Finitary simple Lie algebras

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Abstract

An algebra is called finitary if it consists of finite-rank transformations of a vector space. We classify finitary simple Lie algebras over a field of zero characteristic. We also describe finitary irreducible Lie algebras.

1 Introduction

Let V be a vector space over a field F. An element $x \in \mathrm{End}_F V$ is called *finitary* if $\dim xV < \infty$. The finitary transformations of V form an ideal $\mathfrak{fgl}(V)$ of the Lie algebra $\mathfrak{gl}(V)$ of all linear transformations of V. A Lie algebra is called *finitary* if it is isomorphic to a subalgebra of $\mathfrak{fgl}(V)$ for some V. The aim of this paper is to classify infinite dimensional finitary simple Lie algebras over a field of characteristic 0. We also describe finitary irreducible Lie algebras. Since any simple Lie algebra can be considered as an algebra over its centroid and the latter is central simple, one can restrict ourselves to the case of central simple Lie algebras. We prove the following main theorem.

Theorem 1.1 *Let F be a field of characteristic 0. Then any infinite dimensional central finitary simple Lie algebra over F is isomorphic to one of the following algebras:*

(1) a finitary special linear algebra $\mathfrak{fsl}(V, \Pi)$;

(2) a finitary special unitary algebra $\mathfrak{fsu}(V, \Phi^\sigma)$;

(3) a finitary orthogonal algebra $\mathfrak{fo}(V, \Psi^\sigma)$;

(4) a finitary symplectic algebra $\mathfrak{fsp}(V, \Theta^\sigma)$.

Here V is a right vector space over a finite dimensional division F-algebra Δ; Δ is central, except in the case (2) where the center of Δ is a quadratic extension of F; σ is an involution (an antiautomorphism of order 1 or 2) of Δ such that F is the set of central σ-invariant elements of Δ; Φ^σ, Ψ^σ, and Θ^σ are nondegenerate forms on V; and Π is a total subspace of the dual V^*.

The definition of the algebras (1)–(4) (for arbitrary F) is given in Section 6. They are central finitary simple Lie algebras. Note that Theorem 1.1 is similar to Hall's classification [9] of simple locally finite groups of finitary linear transformations.

Theorem 8.2 describes finitary modules for the algebras (1)–(4). It turns out that any such module is a direct sum of a trivial submodule, modules isomorphic to V, and (for the case of finitary special linear algebras only) modules isomorphic to Π.

If F is algebraically closed, then there are no nontrivial finite dimensional division algebras, so we have the following corollary.

Corollary 1.2 *Let F be an algebraically closed field of characteristic 0. Then any infinite dimensional finitary simple Lie algebra over F is isomorphic to one of the following:*

(1) a finitary special linear algebra $\mathfrak{fsl}(V,\Pi)$;

(2) a finitary orthogonal algebra $\mathfrak{fo}(V,\Psi)$;

(3) a finitary symplectic algebra $\mathfrak{fsp}(V,\Theta)$.

Here V is a vector space over F; Ψ (resp. Θ) is a nondegenerate symmetric (resp. skew-symmetric) form on V; and Π is a total subspace of the dual V^*.

Corollary 1.2 have been proven in [5]. Note that there we use the notation $\mathfrak{t}(V,\Pi)$ instead of $\mathfrak{fsl}(V,\Pi)$. It easily follows from Corollary 1.2 (see also [4, Theorem 1.3]) that any finitary simple Lie algebra of (infinite) countable dimension over an algebraically closed field F of characteristic 0 is isomorphic to one of the following algebras: $\mathfrak{sl}_\infty(F)$, $\mathfrak{o}_\infty(F)$, and $\mathfrak{sp}_\infty(F)$. These algebras can be represented as the direct limits of natural embeddings of the relevant classical simple Lie algebras.

Assume now that $F=\mathbb{R}$ is the field of real numbers. By Frobenius theorem, there are only three finite dimensional real division algebras: $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, and the algebra of quaternions $\mathbb{H}$.

Theorem 1.3 *Any real infinite dimensional central finitary simple Lie algebra is isomorphic to one of the following algebras.*

$\Delta=\mathbb{R}$: $\mathfrak{fsl}(V,\Pi)$, $\mathfrak{fo}(V,\Psi)$, $\mathfrak{fsp}(V,\Theta)$.

$\Delta=\mathbb{C}$: $\mathfrak{fsu}(V,\Phi^\sigma)$.

$\Delta=\mathbb{H}$: $\mathfrak{fsl}(V,\Pi)$, $\mathfrak{fo}(V,\Psi^\sigma)$, $\mathfrak{fsp}(V,\Theta^\sigma)$.

Here V is a right vector space over $\Delta=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$; σ is the standard involution in $\Delta=\mathbb{C},\mathbb{H}$; Ψ, Θ, Φ^σ, Ψ^σ, and Θ^σ are nondegenerate forms; Π is a total subspace of the dual V^*.

For the case of countable dimension one can get the complete classification. Denote by $M_n(F)$ the algebra of $n\times n$ matrices over $F=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$; and by $M_\infty(F)=\cup_{n=1}^\infty M_n(F)$ the algebra of infinite matrices with finite numbers of nonzero entries. Let $\delta\mapsto\bar\delta$ be the standard involution in $F=\mathbb{C},\mathbb{H}$. For $X\in M_\infty(F)$ set $X^*=\bar X^t$ where X^t is the matrix transpose to X. Set

$$
\begin{aligned}
J &= \operatorname{diag}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix},\dots,\begin{pmatrix}0&1\\-1&0\end{pmatrix},\dots\right);\\
I_k &= \operatorname{diag}(\underbrace{-1,\dots,-1}_{k},1,1,\dots),\quad k=0,1,2,\dots;\\
I_\infty &= \operatorname{diag}(-1,1,-1,1,\dots,-1,1,\dots).
\end{aligned}
$$

The algebras

(1) $\mathfrak{sl}_\infty(\mathbb{C})=\{X\in M_\infty(\mathbb{C})\mid \operatorname{tr}X=0\}$,

(2) $\mathfrak{o}_\infty(\mathbb{C})=\{X\in M_\infty(\mathbb{C})\mid X^t+X=0\}$,

(3) $\mathfrak{sp}_\infty(\mathbb{C}) = \{X \in M_\infty(\mathbb{C}) \mid X^t J + JX = 0\}$

are finitary and simple as $\mathbb{R}$-algebras, and any other real finitary simple Lie algebra of countable dimension is central (see Proposition 7.4).

Theorem 1.4 *The algebras*

(1) $\mathfrak{sl}_\infty(\mathbb{R}) = \{X \in M_\infty(\mathbb{R}) \mid \operatorname{tr} X = 0\}$,

(2) $\mathfrak{o}_\infty(\mathbb{R}, k) = \{X \in M_\infty(\mathbb{R}) \mid X^t I_k + I_k X = 0\}$, $k = 0, 1, \ldots, \infty$,

(3) $\mathfrak{sp}_\infty(\mathbb{R}) = \{X \in M_\infty(\mathbb{R}) \mid X^t J + JX = 0\}$,

(4) $\mathfrak{su}_\infty(\mathbb{C}, k) = \{X \in M_\infty(\mathbb{C}) \mid \operatorname{tr} X = 0,\ X^* I_k + I_k X = 0\}$, $k = 0, 1, \ldots, \infty$,

(5) $\mathfrak{sl}_\infty(\mathbb{H}) = \{X \in M_\infty(\mathbb{H}) \mid \operatorname{tr}(X + X^*) = 0\}$,

(6) $\mathfrak{o}_\infty(\mathbb{H}, k) = \{X \in M_\infty(\mathbb{H}) \mid X^* I_k + I_k X = 0\}$, $k = 0, 1, \ldots, \infty$,

(7) $\mathfrak{sp}_\infty(\mathbb{H}) = \{X \in M_\infty(\mathbb{H}) \mid X^* J + JX = 0\}$

are exactly all, pairwise nonisomorphic, central real finitary simple Lie algebras of countable dimension.

