathbb{F}$. We denote by

$$\mathfrak{F}(V) = \{x \in \mathrm{End}_{\mathbb{F}} V \mid \dim xV < \infty\}$$

the algebra of all finitary transformations of the vector space $V$. A subalgebra $L$ of $\mathrm{End}_{\mathbb{F}} V$ is called *finitary* if $L \subseteq \mathfrak{F}(V)$. Denote by $V^*$ the vector space dual to $V$. We identify $V^*$ with the space of linear functions on $V$. Recall that $V^*$ is a natural *right* $\mathrm{End}_{\mathbb{F}} V$-module: for $\varphi \in V^*$ and $x \in \mathrm{End}_{\mathbb{F}} V$, the function $\varphi x$ is defined as $\varphi x : v \mapsto \varphi(xv)$, $v \in V$. A subspace $\Pi$ of $V^*$ is called *total* if

$$\mathrm{Ann}_V \Pi = \{v \in V \mid \varphi v = 0 \text{ for all } \varphi \in \Pi\} = 0.$$

Let $\Pi$ be a total subspace of $V^*$. Set

$$\begin{aligned} \mathfrak{L}(V,\Pi) &= \{x \in \mathrm{End}_{\mathbb{F}} V \mid \Pi x \subseteq \Pi\}; \\ \mathfrak{F}(V,\Pi) &= \{x \in \mathfrak{F}(V) \mid \Pi x \subseteq \Pi\} = \mathfrak{L}(V,\Pi) \cap \mathfrak{F}(V). \end{aligned}$$

One can easily show that $\mathfrak{F}(V,\Pi)$ is the smallest ideal of the algebra $\mathfrak{L}(V,\Pi)$. To simplify the exposition let us assume that $\mathbb{F}$ is algebraically closed. Then the Jacobson's result can be stated as follows (see also more general Theorem 2.4 and Corollary 2.5).

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**Theorem 1.2 (Jacobson)** *Let $V$ be a vector space over an algebraically closed field $\mathbb{F}$ and let $A$ be an irreducible associative subalgebra of $\operatorname{End}_{\mathbb{F}} V$ such that $A_{\mathfrak{F}} = A \cap \mathfrak{F}(V) \neq 0$. Then there exists a total subspace $\Pi$ of $V^*$ such that*

$$\mathfrak{F}(V, \Pi) = A_{\mathfrak{F}} \subseteq A \subseteq \mathfrak{L}(V, \Pi).$$

Let $V$ be a vector space, $\Pi$ be a total subspace of the dual $V^*$. We denote by $\mathfrak{gl}(V, \Pi)$, $\mathfrak{fgl}(V, \Pi)$, and $\mathfrak{fgl}(V)$ the Lie algebras $\mathfrak{L}(V, \Pi)$, $\mathfrak{F}(V, \Pi)$, and $\mathfrak{F}(V)$, respectively, under the usual bracket multiplication. Let $\Phi$ (resp., $\Psi$) be a nondegenerate bilinear symmetric (resp., skew-symmetric) form on $V$. We define the following Lie algebras:

$$\begin{aligned}
\mathfrak{o}(V, \Phi) &= \{x \in \mathfrak{gl}(V) \mid \Phi(xv, w) + \Phi(v, xw) = 0 \quad \forall\, v, w \in V\}; \\
\mathfrak{sp}(V, \Psi) &= \{x \in \mathfrak{gl}(V) \mid \Psi(xv, w) + \Psi(v, xw) = 0 \quad \forall\, v, w \in V\};
\end{aligned}$$

$$\begin{aligned}
\mathfrak{fo}(V, \Phi) &= \mathfrak{o}(V, \Phi) \cap \mathfrak{fgl}(V); \\
\mathfrak{fsp}(V, \Psi) &= \mathfrak{sp}(V, \Psi) \cap \mathfrak{fgl}(V).
\end{aligned}$$

Set also $\mathfrak{fsl}(V, \Pi) = [\mathfrak{fgl}(V, \Pi), \mathfrak{fgl}(V, \Pi)]$. Denote by $\mathrm{id}_V$ the identical transformation of $V$. We prove the following theorem.

**Theorem 1.3** *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. Let $W$ be an infinite dimensional vector space over $\mathbb{F}$ and let $L$ be an irreducible Lie subalgebra of $\mathfrak{gl}(W)$. Assume that $L_{\mathfrak{F}} = L \cap \mathfrak{fgl}(W) \neq 0$. Then the following holds.*

(1) *There exist an irreducible $L_{\mathfrak{F}}$-module $V$ and an integer $n$ such that the $L_{\mathfrak{F}}$-module $W$ is the direct sum of $n$ copies of $V$.*

(2) *The Lie algebra $L'_{\mathfrak{F}} = [L_{\mathfrak{F}}, L_{\mathfrak{F}}]$ is simple.*

(3) *$L \subseteq \mathfrak{sl}_n(\mathbb{F}) \oplus \mathfrak{gl}(V) \subseteq \mathfrak{gl}(W)$ with the natural action of $\mathfrak{sl}_n(\mathbb{F}) \oplus \mathfrak{gl}(V)$ on $W = \mathbb{F}^n \otimes_{\mathbb{F}} V$; in particular, $L_{\mathfrak{F}} \subseteq \mathfrak{fgl}(V)$.*

(4) *Let $M$ and $\bar{L}_V$ be the projections of $L$ to $\mathfrak{sl}_n(\mathbb{F})$ and $\mathfrak{gl}(V)$, respectively, and let $L_V = L \cap \mathfrak{gl}(V)$ (so $L \subseteq M \oplus \bar{L}_V$ and $L/L_V \cong M$). Then $M$ is an irreducible subalgebra of $\mathfrak{sl}_n(\mathbb{F})$ and one of the following conditions holds:*

$$\begin{aligned}
\mathfrak{fsl}(V, \Pi) &= L'_{\mathfrak{F}} \subseteq L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq \ \mathfrak{gl}(V, \Pi) \quad \text{and } L_{\mathfrak{F}} \subseteq \mathfrak{fgl}(V, \Pi); \\
\mathfrak{fo}(V, \Psi) &= L'_{\mathfrak{F}} = L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq \ \mathfrak{o}(V, \Psi) \oplus \mathbb{F} \cdot \mathrm{id}_V; \\
\mathfrak{fsp}(V, \Theta) &= L'_{\mathfrak{F}} = L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq \ \mathfrak{sp}(V, \Theta) \oplus \mathbb{F} \cdot \mathrm{id}_V.
\end{aligned}$$

*Here $\Psi$ (resp. $\Theta$) is a nondegenerate symmetric (resp. skew-symmetric) form on $V$, and $\Pi$ is a total subspace of the dual $V^*$.*

Theorem 1.3 is a particular case of Theorem 1.4, where $\mathbb{F}$ is an arbitrary field of characteristic 0. Let $L$ be a Lie algebra and let $V$ be an $L$-module. The *centralizer* $\Delta$ of $V$ is the algebra $(\operatorname{End}_L V)^\circ$ opposite to the algebra of endomorphisms of the $L$-module $V$, so $V$ can be viewed as a right $\Delta$-module. If $V$ is irreducible, then by Shur's lemma, $\Delta$ is a division $\mathbb{F}$-algebra and $V$ can be viewed as a right $\Delta$-space $V_\Delta$.

