Classification of the direct limits of involution simple associative algebras and the corresponding dimension groups

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Abstract

A classification of the (countable) direct limits of finite dimensional involution simple associative algebras over an algebraically closed field of arbitrary characteristic is obtained. This also classifies the corresponding dimension groups. The set of invariants consists of two supernatural numbers and two real parameters.

1 Introduction

The ground field $\mathbb{F}$ is algebraically closed of arbitrary characteristic. Let A be an associative algebra over $\mathbb{F}$ (not necessarily containing an identity element). Assume A has an involution, that is, a linear transformation $*$ of A such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. We will sometimes denote this algebra by $(A, *)$ to reflect the fact that A is an algebra with involution. Note that our involution is $\mathbb{F}$-linear, i.e. we consider involutions of the first kind only. The algebra A is called *involution simple* if $A^2 \neq 0$ and it has no non-trivial $*$-invariant ideals.

We say that an infinite dimensional algebra A is *locally (semi)simple* if any finite subset of A is contained in a finite dimensional (semi)simple subalgebra. Note that we do not require A to have an identity element. If A has an involution and these subalgebras can be chosen involution simple with respect to the inherited involution then A is called *locally involution simple*. Observe that A itself is involution simple in that case. The aim of this paper is to classify locally involution simple associative algebras over $\mathbb{F}$ of countable dimension.

Let A be a locally simple associative algebra of countable dimension over $\mathbb{F}$. It follows from the definition that there is a chain of simple subalgebras $A_1 \subset A_2 \subset A_3 \subset \dots$ of A such that $A = \cup_{i=1}^{\infty} A_i$. One can also view A as the direct limit $\varinjlim A_i$ for the sequence

$$A_1 \to A_2 \to A_3 \to \dots \tag{1}$$

of injective homomorphisms of finite dimensional simple associative algebras A_i. Since $\mathbb{F}$ is algebraically closed, each A_i can be identified with the algebra $M_{n_i}(\mathbb{F})$ of all $n_i \times n_i$ matrices over $\mathbb{F}$ for some n_i. Moreover, each embedding $A_i \to A_{i+1}$ can be written in the following matrix form

$$M \mapsto \mathrm{diag}(M, \dots, M, 0, \dots, 0), \quad M \in M_{n_i}(\mathbb{F}). \tag{2}$$

Therefore in order to describe locally simple associative algebras of countable dimension one needs to classify the direct limits of the sequences of matrix algebras (1). Elliot [6] did this in terms of systems of idempotents. It has been shown later that Elliot's invariant can be interpreted in terms of the K_0-functor (see Theorem 3.8). As a particular case of our main results we get another parametrization of these algebras.

Assume now that the algebra A is locally involution simple, i.e. we have a sequence (1) of involution simple finite dimensional algebras A_i and $A = \varinjlim A_i$. Note that all homomorphisms in (1) respect the involution but do not necessarily preserve the identity element. It is well known that every involution simple finite dimensional $\mathbb{F}$-algebra is either a full matrix algebra or the direct sum of two isomorphic matrix algebras. Therefore the combinatorial picture is much more complicated than in (2). However it is still possible to provide an explicit parametrization (see our main Theorems 4.1 and 5.2).

In Section 3 we prove that two locally involution simple algebras of the same type (orthogonal, symplectic or special) are isomorphic if and only if they are isomorphic as associative algebras (Theorem 3.4). This partially reduces the classification problem to locally semisimple associative algebras. These algebras are normally classified by ordered dimension groups (see Theorem 3.8). However, as it is pointed out in [4], although dimension groups are relatively easy objects, their isomorphism classes are not, and general classification is not available. Some examples of known isomorphism classes of the dimension groups can be found in [4]. Our main Theorem 4.1 gives complete classification of the dimension groups which correspond to the locally involution simple algebras. These are the direct limits of the sequences $\mathbb{Z}^3 \to \mathbb{Z}^3 \to \cdots \to \mathbb{Z}^3 \to \ldots$ where the embeddings are given by the Bratteli diagrams (21).

Another approach (K-theoretical in nature) to the classification of locally involution simple associative algebras can be found within the general theory of compact group actions on locally semisimple algebras, see [11, 3]. In the case of order 2 automorphisms this was done by Fack and Maréchal [9] (unital embeddings given by the Bratteli diagrams (20)) and Elliott and Su [7] (in terms of K-theoretical invariants).

Our approach uses some technique developed by Baranov and Zhilinskii for the classification of the diagonal direct limits of finite dimensional simple Lie algebras over an algebraically closed field of characteristic zero [2]. It is shown in [1] that there is a natural bijective correspondence between such Lie algebras and locally involution simple associative algebras, so the classification should be similar. Unfortunately the proofs in [1, 2] are very dependent on characteristic zero and fail to work in positive characteristic. In the present paper we provide new, characteristic free, proofs. However, the case of characteristic 2 still requires special attention and the classification is slightly different in that case.

Note that our results do not exhaust the problem of classification of all involution simple locally finite dimensional associative algebras (of countable dimension), since there are examples of such algebras which are not locally semisimple (see [8, 15]).

2 Preliminaries

Recall that an associative algebra A with involution is called *involution simple* if $A^2 \neq 0$ and it has no non-trivial $*$-invariant ideals. The following is well-known.

Proposition 2.1 *Let A be an involution simple associative algebra. Then either A is simple as an algebra or A has exactly two non-zero proper ideals B_1 and B_2. Moreover both B_1 and B_2*

are simple algebras, $B_1^* = B_2$ *and* $A = B_1 \oplus B_2$.

Proof. Assume A is not simple. Let B_1 be a non-zero proper ideal of A. Then $B_2 = B_1^*$ is also an ideal of A. Since $B_1 + B_2$ and $B_1 \cap B_2$ are $*$-invariant ideals of A and A is involution simple, one has $B_1 + B_2 = A$ and $B_1 \cap B_2 = 0$, i.e. $A = B_1 \oplus B_2$. Now, if B is a non-zero proper ideal of B_1 then $B \oplus B^*$ is a non-zero proper $*$-invariant ideal of $B_1 \oplus B_2 = A$. Therefore $B = B_1$ and both B_1 and B_2 are simple algebras.

Assume now that C is another non-zero proper ideal of A. Then by the above argument, $A = C \oplus C^*$. If $B_1 \subseteq C$ or $B_2 \subseteq C$ then it is easy to see that $C = B_1$ or B_2. Assume this is not the case. Let $B = B_1 \cap C$. Then $B + B^*$ is a proper $*$-invariant ideal of A, so $B = 0$. In particular, $B_1 C \subseteq B_1 \cap C = 0$. Similarly, $B_2 C = 0$ and $B_1 C^* = B_2 C^* = 0$. This implies $AA = (B_1 + B_2)(C + C^*) = 0$, which is a contradiction. □

Let A be a finite dimensional associative algebra over $\mathbb{F}$ with involution $*$. Assume that A is involution simple. Then by Proposition 2.1, A is either simple or $A = B \oplus B^*$ the sum of two (anti)isomorphic simple subalgebras. Thus, we can identify A with either $\operatorname{End} V$ or $\operatorname{End} V_1 \oplus \operatorname{End} V_2$ for some finite dimensional vector spaces V, V_1, and V_2 over $\mathbb{F}$ with $\dim V_1 = \dim V_2$. By fixing bases of V, V_1, and V_2, one can represent the algebras $\operatorname{End} V$ and $\operatorname{End} V_1 \oplus \operatorname{End} V_2$ in the matrix forms $M_n(\mathbb{F})$ and $M_m(\mathbb{F}) \oplus M_m(\mathbb{F})$, respectively, where $n = \dim V$ and $m = \dim V_1 = \dim V_2$. We say that these are their *matrix realizations*. We say that a matrix realization of $(\operatorname{End} V, *)$ is *canonical* if the involution in the chosen basis has one of the following two forms:

$$X \mapsto X^t, \qquad X \in M_n(\mathbb{F}) \quad \text{(transpose)}; \tag{3}$$

$$X \mapsto X^\tau, \qquad X \in M_n(\mathbb{F}) \quad \text{(symplectic transpose)}. \tag{4}$$

In the latter case n is even and $X^\tau = -JX^tJ$ where $J = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ ($n/2$ blocks). We say that a matrix realization of $(\operatorname{End} V_1 \oplus \operatorname{End} V_2, *)$ is *canonical* if the involution in the chosen basis has the following form:

$$(X_1, X_2) \mapsto (X_2^t, X_1^t), \quad X_1, X_2 \in M_m(\mathbb{F}). \tag{5}$$

It is well known that any finite dimensional involution simple algebra over an algebraically closed field has a canonical matrix realization. Indeed, let us first consider the algebra $\operatorname{End} V$. Let $b : V \times V \to \mathbb{F}$ be a nondegenerate symmetric or skew-symmetric bilinear form on V. For each $x \in \operatorname{End} V$ define $\alpha_b(x)$ by the following property

$$b(\alpha_b(x)v, w) = b(v, xw) \quad \text{for all } v, w \in V.$$

Then the map

$$\alpha_b : \operatorname{End} V \to \operatorname{End} V$$

is an involution of the algebra $\operatorname{End} V$, called the *adjoint involution* with respect to b. More exactly we have the following fact.

Theorem 2.2 ([12, Ch.1, Introduction]) *The map* $b \mapsto \alpha_b$ *induces a one-to-one correspondence between the equivalence classes of nondegenerate symmetric and skew-symmetric bilinear forms on* V *modulo multiplication by a factor in* $\mathbb{F}^\times$ *and involutions (of the first kind) on* $\operatorname{End} V$.

Recall that a bilinear form is called *alternating* if $b(v, v) = 0$ for all $v \in V$. Obviously, if $\operatorname{char} \mathbb{F} \neq 2$, then the form b is alternating if and only if it is skew-symmetric. If $\operatorname{char} \mathbb{F} = 2$, then b is alternating if and only if it is symmetric and for any choice of basis of V, all diagonal entries of the matrix of b are zeros. An involution α of $\operatorname{End} V$ is called *symplectic* (resp. *orthogonal*) if it is adjoint to an alternating (resp. symmetric non-alternating) bilinear form on V. Recall that each finite dimensional orthogonal (resp. symplectic) vector space over an algebraically closed field has an orthonormal (resp. hyperbolic) basis. That is, the matrix of b in this basis is either the identity (in the orthogonal case) or J (see above) in the symplectic case (see for example [14, Theorems 11.10 and 11.14]). It is easy to see that the adjoint involution in this basis is canonical, i.e. of the forms (3) and (4), respectively. Thus, we get the following well-known fact.

Proposition 2.3 *Let V be a vector space of dimension n over $\mathbb{F}$ and let $*$ be an involution of $\operatorname{End} V$. Then the algebra $(\operatorname{End} V, *)$ has a canonical matrix realization.*

To prove a similar result for the algebra $\operatorname{End} V_1 \oplus \operatorname{End} V_2$, we need the following simple fact.

Proposition 2.4 *Each involution of the matrix algebra $M_n(\mathbb{F})$ is of the following form: $X \mapsto CX^tC^{-1}$ where C is an invertible matrix.*

Proof. The matrix transpose $X \mapsto X^t$ is a natural involution of $M_n(\mathbb{F})$. Thus the map $X \mapsto (X^*)^t$ is an automorphism of $M_n(\mathbb{F})$. By Skolem-Noether theorem each automorphism of $M_n(\mathbb{F})$ is inner, i.e. there exists an invertible matrix K such that $(X^*)^t = K^{-1}XK$. Therefore $X^* = K^tX^t(K^{-1})^t = K^tX^t(K^t)^{-1}$, as required. □

Proposition 2.5 *Let V_1 and V_2 be vector spaces of dimension m and let $*$ be an involution of the algebra $\operatorname{End} V_1 \oplus \operatorname{End} V_2$ such that $(\operatorname{End} V_1)^* = \operatorname{End} V_2$. Then for every matrix realization of $\operatorname{End} V_1$ there is a matrix realization of $\operatorname{End} V_2$ such that the corresponding matrix realization of $(\operatorname{End} V_1 \oplus \operatorname{End} V_2, *)$ is canonical.*

Proof. Fix any matrix realizations of $\operatorname{End} V_1$ and $\operatorname{End} V_2$, i.e. identify these algebras with the algebra $M_m(\mathbb{F})$. Then the map $* : \operatorname{End} V_1 \to \operatorname{End} V_2$ gives an involution $X \mapsto X^*$ of $M_m(\mathbb{F})$. By Proposition 2.4, $X^* = CX^tC^{-1}$. It remains to change basis of V_2 (i.e. matrix realization of $\operatorname{End} V_2$), to eliminate C. □

Let A be an involution simple finite dimensional algebra over $\mathbb{F}$. We say that A is of type **S**, or of *symplectic* type, if A is simple as an algebra and the involution is symplectic. Similarly we define the *orthogonal* type **O**. If A is not simple, then we say that A is of type **A**, or of *special* type. Note that algebras of type **S** are not isomorphic to those of type **O** (as algebras with involution). Thus the canonical matrix realizations (as in Propositions 2.3 and 2.5) give a complete classification of finite dimensional involution simple algebras over an algebraically closed field.