All these algebras can be constructed as the direct limits of natural embeddings of relevant classical real simple Lie algebras. Note that the algebras $\mathfrak{o}_\infty(\mathbb{H}, k)$ and $\mathfrak{sp}_\infty(\mathbb{H})$ are often denoted as $\mathfrak{sp}_\infty(\mathbb{H}, k) = \varinjlim \mathfrak{sp}_n(\mathbb{H}, k)$ and $\mathfrak{u}^*_\infty(\mathbb{H}) = \varinjlim \mathfrak{u}^*_n(\mathbb{H})$, respectively.

The algebras (1)–(7) from Theorem 1.4 as well as the algebras $\mathfrak{sl}_\infty(\mathbb{C})$, $\mathfrak{o}_\infty(\mathbb{C})$, and $\mathfrak{sp}_\infty(\mathbb{C})$ often appear in the literature. For instance, Bahturin and Benkart [1] described their weight and root lattices. Natarajan [16] studied unitary questions for their highest weight modules. Olshanskii [17] investigated representations of the corresponding infinite dimensional classical groups. Finitary simple Lie algebras form a natural subclass of so-called diagonal locally finite Lie algebras. The latter are studied in [3, 4, 6], see also Zalesskii's survey [18].

Theorem 1.1 is certainly true for any field F of positive odd characteristic. So we would like to state the following conjecture.

Conjecture 1.5 *Theorem 1.1 holds for any field F of characteristic $\neq 2$.*

In the last section we classify finitary irreducible Lie algebras. The theorem below holds for any field of characteristic $\neq 2$ provided Theorem 1.1 and Theorem 8.2 (for irreducible finitary modules) are valid in the case of positive odd characteristic. In the proof we use recent results of Leinen and Puglisi [13, 14, 15] for finitary irreducible Lie algebras.

Theorem 1.6 *Let F be a field of characteristic 0 and and $\mathcal{L} \subseteq \mathfrak{fgl}(V)$ be an infinite dimensional finitary irreducible Lie algebra over F. Let Δ be the centralizer of the $\mathcal{L}$-module V. Then Δ is a finite dimensional division F-algebra, V can be considered as a right vector space over Δ, $[\mathcal{L}, \mathcal{L}]$ is simple and one of the following holds:*

(1) $[\mathcal{L}, \mathcal{L}] = \mathfrak{fsl}(V, \Pi) \subseteq \mathcal{L} \subseteq \mathfrak{fgl}(V, \Pi)$;

(2) $[\mathcal{L}, \mathcal{L}] = \mathfrak{fsu}(V, \Phi^\sigma) \subseteq \mathcal{L} \subseteq \mathfrak{fu}(V, \Phi^\sigma)$;

(3) $[\mathcal{L},\mathcal{L}] = \mathcal{L} = \mathfrak{fo}(V,\Psi^\sigma)$*;*

(4) $[\mathcal{L},\mathcal{L}] = \mathcal{L} = \mathfrak{fsp}(V,\Theta^\sigma)$*.*

Moreover, if $[\mathcal{L},\mathcal{L}]$ *is central simple, then either* $\mathcal{L} = [\mathcal{L},\mathcal{L}]$ *is central simple or* $\mathcal{L} = \mathfrak{fgl}(V,\Pi)$ *or* $\mathfrak{fu}(V,\Phi^\sigma)$*.*

Notation

Throughout the paper F is the ground field; $\operatorname{char} F = 0$ except in Sections 5, 6, and 10 where F is arbitrary; P is a field extension of F; $\bar{F}$ is the algebraic closure of F; Δ is a finite dimensional division algebra over F; and Γ is the center of Δ, $[\Gamma : F] = 1, 2$. Let V be a vector space over F. We shall denote $V_P \cong V \otimes_F P$ and $\bar{V} = V_{\bar{F}} = V \otimes_F \bar{F}$. We identify V with the subset $V \otimes_F 1$ in V_P. Unless otherwise stated, modules considered in the paper are assumed to be left. Let A be an algebra and M be a (left) A-module. The *centralizer* Δ of M is the algebra $(\operatorname{End}_A M)^{op}$, so M can be considered as a right Δ-module. In view of this, vector spaces over skew fields are usually assumed to be right in the paper. For a finite dimensional Lie algebra L we denote by $L^{(\infty)}$ the smallest member of the derived series of L. Note that $L^{(\infty)}$ is *perfect*, i.e. $[L,L] = L$. All finitary (or, more generally, locally finite Lie algebras) are assumed to be infinite dimensional. $A(L)$ is the *augmentation* ideal of the universal enveloping algebra $U(L)$ for a Lie algebra L. This is the ideal of codimension 1 generated by all $x \in L$.

2 Finite dimensional central simple Lie algebras

The aim of this section is to recall Jacobson's classification of finite dimensional central simple Lie algebras of classical type over a field of characteristic 0 [12, Ch. X] and to prove some auxiliary lemmas. First of all we need to recall the notion of a centroid. Let A be an arbitrary (not necessary associative or Lie) algebra. The algebra of multiplications $\mathcal{T}(A)$ is the subalgebra of $\operatorname{End} A$ generated by all left and right multiplications $l_a : x \mapsto ax$ and $r_a : x \mapsto xa$. The *centroid* Γ of A is the centralizer of $\mathcal{T}(A)$ in the algebra $\operatorname{End} A$. If A is simple, then Γ is a field and A can be represented as a Γ-algebra. Moreover, the Γ-algebra A is *central*, that is, its centroid coincides with Γ. We shall often use the following well known fact (see [12, Theorem 10.1.3].

Proposition 2.1 *Let A be a central simple algebra over F and let P be a field extension of F. Then A_P is central simple.*

Throughout the rest of the section all algebras considered are finite dimensional.

Let L be a central simple Lie algebra over a field F. Then by Proposition 2.1, $\bar{L}$ is a simple Lie algebra over $\bar{F}$. Since $\bar{F}$ is algebraically closed, we know all simple Lie algebras over $\bar{F}$:

$$A_n\ (n \geq 1),\ B_n\ (n \geq 2),\ C_n\ (n \geq 3),\ D_n\ (n \geq 4),\ F_4,\ G_2,\ E_6,\ E_7,\ \text{and}\ E_8.$$

If $\bar{L}$ is in this list, we say that L is of type X_n ($X = A, B, \dots$). Usually the subscript n will be omitted and we shall say that L is of type A, of type B, and so on. It is often convenient do not distinguish types B and D, so we shall say that L is of type BD if it is of type either B or D. The algebras of types A, B, C, and D are called *classical*. Note that by Jacobson's classification the algebras of type A should be subdivided into two subclasses: A^I and A^{II}. To expose this classification we need to recall some properties of finite dimensional central simple associative algebras.