**Theorem 1.4** *Let $\mathbb{F}$ be a field of chracteristic 0. Let $W$ be an infinite dimensional vector space over $\mathbb{F}$ and let $L$ be an irreducible Lie subalgebra of $\mathfrak{gl}(W)$. Assume that $L_{\mathfrak{F}} = L \cap \mathfrak{fgl}(W) \neq 0$. Then the following holds.*

(1) *There exist an irreducible $L_{\mathfrak{F}}$-module $V$ and an integer $n$ such that the $L_{\mathfrak{F}}$-module $W$ is a direct sum of $n$ copies of $V$.*

(2) *The centralizer* $\Delta$ *of the* $L_{\mathfrak{F}}$*-module* $V$ *is a finite dimensional division* $\mathbb{F}$*-algebra.*

(3) *The Lie algebra* $L'_{\mathfrak{F}} = [L_{\mathfrak{F}}, L_{\mathfrak{F}}]$ *is simple.*

(4) $L \subseteq \mathfrak{sl}_n(\Delta^\circ) \oplus \mathfrak{gl}(V_\Delta) \subseteq \mathfrak{gl}(W)$ *with the natural action of* $\mathfrak{sl}_n(\Delta^\circ) \oplus \mathfrak{gl}(V_\Delta)$ *on* $W = \mathbb{F}^n \otimes_{\mathbb{F}} V_\Delta = (\Delta^\circ)^n \otimes_{\Delta^\circ} V$*; in particular,* $L_{\mathfrak{F}} \subseteq \mathfrak{fgl}(V_\Delta)$*.*

(5) *Let* $M$ *and* $\bar{L}_V$ *be the projections of* $L$ *to* $\mathfrak{sl}_n(\Delta^\circ)$ *and* $\mathfrak{gl}(V_\Delta)$*, respectively, and let* $L_V = L \cap \mathfrak{gl}(V_\Delta)$ *(so* $L \subseteq M \oplus \bar{L}_V$ *and* $L/L_V \cong M$*). Then there is no nontrivial (right)* $\Delta^\circ$*-subspace of* $(\Delta^\circ)^n$ *invariant under* $M$ *and one of the following conditions holds:*

$$\begin{array}{rll} \mathfrak{fsl}(V_\Delta, \Pi) & = L'_{\mathfrak{F}} \subseteq L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq & \mathfrak{gl}(V_\Delta, \Pi) \quad \text{and } L_{\mathfrak{F}} \subseteq \mathfrak{fgl}(V_\Delta, \Pi); \\ \mathfrak{fsu}(V_\Delta, \Phi) & = L'_{\mathfrak{F}} \subseteq L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq & \mathfrak{u}(V_\Delta, \Phi) + Z(\Delta^\circ) \quad \text{and } L_{\mathfrak{F}} \subseteq \mathfrak{fu}(V_\Delta, \Phi); \\ \mathfrak{fo}(V_\Delta, \Psi) & = L'_{\mathfrak{F}} = L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq & \mathfrak{o}(V_\Delta, \Psi) \oplus Z(\Delta^\circ); \\ \mathfrak{fsp}(V_\Delta, \Theta) & = L'_{\mathfrak{F}} = L_{\mathfrak{F}} \subseteq L_V \subseteq \bar{L}_V \subseteq & \mathfrak{sp}(V_\Delta, \Theta) \oplus Z(\Delta^\circ). \end{array}$$

*Here* $Z(\Delta^\circ)$ *is the center of* $\Delta^\circ = \operatorname{End}_L V$*;* $\Phi$*,* $\Psi$*, and* $\Theta$ *are nondegenerate forms on* $V_\Delta$*; and* $\Pi$ *is a total subspace of the dual* $V_\Delta^*$*.*

The definitions of the algebras from Theorem 1.4(5) are given in Section 3. The algebras $\mathfrak{fsl}(V_\Delta, \Pi)$, $\mathfrak{fsu}(V_\Delta, \Phi)$, $\mathfrak{fo}(V_\Delta, \Psi)$, and $\mathfrak{fsp}(V_\Delta, \Theta)$ are simple and (for various $\Delta$, $V_\Delta$, $\Pi$, $\Phi$, $\Psi$, and $\Theta$) exhaust all finitary simple Lie algebras over fields of characteristic 0 (see [2]). In a subsequent paper the author plan to consider the case of positive characteristic of the ground field $\mathbb{F}$. There is an analog of Theorem 1.4 for infinite dimensional irreducible groups of linear transformations containing “finitary” transformations [3].

## 2 Irreducible associative rings of linear transformations

Let $\Delta$ be a skew field and let $V$ be a right vector space over $\Delta$. Denote by $V^*$ the set of all $\Delta$-linear functions $\varphi : V \to \Delta$. Then $V^*$ is a *left* vector space over $\Delta$ under the operations:

$$(\varphi + \psi)v = \varphi v + \psi v, \quad (\delta\varphi)v = \delta(\varphi v)$$

for all $v \in V$, $\varphi, \psi \in V^*$, and $\delta \in \Delta$. On the other hand, $V^*$ is a *right* $\operatorname{End}_\Delta V$-module: for $\varphi \in V^*$ and $x \in \operatorname{End}_\Delta V$, the function $\varphi x$ is defined as

$$\varphi x : v \mapsto \varphi(xv) \quad (v \in V).$$

Clearly, $\Delta^\circ$ is the centralizer of the right $\operatorname{End}_\Delta V$-module $V^*$.

**Definition 2.1** A subspace $\Pi$ of $V^*$ is called *total* if $\operatorname{Ann}_V \Pi = \{v \in V \mid \varphi v = 0 \text{ for all } \varphi \in \Pi\} = 0$.