Remark 2.6 We will also use other canonical forms for involutions. Let n be even. Define the following $n \times n$ matrices:

$$J_+ = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad Q_\pm = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix}$$

where I is the identity $n/2 \times n/2$ matrix. Then J_+ and Q_+ define nondegenerate bilinear symmetric forms on the natural $M_n(\mathbb{F})$-module. If $\operatorname{char} \mathbb{F} \neq 2$, these forms are non-alternating, so induce orthogonal involutions on $M_n(\mathbb{F})$: $\tau_+ : X \mapsto J_+ X^t J_+$ and $\theta_+ : X \mapsto Q_+ X^t Q_+$. In view of Proposition 2.3, by choosing an appropriate basis, these involutions can be represented as matrix transpose. Thus each orthogonal involution (for $\operatorname{char} \mathbb{F} \neq 2$ and algebras of even degree) can be represented as τ_+ (resp. θ_+) in a suitable basis. Similarly, the involution $\theta_- : X \mapsto -Q_- X^t Q_-$ is symplectic and each symplectic involution (for any characteristic) can be represented in this form.

The following simple fact will be used later.

Lemma 2.7 *Let B be a finite dimensional involution simple algebra of even dimension. Let e be the identity element of B. If $\operatorname{char} \mathbb{F} = 2$, assume that B is not of type* **O**. *Then B has idempotents f and g such that $e = f + g$, $fg = gf = 0$, and $f^* = g$.*

Proof. This is obvious if B is of type **A**. Assume that B is symplectic. Represent B as in Proposition 2.3. Then one can easily check that $f = \operatorname{diag}(1, 0, 1, 0, \dots, 1, 0)$ and $g = \operatorname{diag}(0, 1, 0, 1, \dots, 0, 1)$ are the required idempotents. If the involution $*$ of B is orthogonal, then by Remark 2.6, it can be represented as $X^* = J_+ X^t J_+$, $X \in M_n(\mathbb{F})$. Then it is easy to check that the same idempotents f and g as in the symplectic case satisfy the required conditions. $\square$

Now we are going to study embeddings of involution simple algebras, i.e. injective homomorphisms $\varepsilon : A_1 \to A_2$ which respect involution. We do not require these embeddings to preserve the identity element. Since the embeddings respect involution we often use the same symbol "$*$" to denote the involution of A_1 and A_2. We usually identify A_1 with its image $\varepsilon(A_1)$ in A_2. If A_i is of type **A**, we denote by B_i and C_i its simple components (so $A_i = B_i \oplus C_i$ and $B_i \cong C_i$). It is convenient to assume $B_i = A_i$ if A_i is of type **S** or **O**. We denote by e_i, f_i, and g_i the identities of A_i, B_i, and C_i, respectively. Thus $e_i = f_i + g_i$ if A_i is of type **A**, and $e_i = f_i$ otherwise. Note that $f_i^* = g_i$ if A_i is of type **A**.

Recall that $B_i \cong M_{n_i}(\mathbb{F})$ for some $n_i \in \mathbb{N}$. We say that n_i is the *degree* of A_i. Denote by V_i the natural B_i-module of dimension n_i and by W_i the natural module for C_i (if $C_i \neq 0$). We consider these modules as A_i-modules in a natural way. If A_i is not of type **A**, we denote by b_i a nondegenerate bilinear form on V_i corresponding to the involution $*$ on A_i (see Theorem 2.2).

Denote by T_i the trivial one-dimensional A_i-module (with zero action). Now the restriction of the A_2-module V_2 to A_1 is completely reducible, so can be described as follows.

$$V_2{\downarrow}A_1 = \underbrace{V_1 \oplus \cdots \oplus V_1}_{l} \oplus \underbrace{W_1 \oplus \cdots \oplus W_1}_{r} \oplus \underbrace{T_1 \oplus \cdots \oplus T_1}_{z} \tag{6}$$

where $l, r, z \in \mathbb{N} \cup \{0\}$ and $r = 0$ if A_1 is not of type **A**.

Definition 2.8 The triple (l, r, z) in (6) is called the *signature* of the embedding $\varepsilon : A_1 \to A_2$.

Remark 2.9 If both A_1 and A_2 are of type **A**, then the signature depends on the choice of the simple components of A_1 and A_2, e.g. by swapping B_1 and C_1 (or B_2 and C_2, see (8) below), the signature (l, r, z) is replaced by (r, l, z). Thus we can and will assume that $l \geq r$.

Definition 2.10 We say that a homomorphism $\varepsilon : M_{n_1} \to M_{n_2}$ of signature $(l, 0, z)$ of two matrix algebras is *canonical* if

$$\varepsilon(M) = \mathrm{diag}(\underbrace{M, \ldots, M}_{l}, \underbrace{0, \ldots, 0}_{z}), \quad M \in M_{n_1}(\mathbb{F}). \tag{7}$$

We say that a homomorphism $\varepsilon : M_{n_1} \oplus M_{n_1} \to M_{n_2} \oplus M_{n_2}$ of signature (l, r, z) is *canonical* if

$$\varepsilon(M, N) = (\mathrm{diag}(\underbrace{M, \ldots, M}_{l}, \underbrace{N, \ldots, N}_{r}, \underbrace{0, \ldots, 0}_{z}), \mathrm{diag}(\underbrace{N, \ldots, N}_{l}, \underbrace{M, \ldots, M}_{r}, \underbrace{0, \ldots, 0}_{z})) \tag{8}$$

for all $M, N \in M_{n_1}(\mathbb{F})$.

We say that an embedding $\varepsilon : A_1 \to A_2$ of finite dimensional involution simple algebras over $\mathbb{F}$ of the same type (**A**, **O**, or **S**) is (canonically) *representable* if for every canonical matrix realization of A_1 there exists a canonical matrix realization of A_2 such that the matrix embedding ε is canonical.

Remark 2.11 (1) It is easy to see that canonical matrix homomorphisms (7)-(8) commute with the canonical matrix involutions (3)-(5) (e.g. in type **O** the canonical involution is just matrix transpose).

(2) Note that compositions of canonical matrix homomorphisms are canonical.

We are going to show that all embeddings of involution simple algebras of the same type are representable, except for types **O** and **S** in characteristic 2.

Proposition 2.12 *Let $\varepsilon : A_1 \to A_2$ be an embedding of finite dimensional involution simple algebras over $\mathbb{F}$ of type* **A**. *Then ε is* representable.

Proof. Let n_i be the degree of A_i. Fix any bases of V_1 and W_1 such that the corresponding matrix realization $M_{n_1}(\mathbb{F}) \oplus M_{n_1}(\mathbb{F})$ of A_1 is canonical (i.e. the involution has the form (5)). Let π_B (resp. π_C) denote the projection $A_2 \to B_2$ (resp. $A_2 \to C_2$). Fix any basis of V_2 which agree with the bases of V_1 and W_1 and the decomposition (6), i.e. the projection $\pi_B\varepsilon(A_1)$ has the following matrix form.

$$\pi_B\varepsilon(M, N) = \mathrm{diag}(\underbrace{M, \ldots, M}_{l}, \underbrace{N, \ldots, N}_{r}, \underbrace{0, \ldots, 0}_{z}), \quad M, N \in M_{n_1}(\mathbb{F}).$$

Fix a basis of W_2 such that the corresponding matrix realization of $(A_2, *)$ is canonical (see Proposition 2.5). Then

$$\varepsilon(M, N) = \varepsilon((N^t, M^t)^*) = (\varepsilon(N^t, M^t))^* = ((\pi_C\varepsilon(N^t, M^t))^t, (\pi_B\varepsilon(N^t, M^t))^t),$$

so

$$\pi_C\varepsilon(M, N) = (\pi_B\varepsilon(N^t, M^t))^t = \mathrm{diag}(\underbrace{N, \ldots, N}_{l}, \underbrace{M, \ldots, M}_{r}, \underbrace{0, \ldots, 0}_{z}).$$

Therefore

$$\varepsilon(M, N) = (\mathrm{diag}(\underbrace{M, \ldots, M}_{l}, \underbrace{N, \ldots, N}_{r}, \underbrace{0, \ldots, 0}_{z}), \mathrm{diag}(\underbrace{N, \ldots, N}_{l}, \underbrace{M, \ldots, M}_{r}, \underbrace{0, \ldots, 0}_{z}))$$

as required. □

Our aim now is to prove a similar result for orthogonal and symplectic algebras in characteristic $\neq 2$. We need some auxiliary lemmas.

Lemma 2.13 ([12, 2.23]) *Let D_1 and D_2 be finite dimensional simple algebras over $\mathbb{F}$ with involutions α_1 and α_2, respectively. Then $\alpha = \alpha_1 \otimes \alpha_2$ is an involution of $D_1 \otimes_{\mathbb{F}} D_2$.*

(i) If α_1 and α_2 are orthogonal, then α is orthogonal.

(ii) If α_1 is orthogonal and α_2 is symplectic, then α is symplectic.

(iii) If α_1 and α_2 are symplectic, then α is orthogonal in the case of $\operatorname{char}\mathbb{F} \neq 2$ *and symplectic otherwise.*

Recall that e_1 is the identity element of A_1. We will use the notation $\bar{A}_1 = e_1 A_2 e_1$ and $\bar{V}_1 = e_1 V_2$. Let $\bar{b}_1$ be the restriction of the form b_2 to $\bar{V}_1$.

Lemma 2.14 *Assume that A_2 is not of type* **A**. *Then*

(i) $\bar{A}_1$ is a $*$-invariant simple subalgebra of A_2;

(ii) $\bar{V}_1$ is an irreducible $\bar{A}_1$-module and $\bar{V}_1 = \bar{A}_1 V_2$;

(iii) the form $\bar{b}_1$ on $\bar{V}_1$ is nondegenerate and corresponds to the involution $*$ on $\bar{A}_1$; moreover, $\bar{b}_1$ has the same type as b_2 except in the case when $\operatorname{char}\mathbb{F} = 2$ *and A_2 is of type* **O**.

Proof. Note that e_1 is an idempotent of A_2 and $e_1^* = e_1$, so (i) and (ii) are clear. Now assume that $\bar{b}_1$ is degenerate, i.e. there exists $v \in V_2$ such that $e_1 v \neq 0$ and $b_2(e_1 v, e_1 w) = 0$ for all $w \in V_2$. Then

$$b_2(e_1 v, w) = b_2(e_1 e_1 v, w) = b_2(e_1 v, e_1 w) = 0 \quad \text{for all } w \in V_2,$$

which contradicts to nondegeneracy of b_2. It remains to note that if b_2 is alternating (resp. symmetric), then $\bar{b}_1$ is alternating (resp. symmetric). □

Lemma 2.15 *Let $\varepsilon : A_1 \to A_2$ be an embedding of involution simple algebras of types different from* **A** *and let α_i denotes the involution of A_i. Fix any canonical matrix realization $(M_{n_1}(\mathbb{F}), \alpha_1)$ of A_1. Then there exists a matrix realization $(M_{n_2}(\mathbb{F}), \alpha_2)$ of A_2 such that the following hold.*

(i) The embedding ε is the composition of the following embeddings of algebras with involution.

$$(M_{n_1}(\mathbb{F}), \alpha_1) \xrightarrow{\eta} (M_{n_1}(\mathbb{F}) \otimes_{\mathbb{F}} M_k(\mathbb{F}), \alpha_1 \otimes \beta_1) \xrightarrow{\iota} (M_{kn_1}(\mathbb{F}), \beta_2) \xrightarrow{\zeta} (M_{n_2}(\mathbb{F}), \alpha_2)$$

where $\eta(X) = X \otimes e$ with e the identity element of $M_k(\mathbb{F})$, ι is the natural isomorphism, and ζ is a natural embedding (i.e. of signature $(1, 0, z)$).