Let A be a finite dimensional central simple algebra over a field Γ of characteristic 0, i.e. the center of A coincides with Γ. Then by Wedderburn theorem, A can be identified with an algebra $M_n(\Delta)$ of all $n \times n$ matrices over a skew field Δ. Moreover, Δ is a finite dimensional central division algebra over Γ. Considering $M_n(\Delta)$ as a Lie algebra under the usual commutator, we get the algebra $\mathfrak{gl}_n(\Delta)$. Its commutant

$$\mathfrak{sl}_n(\Delta) = [\mathfrak{gl}_n(\Delta), \mathfrak{gl}_n(\Delta)] = \{x \in \mathfrak{gl}_n(\Delta) \mid \operatorname{tr} x \in [\Delta, \Delta]\}$$

is a central simple Lie Γ-algebra of type A^I (or, as we shall also denote, $A^I(\Delta)$).

Assume now that the ring A has an involution α, i.e. an antiautomorphism of order 1 or 2. The set F of α-invariant elements of Γ is a subfield. If $F = \Gamma$, then α is an involution of the *first kind*. Otherwise, Γ is a quadratic extension of F and α is of the *second kind*. The set

$$\mathfrak{h}^\alpha(A) = \{a \in A \mid a^\alpha = -a\}$$

of skew-Hermitian elements of A is a Lie F-algebra under the usual commutator. If α is of the first kind, then $\mathfrak{h}^\alpha(A)$ is a central simple Lie algebra of type either $B(\Delta)$, or $C(\Delta)$, or $D(\Delta)$. If α is of the second kind, then the commutant

$$\mathfrak{sh}^\alpha(A) = [\mathfrak{h}^\alpha(A), \mathfrak{h}^\alpha(A)] = \{x \in \mathfrak{h}^\alpha(A) \mid \operatorname{tr} x \in [\Delta, \Delta]\}$$

is a central simple Lie F-algebra of type $A^{II}(\Delta)$.

Let L be a central simple Lie algebra. Recall that a field $P \supseteq F$ is called *splitting* for L if L_P is *split*, i.e. the set of diagonal matrices in the adjoint representation for some choice of a basis in L_P is a Cartan subalgebra of L_P. Note that the algebraic closure $\bar{F}$ is a splitting field for L. It is also known that L_P has the same root system and representation theory as $\bar{L}$. Since always there exists a finite normal field extension P of L such that L_P is split, we often prefer to work with L_P instead of $\bar{L}$.

Let now L be classical and P be a splitting field for L. Let us fix a splitting Cartan subalgebra of L_P and a base $\alpha_1, \dots, \alpha_n$ of the root system. Let $\omega_1, \dots, \omega_n$ be the relevant fundamental weights. The irreducible L_P-module of highest weight ω_1 is called *standard*. We say that an L_P-module is *natural*, if it is either standard or dual to it. One can check that the latter notion does not depend on the choice of a splitting Cartan subalgebra and a base of the root system.

We shall always identify L with the subset $L \otimes_F 1_P$ in L_P. So any L_P-module W can be regarded as the L-module $W{\downarrow}L$ (over the field F). The following is well known.

Lemma 2.2 *Let V be an irreducible L_P-module. Then $V{\downarrow}L$ is a sum of isomorphic irreducible L-modules.*

Proof. Fix any irreducible submodule M of $V{\downarrow}L$. Then $\bar{F}M = V$. It remains to note that $\bar{F}M$ is a sum of L-modules isomorphic to M. □

In view of Lemma 2.2, one can make the following definition. An irreducible L-module is called *standard* (resp. *natural*) if it is isomorphic to a submodule of the restriction of the standard (resp. natural) $\bar{L}$-module to L. We have $L \subseteq L_P \subseteq \bar{L}$ for any algebraic field extension P of F. Since L is central, L_P is simple. One can easily see that a simple L-module V is standard (resp. natural) if and only if it is isomorphic to a submodule of the restriction of the standard (resp. natural) L_P-module to L. One can also check that an L-module is natural if and only if it is either standard or dual to standard.

Let L be a central simple Lie algebra over F and let P be a finite normal extension of F such that L_P is a split Lie algebra. Assume that L_P is of type either A_n $(n \geq 1)$, or B_n $(n \geq 2)$, or C_n $(n \geq 3)$, or D_n $(n \geq 5)$ (the case D_4 is exceptional). Denote by W the standard L_P-module and by E the associative F-algebra generated by L in $\operatorname{End} W$. Note that by Lemma 2.2, $W{\downarrow}L$ is a sum of isomorphic irreducible L-modules, so E is simple. The following can be derived from Jacobson's results. Assume that L is of type A. Then either E is central simple and $L = [E, E]$ (type A^I) or the center $\Gamma \subseteq P$ of E is a quadratic extension of F, there exists an involution α of E of the second kind, and $L = \mathfrak{sh}^\alpha(E)$ (type A^{II}). If L is of type B, C, or D, then E is central simple, there exists an involution α of E of the first kind, and $L = \mathfrak{h}^\alpha(E)$.

Let L be a Lie algebra and A be an associative enveloping algebra of L, that is, a quotient of the *augmentation* ideal $A(L)$ (the ideal generated by all $x \in L$) of the universal enveloping algebra $U(L)$. (We prefer to work with $A(L)$ instead of $U(L)$ and do not require the existence of the identity in A.) Denote by I the kernel of the natural homomorphism $A(L) \to A$. Assume that A has an involution α. We say that α is *compatible* if $x^\alpha = -x$ for all $x \in L$. Since A is generated by L, there exists at least one compatible involution of A. One easily checks that $A(L)$ has a compatible involution (we denote it by α). Assume that A has a compatible involution. Then it is clear that $I^\alpha = I$ and this involution is inherited from $A(L)$. Conversely, if $I^\alpha = I$, then A inherits a compatible involution. We summarize our results in the following lemma.

Lemma 2.3 *Let L be a finite dimensional central simple Lie algebra over F. Assume that $\bar{L}$ is classical and* $\operatorname{rk} \bar{L} > 4$. *Let V be the standard $\bar{L}$-module. Denote by E the F-algebra generated by $L \subseteq \operatorname{End} V$ and by I the annihilator of V in $A(L)$. Then E is simple and one of the following holds.*

(i) *L is of type A^I, E is central simple, and $L = [E, E]$;*

(ii) *L is of type A^{II}, $I^\alpha = I$, the center Γ of E is a quadratic extension of F, F is the set of α-invariant elements of Γ, and $L = \mathfrak{sh}^\alpha(E)$;*

(iii) *L is of type B, C, or D; $I^\alpha = I$, E is central simple, and $L = \mathfrak{h}^\alpha(E)$.*

Since E is simple, by Wedderburn theorem, there exists a (unique) number k and a (unique) skew field $\Delta = \Delta(L)$ such that E is isomorphic to the matrix algebra $M_k(\Delta)$. Moreover, the center Γ of E coincides with the center of Δ. Recall that $\Gamma = F$ if L is not of type A^{II} and Γ is a quadratic extension of F if L is of type A^{II}. Recall also that the dimension of any finite dimensional central simple associative algebra is a square.