As in Introduction we denote by $\mathfrak{F}(V)$ the subring of all $x \in \operatorname{End}_\Delta V$ such that $\dim xV < \infty$. Let $\Pi$ be a total subspace of $V^*$. Set

$$\begin{array}{rcl} \mathfrak{L}(V, \Pi) & = & \{x \in \operatorname{End}_\Delta V \mid \Pi x \subseteq \Pi\}; \\ \mathfrak{F}(V, \Pi) & = & \{x \in \mathfrak{F}(V) \mid \Pi x \subseteq \Pi\} = \mathfrak{L}(V, \Pi) \cap \mathfrak{F}(V). \end{array}$$

Observe that $\mathfrak{F}(V, \Pi)$ is the ring of all transformations $x \in \operatorname{End}_\Delta V$ of the form

$$v \mapsto e_1(\varphi_1 v) + \cdots + e_n(\varphi_n v)$$

where $n$ is an integer, $e_1, \ldots, e_n \in V$, and $\varphi_1, \ldots, \varphi_n \in \Pi$. We also have $\mathfrak{F}(V) = \mathfrak{F}(V, V^*)$.

An element $t \in \operatorname{End}_\Delta V$ is called a transvection if its rank (the dimension of $tV$) is 1, i.e. there exists $u \in V$ and $\varphi \in V^*$ such that $tv = u(\varphi v)$ for all $v \in V$. We shall write $t = t_{u\varphi}$. Note that we use the term ”transvection” in a nonstandard way (in the standard definition one takes $t - 1$ rather than $t$).

The following properties of transvections can be checked by the direct calculations.

**Lemma 2.2** *For all $v, u \in V$, $\varphi, \psi \in \Pi$, and $\delta \in \Delta$ we have*
(1) $t_{v+u,\varphi} = t_{v\varphi} + t_{u\varphi}$;
(2) $t_{u,\varphi+\psi} = t_{u\varphi} + t_{u\psi}$;
(3) $t_{u\delta,\varphi} = t_{u,\delta\varphi}$;
(4) $t_{u\varphi}t_{v\psi} = t_{u(\varphi v),\psi} = t_{u,(\varphi v)\psi}$.

Note that any element $x \in \mathfrak{F}(V, \Pi)$ can be represented as a sum of $n = \operatorname{rk} x$ transvections. We shall need the following simple lemma from [2].

**Lemma 2.3** *([2, Lemma 10.2]) Let $x \in \mathfrak{F}(V, \Pi)$ and $n = \operatorname{rk} x$. Assume that $x$ is represented as a sum of $n$ transvections $x = t_{u_1\varphi_1} + \cdots + t_{u_n\varphi_n}$. Then*
(1) $u_1, \ldots, u_n$ *are linearly independent over* $\Delta$;
(2) $\varphi_1, \ldots, \varphi_n$ *are linearly independent over* $\Delta$;
(3) $\varphi_1, \ldots, \varphi_n \in \Pi$.

**Theorem 2.4 (Jacobson)** *Let $A$ be an associative ring and let $V$ be a faithful irreducible $A$-module. Let $\Delta$ be the centralizer of $V$. Assume that $A_{\mathfrak{F}} = A \cap \mathfrak{F}(V_\Delta) \neq 0$. Then there exists a total subspace $\Pi$ of $V_\Delta^*$ such that*

$$\mathfrak{F}(V_\Delta, \Pi) = A_{\mathfrak{F}} \subseteq A \subseteq \mathfrak{L}(V_\Delta, \Pi).$$

*Proof.* This follows from [5, Theorem 3.10.6 and General density theorem in §9.13]. □

**Corollary 2.5** *Let $V$ be a vector space over a field $\mathbb{F}$ and let $A$ be an irreducible associative $\mathbb{F}$-subalgebra of $\operatorname{End}_{\mathbb{F}} V$. Assume that $A_{\mathfrak{F}} = A \cap \mathfrak{F}(V) \neq 0$. Then the centralizer $\Delta$ of the $A$-module $V$ is a finite dimensional division algebra over $\mathbb{F}$ and there exists a total subspace $\Pi$ of $V_\Delta^*$ such that*

$$\mathfrak{F}(V_\Delta, \Pi) = A_{\mathfrak{F}} \subseteq A \subseteq \mathfrak{L}(V_\Delta, \Pi).$$

*Proof.* By [5, §9.14], $A$ is a dense subring of $\operatorname{End}_\Delta V$ and $\Delta$ is an $\mathbb{F}$-algebra. Let $a \in A_{\mathfrak{F}}$. Then for all $v \in V$ the vector space $(av)\Delta = a(v\Delta)$ is finite dimensional over $\mathbb{F}$. Therefore, $\Delta$ is finite dimensional, so $A_{\mathfrak{F}} = A \cap \mathfrak{F}(V_\Delta)$. It remains to apply Theorem 2.4. □

Assume that $\Delta$ has an involution (an involution of order 1 or 2) $\delta \mapsto \bar{\delta}$. A function $\Phi : V \times V \to \Delta$ is called a *Hermitian* form on $V$ if

$$\begin{aligned} \Phi(u+v, w) &= \Phi(u, w) + \Phi(v, w); \\ \Phi(v\delta, w\gamma) &= \bar{\gamma}\Phi(v, w)\delta; \\ \Phi(w, v) &= \overline{\Phi(v, w)} \end{aligned}$$

for all $u, v, w \in V$ and $\delta, \gamma \in \Delta$. If we have $\Phi(w, v) = -\overline{\Phi(v, w)}$ instead of $\Phi(w, v) = \overline{\Phi(v, w)}$, then $\Phi$ is called *skew-Hermitian*. We say that $\Phi$ is of the *first kind* if the involution $\delta \mapsto \bar{\delta}$ acts trivially on the center of $\Delta$, otherwise we say that $\Phi$ is of the *second kind*. We shall call Hermitian forms of the first kind *orthogonal* and skew-Hermitian those *symplectic*. We say that $\Phi$ is *unitary* if it is a Hermitian form of the second kind.

Assume that $\Phi$ is a nondegenerate Hermitian or skew-Hermitian form on $V$. Denote by $\Pi_\Phi$ the set of all functions $\varphi_u : v \mapsto \Phi(v, u)$. One easily checks that $\Pi_\Phi$ is a total subspace of $V^*$. Moreover, the map $u \mapsto \varphi_u$ produces an isomorphism between the left $\Delta$-spaces $V$ and $\Pi_\Phi$ ($V$ can be considered as a left vector space setting $\delta v = v\bar{\delta}$ for $\delta \in \Delta$ and $v \in V$), so we shall denote the algebras $\mathfrak{L}(V, \Pi_\Phi)$ and $\mathfrak{F}(V, \Pi_\Phi)$ by $\mathfrak{L}_\Phi(V, V)$ and $\mathfrak{F}_\Phi(V, V)$, respectively. For each $x \in \mathfrak{L}_\Phi(V, V)$ there exists a unique $x^* \in \mathfrak{L}_\Phi(V, V)$ such that $\Phi(xv, w) = \Phi(v, x^*w)$ for all $v, w \in V$. Moreover, $x \in \mathfrak{F}_\Phi(V, V)$ if and only if $x^* \in \mathfrak{F}_\Phi(V, V)$. The map $x \mapsto x^*$ is called the *adjoint map with respect to* $\Phi$.