(ii) If $\operatorname{char}\mathbb{F} \neq 2$ *and A_1 and A_2 are both of type* **O**, *then $\alpha_1 = \beta_1 = \beta_2 = \alpha_2 = t$ (matrix transpose)*

(iii) If $\operatorname{char}\mathbb{F} \neq 2$ *and A_1 and A_2 are both of type* **S**, *then $\alpha_1 = \beta_2 = \alpha_2 = \tau$ (symplectic transpose) and $\beta_1 = t$.*

Proof. Let $\bar{A}_1$ be as in Lemma 2.14 and let β_2 be the restriction of α_2 to $\bar{A}_1$. Let B be the centralizer of A_1 in $\bar{A}_1$. Note that A_1 and $\bar{A}_1$ are simple and have the same identity element. Therefore B is simple and $\bar{A}_1 = A_1 B \cong A_1 \otimes_{\mathbb{F}} B$ (see e.g. [12, 1.5]). Clearly, B is β_2-invariant. Denote by β_1 the restriction of β_2 to B. Then $(\bar{A}_1, \beta_2) \cong (A_1 \otimes_{\mathbb{F}} B, \alpha_1 \otimes \alpha_2)$. We get the following chain of embeddings of algebras with involution:

$$A_1 \longrightarrow A_1 \otimes_{\mathbb{F}} B \simeq \bar{A}_1 \longrightarrow A_2.$$

Identifying A_1 with $M_{n_1}(\mathbb{F})$, B with $M_k(\mathbb{F})$ for some k, $\bar{A}_1$ with $M_{kn_1}(\mathbb{F})$, and A_2 with $M_{n_2}(\mathbb{F})$, we prove (i).

Assume now that $\operatorname{char}\mathbb{F} \neq 2$ and A_1 and A_2 are of the same type **S** (resp. **O**), i.e. α_1 and α_2 are of type **S** (resp. **O**). Then by Lemma 2.14, β_2 is of type **S** (resp. **O**). Therefore by Lemma 2.13, β_1 is of type **O**. Fixing an appropriate isomorphism $B \cong M_k(\mathbb{F})$, by Lemma 2.3, we can assume that β_1 is a matrix transpose. Using the same lemma we get that β_2 can be represented as τ (resp. t). Now by Lemma 2.14, the restriction of the form b_1 to $\bar{V}_1$ is nondegenerate. Thus $V_2 = \bar{V}_1 \oplus \bar{V}_1^{\perp}$. By choosing a suitable basis in $\bar{V}_1^{\perp}$, we can easily represent α_2 as τ (resp. t). □

As a corollary we get the following analogue of Proposition 2.12 for symplectic and orthogonal algebras.

Proposition 2.16 *Let $\varepsilon : A_1 \to A_2$ be an embedding of finite dimensional involution simple algebras over $\mathbb{F}$ of the same type* **S** *or* **O**. *Assume that* $\operatorname{char}\mathbb{F} \neq 2$. *Then ε is* representable. *That is, for every canonical matrix realization of A_1 there exists a canonical matrix realization of A_2 such that the embedding ε is of the form (7).*

Proposition 2.19 below shows that the case of characteristic 2 is exceptional indeed.

We will also need the following result, which describes embeddings of involution simple algebras of different types. Recall that (l, r, z) is the signature of the embedding $\varepsilon : A_1 \to A_2$.

Proposition 2.17 *Let $\varepsilon : A_1 \to A_2$ be an embedding of finite dimensional involution simple algebras over $\mathbb{F}$. Assume that* $\operatorname{char}\mathbb{F} \neq 2$.

(i) If A_1 is of type **A** *and A_2 is not of type* **A**, *then $l = r$.*

(ii) If A_1 is of type **S** *(resp.,* **O***) and A_2 is of type* **O** *(resp.,* **S***), then l is even.*

(iii) If A_1 and A_2 are both not of type **A** *and l is even, then there exist an algebra D of type* **A**, *an embedding $\eta : A_1 \to D$ with the signature $(l/2, 0, 0)$ and an embedding $\zeta : D \to A_2$ with the signature $(1, 1, z)$ such that $\varepsilon = \zeta\eta$.*

(iv) If A_1 and A_2 are of type **A** *and $l = r$, then there exist an algebra D of type* **O** *(resp.,* **S***), embeddings $\eta : A_1 \to D$ of signature $(l, l, 0)$ and $\zeta : D \to A_2$ of signature $(1, 0, z)$ such that $\varepsilon = \zeta\eta$.*

Proof. (i) Recall that $A_i = B_i \oplus C_i$ where B_i and C_i are the simple components of A_i and $B_i^* = C_i$. And f_i and $g_i = f_i^*$ are the identities of B_i and C_i, respectively. Obviously, $l = (\dim f_1 A_2 f_1)/n_1$ and $r = (\dim g_1 A_2 g_1)/n_1$ where $n_1 = \dim V_1 = \dim W_1$. Since $(f_1 A_2 f_1)* = g_1 A_2 g_1$, we get that $l = r$. Note that this is valid for the case of $\operatorname{char}\mathbb{F} = 2$ as well.

(ii) Represent the embedding $A_1 \to A_2$ as in Lemma 2.15(i). Note that $k = l$. By Lemma 2.14(iii), β_2 has the same type as α_2. Thus the types of α_1 and β_2 are different. By Lemma 2.13, β_1 must be symplectic. Therefore $k = l$ is even.

(iii) Represent the embedding $A_1 \to A_2$ as in Lemma 2.15(i). Denote by B the algebra $M_k(\mathbb{F})$. By assumption, $k = l$ is even. Let e be the identity element of B. By Lemma 2.7, B has two idempotents f and g such that $e = f + g$, $fg = gf = 0$, and $f^* = g$. Then $B_f = fBf$ and $B_g = gBg$ are simple subalgebras of B, $B_f \cap B_g = 0$, $B_f B_g = B_g B_f = 0$, and $B_f^* = B_g$. Thus $B' = B_f \oplus B_g$ is an involution simple subalgebra of B of type **A**. Therefore $D = A_1 \otimes_{\mathbb{F}} B'$ is an involution simple subalgebra of $A_1 \otimes_{\mathbb{F}} B$ of type **A**. Since $e = f + g$, D contains A_1. Clearly the signature of the embedding $A_1 \to D$ is $(l/2, 0, 0)$ and the signature of the embedding $D \to A_2$ is $(1, 1, z)$.

(iv) Let $A = M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$ be an involution simple algebra with standard involution $(X, Y)^* = (Y^t, X^t)$. Let $k \le n$ and let the algebra $D = M_k(\mathbb{F})$ have an involution α. Define a "corner" embedding $\varphi : D \to A$ via $\varphi(Z) = (\bar{Z}, \overline{(Z^\alpha)^t})$ where $\bar{Z} = \mathrm{diag}(Z, 0 \dots, 0)$. Since $t \circ \alpha$ is an automorphism of D, φ is an algebra homomorphism. Moreover, one can easily check that φ respects involution:

$$\varphi(Z^\alpha) = (\overline{Z^\alpha}, \overline{Z^t}) = (\bar{Z}, \overline{(Z^\alpha)^t})^* = \varphi(Z)^*$$

By Proposition 2.12, the embedding ε can be represented as in (8) with $l = r$ and involution $*$ acting as $(X, Y)^* = (Y^t, X^t)$ on both algebras. Now let α be either symplectic involution θ_- or orthogonal involution θ_+ (see Remark 2.6) of the algebra $D = M_{2l}(\mathbb{F})$. Let $\zeta = \varphi : D \to A_2$ be the corner embedding of algebras with involution described above. Note that it is an embedding of signature $(1, 0, z)$. Observe that

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\theta_\pm} \right)^t = \begin{pmatrix} d^t & \pm b^t \\ \pm c^t & a^t \end{pmatrix}^t = \begin{pmatrix} d & \pm c \\ \pm b & a \end{pmatrix}, \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2l}(\mathbb{F}),$$

where a, b, c, d are square matrices of size l. Therefore, it is easy to see from formula (8) that $\varepsilon(A_1) \subset \zeta(D)$. Define by η the following composition of embeddings:

$$A_1 \longrightarrow \varepsilon(A_1) \longrightarrow \varphi(D) \xrightarrow{\zeta^{-1}} D.$$

Then η is of signature $(l, r, 0)$ and $\varepsilon = \zeta\eta$, as required. $\square$

It remains to consider the case of characteristic 2, which is a bit more complicated.

Lemma 2.18 *Let* $\mathrm{char}\,\mathbb{F} = 2$ *and let* $\varepsilon : A_1 \to A_2$ *be an embedding of involution simple algebras preserving the identity element (i.e.* $\varepsilon(e_1) = e_2$*). Assume that* A_2 *is of type* **O**. *Then* A_1 *is of type* **O**.

Proof. Assume that A_1 is of type **A** or **S**. Then by Lemma 2.7, A_1 has idempotents f and g such that $e_1 = f + g$, $fg = gf = 0$, and $f^* = g$. Let b be a symmetric nondegenerate form on V_2 corresponding to the involution. Then for all $v \in V_2$ we have

$$b(v, v) = b((f + g)v, v) = b(fv, v) + b(v, fv) = 0,$$

as b is symmetric. Therefore b is alternating, so A_2 is symplectic, which contradicts the assumption. $\square$

Proposition 2.19 *Let* $\operatorname{char}\mathbb{F} = 2$ *and let* $\varepsilon : A_1 \to A_2$ *be an embedding of involution simple algebras of the same type* $X = \mathbf{O}$ *or* $\mathbf{S}$*. Then the following conditions are equivalent.*

(i) The embedding ε *is representable.*

(ii) Each $*$*-invariant involution simple subalgebra* D *of* A_2 *containing* A_1 *is of type* X*.*

Moreover, if the embedding ε *is not representable, then there exists a* $*$*-invariant involution simple subalgebra* D *of* A_2 *which is of type* $\mathbf{A}$ *and contains* A_1*.*

Proof. By Lemma 2.15(i), the embedding ε can be represented as the composition of embeddings $A_1 \to C \to A_2$ where $C = e_1A_2e_1 \cong A_1 \otimes M_k(\mathbb{F})$ is involution simple of type $\mathbf{O}$ or $\mathbf{S}$ and has the same identity element e_1 as A_1, and the embedding $C \to A_2$ is natural (of signature $(1, 0, z)$).

(i) $\Rightarrow$ (ii) ($X = \mathbf{O}$): Assume that ε is representable and there exists a $*$-invariant involution simple subalgebra D of A_2 containing A_1 which is not of type $\mathbf{O}$. The matrix presentation (7) shows that C is of type $\mathbf{O}$. Let e_D be the identity element of D. Since e_D is an idempotent, the algebra $F = e_DA_2e_D$ is a $*$-invariant simple subalgebra of A_2 containing D. By Lemma 2.18, it cannot be orthogonal. Therefore F is of type $\mathbf{S}$. Note that F contains C and $e_1Fe_1 = C$. Therefore by Lemma 2.14(iii), C must be of the same type $\mathbf{S}$, which is a contradiction.

(i) $\Rightarrow$ (ii) ($X = \mathbf{S}$): Assume that ε is representable and there exists a $*$-invariant involution simple subalgebra D of A_2 containing A_1 which is not of type $\mathbf{S}$. First assume that D is of type $\mathbf{A}$. Then e_1De_1 is an involution simple subalgebra of $C = e_1A_2e_1$ of type $\mathbf{A}$ containing A_1. Recall that $C \cong A_1 \otimes M_k(\mathbb{F})$. Therefore $e_1De_1 \cong A_1 \otimes E$ where E is an involution simple subalgebra of $M_k(\mathbb{F})$ of type $\mathbf{A}$ with the same identity element. Since ε is representable, the involution on $M_k(\mathbb{F})$ is orthogonal, which contradicts to Lemma 2.18.

Suppose now that D is of type $\mathbf{O}$. As in the case $X = \mathbf{O}$, the algebra $F = e_DA_2e_D$ is a $*$-invariant simple subalgebra of A_2 containing C. By Lemma 2.14(iii), F is symplectic. Therefore $F \cong D \otimes_{\mathbb{F}} M_q(\mathbb{F})$ with a symplectic involution on $M_q(\mathbb{F})$ (Lemma 2.13(i)). By Lemma 2.7, $M_q(\mathbb{F})$ has two idempotents f and g such that $f + g$ is the identity element of $M_q(\mathbb{F})$, $fg = gf = 0$, and $f^* = g$. Therefore $D' = D \otimes f \oplus D \otimes g$ is an involution simple subalgebra of A_2 of type $\mathbf{A}$ containing A_1. However the case of type $\mathbf{A}$ subalgebra containing A_1 has been already considered in the previous paragraph.