Proposition 2.4 *Let L be a classical central simple Lie algebra over F and V be the standard L-module. Let n be the dimension of the standard $\bar{L}$-module, and $l^2 = [\Delta(L) : \Gamma]$. Then* $\dim V = 2ln$ *if L is of type A^{II}, and* $\dim V = ln$, *otherwise.*

Proof. We have $E \otimes_\Gamma \bar{F} \cong M_n(\bar{F})$, so $[E(L) : \Gamma] = [E \otimes_\Gamma \bar{F} : \bar{F}] = n^2$. On the other hand $E(L) \cong M_k(\Delta)$ for some k. Comparing dimensions of these Γ-algebras, we conclude that $n = kl$. Since V is the standard $M_k(\Delta)$-module, V can be considered as a right Δ-module and $[V : \Delta] = k = n/l$. It remains to note that $[V : F] = [V : \Delta][\Delta : \Gamma][\Gamma : F]$. □

We say that an embedding of classical split simple Lie algebras (over the same field) is *natural* if the restriction of the standard Q_2-module to Q_1 contains a unique nontrivial composition factor and the latter is natural. Clearly, this implies that Q_1 and Q_2 have the same type A, BD, or C.

Lemma 2.5 *Let $L_1 \subset L_2$ be classical central simple Lie algebras of the same type (A^I, A^{II}, BD, or C). Assume that $\bar{L}_1 \subset \bar{L}_2$ is a natural embedding. Then the following conditions are equivalent.*

(1) $\Delta(L_1) = \Delta(L_2)$;

(2) the restriction of a natural L_2-module to L_1 contains a unique nontrivial composition factor.

Proof. Let P be a finite normal extension of F such that L_{1P} and L_{2P} are split Lie algebras. Clearly, the embedding $L_{1P} \subset L_{2P}$ is natural. Set $r = [P : F]$. Let W_2 be a natural L_{2P}-module and W_1 be a natural submodule of $W_2{\downarrow}L_{1P}$. Set $n_i = \dim_P W_i$. Clearly, n_i is the dimension of the standard $\bar{L}_i$-module. Let l_i^2 be the dimension of $\Delta(L_i)$ over its center Γ_i. Then by Proposition 2.4, the dimension of a natural L_i-module is either $2l_in_i$ or l_in_i, $i = 1, 2$. It follows that $W_i{\downarrow}L_i$ is a sum of $r/2l_i$ (or r/l_i) natural L_i-modules. Assume that (1) holds. Then $l_1 = l_2$. Therefore the restriction of a natural L_2-module to L_1 contains exactly one nontrivial composition factor and the latter is natural, so (2) holds. Conversely, assume that (2) holds. Then obviously, $l_1 = l_2$. Let V_2 be a natural L_2-module and V_1 be the natural submodule of $V_2{\downarrow}L_1$. Denote by E_i the algebra generated by L_i in $\operatorname{End} V_i$. We identify E_1 with the corresponding subalgebra in E_2. Recall that $E_i \cong M_{k_i}(\Delta_i)$ for some k_i, and $\Delta_i = \Delta(L_i)$ is the centralizer of the E_i-module V_i. Therefore V_i can be considered as a right vector space over Δ_i. Recall that the action of Δ_2 commutes with the action of E_2. Therefore for each $\delta \in \Delta_2$, the space $V_1\delta$ is a E_1-submodule of V_2 isomorphic to V_1. In view of uniqueness, $V_1\Delta_2 = V_1$. It follows that Δ_2 centralizes E_1 in $\operatorname{End} V_1$, so $\Delta_2 \subseteq \Delta_1$. Now since $\dim \Delta_2 = \dim \Delta_1$, we have $\Delta_2 = \Delta_1$, as required. □

Definition 2.6 An embedding $L_1 \subseteq L_2$ of classical central simple Lie algebras of the same type (A^I, A^{II}, BD, or C) is called *natural* if the embedding $\bar{L}_1 \subseteq \bar{L}_2$ is natural and one of the equivalent conditions (1) and (2) of Lemma 2.5 holds.

Lemma 2.7 *Let $L_1 \subseteq L_2$ be central simple Lie algebras of type A. Assume that L_1 has type A^{II}, $\bar{L}_1 \subseteq \bar{L}_2$ is a natural embedding, and the restriction of a natural L_2-module to L_1 contains a unique nontrivial composition factor. Then L_2 is of type A^{II}.*

Proof. We use the notation in the proof of Lemma 2.5. Assume that L_2 is of type A^I. Then we have $r/2l_1 = r/l_2$, so $l_2 = 2l_1 > l_1$. On the other hand, we have as above $\Delta_2 \subseteq \Delta_1$, i.e. $l_2 \le l_1$. The contradiction obtained proves the theorem. □

Lemma 2.8 *Let $L_1 \subseteq M \subseteq L_2$ be finite dimensional simple Lie algebras. Assume that L_1 and L_2 are central simple Lie algebras of the same type X ($= A^I$, A^{II}, BD, or C), the embedding $L_1 \subseteq L_2$ is natural, and $\operatorname{rk} \bar{L}_1 > 10$. Then M is a central simple Lie algebra of type X and the embeddings $L_1 \subseteq M$ and $M \subseteq L_2$ are natural.*

Proof. The central simplicity of M follows from Lemma 3.3 below, so $\bar{M}$ is simple. By [4, Lemma 5.2], the embeddings $\bar{L}_1 \subseteq \bar{M}$ and $\bar{M} \subseteq \bar{L}_2$ are natural, so $\bar{M}$ has the same type as $\bar{L}_1$ and $\bar{L}_2$ (A, BD, or C). Observe that the restrictions of a natural L_2-module to M and of a natural M-module to L_1 have unique nontrivial composition factors. Since L_1 and L_2 are of the same type X, it follows from Lemma 2.7 that M is also of type X. Therefore the embeddings $L_1 \subseteq M$ and $M \subseteq L_2$ are natural. □

3 Some lemmas on embeddings

In this section we prove some auxiliary lemmas on embeddings of Lie algebras. All Lie algebras considered are finite dimensional. The following is well known (see, for instance, [7, 1.5.6].

Lemma 3.1 *Let L be a Lie algebra over F and let S be a Levi subalgebra of L. Then* $\operatorname{Rad}\bar{L} = \overline{\operatorname{Rad} L}$ *and $\bar{S}$ is a Levi subalgebra of $\bar{L}$.*

Lemma 3.2 *Let M be a simple Lie algebra over F. Then*

(i) *$\bar{M}$ is semisimple;*

(ii) *$\bar{M}$ is simple if and only if M is central simple;*

(iii) *the projection of any nonzero element $x \in M$ to each simple component of $\bar{M}$ is nonzero.*

Proof. (i) By Lemma 3.1, $\operatorname{Rad}\bar{M} = \overline{\operatorname{Rad} M} = 0$, so $\bar{M}$ is semisimple.

(ii) This follows from [12, Theorem 10.1.3].

(iii) Let $M_1, \ldots, M_k$ be the simple components of $\bar{M}$. Assume that the projection of x to M_1 is zero. Then $x \in \bar{M}' = M_2 \oplus \cdots \oplus M_k$. Note that $\bar{M}'$ is an ideal in $\bar{M}$, so $\bar{M}' \cap M$ is a nontrivial ideal in M. Since M is simple, we have $M \subseteq \bar{M}'$, so $\bar{M} \subseteq \bar{M}'$. The contradiction obtained proves the lemma. □

Let L be a perfect Lie algebra over F. Let $R = \operatorname{Rad} L$ and $S = S_1 \oplus \cdots \oplus S_k$ be a Levi subalgebra of L where $S_1, \ldots, S_k$ are the simple components of S.