**Theorem 2.6 ([5, Theorem 9.12.9],[6, §4.12])** *Let $V$ be a vector space over a skew field $\Delta$ and let $\Phi$ be a nondegenerate Hermitian or skew-Hermitian form on $V$. Then the adjoint map with respect to $\Phi$ is an involution of the ring $\mathfrak{L}_\Phi(V,V)$.*

Let $A$ be a ring with involution $\alpha$. Then one can easily check that the set $\mathfrak{h}^\alpha(A) = \{a \in A \mid a^\alpha = -a\}$ is a Lie ring under the usual commutator. Let now we have a vector space $V$ over a skew field $\Delta$ with a nondegenerate Hermitian or skew-Hermitian form $\Phi$. Then we define the following Lie rings:

$$\begin{aligned}
\mathfrak{h}(V,\Phi) &= \{x \in \operatorname{End}_\Delta V \mid \Phi(xv,w)+\Phi(v,xw)=0 \text{ for all } v,w\in V\};\\
\mathfrak{fh}(V,\Phi) &= \{x \in \mathfrak{F}(V) \mid \Phi(xv,w)+\Phi(v,xw)=0 \text{ for all } v,w\in V\};\\
\mathfrak{fsh}(V,\Phi) &= [\mathfrak{fh}(V,\Phi),\mathfrak{fh}(V,\Phi)].
\end{aligned}$$

**Proposition 2.7** *Let $V$ be a vector space with a nondegenerate Hermitian or skew-Hermitian form $\Phi$ and $\alpha$ be the involution of $\mathfrak{L}_\Phi(V,V)$ given by the adjoint map with respect to $\Phi$. Then $\mathfrak{h}^\alpha(\mathfrak{L}_\Phi(V,V)) = \mathfrak{h}(V,\Phi)$ and $\mathfrak{h}^\alpha(\mathfrak{F}_\Phi(V,V)) = \mathfrak{fh}(V,\Phi)$.*

*Proof.* It suffices to prove the first equality. Recall that $\mathfrak{L}_\Phi(V,V) = \mathfrak{L}(V,\Pi_\Phi)$ where $\Pi_\Phi$ is the space of all functions $\varphi_u : v \mapsto \Phi(v,u)$. Let $x \in \mathfrak{h}(V,\Phi)$. Then for all $u,v \in V$ we have

$$(\varphi_u x)(v) = \varphi_u(xv) = \Phi(xv,u) = \Phi(v,-xu) = \varphi_{-xu}(v).$$

Therefore, $x \in \mathfrak{h}^\alpha(\mathfrak{L}_\Phi(V,V))$. Conversely, if $x \in \mathfrak{h}^\alpha(\mathfrak{L}_\Phi(V,V))$, then $\Phi(xv,u) = \Phi(v,-xu)$ for all $u,v \in V$, so $x \in \mathfrak{h}(V,\Phi)$. □

## 3 Finitary simple Lie algebras and their normalizers

Let $\Delta$ be a skew field and let $V$ be a right vector space over $\Delta$. Let $\Pi$ be a total subspace of the dual $V^*$. We denote by $\mathfrak{gl}(V,\Pi)$, $\mathfrak{fgl}(V,\Pi)$, and $\mathfrak{fgl}(V)$ the Lie rings $\mathfrak{L}(V,\Pi)$, $\mathfrak{F}(V,\Pi)$, and $\mathfrak{F}(V)$, respectively, under the usual bracket multiplication. Set also $\mathfrak{fsl}(V,\Pi) = [\mathfrak{fgl}(V,\Pi),\mathfrak{fgl}(V,\Pi)]$. If $\mathbb{F}$ is a subfield of the center of $\Delta$, these Lie rings can be considered as algebras over $\mathbb{F}$. For finite dimensional $V$ we have $\mathfrak{fgl}(V) = \mathfrak{gl}(V) \cong \mathfrak{gl}_n(\Delta^\circ)$ where $n = \dim V$. If $\operatorname{char}\Delta = 0$ and $\Delta$ is finite dimensional over its center $Z(\Delta)$, then

$$\mathfrak{gl}_n(\Delta^\circ) = \mathfrak{sl}_n(\Delta^\circ) \oplus Z(\Delta^\circ). \tag{1}$$

Indeed, let $\mathbb{K}$ be a maximal subfield of $\Delta^\circ$. Then $\mathbb{K}$ slits $\mathfrak{gl}_n(\Delta^\circ)$, i.e. for some $m$ we have $\mathfrak{gl}_n(\Delta^\circ) \otimes_{Z(\Delta^\circ)} \mathbb{K} = \mathfrak{gl}_m(\mathbb{K})$ and

$$\mathfrak{sl}_n(\Delta^\circ) \otimes_{Z(\Delta^\circ)} \mathbb{K} = [\mathfrak{gl}_n(\Delta^\circ),\mathfrak{gl}_n(\Delta^\circ)] \otimes_{Z(\Delta^\circ)} \mathbb{K} = [\mathfrak{gl}_m(\mathbb{K}),\mathfrak{gl}_m(\mathbb{K})] = \mathfrak{sl}_m(\mathbb{K}).$$

Therefore, the quotient $\mathfrak{sl}_n(\Delta^\circ)/\mathfrak{sl}_n(\Delta^\circ)$ is a 1-dimensional Lie $Z(\Delta^\circ)$-algebra. Since $\mathfrak{sl}_n(\Delta^\circ) \cap Z(\Delta^\circ) = 0$, this imply (1).

If $\Phi$ is a Hermitian or skew-Hermitian nondegenerate form on $V$, then we can consider the Lie rings $\mathfrak{h}(V,\Phi)$, $\mathfrak{fh}(V,\Phi)$, and $\mathfrak{fsh}(V,\Phi)$ (see Section 2). If $\Phi$ is unitary, then these rings are called unitary and denoted by $\mathfrak{u}(V,\Phi)$, $\mathfrak{fu}(V,\Phi)$, and $\mathfrak{fsu}(V,\Phi)$, respectively. Similarly, we get for an orthogonal form $\Psi$ the rings $\mathfrak{o}(V,\Psi)$ and $\mathfrak{fo}(V,\Psi)$ and for a symplectic form $\Theta$ the rings $\mathfrak{sp}(V,\Theta)$ and $\mathfrak{fsp}(V,\Theta)$.