(ii) $\Rightarrow$ (i) and "Moreover" part: Assume that the embedding ε is not representable. We are going to show that A_2 contains an involution simple subalgebra of type $\mathbf{A}$ containing A_1. Recall that $C \cong A_1 \otimes M_k(\mathbb{F})$ and the restriction of the involution $*$ on C has the form $\alpha_1 \otimes \alpha_2$ where α_1 is the involution of A_1 and α_2 is an involution of $M_k(\mathbb{F})$. Clearly if α_2 is orthogonal, then ε is representable (see the proof of Lemma 2.15). Therefore α_2 is symplectic. Then, as above, $M_k(\mathbb{F})$ has two idempotents f and g such that $f + g$ is the identity element of $M_k(\mathbb{F})$, $fg = gf = 0$, and $f^* = g$. Therefore $D = A_1 \otimes f \oplus A_1 \otimes g$ is an involution simple subalgebra of A_2 of type $\mathbf{A}$ containing A_1. The proposition follows. □

The following results show how embedding signatures behave under compositions.

Proposition 2.20 *Let* $\varepsilon_1 : A_1 \to A_2$ *and* $\varepsilon_2 : A_2 \to A_3$ *be embeddings of involution simple algebras of the same type with the signatures* (l_1, r_1, z_1) *and* (l_2, r_2, z_2)*, respectively. Denote by* (l, r, z) *the signature of* $\varepsilon = \varepsilon_2\varepsilon_1$*. Then*

$$\begin{aligned} l &= l_1l_2 + r_1r_2, && (9)\\ r &= r_1l_2 + l_1r_2, && (10)\\ z &= z_1(l_2 + r_2) + z_2. \end{aligned}$$

Proof. For types **S** and **O** one has $r = r_1 = r_2 = 0$, so the statement immediately follows from (6). For type **A**, the embeddings are representable so one can use (8). $\square$

Note that $l + r = (l_1 + r_1)(l_2 + r_2)$ and $l - r = (l_1 - r_1)(l_2 - r_2)$. Thus, the following is true.

Corollary 2.21 *Let $A_1 \to \cdots \to A_k$ be a sequence of embeddings of involution simple algebras of the same type. Let (l_i, r_i, z_i) be the signature of $A_i \to A_{i+1}$, (l, r, z) the signature of $A_1 \to A_k$, $s_i = l_i + r_i$, $c_i = l_i - r_i$, $s = l + r$, $c = l - r$. Then $s = s_1 \dots s_{k-1}$ and $c = c_1 \dots c_{k-1}$.*

Recall that n_i is the degree of A_i (so $A_i \cong M_{n_i}(\mathbb{F})$ or $M_{n_i}(\mathbb{F}) \oplus M_{n_i}(\mathbb{F})$).

Lemma 2.22 *Let $\varepsilon_1 : A_1 \to A_2$ and $\varepsilon : A_1 \to A_3$ be representable embeddings of involution simple algebras of the same type (**A**, **S** or **O**) with the signatures (l_1, r_1, z_1) and (l, r, z), respectively. Assume that a triple of non-negative integers (l_2, r_2, z_2) satisfies the following conditions*

$$l + r = (l_1 + r_1)(l_2 + r_2), \tag{11}$$

$$l - r = (l_1 - r_1)(l_2 - r_2), \tag{12}$$

$$n_3 = n_2(l_2 + r_2) + z_2 \tag{13}$$

where n_i is the degree of A_i. Then there exists a representable embedding $\varepsilon_2 : A_2 \to A_3$ with the signature (l_2, r_2, z_2) such that $\varepsilon = \varepsilon_2\varepsilon_1$.

Proof. Fix any canonical matrix realizations of A_1, A_2, A_3 such that the matrix embeddings ε_1 and ε become canonical (see Definition 2.10). Consider the canonical matrix embedding $\varepsilon_2 : A_2 \to A_3$ with signature (l_2, r_2, z_2). The embedding ε_2 is well-defined because of (13) and respects the involution (see Remark 2.11(1)). By Remark 2.11(2), the matrix homomorphism $\varepsilon_2\varepsilon_1$ is canonical. By rewriting the conditions (11) and (12) in the form (9) and (10) we see that both canonical homomorphisms ε and $\varepsilon_2\varepsilon_1$ have the same signature, so $\varepsilon = \varepsilon_2\varepsilon_1$. $\square$

Our classification will be given in terms of so-called supernatural (or Steinitz) numbers. They are defined as follows. Let $(p_1, p_2, \dots)$ be the increasing sequence of all prime numbers. The set of all mappings from $\{p_1, p_2, \dots\}$ into the set $\{0, 1, 2, \dots\} \cup \{\infty\}$ is called the set of *supernatural* (or *Steinitz*) numbers. If a supernatural number takes a value α_1 at p_1, α_2 at p_2,..., this element will be denoted by $p_1^{\alpha_1} p_2^{\alpha_2} \dots$. The supernatural numbers can be regarded as formal products of powers of primes where infinity is permitted as a power. The set of natural numbers $\mathbb{N}$ can be identified in an evident way with a subset of supernatural numbers. If $\Pi = p_1^{\alpha_1} p_2^{\alpha_2} \dots$ and $\Pi' = p_1^{\alpha'_1} p_2^{\alpha'_2} \dots$ are two supernatural numbers, we set $\Pi\Pi' = p_1^{\alpha_1 + \alpha'_1} p_2^{\alpha_2 + \alpha'_2} \dots$. We say that Π divides Π' if and only if $\alpha_1 \le \alpha'_1, \alpha_2 \le \alpha'_2 \dots$. Let $q \in \mathbb{Q}$. We write $\Pi = q\Pi'$ (or $q \in \frac{\Pi}{\Pi'}$) if there exists $n \in \mathbb{N}$ such that $nq \in \mathbb{N}$ and $n\Pi = nq\Pi'$. If there exists non-zero $q \in \mathbb{Q}$ such that $\Pi = q\Pi'$, then we say that Π and Π' are $\mathbb{Q}$-*equivalent* and denote this relation by $\Pi \overset{\mathbb{Q}}{\sim} \Pi'$. Let $\mathcal{S} = (s_1, s_2, \dots)$ be a sequence of natural numbers. Denote by $\Pi(\mathcal{S})$ the supernatural number $s_1 s_2 s_3 \dots$. We will use the following simple observation.

Proposition 2.23 ([2, Proposition 3.2]) *Let $\mathcal{S} = (s_i)_{i \in I}$ and $\mathcal{S}' = (s'_j)_{j \in J}$ be sequences of natural numbers. Then $q \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$ if and only if for each $i \in I$ and $k \in J$ there exist $j = j(i) \in J$ and $l = l(k) \in I$ such that $s_1 \dots s_i$ divides $qs'_1 \dots s'_j$ (over $\mathbb{Z}$) and $qs'_1 \dots s'_k$ divides $s_1 \dots s_l$ (over $\mathbb{Z}$).*

3 Bratteli diagrams and dimension groups

Let

$$A_1 \to A_2 \to A_3 \to \dots \to A_i \to A_{i+1} \to \dots \tag{14}$$

be a sequence of embeddings of finite dimensional involution simple algebras over $\mathbb{F}$. Assume that all A_i are of the same type and $\operatorname{char}\mathbb{F} \neq 2$. Then, as we proved in Propositions 2.12 and 2.16, all embeddings $A_i \to A_{i+1}$ are representable, so one can assume that all A_i are matrix (or double matrix) algebras and the embeddings and involutions are canonical. This justifies the following definition.

Definition 3.1 Let A be a locally involution simple associative algebra of countable dimension. We say that A is *canonically representable* if it is isomorphic to the direct limit of the sequence (14) where all A_i are matrix (resp. double matrix) algebras with canonical involutions of the same type X ($= \mathbf{A}, \mathbf{S}$ or $\mathbf{O}$) and all embeddings are canonical. In that case we say that the sequence (14) is a *canonical representation* for A and A is of *type* X.

Note that the type X of the algebra A may not be unique.

Proposition 2.19 shows that some of the embeddings $A_i \to A_{i+1}$ may not be representable in characteristic 2. Fortunately, there is a way to modify the sequence (14), without changing the limit algebra, in order to get representable embeddings even in characteristic 2.

Theorem 3.2 *Let A be a locally involution simple associative algebra over $\mathbb{F}$ of countable dimension. Then A is canonically representable.*

Proof. Let $A_1 \to A_2 \to A_3 \to \dots$ be a sequence of embeddings of involution simple finite dimensional associative algebras such that $A = \varinjlim A_i$. Choose an infinite subsequence of algebras of the same type. If $\operatorname{char}\mathbb{F} \neq 2$, or $\operatorname{char}\mathbb{F} = 2$ and all algebras are of type $\mathbf{A}$, then all embeddings are representable by Propositions 2.12 and 2.16. Assume $\operatorname{char}\mathbb{F} = 2$. If there is an infinite number of non-representable embeddings, then by Proposition 2.19, we can replace the subsequence by a sequence of embeddings of algebras of type $\mathbf{A}$. Otherwise, we get the result by removing a finite number of algebras in the beginning of the sequence. □

Theorem 3.2 reduces classification of locally involution simple algebras to the following two problems:

(a) classification of the direct limits of canonical sequences of the same type;

(b) classification of intertype isomorphisms.

We are going to simplify Problem (a) even further and reduce it to the algebras without involution. We need the following trivial observation.

Proposition 3.3 *Let $A_1 \to A_2 \to A_3 \to \dots$ and $A'_1 \to A'_2 \to A'_3 \to \dots$ be two sequences of embeddings of algebras (or algebras with involution). Then $\varinjlim A_i \cong \varinjlim A'_j$ if and only if there exist sequences of indices $i_1 < i_2 < \dots$ and $j_1 < j_2 < \dots$ and homomorphisms $\varphi_k : A_{i_k} \to A'_{j_k}$ and $\varphi'_k : A'_{j_k} \to A_{i_{k+1}}$ ($k = 1, 2, \dots$) such that the following diagram commutes.*

$$\begin{array}{ccccccccccc}
A_{i_1} & \longrightarrow & A_{i_2} & \longrightarrow & \dots & A_{i_k} & \longrightarrow & A_{i_{k+1}} & \longrightarrow & \dots \\
\downarrow\varphi_1 & \nearrow\varphi'_1 & \downarrow\varphi_2 & \nearrow\varphi'_2 & & \downarrow\varphi_k & \nearrow\varphi'_k & \downarrow\varphi_{k+1} & \nearrow\varphi'_{k+1} & \\
A'_{j_1} & \longrightarrow & A'_{j_2} & \longrightarrow & \dots & A'_{j_k} & \longrightarrow & A'_{j_{k+1}} & \longrightarrow & \dots
\end{array} \tag{15}$$

Proof. Set $A = \varinjlim A_i$ and $A' = \varinjlim A'_j$. Assume that there exists an isomorphism $\varphi : A \to A'$. Fix any index i_1. Then there exists j_1 such that $\varphi(A_{i_1}) \subseteq A'_{j_1}$. Similarly, there exists i_2 such that $\varphi^{-1}(A'_{j_1}) \subseteq A_{i_2}$, and so on. Denote by φ_k the restriction of φ to A_{i_k}, and by φ'_k the restriction of φ^{-1} to A'_{j_k}, $k = 1, 2, \ldots$. Then the diagram above commutes. The converse statement is obvious. □

Theorem 3.4 *Two locally involution simple associative algebras of the same type over $\mathbb{F}$ of countable dimension are isomorphic if and only if they are isomorphic as associative algebras.*

Proof. Let A and A' be two locally involution simple associative algebras and let $A = \varinjlim A_i$ and $A' = \varinjlim A'_j$ be their canonical representations. Assume that A and A' are isomorphic as associative algebras. Using Proposition 3.3, we get a commutative diagram (15), where $\varphi_k : A_{i_k} \to A'_{j_k}$ and $\varphi'_k : A'_{j_k} \to A_{i_{k+1}}$ are algebra homomorphisms, not necessarily respecting the involution. Let $\varepsilon_k : A_{i_k} \to A_{i_{k+1}}$ and $\varepsilon'_k : A'_{j_k} \to A'_{j_{k+1}}$ be the horizontal maps. Note that they are canonical and respect the involution. Denote by ψ_k (resp. ψ'_k) the canonical map $A_{i_k} \to A'_{j_k}$ (resp. $A'_{j_k} \to A_{i_{k+1}}$) of the same signature as φ_k (resp. φ'_k). Then by Remark 2.11(1) these maps respect the involution. It remains to show that they make the diagram (15) commutative. Note that the signature of $\psi'_k\psi_k$ equals to the signature of $\varphi'_k\varphi_k = \varepsilon_k$. Since both $\psi'_k\psi_k$ and ε_k are canonical, we get that $\psi'_k\psi_k = \varepsilon_k$. Similarly, one proves that $\psi_{k+1}\psi'_k = \varepsilon'_k$. Therefore the diagram (15) commutes with respect to the maps ψ_k and ψ'_k. Proposition 3.3 implies that A and A' are isomorphic as algebras with involution.