Lemma 3.3 *Assume that there exist simple Lie algebras Q_1 and Q_2 of the same type over $\bar{F}$ such that $Q_1 \subseteq \bar{L} \subseteq Q_2$ and the embedding $Q_1 \subseteq Q_2$ is natural. Then there exists i such that $Q_1 \subseteq \bar{S}_i \oplus \bar{R}$. Moreover, if $Q_1 \cap L \neq 0$, then S_i is central simple.*

Proof. By Levi-Malcev theorem, there exists a Levi subalgebra T of $\bar{L}$ such that $Q_1 \subseteq T \cong \bar{S}$. Let $T_1, \ldots, T_n$ be the simple components of T. Note that Q_1 is a simple submodule of the Q_1-module Q_2 under the adjoint action. Moreover, since the embedding $Q_1 \subseteq Q_2$ is natural, Q_2 has a unique composition factor isomorphic to Q_1. Since Q_1 is a submodule of $T = T_1 \oplus \cdots \oplus T_n \subseteq Q_2$, we have $Q_1 \subseteq T_j$ for some j. It follows that $Q_1 \subseteq T_j \oplus \bar{R} = Q \oplus \bar{R}$ where Q is a simple component of some $\bar{S}_i$. Assume now that there is a nonzero $x \in Q_1 \cap L$. Since Q_1 is simple, $x \notin \operatorname{Rad} L$, so x has a nonzero projection x_1 to S_i. Therefore by Lemma 3.2(iii), x_1 (and x) has a nonzero projection to each simple component of $\bar{S}_i$. Since $x \in Q \oplus \bar{R}$, we have $Q = \bar{S}_i$, so by Lemma 3.2(ii), S_i is central simple. □

The following lemma for the case of algebraically closed fields have been proved in [4, Lemma 4.6].

Lemma 3.4 *Let $L_1 \subseteq L_2$ be perfect subalgebras of a simple Lie algebra S_3 such that $L_1 \cap \operatorname{Rad} L_2 = 0$. Let also $S_1 \subseteq S_2$ be Levi subalgebras of L_1, L_2, respectively. Assume that the algebras S_1, S_2, and S_3 are central simple and $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \bar{S}_3$ are natural embeddings of classical simple Lie algebras of the same type (A, B, C, or D). Then there exists a Levi subalgebra S_2' of L_2 such that $L_1 \subseteq S_2'$.*

Proof. Set $R_i = \operatorname{Rad} L_i$ and $R'_i = [R_i, R_i]$ for $i = 1, 2$. One can assume that $R_2 \neq 0$. Recall that for each element $r \in R_2$ one can define a (special) automorphism θ_r of L_2 via

$$\theta_r = e^{\operatorname{ad} r} = 1 + \operatorname{ad} r + \frac{1}{2}(\operatorname{ad} r)^2 + \cdots + \frac{1}{n!}(\operatorname{ad} r)^n + \ldots$$

Clearly, θ_r permutes Levi subalgebras. Moreover, by Levi-Malcev theorem, any Levi subalgebra of L_2 has the form $\theta_r(S_2)$ for some $r \in R_2$. We shall prove that there exists a special automorphism θ of L_2 such that $\theta(L_1) \subseteq S_2 \oplus R'_2$. This will imply our assertion. Indeed, note that $\operatorname{Rad}(S_2 \oplus R'_2)^{(\infty)} \subseteq R'_2 \subset R_2$. Therefore applying induction on $\dim R_2$ to the sequence

$$\theta(L_1) \subseteq (S_2 \oplus R'_2)^{(\infty)} \subseteq S_3,$$

we find θ_1 such that $\theta_1(\theta(L_1)) \subseteq S_2$. Hence $L_1 \subseteq S'_2 = \theta^{-1}(\theta_1^{-1}(S_2))$, as required.

We have $\bar{L}_i = \bar{S}_i \oplus \bar{R}_i$. Note also that $[\bar{R}_2, \bar{R}_2] = \bar{R}'_2$ and $\bar{L}_1 \cap \bar{R}_2 = 0$. Since the embeddings $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \bar{S}_3$ are natural, by [4, Lemma 4.6], there exists $r \in \bar{R}_2$ such that $\theta_r(\bar{L}_1) \subseteq \bar{S}_2$. In particular, $\theta_r(L_1) \subseteq \bar{S}_2$. Each element $l \in L_1$ can be uniquely represented in the form $l = s_l + n_l$ where $s_l \in S_2$ and $n_l \in R_2$. We have

$$\theta_r(l) = s_l + n_l + [r, s_l] + r'$$

where $r' \in \bar{R}'_2$. It follows that $n_l + [r, s_l] \in \bar{R}'_2$ for all $l \in L_1$. Represent r in the form $r = r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n$ where $r_1, \ldots, r_n \in R_2$ and $\{1, \alpha_2, \ldots, \alpha_n\}$ are linearly independent over F elements of $\bar{F}$. Then we have $n_l + [r_1, s_l] \in R'_2$ for all $l \in L_1$. Therefore

$$\theta_{r_1}(l) = s_l + n_l + [r_1, s_l] + \cdots \in s_l + R'_2 \text{ for all } l \in L_1,$$

so $\theta_{r_1}(L_1) \subseteq S_2 + R'_2$, as required. The lemma follows. □

Lemma 3.5 *Let $L_1 \subset L_2$ be a natural embedding of classical simple Lie algebras of the same type of ranks $m \geq 4$ and n, respectively, over an algebraically closed field. Let V be a nontrivial irreducible L_2-module different from the standard one and dual to it. Then* $\dim L_1 V \geq [(n-m)/2]$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ and $\omega_1, \ldots, \omega_n$ be the simple roots and the fundamental weights of L_2; $\alpha_{n-m+1}, \ldots, \alpha_n$ and $\omega_{n-m+1}, \ldots, \omega_n$ be those of L_1. Let $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ be the highest weight of V. By assumptions, $\lambda \neq \omega_1$ and for type A, $\lambda \neq \omega_1, \omega_n$. For a root α denote by X_α the corresponding root element of L_2. Set $s = [(n-m)/2] - 1$. We shall prove that there exist weights $\mu_0, \ldots, \mu_s$ of V and $i \geq n-m+1$ such that $\langle \mu_k, \alpha_i \rangle > 0$ for $0 \leq k \leq s$. (Here $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ where $(\ , \ )$ is the standard nondegenerate symmetric bilinear form on the weight system associated with the Killing form.) This will imply the lemma. Indeed, fix any nonzero vector v_k of weight μ_k. Since $\langle \mu_k, \alpha_i \rangle > 0$, we have $w_k = X_{-\alpha_i} v_k \neq 0$ (see, for instance, [8, Proposition 8.7.3(iii)]). As all w_k are in different weight spaces, they are linearly independent, so $\dim L_1 V \geq s + 1 = [(n-m)/2]$, as required.

We have use the following fact (see [8, Corollary 8.7.1]): if μ is a weight of V and $\langle \mu, \alpha_j \rangle > 0$, then $\mu - \alpha_j$ is a weight of V. Set $\alpha_{ii} = \alpha_i$, $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i < j$, and $\alpha_{ij} = \alpha_i + \alpha_{i-1} + \cdots + \alpha_j$ for $i > j$. It easily follows from the above fact that if $a_i \neq 0$, then $\lambda - \alpha_{ij}$ is a weight of V for any $1 \leq j \leq n$, except possibly in the cases when L_2 is of type D_n and either $i = n$ or $j = n$. Consider the following cases.

Case 1: $a_i \neq 0$ for some $n-m+1 \leq i < n$. Then $\mu_k = \lambda - \alpha_{i,i-1-k}$, $0 \leq k \leq s$, are weights of V and $\langle \mu_k, \alpha_{i+1} \rangle > 0$ for all k, as required.