**Theorem 3.1 ([2, Theorem 1.6])** *Let $\mathbb{F}$ be a field of characteristic 0 and let $V$ be an infinite dimensional vector space over $\mathbb{F}$. Let $L \subseteq \mathfrak{fgl}(V)$ be a finitary irreducible Lie subalgebra and let $\Delta$ be the centralizer of the $L$-module $V$. Then $\Delta$ is a finite dimensional division $\mathbb{F}$-algebra, $[L,L]$ is simple, and one of the following holds:*

$$\begin{aligned} \mathfrak{fsl}(V_\Delta,\Pi) &= [L,L] \subseteq L \subseteq \mathfrak{fgl}(V_\Delta,\Pi);\\ \mathfrak{fsu}(V_\Delta,\Phi) &= [L,L] \subseteq L \subseteq \mathfrak{fu}(V_\Delta,\Phi);\\ \mathfrak{fo}(V_\Delta,\Psi) &= [L,L] = L;\\ \mathfrak{fsp}(V_\Delta,\Theta) &= [L,L] = L. \end{aligned}$$

*Here $\Phi$, $\Psi$, and $\Theta$ are nondegenerate forms on $V_\Delta$; and $\Pi$ is a total subspace of the dual $V^*_\Delta$.*

We shall denote by $N_M(L) = \{m \in M \mid [m,L] \subseteq L\}$ the normalizer of a subalgebra $L$ of a Lie algebra $M$.

**Proposition 3.2** *Let $V$ be a right infinite dimensional vector space over a skew field $\Delta$.*

(i) *Let $\Pi$ be a total subspace of $V^*$. Then*

$$N_{\mathfrak{gl}(V)}(\mathfrak{fsl}(V,\Pi)) = N_{\mathfrak{gl}(V)}(\mathfrak{fgl}(V,\Pi)) = \mathfrak{gl}(V,\Pi).$$

(ii) *Let* $\operatorname{char}\Delta \neq 2$ *and let $\Phi$ be a nondegenerate Hermitian or skew-Hermitian form on $V$. Then*

$$N_{\mathfrak{gl}(V)}(\mathfrak{fsh}(V,\Phi)) = N_{\mathfrak{gl}(V)}(\mathfrak{fh}(V,\Phi)) = \mathfrak{h}(V,\Phi) + Z(\Delta^\circ).$$

*Proof.* (i) Let $x \in N_{\mathfrak{gl}(V)}(\mathfrak{fsl}(V,\Pi))$. Fix any $\varphi \in \Pi$ and take arbitrary $v \in \operatorname{Ker}(\varphi) \cap \operatorname{Ker}(\varphi x)$. For the transvection $t_{v\varphi}$ we have

$$[x, t_{v\varphi}] = x t_{v\varphi} - t_{v\varphi} x = t_{(xv)\varphi} - t_{v(\varphi x)}.$$

Since $\varphi(v) = \varphi(xv) = 0$, and $\varphi \in \Pi$, the transvections $t_{v\varphi}$ and $t_{(xv)\varphi}$ belong to $\mathfrak{fsl}(V,\Pi)$. Therefore $t_{v(\varphi x)} \in \mathfrak{fsl}(V,\Pi)$. This imply that $\varphi x \in \Pi$, so $x \in \mathfrak{gl}(V,\Pi)$. Since $\mathfrak{fsl}(V,\Pi)$ is the commutant of $\mathfrak{fgl}(V,\Pi)$, we have

$$N_{\mathfrak{gl}(V)}(\mathfrak{fgl}(V,\Pi)) \subseteq N_{\mathfrak{gl}(V)}(\mathfrak{fsl}(V,\Pi)) \subseteq \mathfrak{gl}(V,\Pi).$$

On the other hand, for each $x \in \mathfrak{gl}(V,\Pi)$ and each $t \in \mathfrak{fgl}(V,\Pi)$ the transformation $[x,t]$ belongs to $\mathfrak{gl}(V,\Pi) \cap \mathfrak{F}(V) = \mathfrak{fgl}(V,\Pi)$, so $\mathfrak{gl}(V,\Pi)$ normalizes $\mathfrak{fgl}(V,\Pi)$ and its commutant $\mathfrak{fsl}(V,\Pi)$, as required.

(ii) Let $x \in N_{\mathfrak{gl}(V)}(\mathfrak{fsh}(V,\Phi))$. By [2, Lemma 8.1], the associative subalgebra generated by $\mathfrak{fsh}(V,\Phi)$ in $\operatorname{End}_\Delta(V)$ is $\mathfrak{F}(V,V) = \mathfrak{F}(V,\Pi_\Phi)$. Hence $x$ normalizes $\mathfrak{F}(V,\Pi_\Phi)$ (and $\mathfrak{fgl}(V,\Pi_\Phi)$). Therefore by (i), $x \in \mathfrak{L}(V,\Pi_\Phi) = \mathfrak{L}_\Phi(V,V)$.

Let $\alpha$ be the adjoint map on $\mathfrak{L}_\Phi(V,V)$ with respect to $\Phi$. Then by Proposition 2.7,

$$\mathfrak{fh}(V,\Phi) = \mathfrak{h}^\alpha(\mathfrak{F}_\Phi(V,V)) = \{t \in \mathfrak{F}_\Phi(V,V) \mid t^\alpha = -t\}.$$

Let $y \in \mathfrak{fsh}(V,\Phi)$. Since $y$ and $z = [x,y]$ lie in $\mathfrak{fsh}(V,\Phi) \subseteq \mathfrak{fh}(V,\Phi)$, we have $y^\alpha = -y$ and $z^\alpha = -z$. Therefore

$$[x + x^\alpha, y] = xy + x^\alpha y - yx - yx^\alpha = (xy - yx) + (xy - yx)^\alpha = z + z^\alpha = 0.$$

It follows that $c = x + x^\alpha$ centralizes $\mathfrak{F}_\Phi(V,V)$, so $c \in \Delta^\circ$. On the other hand, $c$ is $\Delta$-linear, so $c \in Z(\Delta^\circ)$. Observe that $x - (1/2)c \in \mathfrak{h}^\alpha(\mathfrak{L}_\Phi(V,V)) = \mathfrak{h}(V,\Phi)$. Since $\mathfrak{fsh}(V,\Phi)$ is the commutant of $\mathfrak{fh}(V,\Phi)$, we have

$$N_{\mathfrak{gl}(V)}(\mathfrak{fh}(V,\Phi)) \subseteq N_{\mathfrak{gl}(V)}(\mathfrak{fsh}(V,\Phi)) \subseteq \mathfrak{h}(V,\Phi) + Z(\Delta^\circ).$$

It remains to observe that $\mathfrak{h}(V,\Phi)$ normalizes $\mathfrak{fh}(V,\Phi)$. □

## 4 Some lemmas on normalizers

Almost all results of this section are known. However, for completeness of the paper we give here their proofs because of the lack of reference. Let $A$ be an associative algebra over a field $\mathbb{F}$. For an integer $n$ we denote by $M_n(A)$ the algebra of $n \times n$ matrices with coefficients in $A$ and by $\operatorname{Der}_{\mathbb{F}} A$ the Lie algebra of all derivations of $A$ over $\mathbb{F}$. Recall that a derivation $\zeta$ of $A$ is called *internal* if there exists $\theta \in A$ such that $\zeta(a) = [\theta, a]$ for all $a \in A$.