The converse statement is trivial. □

Theorem 3.4 reduces Problem (a) to classifying the direct limits of finite dimensional semisimple algebras. This is usually done in terms of Bratteli diagrams, the K_0-functor and dimension groups. To make the statements of the results a little bit clearer it is best to work in the category of unital algebras (i.e. algebras with identity elements and with identity preserving homomorphisms). In our case this can be easily achieved by adjoining an external identity element.

Definition 3.5 Let A be an associative algebra. Define the algebra $\hat{A}$ as follows. If A has an identity element, put $\hat{A} = A$. Otherwise, put $\hat{A} = A + \mathbb{F}\mathbf{1}_{\hat{A}}$ where $\mathbf{1}_{\hat{A}}$ is the identity element of $\hat{A}$.

Note that if A has an involution then this involution trivially extends to $\hat{A}$.

Lemma 3.6 *Let A be a locally semisimple associative algebra. Then $\hat{A}$ is locally semisimple in the category of unital algebras.*

Proof. Let $A = \varinjlim A_i$ with A_i finite dimensional semisimple. If A contains an identity element $\mathbf{1}_A$, then A is the direct limit of those A_i which contain $\mathbf{1}_A$, as required. If A has no identity element, then $\hat{A} = A + \mathbb{F}\mathbf{1}_{\hat{A}} = \varinjlim B_i$ where $B_i = A_i + \mathbb{F}\mathbf{1}_{\hat{A}}$ are obviously finite dimensional and semisimple. □

Proposition 3.7 *Let A and A' be involution simple associative algebras. Then $A \cong A'$ as associative algebras if and only if $\hat{A} \cong \hat{A}'$ as associative algebras.*

Proof. By construction, $A \cong A'$ implies $\hat{A} \cong \hat{A}'$. Assume now that $\hat{A} \cong \hat{A}'$. We need to show that $A \cong A'$. Denote by $\mathrm{Soc}(A)$ the sum of all minimal ideals of A. Then by Proposition 2.1, $\mathrm{Soc}(A) = A$. Obviously, $\mathrm{Soc}(A) \subseteq \mathrm{Soc}(\hat{A})$. We claim that $\mathrm{Soc}(A) = \mathrm{Soc}(\hat{A})$. Indeed, this is obvious if $A = \hat{A}$. Assume $A \neq \hat{A}$, i.e. A has no identity element. Let M be a minimal ideal of $\hat{A}$ such that $M \not\subseteq \mathrm{Soc}(A)$. Then $M \cap \mathrm{Soc}(A) = 0$. But $\mathrm{Soc}(A) = A$ is an ideal of codimension 1 in $\hat{A}$. Therefore M is one-dimensional and $\hat{A} = A \oplus M$. Write $\mathbf{1}_{\hat{A}} = a + m$ where $a \in A$ and $m \in M$. Then obviously a is an identity element of A, which is a contradiction. Therefore, $A = \mathrm{Soc}(\hat{A}) \cong \mathrm{Soc}(\hat{A}') = A'$, as required. □

Locally semisimple algebras are best described in terms of their Bratteli diagrams. These are defined as follows. Let B be the direct limit of the infinite sequence

$$B_1 \to B_2 \to B_3 \to \dots \tag{16}$$

where the B_i are finite dimensional semisimple algebras over $\mathbb{F}$. Let $S_i^1, S_i^2, \dots S_i^{k_i}$ be the simple components of B_i, i.e. $B_i = S_i^1 \oplus S_i^2 \oplus \dots \oplus S_i^{k_i}$. Let V_i^j be the natural S_i^j-module. Then V_i^j can be considered as an B_i-module. Denote by m_i^{jq} the multiplicity of V_i^j in the restriction of V_{i+1}^q to S_i^j (i.e. m_i^{jq} is the number of copies of S_i^j that are mapped to S_{i+1}^q). The Bratteli diagram of the sequence (16) consists of the vertices $V = \{V_i^j \mid i = 1, 2, 3, \dots;\ 1 \leq j \leq k_i\}$ and edges. Two vertices V_i^j and V_{i+1}^q are connected by an edge if and only if $m_i^{jq} > 0$. In that case the edge is labelled by the number m_i^{jq}. Let $n_i^j = \dim V_i^j$ be the degree of S_i^j. Then obviously

$$\sum_{j=1}^{k_i} m_i^{jq} n_i^j \leq n_{i+1}^q \tag{17}$$

Moreover, if all homomorphisms in (16) are unital then we have equality in (17) for all i and q, so the whole sequence (16) can be reconstructed from its Bratteli diagram provided the degrees of the simple components of the first term B_1 are known (in the case of non-unital embeddings extra data is needed).

Now let A be a locally involution simple associative algebra of type X ($= \mathbf{A}, \mathbf{S}$ or $\mathbf{O}$) over $\mathbb{F}$ of countable dimension. By Theorem 3.2, A is the direct limit of the sequence (14) where all A_i are matrix (resp. double matrix) algebras with canonical involutions of the same type X and all embeddings are canonical.

We will denote by (l_i, r_i, z_i) the signature of the embedding $A_i \to A_{i+1}$ and by n_i the degree of A_i (i.e. $A_i = M_{n_i}(\mathbb{F})$ and $r_i = 0$ for $X = \mathbf{S}, \mathbf{O}$ and $A_i = M_{n_i}(\mathbb{F}) \oplus M_{n_i}(\mathbb{F})$ for $X = \mathbf{A}$). By Remark 2.9, for type $\mathbf{A}$ algebras we can and will assume that $l_i \geq r_i$ for all i. It is convenient to add to the sequence an algebra of degree 1 (the 1-dimensional algebra $\mathbb{F}$ is considered to be of both types $\mathbf{O}$ and $\mathbf{S}$), so we will assume that $n_1 = 1$, $l_1 = n_2$ and $r_1 = z_1 = 0$. Denote by $\mathcal{T}$ the triple sequence $(l_i, r_i, z_i)_{i \in \mathbb{N}}$. Since $n_{i+1} = (l_i + r_i)n_i + z_i$ for all i, the canonical sequence (14) is uniquely determined by the triple sequence $\mathcal{T}$ and type X. We will denote by $A(\mathcal{T}, X)$ the corresponding locally involution simple associative algebra over $\mathbb{F}$, by $A(\mathcal{T})$ the corresponding locally semisimple algebra (i.e. the direct limit of the associative algebras (14) disregarding the involution) and by $\hat{A}(\mathcal{T})$ the corresponding algebra with an identity element (see Definition 3.5). Recall that by Theorem 3.4 and Proposition 3.7, $A(\mathcal{T}, X) \cong A(\mathcal{T}', X)$ if and only if $A(\mathcal{T}) \cong A(\mathcal{T}')$ (equivalently, $\hat{A}(\mathcal{T}) \cong \hat{A}(\mathcal{T}')$).

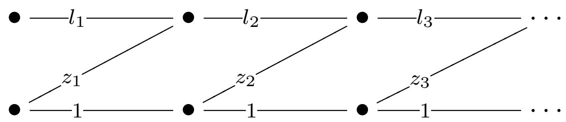
If A has an identity element $\mathbf{1}_A$ (i.e. $A = \hat{A}$) then we can and will assume that $\mathbf{1}_A \in A_i$ for all i. Put $B_i = A_i$ if $\mathbf{1}_A \in A$ and $B_i = A_i + \mathbb{F}\mathbf{1}_{\hat{A}}$ otherwise, see the proof of Lemma 3.6. Then B_i is semisimple, with possibly one extra 1-dimensional simple component. Moreover, all embeddings $B_i \to B_{i+1}$ are unital and $\hat{A} = \varinjlim B_i$. Recall that X is the type of A. We will denote by $\mathcal{B}(\mathcal{T})$ the Bratteli diagram $\mathcal{B}(\hat{A})$ of the algebra $\hat{A}$ with respect to the sequence $B_1 \to B_2 \to B_3 \to \dots$.

If $\mathbf{1}_A \in A$ and the type $X = \mathbf{S}, \mathbf{O}$, then all $z_i = 0$ and it is easy to see that $\mathcal{B}(\mathcal{T})$ is

$$\bullet \overset{l_1}{\text{———}} \bullet \overset{l_2}{\text{———}} \bullet \overset{l_3}{\text{———}} \cdots \tag{18}$$

The locally semisimple algebras of this type are just the limits of "pure diagonal" matrix embeddings $M_{n_i} \to M_{n_{i+1}}$ given by $M \mapsto \mathrm{diag}(M, \dots, M)$ (l_i blocks), $M \in M_{n_i}(\mathbb{F})$. They were first classified by Glimm [10] (in $\mathbb{C}^*$-algebras setting). It is easy to see that two algebras of this type are isomorphic if and only if their corresponding supernatural numbers $\Pi = l_1 l_2 l_3 \dots$ are equal.

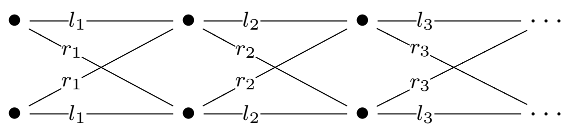
If $\mathbf{1}_A \notin A$ and the type $X = \mathbf{S}, \mathbf{O}$, then $\mathcal{B}(\mathcal{T})$ is



(19)

The corresponding locally semisimple algebras $A(\mathcal{T})$ are the direct limits of matrix embeddings of the shape (2). They were first classified by Dixmier [5] (in $\mathbb{C}^*$-algebras setting). Dixmier's parametrization consists of the supernatural number $\Pi = l_1 l_2 l_3 \dots$ and one real parameter θ, which is in fact the inverse of our density index δ, see below. The diagrams of this shape also parametrize so-called "diagonal" direct limits of finite symmetric and alternating groups [13].

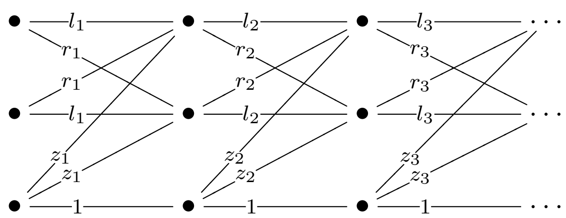
If $\mathbf{1}_A \in A$ and the type $X = \mathbf{A}$, then $\mathcal{B}(\mathcal{T})$ is



(20)

The corresponding algebras were first classified by Fack and Maréchal [9] (in $\mathbb{C}^*$-algebras setting).

If $\mathbf{1}_A \notin A$ and the type $X = \mathbf{A}$, then $\mathcal{B}(\mathcal{T})$ is



(21)

This is the most general case. We parametrize the corresponding algebras by two supernatural numbers and two real parameters (see Theorem 4.1).

Let $B = \varinjlim B_i$ be a unital locally semisimple algebra and let $K_0(B)$ be its Grothendieck group with positive cone $K_0(B)^+$. Note that the homomorphism $B_i \to B_{i+1}$ induces the homomorphism of the abelian groups $K_0(B_i) \to K_0(B_{i+1})$ and $K_0(B)$ can be obtained as the direct limit $\varinjlim K_0(B_i)$. Since B_i are finite dimensional and semisimple, one has $(K_0(B_i), K_0(B_i)^+) = (\mathbb{Z}^{k_i}, \mathbb{Z}_+^{k_i})$ where k_i is the number of the simple components of B_i. Therefore the abelian group $K_0(B)$ is the direct limit of the sequence

$$\mathbb{Z}^{k_1} \to \mathbb{Z}^{k_2} \to \dots \to \mathbb{Z}^{k_i} \to \mathbb{Z}^{k_{i+1}} \to \dots$$

Moreover, the embedding on the ith level is given by the adjacency (or multiplicities) matrix of the ith level of the Bratteli diagram of B. For example, for the algebra $\hat{A}$ in (21) the group $K_0(\hat{A})$ is the direct limit of the sequence $\mathbb{Z}^3 \to \mathbb{Z}^3 \to \cdots \to \mathbb{Z}^3 \to \ldots$ were the embedding on the ith level is given by the matrix $\begin{pmatrix} l_i & r_i & z_i \\ r_i & l_i & z_i \\ 0 & 0 & 1 \end{pmatrix}$.

Let $\mathbf{1}_B$ be the identity element of B and let $[\mathbf{1}_B]$ be the corresponding element of $K_0(B)^+$. The triple $(K_0(B), K_0(B)^+, [\mathbf{1}_B])$ is called the *dimension group* of B and is a complete invariant for unital locally semisimple algebras. More exactly the following is true.