Case 2: $a_n \neq 0$. Set $\mu_k = \lambda - \alpha_{n,n-3-k}$, $0 \leq k \leq s$. Then for L of types B_n, C_n, and D_n for all k we have $\langle \mu_k, \alpha_n \rangle > 0$, $\langle \mu_k, \alpha_{n-1} \rangle > 0$, and $\langle \mu_k, \alpha_{n-2} \rangle > 0$, respectively, as required. Assume that L is of type A_n. If $a_n \geq 2$, then we have $\langle \mu_k, \alpha_n \rangle > 0$, as desired. So one can suppose that $a_n = 1$. Then there exists $i < n$ such that $a_i > 0$. For $k = 0, 1, \ldots, s$ set $\mu_k = \lambda - \alpha_{i,i-k}$ if $i > s$, and $\mu_k = \lambda - \alpha_{i,i+k}$, otherwise. (Note that in the latter case $i + k \leq 2s < n - m$.) Then $\mu_0, \ldots, \mu_s$ are weights of V and $\langle \mu_k, \alpha_n \rangle > 0$ for all k, as required.

Case 3: $a_i \neq 0$ for some $i < n - m + 1$, and $a_{i+1} = \cdots = a_n = 0$. Then $\mu = \lambda - \alpha_{i,n-m}$ is a weight of V. Assume that $i > 1$. Then $\mu = \lambda + \omega_{i-1} - \omega_i - \omega_{n-m} + \omega_{n-m+1}$. In particular, $\langle \mu, \alpha_{i-1} \rangle > 0$. Set $\mu_k = \mu - \alpha_{i-1,i-1-k}$ if $i > s + 1$, and $\mu_k = \mu - \alpha_{i-1,i-1+k}$, otherwise. (Note that in the latter case $i - 1 + k \leq 2s < n - m$.) Then $\mu_0, \ldots, \mu_s$ are weights of V and $\langle \mu_k, \alpha_{n-m+1} \rangle > 0$ for all k, as required. Assume now that $i = 1$. Then $a_1 \geq 2$ and $\mu = \lambda - \omega_1 - \omega_{n-m} + \omega_{n-m+1}$. Since $\langle \mu, \alpha_1 \rangle > 0$, we have that $\mu_k = \mu - \alpha_{1,1+k}$, $0 \leq k \leq s$, are weights of V. It remains to note that $\langle \mu_k, \alpha_{n-m+1} \rangle > 0$ for all k, as required. □

4 Natural local systems

Note that any finitary Lie algebra is *locally finite*, that is, each finitely generated subalgebra is finite dimensional. Let $\mathcal{L}$ be a locally finite Lie algebra. A set $\mathfrak{A}$ of finite dimensional subalgebras of $\mathcal{L}$ is called a *local system* if $\mathcal{L} = \cup_{L \in \mathfrak{A}} L$ and for each pair $L, M \in \mathfrak{A}$ there is $Q \in \mathfrak{A}$ such that $L, M \subseteq Q$. The latter property says that $\mathfrak{A}$ is a *directed* set. Observe that $\mathcal{L}$ is isomorphic to the corresponding direct limit of algebras from $\mathfrak{A}$. In this section we construct "natural" local systems for finitary simple Lie algebras. First of all we recall some well known facts on local systems (see, for example, [2]).

Lemma 4.1 *Let $\mathfrak{A}$ be a local system of a simple locally finite Lie algebra $\mathcal{L}$ and let Q be a finite dimensional subspace of $\mathcal{L}$. Then the following hold.*

(i) *$\{L^{(\infty)} \mid L \in \mathfrak{A}\}$ is a local system of $\mathcal{L}$ consisting of perfect Lie algebras.*

(ii) *$\{L \mid L \in \mathfrak{A},\ Q \subseteq L\}$ is a local system of $\mathcal{L}$.*

(iii) *There exists $L \in \mathfrak{A}$ and a maximal ideal M of L such that $Q \subseteq L$ and $Q \cap M = 0$.*

(iv) *If $\mathfrak{A} = \mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k$, then one of the $\mathfrak{N}_i$ is a local system of $\mathcal{L}$.*

Definition 4.2 A local system $\mathfrak{N}$ of a finitary central simple Lie algebra $\mathcal{L}$ is called *natural* (of type $X(\Delta) = A^I(\Delta), A^{II}(\Delta), BD(\Delta)$, or $C(\Delta)$) if all $L \in \mathfrak{N}$ are classical central simple Lie algebras of the same type $X(\Delta)$ naturally embedded each into another. We say that $\mathcal{L}$ is *of type* $X(\Delta)$ if $\mathcal{L}$ has a natural local system of type $X(\Delta)$.

The following proposition shows that the type of a central finitary simple Lie algebra is determined uniquely.

Proposition 4.3 *Let $\mathcal{L}$ be a central finitary simple Lie algebra having natural local systems $\mathfrak{N}_1$ and $\mathfrak{N}_2$ of types $X_1(\Delta_1)$ and $X_2(\Delta_2)$, respectively. Then $X_1 = X_2$ and $\Delta_1 = \Delta_2$.*

Proof. Fix any $N_1 \in \mathfrak{N}_1$ such that $\operatorname{rk} \bar{N}_1 > 10$. Find $N_2 \in \mathfrak{N}_2$ and $N_1' \in \mathfrak{N}_1$ such that $N_1 \subset N_2 \subset N_1'$. Then by Lemma 2.8, $X_1 = X_2$ and the embedding $N_1 \subset N_2$ is natural. By definition, this implies that $\Delta_1 = \Delta_2$, as required. □

Theorem 4.4 *Let $\mathcal{L} \subset \mathfrak{fgl}(V)$ be a finitary central simple Lie algebra. Then $\mathcal{L}$ possesses a natural local system $\mathfrak{N}$. Moreover, there exist submodules W^0, W^1, and W^2 of V such that*

(1) $V = W^0 \oplus W^1 \oplus W^2$;

(2) $\mathcal{L}W^0 = 0$, $\mathcal{L}W^1 = W^1$, and $\mathcal{L}W^2 = W^2$;

(3) for each $L \in \mathfrak{N}$, LW^i $(i = 1, 2)$ is a finite direct some of isomorphic natural L-modules;

(4) if $W^2 \neq 0$, then there exists $L \in \mathfrak{N}$ such that the natural submodules of LW^1 are not isomorphic to those of LW^2; in particular, $W^2 = 0$ whenever $\mathcal{L}$ is not of type A^I.