**Lemma 4.1** *Let $\mathbb{K}$ be an algebraic separable field extension of $\mathbb{F}$. Then $\operatorname{Der}_{\mathbb{F}} \mathbb{K} = 0$.*

*Proof.* Let $\zeta \in \operatorname{Der}_{\mathbb{F}} \mathbb{K}$. Obviously, $\zeta(\mathbb{F}) = 0$. Let now $k \in \mathbb{K}\backslash\mathbb{F}$ and let $f(x)$ be the minimal polynomial of $k$ over $\mathbb{F}$. We have $f(k) = 0$. Applying $\zeta$ to both sides, we get $f'(k)\zeta(k) = 0$. Since $\mathbb{K}$ is a separable extension of $\mathbb{F}$, $f'(k) \neq 0$, so $\zeta(k) = 0$. This implies that $\operatorname{Der}_{\mathbb{F}} \mathbb{K} = 0$. □

**Lemma 4.2** *Let $\mathbb{F}$ be a field and let $\Delta$ be a division algebra over $\mathbb{F}$. Assume that the center $\mathbb{K}$ of $\Delta$ is an algebraic separable field extension of $\mathbb{F}$. Then each $\zeta \in \operatorname{Der}_{\mathbb{F}} \Delta$ is $\mathbb{K}$-linear, i.e. $\operatorname{Der}_{\mathbb{F}} \Delta = \operatorname{Der}_{\mathbb{K}} \Delta$. Moreover, if $\Delta$ is finite dimensional over $\mathbb{K}$, then each $\zeta \in \operatorname{Der}_{\mathbb{F}} \Delta$ is internal.*

*Proof.* Let $k \in \mathbb{K}$. Then for each $\zeta \in \operatorname{Der}_{\mathbb{F}} \Delta$ and each $\delta \in \Delta$ we have

$$[\zeta(k), \delta] = \zeta([k, \delta]) - [k, \zeta(\delta)] = 0.$$

Therefore, $\zeta(\mathbb{K}) \subseteq \mathbb{K}$. By Lemma 4.1, $\zeta(\mathbb{K}) = 0$. Hence $\zeta(k\delta) = k\zeta(\delta)$ for all $k \in \mathbb{K}$ and $\delta \in \Delta$, so $\zeta$ is $\mathbb{K}$-linear and $\operatorname{Der}_{\mathbb{F}} \Delta = \operatorname{Der}_{\mathbb{K}} \Delta$. It remains to recall that all derivations of a finite dimensional central simple algebra are internal (this is a simple corollary from Noether-Skolem theorem). □

Recall that $N_M(L)$ is the normalizer of a subalgebra $L$ of a Lie algebra $M$.

**Proposition 4.3** *Let $V$ be a vector space over a field $\mathbb{F}$ and let $L$ be an irreducible subalgebra of $\mathfrak{gl}(V)$. Assume that the centralizer $\Delta = (\operatorname{End}_L V)^\circ$ of the $L$-module $V$ is a finite dimensional division $\mathbb{F}$-algebra and the center of $\Delta$ is a separable extension of $\mathbb{F}$. Then*

$$N_{\mathfrak{gl}(V)}(L) = N_{\mathfrak{gl}(V_\Delta)}(L) + \Delta^\circ.$$

*Proof.* Let $x \in N_{\mathfrak{gl}(V)}(L)$, $l \in L$, and $\delta \in \Delta^\circ = \operatorname{End}_L V$. Since $\delta$ centralizes $L$, we have $[\delta, L] = 0$ and $[x, L] \subseteq L$. Therefore

$$[[x, \delta], L] = [[x, L], \delta] + [x, [\delta, L]] = 0.$$

Since $[x, \delta]$ centralizes $L$, it belongs to $\Delta^\circ$. Therefore $\operatorname{ad} x : \delta \mapsto [x, \delta]$ is a derivation of the $\mathbb{F}$-algebra $\Delta^\circ$. By Lemma 4.2, there exists $\theta \in \Delta^\circ$ such that $[x, \delta] = [\theta, \delta]$ for all $\delta \in \Delta^\circ$. Therefore, $x - \theta$ is $\Delta$-linear. Since $[x - \theta, L] \subseteq L$, we have $x - \theta \in N_{\mathfrak{gl}(V_\Delta)}(L)$, as required. □

Let $L$ be a Lie algebra over $\mathbb{F}$ and $V$ be a faithful $L$-module. We denote by $V^{(n)}$ the direct sum of $n$ copies of $V$. We identify the algebra $\operatorname{End}_{\mathbb{F}} V^{(n)}$ with the algebra $M_n(\operatorname{End}_{\mathbb{F}} V)$ of $n \times n$ matrices with coefficients in $\operatorname{End}_{\mathbb{F}} V$. Obviously, each subalgebra $A \subseteq \operatorname{End}_{\mathbb{F}} V$ can be considered as the subalgebra $\{\operatorname{diag}(a, \dots, a) \mid a \in A\}$ of $M_n(\operatorname{End}_{\mathbb{F}} V)$.

**Proposition 4.4** *Let $V$ be a vector space over a field $\mathbb{F}$ and let $L$ be an irreducible subalgebra of $\mathfrak{gl}(V)$. Let $\Delta = (\operatorname{End}_L V)^\circ$ be the centralizer of the $L$-module $V$. Then*
(i) $\operatorname{End}_L V^{(n)} = M_n(\Delta^\circ)$;
(ii) $N_{\mathfrak{gl}(V^{(n)})}(L) = \mathfrak{gl}_n(\Delta^\circ) + N_{\mathfrak{gl}(V)}(L)$.