Theorem 3.8 *[6] Let B_1 and B_2 be unital locally semisimple algebras. Then $B_1 \cong B_2$ if and only if there is an order-isomorphism $\varphi : K_0(B_1) \to K_0(B_2)$ such that $\varphi([\mathbf{1}_{B_1}]) = [\mathbf{1}_{B_2}]$.*

A similar result holds for non-unital algebras if one replaces $[\mathbf{1}_B]$ by the *scale* of $K_0(B)$.

For a triple sequence $\mathcal{T}$ we denote by $G(\mathcal{T})$ the dimension group of the unital locally semisimple algebra $\hat{A}(\mathcal{T})$.

4 The classification of algebras of the same type and the corresponding dimension groups

In this section, $\mathcal{T} = (l_i, r_i, z_i)_{i\in\mathbb{N}}$ is the triple sequence of the canonically represented locally involution simple algebra $A = A(\mathcal{T}, X)$ of type X. Recall that $l_i \geq r_i$ and $l_i + r_i \geq 1$ for all i. The degrees n_i of the subalgebras A_i satisfy the following: $n_1 = 1$ and $n_{i+1} = (l_i + r_i)n_i + z_i$ for all $i \geq 1$.

Set $s_i = l_i + r_i$, $c_i = l_i - r_i$ $(i = 1, 2, \ldots)$, $\mathcal{S} = (s_i)_{i\in\mathbb{N}}$, $\mathcal{C} = (c_i)_{i\in\mathbb{N}}$, $s_i^k = s_i \ldots s_{k-1}$ and $c_i^k = c_i \ldots c_{k-1}$. Put $\delta_i = s_1^i/n_i$. Then

$$\delta_{i+1} = \frac{s_1^{i+1}}{n_{i+1}} = \frac{s_1^i s_i}{n_i s_i + z_i} = \frac{s_1^i}{n_i + (z_i/s_i)} \leq \delta_i. \tag{22}$$

The limit

$$\delta = \lim_{i\to\infty} \delta_i$$

is called the *density index* of $\mathcal{T}$ and is denoted by $\delta(\mathcal{T})$. Since $\delta_2 = s_1/n_2 = 1$, we have $0 \leq \delta \leq 1$. If $\delta = 0$, then the triple sequence is called *sparse*. If there exists i such that for all $j > i$ we have $\delta_j = \delta_i \neq 0$, then the triple sequence is called *pure*. In view of (22) this is equivalent to the following. There exists i such that for all $j \geq i$ we have $z_j = 0$. In this case, by removing a finite number of terms from the canonically represented sequence without changing the limit algebra, we may and will assume that $z_i = 0$ for all i. We say that the triple sequence is *dense* if and only if $0 < \delta < \delta_i$ for all i.

If there exists i such that $c_j = s_j$ (equivalently, $r_j = 0$) for all $j \geq i$, then $\mathcal{T}$ is called *one-sided*. Otherwise, it is called *two-sided*. If for each i there exists $j > i$ such that $c_j = 0$ (equivalently, $l_j = r_j$), then $\mathcal{T}$ is called (two-sided) *symmetric*. Otherwise it is called *non-symmetric*. In the latter case we may and will assume that $c_i > 0$ for all $i \in \mathbb{N}$. Set $\sigma_i = \frac{c_1 \ldots c_i}{s_1 \ldots s_i}$. The limit

$$\sigma = \lim_{i\to\infty} \sigma_i$$

is called the *symmetry index* of $\mathcal{T}$ and is denoted by $\sigma(\mathcal{T})$. Observe that $0 \le \sigma \le 1$. Two-sided non-symmetric triple sequences with $\sigma = 0$ are called *weakly non-symmetric*, and those with $\sigma \neq 0$ are called *strongly non-symmetric*.

Thus all triple sequences can be partitioned into three classes with respect to density and into four classes with respect to symmetry.

Density types

(D1) Sparse ($\delta = 0$).

(D2) Dense ($\delta_i > \delta > 0$ for all i).

(D3) Pure ($\delta_i = \delta > 0$ for some i).

Symmetry types

(S1) One-sided ($r_j = 0$ for all $j \gg 1$).

(S2) Two-sided symmetric ($l_j = r_j$ for an infinite set of j).

(S3) Two-sided weakly non-symmetric ($r_j > 0$ for an infinite set of j, $l_k > r_k$ for all $k \gg 1$, and $\sigma = 0$).

(S4) Two-sided strongly non-symmetric ($r_j > 0$ for an infinite set of j, $l_k > r_k$ for all $k \gg 1$, and $\sigma \neq 0$).

Now we are ready to prove our main classification result for algebras of the same type.

Theorem 4.1 *Let $\mathcal{T} = \{(l_i, r_i, z_i) \mid i \in \mathbb{N}\}$ and $\mathcal{T}' = \{(l'_i, r'_i, z'_i) \mid i \in \mathbb{N}\}$ be triple sequences and let $X = \mathbf{A}, \mathbf{S}$ or $\mathbf{O}$. Set $\delta = \delta(\mathcal{T})$, $\sigma = \sigma(\mathcal{T})$, $\delta' = \delta(\mathcal{T}')$ and $\sigma' = \sigma(\mathcal{T}')$. Then the locally involution simple algebras $A(\mathcal{T}, X)$ and $A(\mathcal{T}', X)$ (respectively, the locally semisimple algebras $A(\mathcal{T})$ and $A(\mathcal{T}')$; respectively, the dimension groups $G(\mathcal{T})$ and $G(\mathcal{T}')$) are isomorphic if and only if the following conditions hold.*

($\mathcal{A}_1$) *The triple sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same density type.*

($\mathcal{A}_2$) $\Pi(\mathcal{S}) \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{S}')$.

($\mathcal{A}_3$) $\frac{\delta}{\delta'} \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$ *for dense and pure triple sequences (types* (D2) *and* (D3)*).*

($\mathcal{B}_1$) *The triple sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same symmetry type.*

($\mathcal{B}_2$) $\Pi(\mathcal{C}) \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{C}')$ *for two-sided non-symmetric triple sequences (types* (S3) *and* (S4)*).*

($\mathcal{B}_3$) *There exists $\alpha \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$ such that $\alpha\frac{\sigma}{\sigma'} \in \frac{\Pi(\mathcal{C})}{\Pi(\mathcal{C}')}$ for two-sided strongly non-symmetric triple sequences (type* (S4)*). Moreover, $\alpha = \frac{\delta}{\delta'}$ if in addition the triple sequences are dense or pure (types* (D2) *and* (D3)*).*

Proof. The proof is similar to that in the case of Lie algebras in characteristic zero (see [2]). First we will prove necessity. By Theorem 3.4 and Propositions 3.7 and 3.8, it is enough to prove the result for the locally semisimple algebras $A(\mathcal{T})$ and $A(\mathcal{T}')$. We will prove the following more general statement (which will later be used for intertype isomorphisms, Theorem 5.2). If $A(\mathcal{T}, X) \cong A(\mathcal{T}', X')$ (we do not demand that $X = X'$), then $\mathcal{T}$ and $\mathcal{T}'$ satisfy the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$. Moreover, if $X = X' = \mathbf{A}$, then the conditions $(\mathcal{B}_1), (\mathcal{B}_2), (\mathcal{B}_3)$ hold. Let $(A_i)_{i\in I}$ and $(A'_j)_{j\in J}$ ($I \cong J \cong \mathbb{N}$) be canonically represented sequences of involution simple algebras of types X and X', corresponding to the triple sequences $\mathcal{T}$ and $\mathcal{T}'$, respectively. We have $A \cong A'$ where $A = \varinjlim A_i$, $A' = \varinjlim A'_i$. By Proposition 3.3, there exist subsequences $i_1 < i_2 < \dots$ of I, $j_1 < j_2 < \dots$ of J, and embeddings $\varepsilon_k : A_{i_k} \to A'_{j_k}$, $\varepsilon'_k : A'_{j_k} \to A_{i_{k+1}}$ ($k = 1, 2, \dots$) such that the following diagram is commutative.

$$\begin{array}{ccccccccccccc}
A_{i_1} & \longrightarrow & \dots & \longrightarrow & A_{i_k} & \longrightarrow & A_{i_{k+1}} & \longrightarrow & \dots & \longrightarrow & A_{i_m} & \longrightarrow & \dots \\
\downarrow\varepsilon_1 & \nearrow\varepsilon'_1 & & \nearrow & \downarrow\varepsilon_k & \nearrow\varepsilon'_k & \downarrow\varepsilon_{k+1} & \nearrow & & \nearrow & \downarrow\varepsilon_m & \nearrow & \\
A'_{j_1} & \longrightarrow & \dots & \longrightarrow & A'_{j_k} & \longrightarrow & A'_{j_{k+1}} & \longrightarrow & \dots & \longrightarrow & A'_{j_m} & \longrightarrow & \dots
\end{array} \tag{23}$$

Let (p_k, q_k, u_k) (resp., (p'_k, q'_k, u'_k)) be the signature of ε_k (resp., ε'_k). Let n_i be the degree of A_i. Set $s_i = l_i + r_i$, $c_i = l_i - r_i$, $\delta_i = s_1^i / n_i$, $\delta = \lim_{i\to\infty} \delta_i$. The numbers n'_j, s'_j,... for the algebra A' are defined similarly. We have

$$n'_{j_m} = (p_m + q_m) n_{i_m} + u_m = (p_m + q_m) s_1^{i_m} \delta_{i_m}^{-1} + u_m = (p_m + q_m) s_1^{i_k} s_{i_k}^{i_m} \delta_{i_m}^{-1} + u_m. \tag{24}$$

On the other hand,

$$n'_{j_m} = s_1'^{j_m} (\delta'_{j_m})^{-1} = s_1'^{j_k} s_{j_k}'^{j_m} (\delta'_{j_m})^{-1}. \tag{25}$$

In view of commutativity of the diagram and by Corollary 2.21 we have

$$s_{i_k}^{i_m} (p_m + q_m) = (p_k + q_k) s_{j_k}'^{j_m}. \tag{26}$$

Dividing (24) and (25) by $s_{j_k}'^{j_m}$, we get $(p_k + q_k) s_1^{i_k} \delta_{i_m}^{-1} + u_m / s_{j_k}'^{j_m} = s_1'^{j_k} (\delta'_{j_m})^{-1}$, so

$$(p_k + q_k) s_1^{i_k} \delta'_{j_m} \le s_1'^{j_k} \delta_{i_m}. \tag{27}$$

Taking $m \to \infty$, we obtain $(p_k + q_k) s_1^{i_k} \delta' \le s_1'^{j_k} \delta$. Similarly, we get $(p'_k + q'_k) s_1'^{j_k} \delta \le s_1^{i_{k+1}} \delta'$. By Corollary 2.21, we have $(p_k + q_k)(p'_k + q'_k) = s_{i_k}^{i_{k+1}}$. Hence

$$(p_k + q_k) s_1^{i_k} \delta' \le s_1'^{j_k} \delta \le (p'_k + q'_k)^{-1} s_1^{i_{k+1}} \delta' = (p_k + q_k) s_1^{i_k} \delta'.$$

Therefore

$$(p_k + q_k) s_1^{i_k} \delta' \quad = \quad s_1'^{j_k} \delta, \tag{28}$$

$$(p'_k + q'_k) s_1'^{j_k} \delta \quad = \quad s_1^{i_{k+1}} \delta'. \tag{29}$$

Clearly $\delta = 0$ if and only if $\delta' = 0$. Therefore $\mathcal{T}$ is sparse if and only if $\mathcal{T}'$ is so. If the triple sequence $\mathcal{T}$ is pure, then $\delta = \delta_{i_m}$ for some m. Subtracting (28) from (27), we get

$$0 \le (p_k + q_k) s_1^{i_k} (\delta'_{j_m} - \delta') \le s_1'^{j_k} (\delta_{i_m} - \delta) = 0.$$

Therefore $\delta'_{j_m} = \delta'$, so $\mathcal{T}'$ is also pure. By symmetry, $\mathcal{T}$ is pure if and only if $\mathcal{T}'$ is pure. So $(\mathcal{A}_1)$ holds.