Proof. Let $\mathfrak{A}$ be a local system of $\mathcal{L}$. Set $\bar{\mathcal{L}} = \mathcal{L} \otimes_F \bar{F}$ and $\bar{V} = V \otimes_F \bar{F}$. Clearly, $\bar{V}$ is a finitary $\bar{\mathcal{L}}$-module, so $\bar{\mathcal{L}}$ is finitary. Since $\mathcal{L}$ is central, by Proposition 2.1, $\bar{\mathcal{L}}$ is simple. Therefore by [4, Theorem 5.1], there exists a natural local system $\mathfrak{T}$ of $\bar{\mathcal{L}}$. Moreover, all algebras in $\mathfrak{T}$ are of type either A, or B, or C. Fix any nonzero element $x \in \mathcal{L}$. By Lemma 4.1(ii), one can assume that all algebras from $\mathfrak{A}$ and $\mathfrak{T}$ contain x. Fix any $Q \in \mathfrak{T}$. Since $\bar{\mathfrak{A}} = \{\bar{L} \mid L \in \mathfrak{A}\}$ and $\mathfrak{T}$ are local systems of $\bar{\mathcal{L}}$, there exist $L_Q \in \mathfrak{A}$ and $Q' \in \mathfrak{T}$ such that $Q \subseteq \bar{L}_Q \subseteq Q'$. As $Q \cap L_Q \ni x \neq 0$, by Lemma 3.3, there exists a subalgebra M_Q of L_Q such that $Q \subseteq \bar{M}_Q$ and $M_Q/\operatorname{Rad} M_Q$ is central simple. Replacing M_Q by $M_Q^{(\infty)}$, one can assume that M_Q is perfect. Note that for each $L \in \mathfrak{A}$ there exists $Q \in \mathfrak{T}$ such that $L \subseteq Q \subseteq \bar{M}_Q$, so $L \subseteq M_Q$. Therefore $\mathfrak{M} = \{M_Q \mid Q \in \mathfrak{T}\}$ is a local system of $\mathcal{L}$. Fix any $M_1 \in \mathfrak{M}$. By Lemma 4.1(iii), there exists $M_2, M_3 \in \mathfrak{M}$ such that $M_1 \subseteq M_2 \subseteq M_3$ and $M_i \cap \operatorname{Rad} M_{i+1} = 0$ for $i = 1, 2$. We have

$$Q \subseteq \bar{M}_1 \subseteq \bar{M}_2 \subseteq \bar{M}_3 \subseteq Q'$$

for some $Q, Q' \in \mathfrak{T}$. By Levi-Malcev theorem, there exists a Levi subalgebra P_1 of $\bar{M}_1$ such that $Q \subseteq P_1$. Similarly, one can choose Levi subalgebras P_2 of $\bar{M}_2$ and P_3 of $\bar{M}_3$ such that we have $Q \subseteq P_1 \subseteq P_2 \subseteq P_3 \subseteq Q'$. Note that P_1, P_2, and P_3 are simple. By Lemma 2.8, the algebras Q, Q', P_1, P_2, and P_3 are of the same type (A, BD, or C) and all the embeddings are natural. Now fix Levi subalgebras S_i of M_i such that $S_1 \subseteq S_2 \subseteq S_3$. Note that the corresponding embeddings $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \bar{S}_3$ are natural. Identifying M_1 and M_2 with their images in $M_3/\operatorname{Rad} M_3 \cong S_3$, and applying Lemma 3.4, one can find a Levi subalgebra S_2' of M_2 containing M_1. It follows that each $M \in \mathfrak{M}$ is contained in a central simple subalgebra S_M of $\mathcal{L}$. Therefore $\mathfrak{S} = \{S_M \mid M \in \mathfrak{M}\}$ is a local system of $\mathcal{L}$. Note that all algebras from $\mathfrak{S}$ are central simple. Moreover, by Lemma 4.1(iv), one can assume that they have the same type: A^I, A^{II}, BD, or C. Note also that for all $L, L' \in \mathfrak{S}$ with $L \subseteq L'$ the embedding $\bar{L} \subseteq \bar{L}'$ is natural.

Fix any $L \in \mathfrak{S}$. Let m be the rank of $\bar{L}$ and $s = \dim \bar{L}\bar{V}$. Find $Q \in \mathfrak{S}$ such that $L \subset Q$ and the rank of $\bar{Q}$ is greater than $2(s+1)+m$. Then it follows from the Lemma 3.5 that all nontrivial composition factors of the $\bar{Q}$-module $\bar{Q}\bar{V}$ are natural. Since the embedding $\bar{L} \subset \bar{Q}$ is natural, all nontrivial composition factors of the $\bar{L}$-module $\bar{L}\bar{V} \subseteq \bar{Q}\bar{V}$ are natural. We now show that $\bar{L}\bar{V} = \bar{L}^2\bar{V}$. Indeed, otherwise there exists $v \in \bar{V}$ such that $\bar{L}v \not\subseteq \bar{L}^2\bar{V}$. Therefore $(\bar{F}v + \bar{L}\bar{V})/\bar{L}^2\bar{V}$ is a nontrivial $\bar{L}$-module. But this is not the case because this module is completely reducible and all composition factors are trivial. It follows that the $\bar{L}$-module $\bar{L}\bar{V} = \bar{L}^2\bar{V}$ has no trivial composition factors, so $\bar{L}\bar{V}$ is a sum of natural $\bar{L}$-modules. Note that LV is an L-submodule of $\bar{L}\bar{V}$. Therefore LV is a sum of natural L-modules. It follows that LV is a sum of natural L-modules for each $L \in \mathfrak{S}$.

Let $L = L_1 \subset L_2 \subset \cdots \subset L_k$ be a chain of algebras from $\mathfrak{S}$ such that $\Delta(L_i) \neq \Delta(L_{i+1})$ for $i = 1, \ldots, k-1$. Then by Lemma 2.5, the restriction of a natural L_{i+1}-module to L_i contains at least two nontrivial (natural) composition factors. It follows that the restriction of a natural L_k-module to L contains a direct sum of at least 2^{k-1} natural L-modules. Since $\dim LV$ is fixed, we conclude that k is bounded. Therefore one can assume that our chain is maximal. Hence for each $N \in \mathfrak{S}$ with $N \supseteq L_k$ we have $\Delta(N) = \Delta(L_k)$, so the embedding $L_k \subseteq N$ is natural. Now by Lemma 4.1(ii), $\mathfrak{N} = \{N \in \mathfrak{S} \mid L_k \subseteq N\}$ is a natural local system of $\mathcal{L}$.

Fix $N \in \mathfrak{N}$ such that there are nonisomorphic natural submodules in NV. If such an algebra does not exist, fix arbitrary $N \in \mathfrak{N}$. We have a unique decomposition $V = W_N^0 \oplus W_N^1 \oplus W_N^2$ where $NW_N^0 = 0$, $W_N^1 \oplus W_N^2 = NV$, W_N^i is a finite direct sum of isomorphic natural N-modules for $i = 1, 2$, and if $W_N^2 \neq 0$ then the irreducible submodules of W_N^1 are not isomorphic to those of W_N^2. For each algebra $M \supset N$ in $\mathfrak{N}$ we have a similar decomposition $V = W_M^0 \oplus W_M^1 \oplus W_M^2$. Moreover, one can assume that $W_N^i \subseteq W_M^i$; $W_M^i{\downarrow}N \subseteq W_N^0 \oplus W_N^i$ for $i = 1, 2$; and $W_M^0 \subseteq W_N^0$. Note that $W_M^i \subseteq W_L^i$, $i = 1, 2$, for all $M, L \in \mathfrak{N}$ with $N \subseteq M \subseteq L$. Set

$$\begin{aligned} W^i &= \cup_{M \supseteq N} W_M^i \subseteq V, \quad (i = 1, 2), \\ W^0 &= \cap_{M \supseteq N} W_M^0 \subset V. \end{aligned}$$

One easily checks that W^0, W^1, and W^2 satisfy the required properties. □

5 Simple associative rings with minimal left ideals

In this section we recall the structure of simple associative rings with minimal left ideals. Observe that any such ring possesses a faithful irreducible module, i.e. it is *primitive*. Primitive rings with minimal left ideals are described in [11, ch. IV] and we shall expose this description at the end of the section. First of all we recall some well known simple facts which will be used in Section 7. Throughout the section F is an arbitrary field.