*Proof.* Denote $N = N_{\mathfrak{gl}(V^{(n)})}(L)$. Let $x \in N$ and $a \in L$. Then $a_x = [x, a] = xa - ax \in L$. As we identify $\mathrm{End}_{\mathbb{F}} V^{(n)}$ with the algebra $M_n(\mathrm{End}_{\mathbb{F}} V)$ we can express $x$ in a matrix form $x = (x_{ij})_{i,j=1}^n$ with $x_{ij} \in \mathrm{End}_{\mathbb{F}} V$. The algebra $L$ is identified with $\{\mathrm{diag}(a, \dots, a) \mid a \in L\} \subseteq M_n(\mathrm{End}_{\mathbb{F}} V)$. Therefore, for all $i, j$ we have

$$x_{ij}a - ax_{ij} = \begin{cases} 0, & \text{if } i \neq j; \\ a_x & \text{if } i = j. \end{cases}$$

If $x \in \mathrm{End}_L V^{(n)}$, then $a_x = 0$ for all $a \in L$, so $x_{ij} \in \Delta^\circ$ for all $i$ and $j$. Therefore $\mathrm{End}_L V^{(n)} \subseteq M_n(\Delta^\circ)$. The reverse inclusion is obvious. This proves (i).

Now let us prove (ii). Set $x_L = x_{11} \in \mathrm{End}_{\mathbb{F}} V$. Note that $x_L a - a x_L = a_x \in L$ for all $a \in L$, so $x_L \in N_{\mathfrak{gl}(V)}(L)$ and $[x - x_L, a] = 0$ for all $a \in L$. By (i), $x - x_L \in M_n(\Delta^\circ)$. Therefore, $N \subseteq \mathfrak{gl}_n(\Delta^\circ) + N_{\mathfrak{gl}(V)}(L)$. The reverse inclusion is obvious. □

## 5 Proof of the main result

We start with the following theorem.

**Theorem 5.1** *Let $\mathbb{F}$ be any field and let $W$ be a vector space over $\mathbb{F}$. Let $L$ be an irreducible subalgebra of $\mathfrak{gl}(W)$. Assume that $L_{\mathfrak{F}} = L \cap \mathfrak{fgl}(W) \neq 0$. Then the $L_{\mathfrak{F}}$-module $W$ has a composition series of finite length with isomorphic composition factors. Moreover, if $\mathrm{char}\,\mathbb{F} = 0$, then $W$ is a homogeneous $L_{\mathfrak{F}}$-module.*

*Proof.* One can assume that W has infinite dimension (otherwise $L_{\mathfrak{F}} = L$). We proceed by steps.

*Step 1: $W$ has an irreducible $L_{\mathfrak{F}}$-submodule $V$.*

Let $t = t_{e_1\varphi_1} + \cdots + t_{e_n\varphi_n}$ be a nonzero element of $L_{\mathfrak{F}}$ where $t_{e_i\varphi_i}$ are transvections and $e_1, \dots, e_n \in W$ and $\varphi_1, \dots, \varphi_n \in W^*$ are linearly independent (see Lemma 2.3). Denote by $E$ the $n$-dimensional space spanned by $e_1, \dots, e_n$. Assume that

(∗) $L_{\mathfrak{F}} v \cap E \neq 0$ for all nonzero $v \in W$.

Then $E \cap U \neq 0$ for each nonzero $L_{\mathfrak{F}}$-submodule $U$ of $W$. Therefore for each descending chain $\{U_\nu \mid \nu \in \mathfrak{N}\}$ of nonzero $L_{\mathfrak{F}}$-submodules of $W$, the submodule $\cap_{\nu\in\mathfrak{N}} U_\nu$ is nonzero. Hence by Zorn's lemma, W has minimal $L_{\mathfrak{F}}$-submodules, as required. Therefore, it suffices to prove (∗).

One can assume that $\varphi_i(v) = 0$ for all $i$. Indeed, otherwise

$$0 \neq e_1\varphi_1(v) + \cdots + e_n\varphi_n(v) = tv \in L_{\mathfrak{F}} v \cap E,$$

as desired. Since $L$ is irreducible, $A(L)v = W$ where $A(L)$ is the augmentation ideal of the universal enveloping algebra $U(L)$, i.e. the ideal of codimension 1 generated by $L$. Therefore, there exists a monomial $m = x_1 \dots x_k \in A(L)$ with $x_j \in L$ such that $\varphi_i(mv) \neq 0$ for some $i$. One can assume that its length $k$ is minimal. Set

$$t' = [\dots[[t, x_1], x_2] \dots, x_k] = tm + \sum_\beta m'_\beta t m''_\beta \in L \subset A(L)$$

where each $m''_\beta$ is a monomial of length $\leq k - 1$. Since $L_{\mathfrak{F}}$ is an ideal of $L$, $t' \in L_{\mathfrak{F}}$. In view of minimality of the length of $m$, $\varphi_i(m''_\beta v) = 0$ for all $i$ and $\beta$. Therefore $m'_\beta t m''_\beta v = 0$ for all $i$ and $\beta$. Hence

$$t'v = tmv = e_1\varphi_1(mv) + \cdots + e_n\varphi_n(mv)$$

is a nonzero element of $L_{\mathfrak{F}} v \cap E$, as required.

*Step 2: The $L_{\mathfrak{F}}$-module $W$ has a composition series of finite length and all composition factors are isomorphic to $V$.*

For all $x \in L, t \in L_{\mathfrak{F}}$, and $v \in W$ we have

$$txv = xtv + [t, x]v.$$

Therefore, $V_x = xV + V$ is an $L_{\mathfrak{F}}$-submodule of $W$ and the $L_{\mathfrak{F}}$-module $V_x/V$ is a homomorphic image of $V$. Since $V$ is irreducible, either $V_x = V$ or $V_x/V \cong V$. Therefore, $V^1 = LV + V$ is a submodule of $W$ and $V^1/V$ is a sum of modules isomorphic to $V$. Set $V^0 = V$ and $V^i = LV^{i-1} + V^{i-1}$ for $i \geq 1$. As above, $V^i/V^{i-1}$ is a sum of modules isomorphic to $V$, so each composition factor of $V^i$ is isomorphic to $V$. Since $L_{\mathfrak{F}}$ has an element $t$ acting nontrivially on $V$ (see $(*)$ in Step 1), the composition length of each $V^i$ does not exceed rank of $t$. Therefore, $V^i = V^{i+1} = W$ for some $i$, so $V$ has a finite composition series with each composition factor isomorphic to $V$.