By (26), $s_{i_k}^{i_m}$ divides $(p_k+q_k)s_{j_k}^{\prime j_m}$ for all $m > k$. On the other hand, in view of commutativity of the diagram we have

$$s_{i_k}^{i_{m+1}} = (p_k + q_k)s_{j_k}^{\prime j_m}(p'_m + q'_m), \tag{30}$$

so $(p_k + q_k)s_{j_k}^{\prime j_m}$ divides $s_{i_k}^{i_{m+1}}$. Therefore by Proposition 2.23,

$$\Pi(\mathcal{S}_{i_k}) = (p_k + q_k)\Pi(\mathcal{S}'_{j_k}), \tag{31}$$

where $\mathcal{S}_{i_k} = (s_{i_k}, s_{i_k+1}, \dots)$, $\mathcal{S}'_{j_k} = (s'_{j_k}, s'_{j_k+1}, \dots)$. It follows that $\Pi(\mathcal{S}) \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{S}')$, so $(\mathcal{A}_2)$ holds.

Finally, if δ and δ' are nonzero (dense or pure sequences), then by (28) and (29), $s_1^{i_k}$ divides $(\delta/\delta')s_1^{\prime j_k}$ and $(\delta/\delta')s_1^{\prime j_k}$ divides $s_1^{i_{k+1}}$ for any k. Therefore by Proposition 2.23, $\Pi(\mathcal{S}) = (\delta/\delta')\Pi(\mathcal{S}')$, and $(\mathcal{A}_3)$ holds.

Assume now that $X = X' = \mathbf{A}$. By Corollary 2.21, one can write down equalities for "differences" similar to (26) and (30):

$$c_{i_k}^{i_m}(p_m - q_m) = (p_k - q_k)c_{j_k}^{\prime j_m}; \tag{32}$$

$$c_{i_k}^{i_{m+1}} = (p_k - q_k)c_{j_k}^{\prime j_m}(p'_m - q'_m). \tag{33}$$

If $\mathcal{T}'$ is symmetric, then by definition, for each k there exists m such that $c_{j_k}^{\prime j_m} = 0$. It follows from (33) that $c_{i_k}^{i_{m+1}} = 0$, so $\mathcal{T}$ is symmetric. Therefore, $\mathcal{T}$ is symmetric if and only if $\mathcal{T}'$ is so. Assume that $\mathcal{T}$ is non-symmetric. Recall that in this case one can suppose that all c_i and c'_j are nonzero. Dividing (33) by (30), we get

$$\frac{c_{i_k}^{i_{m+1}}}{s_{i_k}^{i_{m+1}}} = \frac{(p_k - q_k)}{(p_k + q_k)} \frac{c_{j_k}^{\prime j_m}}{s_{j_k}^{\prime j_m}} \frac{(p'_m - q'_m)}{(p'_m + q'_m)}, \tag{34}$$

or equivalently,

$$\sigma_1^{i_{m+1}} \cdot \frac{s_1^{i_k}}{c_1^{i_k}} = \sigma_1^{\prime j_m} \cdot \frac{(p_k - q_k)}{(p_k + q_k)} \frac{s_1^{\prime j_k}}{c_1^{\prime j_k}} \frac{(p'_m - q'_m)}{(p'_m + q'_m)}. \tag{35}$$

Taking $m \to \infty$, we get

$$\sigma \cdot \frac{s_1^{i_k}}{c_1^{i_k}} \le \sigma' \cdot \frac{(p_k - q_k)}{(p_k + q_k)} \frac{s_1^{\prime j_k}}{c_1^{\prime j_k}}. \tag{36}$$

Similarly, dividing (32) by (26) and taking $m \to \infty$, we get

$$\sigma \cdot \frac{s_1^{i_k}}{c_1^{i_k}} \ge \sigma' \cdot \frac{(p_k - q_k)}{(p_k + q_k)} \frac{s_1^{\prime j_k}}{c_1^{\prime j_k}}. \tag{37}$$

Combining with (36), we obtain

$$\sigma \cdot \frac{s_1^{i_k}}{c_1^{i_k}} = \sigma' \cdot \frac{(p_k - q_k)}{(p_k + q_k)} \frac{s_1^{\prime j_k}}{c_1^{\prime j_k}}. \tag{38}$$

It follows that $\sigma = 0$ if and only if $\sigma' = 0$. That is, $\mathcal{T}$ is weakly non-symmetric if and only if $\mathcal{T}'$ is so. Assume that $\mathcal{T}'$ is one-sided. Then $\sigma' = \sigma_1'^{j_m}$ for some m. Subtracting (38) from (35), we have $0 \le (\sigma_1^{i_{m+1}} - \sigma) s_1^{i_k}/c_1^{i_k} \le 0$. Therefore $\sigma_1^{i_{m+1}} = \sigma$, i.e. $\mathcal{T}$ is one-sided. So $(\mathcal{B}_1)$ holds.

Similarly to (31), one can get

$$\Pi(\mathcal{C}_{i_k}) = (p_k - q_k)\Pi(\mathcal{C}'_{j_k}). \tag{39}$$

It follows that $\Pi(\mathcal{C}) \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{C}')$, so $(\mathcal{B}_2)$ holds.

Assume now that $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric, i.e. $\sigma \neq 0$ and $\sigma' \neq 0$. Set $\alpha = (p_k + q_k) s_1^{i_k}/s_1'^{j_k}$. Then (38) can be rewritten in the form

$$\frac{\sigma}{\sigma'} \alpha c_1'^{j_k} = (p_k - q_k) c_1^{i_k}. \tag{40}$$

Observe that $\alpha \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$. Indeed, using (31), we have

$$\alpha \Pi(\mathcal{S}') = (p_k + q_k) s_1^{i_k} \Pi(\mathcal{S}'_{j_k}) = s_1^{i_k} \Pi(\mathcal{S}_{i_k}) = \Pi(\mathcal{S}).$$

Moreover, if $\mathcal{T}$ and $\mathcal{T}'$ are dense or pure, then by (28), $\alpha = \delta/\delta'$. It follows from (40) and (39) that

$$\frac{\sigma}{\sigma'} \alpha \Pi(\mathcal{C}') = (p_k - q_k) c_1^{i_k} \Pi(\mathcal{C}'_{j_k}) = c_1^{i_k} \Pi(\mathcal{C}_{i_k}) = \Pi(\mathcal{C}).$$

Therefore, $\frac{\sigma}{\sigma'}\alpha \in \frac{\Pi(\mathcal{C})}{\Pi(\mathcal{C}')}$. This proves $(\mathcal{B}_3)$. $\square$

To prove the sufficiency in Theorem 4.1, we need the following lemma.

Lemma 4.2 *Let $\mathcal{T}$ and $\mathcal{T}'$ satisfy the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$, $(\mathcal{B}_1), (\mathcal{B}_2)$, $(\mathcal{B}_3)$ of the theorem. Fix $\alpha \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$ ($\alpha = \delta/\delta'$ if $\mathcal{T}$ and $\mathcal{T}'$ are dense or pure), $\beta \in \frac{\Pi(\mathcal{C})}{\Pi(\mathcal{C}')}$ for the case of two-sided non-symmetric triple sequences ($\beta/\alpha = \sigma/\sigma'$ if $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric). Let i, j, a, b be integers such that*

(a) $\alpha s_1'^j = a s_1^i$,

(b) $\beta c_1'^j = b c_1^i$ (for two-sided non-symmetric $\mathcal{T}$ and $\mathcal{T}'$).

Then there exists $k > i$ such that $a' = s_i^k/a$ and $b' = c_i^k/b$ are integers of the same parity (a' is even and $c_i^k = 0$ for the case of symmetric $\mathcal{T}$ and $\mathcal{T}'$), $a' \ge b'$ and $n_k \ge a' n'_j$.

Proof. If otherwise is not specified we assume that $\mathcal{T}$ and $\mathcal{T}'$ are two-sided non-symmetric. The case of one-sided and symmetric sequences can be settled by removing from the proof the arguments with c_i, β, b.

Since $\alpha \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$ and $\alpha s_1'^j = a s_1^i$, we have

$$\Pi(\mathcal{S}_i) = \alpha (s_1^i)^{-1} s_1'^j \Pi(\mathcal{S}'_j) = a \Pi(\mathcal{S}'_j). \tag{41}$$

Similarly, we get

$$\Pi(\mathcal{C}_i) = b \Pi(\mathcal{C}'_j). \tag{42}$$

Therefore there exists $k_1 > i$ such that $a' = s_i^k/a$ and $b' = c_i^k/b$ are integers for all $k \ge k_1$. Since for each m the integers $s'_m = l'_m + r'_m$ and $c'_m = l'_m - r'_m$ have the same parity, 2 divides $\Pi(\mathcal{S}'_j)$ if and only if 2 divides $\Pi(\mathcal{C}'_j)$ (for symmetric sequences 2 divides $\Pi(\mathcal{S}'_j)$ always). Therefore by

(41) and (42), there exists $k_2 \geq k_1$ such that the integers a' and b' have the same parity (a' is even and $c_i^k = 0$ for the case of symmetric $\mathcal{T}$ and $\mathcal{T}'$) for all $k \geq k_2$. Set $\gamma_k = b'/a'$. In view of (a) and (b), we have

$$\gamma_k = \frac{c_i^k}{b} \cdot \frac{a}{s_i^k} = \frac{c_1^i c_i^k}{\beta c_1'^j} \cdot \frac{\alpha s_1'^j}{s_1^i s_i^k} = \frac{\alpha}{\beta} \cdot \frac{\sigma_1^k}{\sigma_1'^j}.$$

If $\mathcal{T}$ and $\mathcal{T}'$ are weakly non-symmetric, then $\sigma_1^k \to 0$ as $k \to \infty$, so $\gamma_k \to 0$. If $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric, then by assumption $\beta/\alpha = \sigma/\sigma'$, so

$$\gamma_k \to \frac{\alpha}{\beta} \cdot \frac{\sigma}{\sigma_1'^j} = \frac{\sigma'}{\sigma_1'^j} < 1$$

as $k \to \infty$. In both cases there exists $k_3 \geq k_2$ such that $\gamma_k \leq 1$ (i.e. $a' \geq b'$) for all $k \geq k_3$.

Set $\nu_k = n_k/a' - n_j'$. We have to show that $\nu_k \geq 0$ for sufficiently large k. One has

$$\nu_k = \frac{n_k}{a'} - n_j' = \frac{s_1^k}{a'\delta_k} - \frac{s_1'^j}{\delta_j'} = \frac{a s_1^i}{\delta_k} - \frac{s_1'^j}{\delta_j'} = s_1'^j(\frac{\alpha}{\delta_k} - \frac{1}{\delta_j'}).$$

(The last equality follows from (a).) If $\mathcal{T}$ and $\mathcal{T}'$ are sparse, then $\delta_k \to 0$ as $k \to \infty$, so $\nu_k \to +\infty$. Therefore there exists $k_4 \geq k_3$ such that $\nu_k \geq 0$ for all $k \geq k_4$. Let $\mathcal{T}$ and $\mathcal{T}'$ be dense. Then $\alpha = \delta/\delta'$ and $\delta_j' > \delta'$. Therefore

$$\nu_k = s_1'^j(\frac{\delta}{\delta_k} \cdot \frac{1}{\delta'} - \frac{1}{\delta_j'}) \to s_1'^j(\frac{1}{\delta'} - \frac{1}{\delta_j'}) > 0,$$

as $k \to \infty$. Hence there exists $k_4 \geq k_3$ such that $\nu_k \geq 0$ for all $k \geq k_4$. Let $\mathcal{T}$ and $\mathcal{T}'$ be pure. Then there exists $k_4 \geq k_3$ such that $\delta = \delta_k$ for all $k \geq k_4$. Therefore

$$\nu_k = s_1'^j(\frac{1}{\delta'} - \frac{1}{\delta_j'}) \geq 0,$$

for all $k \geq k_4$. So each $k \geq k_4$ satisfies the assumptions of the theorem. □

Proof of sufficiency in Theorem 4.1. According to Proposition 3.3 we have to construct sequences $i_1 < i_2 < \dots$, $j_1 < j_2 < \dots$, and embeddings $\varepsilon_k : A_{i_k} \to A'_{j_k}$, $\varepsilon'_k : A'_{j_k} \to A_{i_{k+1}}$ ($k = 1, 2, \dots$) such that the diagram (23) is commutative. Fix $\alpha \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$ ($\alpha = \delta/\delta'$ if $\mathcal{T}$ and $\mathcal{T}'$ are dense or pure) and $\beta \in \frac{\Pi(\mathcal{C})}{\Pi(\mathcal{C}')}$ for the case of two-sided non-symmetric triple sequences ($\beta/\alpha = \sigma/\sigma'$ if $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric). Fix also $j_0 \in J$. Since $\Pi(\mathcal{S}') = \alpha^{-1}\Pi(\mathcal{S})$ and $\Pi(\mathcal{C}') = \beta^{-1}\Pi(\mathcal{C})$, by Proposition 2.23, there exists $i_1 \in I$ such that

(a_0) $\alpha^{-1} s_1^{i_1} = a_0 s_1'^{j_0}$,

(b_0) $\beta^{-1} c_1^{i_1} = b_0 c_1'^{j_0}$ (for two-sided non-symmetric $\mathcal{T}$ and $\mathcal{T}'$)

where $a_0, b_0 \in \mathbb{N}$. Applying Lemma 4.2 (interchanging $\mathcal{T}$ and $\mathcal{T}'$), we find j_1 such that $a_1 = s_{j_0}'^{j_1}/a_0$ and $b_1 = c_{j_0}'^{j_1}/b_0$ are integers of the same parity (a_1 is even if $\mathcal{T}$ and $\mathcal{T}'$ are symmetric), $a_1 \geq b_1$ and $n'_{j_1} \geq a_1 n_{i_1}$. Set $p_1 = (a_1 + b_1)/2$, $q_1 = (a_1 - b_1)/2$, $u_1 = n'_{j_1} - a_1 n_{i_1}$ ($p_1 = q_1 = a_1/2$ for symmetric sequences). Consider the canonical embedding $\varepsilon_1 : A_{i_1} \to A'_{j_1}$ with the signature (p_1, q_1, u_1). We have

(a_1) $\alpha s_1'^{j_1} = a_1 s_1^{i_1}$,

$(b_1)\quad \beta c_1^{\prime j_1} = b_1 c_1^{i_1}$.