*Step 3: If $\operatorname{char} \mathbb{F} = 0$, then the enveloping algebra $\mathcal{E}_{\mathfrak{F}}$ of $L_{\mathfrak{F}}$ in $\operatorname{End}_{\mathbb{F}} W$ acts faithfully on $V$.*

Assume that $tV = 0$ for some $t \in \mathcal{E}_{\mathfrak{F}}$. Let the $L_{\mathfrak{F}}$-modules $V^i$ be as in Step 2 and let $k$ be the maximal integer such that $tV^k = 0$ and $tV^{k+1} \neq 0$. Then there exists $x_1, \dots, x_{k+1} \in L$ and $v \in V$ such that $v' = tx_1 \dots x_{k+1}v \neq 0$. Set $t' = [\dots[t, x_1] \dots, ]x_k] \in \mathcal{E}_{\mathfrak{F}}$. Then $t'V = 0$ as $tV^k = 0$. Note that $t'x_{k+1}v = v' \neq 0$. Since $L_{\mathfrak{F}}$ acts irreducibly on $V$, there exists $t'' \in \mathcal{E}_{\mathfrak{F}}$ such that $t''v' = v$. Set $\tau = t''t' \in \mathcal{E}_{\mathfrak{F}}$ and $x = x_{k+1}$. We have $\tau V = 0$ and $\tau xv = v$. Note that $xv \notin V$. Set $U^{-1} = 0$ and $U^i = V + xV + \dots + x^iV$ for $i \geq 0$. Obviously, $U^i$ is an $L_{\mathfrak{F}}$-submodule of $W$. Denote by $n$ the maximal integer such that $U^{n-1} \neq U^n$. Then we have $U^i/U^{i-1} \cong V$ for $0 \leq i \leq n$ and $xU^n \subseteq U^n$. Note that $[\tau, x]v = v$ and

$$[\tau, x](x^iv + U^{i-1}) = x^i[\tau, x]v + U^{i-1} = x^iv + U^{i-1} \quad \text{for } 0 \leq i \leq n.$$

Let us show by induction that

$$\tau x^iv \in ix^{i-1}v + U^{i-2} \quad \text{for } 1 \leq i \leq n+1.$$

Indeed, this holds for $i = 1$ and

$$\tau x^{i+1}v = x\tau x^iv + [\tau, x]x^iv \in (i+1)x^iv + U^{i-1},$$

as desired. In particular, $\tau x^{n+1}v \notin U^{n-1}$ (here we use that $n+1 \neq 0$ in $\mathbb{F}$). On the other hand, since $x^{n+1}v \in U^{n+1} = U^n$ and $\tau V = 0$, we have $\tau x^{n+1}v \in \tau U^n \subseteq U^{n-1}$. The contradiction obtained proves the assertion.

*Step 4: If $\operatorname{char} \mathbb{F} = 0$, then $W$ is a finite direct sum of copies of $V$.*

By Step 3, $V$ is a faithful irreducible $\mathcal{E}_{\mathfrak{F}}$-module, so the ring $\mathcal{E}_{\mathfrak{F}}$ is primitive. Since $\mathcal{E}_{\mathfrak{F}}$ contains transformations of finite rank, by [2, Lemma 5.2], $\mathcal{E}_{\mathfrak{F}}$ has minimal left ideals. Observe that $L_{\mathfrak{F}}W$ is an $L$-submodule of $V$. Therefore, $L_{\mathfrak{F}}W = W$, so $\mathcal{E}_{\mathfrak{F}}W = W$. Now by [6, Theorem 4.14.1], $W$ is a homogeneous completely reducible $\mathcal{E}_{\mathfrak{F}}$-module, as required. □

*Proof of Theorem 1.4.*

(1) This is proved in Theorem 5.1.

(2)-(5) Observe that $L_{\mathfrak{F}}$ is an irreducible subalgebra of $\mathfrak{fgl}(V)$. Therefore, we can apply Theorem 3.1. We get (2) and (3). Moreover, either there exists a total subspace $\Pi$ of $V_{\Delta}^*$ such that

$$\mathfrak{fsl}(V_{\Delta}, \Pi) = L'_{\mathfrak{F}} \subseteq L_{\mathfrak{F}} \subseteq \mathfrak{fgl}(V_{\Delta}, \Pi)$$

or there exists a nondegenerate Hermitian or skew-Hermitian form $\Phi$ on $V_{\Delta}$ such that

$$\mathfrak{fsh}(V_{\Delta}, \Phi) = L'_{\mathfrak{F}} \subseteq L_{\mathfrak{F}} \subseteq \mathfrak{fh}(V_{\Delta}, \Phi).$$

Observe that $L'_{\mathfrak{F}}$ is an ideal of $L$. Therefore, $L \subseteq N_{\mathfrak{gl}(W)}(L'_{\mathfrak{F}})$. Applying Propositions 4.4(ii), 4.3, and 3.2, we get

$$\begin{aligned} N_{\mathfrak{gl}(W)}(L'_{\mathfrak{F}}) &= \mathfrak{gl}_n(\Delta^\circ) + N_{\mathfrak{gl}(V)}(L'_{\mathfrak{F}}) \\ &= \mathfrak{gl}_n(\Delta^\circ) + N_{\mathfrak{gl}(V_\Delta)}(L'_{\mathfrak{F}}) \\ &= \begin{cases} \mathfrak{gl}_n(\Delta^\circ) + \mathfrak{gl}(V_\Delta, \Pi), & \text{if} \quad L'_{\mathfrak{F}} = \mathfrak{fsl}(V_\Delta, \Pi); \\ \mathfrak{gl}_n(\Delta^\circ) + \mathfrak{h}(V_\Delta, \Phi), & \text{if} \quad L'_{\mathfrak{F}} = \mathfrak{fsh}(V_\Delta, \Phi). \end{cases} \end{aligned}$$

It remains to note that $\mathfrak{gl}_n(\Delta^\circ) = \mathfrak{sl}_n(\Delta^\circ) \oplus Z(\Delta^\circ)$ (see (1)); $\mathfrak{gl}_n(\Delta^\circ) \cap \mathfrak{gl}(V_\Delta) \subseteq Z(\Delta^\circ)$; $Z(\Delta^\circ) \subset \mathfrak{gl}(V_\Delta, \Pi)$; $Z(\Delta^\circ) \cap \mathfrak{o}(V_\Delta, \Psi) = Z(\Delta^\circ) \cap \mathfrak{sp}(V_\Delta, \Theta) = 0$; and there is no nontrivial $\Delta^\circ$-subspaces of $(\Delta^\circ)^n$ invariant under $M$ (otherwise $L$ is not irreducible). □

*Proof of Theorem 1.3.* It suffices to apply Theorem 1.4 and observe that $\Delta = \mathbb{F}$, since there is no nontrivial finite dimensional algebra over an algebraically closed field. □


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