Proceed by induction. Assume that sequences $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ and embeddings $\varepsilon_1, \varepsilon_1', \ldots, \varepsilon_k$ have been constructed, and the following conditions hold.

$(a_k)\quad \alpha s_1^{\prime j_k} = a_k s_1^{i_k}$,

$(b_k)\quad \beta c_1^{\prime j_k} = b_k c_1^{i_k}$

where $a_k = p_k + q_k$, $b_k = p_k - q_k$. Construct an embedding ε_k' as follows. By Lemma 4.2, there exists $i_{k+1} > i_k$ such that $a_k' = s_{i_k}^{i_{k+1}}/a_k$ and $b_k' = c_{i_k}^{i_{k+1}}/b_k$ are integers of the same parity (a_k' is even if $\mathcal{T}$ and $\mathcal{T}'$ are symmetric), $a_k' \geq b_k'$ and $n_{i_{k+1}} \geq a_k' n_{j_k}'$. Set $p_k' = (a_k' + b_k')/2$, $q_k' = (a_k' - b_k')/2$, $u_k' = n_{i_{k+1}} - a_k' n_{j_k}'$ ($p_k' = q_k' = a_k'/2$ for symmetric sequences). Since

$$\begin{aligned}(p_k + q_k)(p_k' + q_k') &= a_k a_k' = s_{i_k}^{i_{k+1}}, \\ (p_k - q_k)(p_k' - q_k') &= b_k b_k' = c_{i_k}^{i_{k+1}},\end{aligned}$$

and $u_k' \geq 0$, by Lemma 2.22 there exists an embedding $\varepsilon_k' : A_{j_k}' \to A_{i_{k+1}}$ such that $\iota_k = \varepsilon_k' \varepsilon_k$ where ι_k denotes the embedding $A_{i_k} \to A_{i_{k+1}}$. Observe that

$(a_k')\quad \alpha^{-1} s_1^{i_{k+1}} = a_k' s_1^{\prime j_k}$,

$(b_k')\quad \beta^{-1} c_1^{i_{k+1}} = b_k' s_1^{\prime j_k}$.

Therefore Lemma 4.2 can be applied once more (interchanging $\mathcal{T}$ and $\mathcal{T}'$). So the result follows by induction. $\square$

Remark 4.3 It is not difficult to see that for pure triple sequences one can always assume that all $z_i = 0$ (by removing a finite number of terms in the sequences). In this case $\delta = 1$, so the condition $(\mathcal{A}_3)$ can be rewritten in the form $\Pi(\mathcal{S}) = \Pi(\mathcal{S}')$.

5 Isomorphisms of algebras of different types

In this section we find conditions under which $A(\mathcal{T}, X) \cong A(\mathcal{T}', X')$ where $\mathcal{T}$ and $\mathcal{T}'$ are triple sequences and $X \neq X'$. We also give a general parametrization of countable locally involution simple algebras.

Lemma 5.1 *Let $\mathcal{T}$ be a two-sided symmetric triple sequence, $\mathcal{S} = \mathcal{S}(\mathcal{T})$. Then 2^∞ divides $\Pi(\mathcal{S})$.*

Proof. By definition, $l_i = r_i$ (in particular, $s_i = l_i + r_i$ is even) for an infinite set of i. Therefore 2^∞ divides $\Pi(\mathcal{S})$. $\square$

Theorem 5.2 *Let $\mathcal{T}$, $\mathcal{T}'$ be triple sequences.*

(i) Let $\operatorname{char} \mathbb{F} \neq 2$. Then $A(\mathcal{T}, \mathbf{A}) \cong A(\mathcal{T}', \mathbf{O})$ (resp., $A(\mathcal{T}, \mathbf{A}) \cong A(\mathcal{T}', \mathbf{S})$) if and only if $\mathcal{T}$ is two-sided symmetric, 2^∞ divides $\Pi(\mathcal{S}')$ and the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ of Theorem 4.1 hold.

(ii) Let $\operatorname{char} \mathbb{F} \neq 2$. $A(\mathcal{T}, \mathbf{O}) \cong A(\mathcal{T}', \mathbf{S})$ if and only if 2^∞ divides both $\Pi(\mathcal{S})$ and $\Pi(\mathcal{S}')$, and the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ of Theorem 4.1 hold.

(iii) Let $\operatorname{char} \mathbb{F} = 2$. If $X, X' \in \{\mathbf{A}, \mathbf{O}, \mathbf{S}\}$ are different, then $A(\mathcal{T}, X)$ is not isomorphic to $A(\mathcal{T}', X')$.

Proof. (i). Set $A = A(\mathcal{T}, \mathbf{A})$ and $A' = A(\mathcal{T}', \mathbf{O})$. Assume that $A \cong A'$. Then, as it was established in the proof of Theorem 4.1, the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ hold. Now denote by (x_k, y_k, z_k) the signature of $A_{i_k} \to A_{i_{k+1}}$ (see diagram (23)). Since the diagram is commutative, we have $x_k = p_k p'_k$ and $y_k = q_k p'_k$ where (p_k, q_k, u_k) and $(p'_k, 0, u'_k)$ are the signatures of ε_k and ε'_k, respectively. By Proposition 2.17(i), $p_k = q_k$, so $x_k = y_k$. Therefore $c_{i_k}^{i_{k+1}} = x_k - y_k = 0$, so $\mathcal{T}$ is two-sided symmetric. By Lemma 5.1, 2^∞ divides $\Pi(\mathcal{S})$. Therefore in view of condition $(\mathcal{A}_2)$, 2^∞ divides $\Pi(\mathcal{S}')$.

Conversely. Let $A = A(\mathcal{T}, \mathbf{A})$ and $A' = A(\mathcal{T}', \mathbf{O})$ be such that $\mathcal{T}$ is two-sided symmetric, 2^∞ divides $\Pi(\mathcal{S}')$ and the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ hold. Then there exists a sequence of indices $j_1 < j_2 < \dots$ such that $s'^{j_{k+1}}_{j_k}$ is even for all $k = 1, 2, \dots$. By Proposition 2.17(iii), there exists an algebra A''_k of type $\mathbf{A}$ and representable embeddings $A'_{j_k} \to A''_k$ and $A''_k \to A'_{j_{k+1}}$ such that the diagram

$$\begin{array}{ccc} A'_{j_k} & \longrightarrow & A'_{j_{k+1}} \\ & \searrow \quad \nearrow & \\ & A''_k & \end{array}$$

is commutative. Set $A'' = \varinjlim A''_k$. Let $\mathcal{T}''$ be the corresponding triple sequence. We have $A'' = A(\mathcal{T}'', \mathbf{A})$. By construction, $A'' \cong A'$. Moreover, by the above arguments (the proof of necessity) $\mathcal{T}''$ is symmetric and the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ (for $\mathcal{T}'$ and $\mathcal{T}''$) hold. Since the same is true for the pair $\mathcal{T}$, $\mathcal{T}'$, we conclude that the pair $\mathcal{T}$, $\mathcal{T}''$ also satisfies these conditions. Indeed, $(\mathcal{A}_1)$ trivially holds. Further, since $\Pi(\mathcal{S}') \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{S}'')$ and $\Pi(\mathcal{S}) \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{S}')$, we have $\Pi(\mathcal{S}) \overset{\mathbb{Q}}{\sim} \Pi(\mathcal{S}'')$. Finally, if $\frac{\delta'}{\delta''} \in \frac{\Pi(\mathcal{S}')}{\Pi(\mathcal{S}'')}$ and $\frac{\delta}{\delta'} \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}')}$, then

$$\Pi(\mathcal{S}') = \frac{\delta'}{\delta''}\Pi(\mathcal{S}'') = \frac{\delta'}{\delta}\Pi(\mathcal{S}),$$

so $\frac{\delta}{\delta''} \in \frac{\Pi(\mathcal{S})}{\Pi(\mathcal{S}'')}$. Consequently, by Theorem 4.1, $A(\mathcal{T}, \mathbf{A}) \cong A(\mathcal{T}'', \mathbf{A})$, i.e. $A \cong A''$. Therefore $A \cong A'$. The proof for the case $A' = A(\mathcal{T}', \mathbf{S})$ is similar.

(ii). Let $A(\mathcal{T}, \mathbf{O}) \cong A(\mathcal{T}', \mathbf{S})$. Using Proposition 2.17 $(ii), (iii)$, it is not difficult to construct an algebra $A(\mathcal{T}'', \mathbf{A}) \cong A(\mathcal{T}, \mathbf{O}) \cong A(\mathcal{T}', \mathbf{S})$. The claim now follows from Theorem 5.2 (i). To prove the converse statement we construct $A(\mathcal{T}'', \mathbf{A})$ isomorphic to $A(\mathcal{T}, \mathbf{O})$ and use Theorem 5.2 (i).

(iii). By definition of $A(\mathcal{T}, X)$, all the corresponding embeddings are representable. Thus the claim follows from Proposition 2.19. $\square$

It remains to discuss the general parametrization. Let A be a locally involution simple associative algebra over $\mathbb{F}$ of countable dimension. Then by Theorem 3.2, A is canonically representable, i.e. A is the direct limit of a sequence $(A_i)_{i \in \mathbb{N}}$ of subalgebras of the same type $X = \mathbf{A}$, $\mathbf{O}$, or $\mathbf{S}$ such that all embeddings are canonical. Fix any such system of subalgebras. This gives the triple sequence $\mathcal{T} = ((l_i, r_i, z_i))_{i \in \mathbb{N}}$, the sequences of "sums" $\mathcal{S} = (l_i + r_i)_{i \in \mathbb{N}}$ and (for $X = \mathbf{A}$ only) "differences" $\mathcal{C} = (l_i - r_i)_{i \in \mathbb{N}}$. Now we can determine the density type D=(D1), (D2) or (D3), the density index $\delta = \delta(\mathcal{T})$, supernatural number $\Pi_{\mathcal{S}} = \Pi(\mathcal{S})$, and (for $X = \mathbf{A}$ only) the symmetry type S=(S1), (S2), (S3), or (S4), the symmetry index $\sigma = \sigma(\mathcal{T})$, and supernatural number $\Pi_{\mathcal{C}} = \Pi(\mathcal{C})$. So one can associate with any algebra A a tuple

$$\mathcal{P}(A) = (X, \mathrm{D}, \mathrm{S}, \delta, \sigma, \Pi_{\mathcal{S}}, \Pi_{\mathcal{C}})$$

where $X, \mathrm{D}, \mathrm{S}$ describe a type of A; δ and σ are real numbers ($0 \leq \delta, \sigma \leq 1$); $\Pi_{\mathcal{S}}$ and $\Pi_{\mathcal{C}}$ are supernatural numbers. For $X = \mathbf{S}, \mathbf{O}$ (and $X = \mathbf{A}$ with one-sided or symmetric $\mathcal{T}$) we use a shorter list of invariants:

$$A \mapsto (X, \mathrm{D}, \delta, \Pi_{\mathcal{S}}).$$

By Theorem 4.1, the tuples associated with two nonisomorphic algebras are distinct. The question under what conditions A and A' with tuples $\mathcal{P}(A)$ and $\mathcal{P}(A')$ are isomorphic has been resolved in Theorems 4.1 and 5.2.

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