Lemma 5.1 *Let V be a vector space over a field F and let $\mathcal{A}$ be a simple associative subalgebra of* $\operatorname{End} V$ *containing an element of finite rank. Then $\mathcal{A}$ is primitive.*

Proof. Observe that $\mathcal{A}$ is finitary and, in particular, locally finite dimensional. Let A be a non-nilpotent finite dimensional subalgebra of $\mathcal{A}$. Then there exist nontrivial composition factors in the A-module AV. Denote by M a minimal submodule of AV containing a nontrivial composition factor. Let U be a maximal submodule of the $\mathcal{A}$-module $W = \mathcal{A}M$. Set $N = U \cap M$. Since the A-module M/N contains a nontrivial composition factor, W/U is a faithful irreducible $\mathcal{A}$-module. □

Lemma 5.2 *Let V be a vector space over a field F and $\mathcal{A}$ be an irreducible associative subalgebra of* $\operatorname{End} V$*. Assume that $\mathcal{A}$ has an element of finite rank. Then $\mathcal{A}$ has a minimal left ideal.*

Proof. Since V is an irreducible A-module, the centralizer Δ of V is a division F-algebra and V can be considered as a right vector space over Δ. Let $a \in \mathcal{A}$ be an element of minimal rank n over Δ. We shall prove that the left ideal $\mathcal{A}a$ is minimal. Fix any $b = ca \in \mathcal{A}a$. It suffices to show that $xb = a$ for some $x \in \mathcal{A}$. Let $\{e_1, \dots, e_n\}$ be a Δ-basis of aV. Observe that $bV = caV = \langle ce_1, \dots, ce_n \rangle_\Delta$. Since $\operatorname{rk} b \geq \operatorname{rk} a$, the elements $ce_1, \dots, ce_n$ are linearly independent over Δ. Hence by Jacobson's Density Theorem, there exists $x \in \mathcal{A}$ such that $xce_i = e_i$ for $1 \leq i \leq n$. Therefore $(xca - a)V = 0$, so $xb = xca = a$, as required. □

Proposition 5.3 *Let V be a vector space over a field F and let $\mathcal{A}$ be a simple associative subalgebra of* $\operatorname{End} V$ *containing an element of finite rank. Then $\mathcal{A}$ is primitive with minimal left ideals.*

Proof. This follows from Lemmas 5.1 and 5.2. □

Lemma 5.4 *Let $\mathcal{A}$ be a simple associative algebra with minimal left ideals. Then $\mathcal{A}$ is a simple ring with minimal left ideals.*

Proof. This is obvious (see, for instance, [11, Proposition 1.9.2 and §5.5]). □

Let V be a right vector space over a division F-algebra Δ. Denote by V^* the set of all Δ-linear functions $\varphi : V \to \Delta$. Then V^* is a *left* vector space over Δ under the operations:

$$(\varphi + \psi)v = \varphi v + \psi v, \quad (\delta\varphi)v = \delta(\varphi v)$$

for all $v \in V$, $\varphi, \psi \in V^*$, and $\delta \in \Delta$.

Definition 5.5 A subspace Π of V^* is called *total* if $\operatorname{Ann}_V \Pi = \{v \in V \mid \varphi v = 0 \text{ for all } \varphi \in \Pi\} = 0$.

Let Π be a total subspace of V^*. Denote by $\mathfrak{F}(V, \Pi)$ the ring of all transformations $x \in \operatorname{End} V$ of the form

$$v \mapsto e_1(\varphi_1 v) + \dots + e_n(\varphi_n v)$$

where n is an integer, $e_1, \dots, e_n \in V$, and $\varphi_1, \dots, \varphi_n \in \Pi$. Set $\mathfrak{F}(V) = \mathfrak{F}(V, V^*)$. Note that $\mathfrak{F}(V)$ is the ring of all finite-rank (over Δ) transformations of V.

Theorem 5.6 ([11, §4.9 and §4.15]) *A ring $\mathcal{A}$ is isomorphic to a ring $\mathfrak{F}(V, \Pi)$ for some vector space V and total subspace Π of V^* if and only if $\mathcal{A}$ is a simple ring with minimal left ideals.*

We say that finite dimensional subspaces $V' \subset V$ and $\Pi' \subset \Pi$ are *compatible* if $\dim V' = \dim \Pi'$ and $\operatorname{Ann}_{V'} \Pi' = 0$.

Lemma 5.7 *Let Π be a total subspace of V^*. Then for any finite dimensional subspaces $V_1 \subset V$ and $\Pi_1 \subset \Pi$ there exists compatible subspaces $V' \subset V$ and $\Pi' \subset \Pi$ such that $V_1 \subseteq V'$ and $\Pi_1 \subseteq \Pi'$.*

Proof. Choose any $V_2 \subset V$ compatible with Π_1. Set $V' = V_1 + V_2$. Since $\operatorname{Ann}_V \Pi = 0$, there exists a subspace $\Pi' \supseteq \Pi_1$ of Π compatible with V'. □

Assume that V' and Π' are compatible, $k = \dim V' = \dim \Pi'$. Denote by $F(V', \Pi')$ the set of all transformations $x : v \mapsto \sum_{i=1}^n e_i(\varphi_i v)$ with $e_i \in V'$ and $\varphi_i \in \Pi'$. Then $F(V', \Pi')$ is a subalgebra of $\mathfrak{F}(V, \Pi)$ isomorphic to the algebra $M_k(\Delta)$ of all $k \times k$ matrices over Δ. We obtain the following corollary from Lemma 5.7.

Corollary 5.8 *Let Π be a total subspace of V^*. Then $\mathfrak{F}(V,\Pi)$ is the direct limit of its subalgebras $F(V_\tau,\Pi_\tau) \cong M_{k_\tau}(\Delta)$ ($k_\tau = \dim V_\tau$) where V_τ and Π_τ run over all compatible pairs of subspaces in V and Π.*

Remark 5.9 Observe that $F(V',\Pi') \subseteq F(V'',\Pi'')$ if and only if $V' \subseteq V''$ and $\Pi' \subseteq \Pi''$. Moreover, the corresponding embedding is *natural*, that is, one can identify these algebras with the matrix algebras $M_k(\Delta)$ and $M_l(\Delta)$ in such a way that any matrix $M \in M_k(\Delta)$ maps to $\mathrm{diag}(M,0,\ldots,0) \in M_l(\Delta)$.

Assume that Δ has an involution $\delta \mapsto \bar{\delta}$. A function $\Phi : V \times V \to \Delta$ is called a *Hermitian* form on V if

$$\begin{aligned} \Phi(v_1+v_2,w) &= \Phi(v_1,w)+\Phi(v_2,w), \\ \Phi(v\delta,w) &= \Phi(v,w)\delta, \\ \Phi(w,v) &= \overline{\Phi(v,w)} \end{aligned}$$

for all $v,w \in V$ and $\delta \in \Delta$. If we have $\Phi(w,v) = -\overline{\Phi(v,w)}$ instead of $\Phi(w,v) = \overline{\Phi(v,w)}$, then Φ is called *skew-Hermitian*. We say that Φ is of the *first kind*, if the involution $\delta \mapsto \bar{\delta}$ acts trivially on the center Γ of Δ, otherwise we say that Φ is of the *second kind*. We shall call Hermitian forms of the first kind *orthogonal* and skew-Hermitian those *symplectic*. We say that Φ is *unitary* if it is a Hermitian form of the second kind.

Assume that Φ is nondegenerate. Denote by Π_Φ the set of all functions $\varphi_u : v \mapsto \Phi(v,u)$. One easily checks that Π_Φ is a total subspace of V^*. Moreover, the map $u \mapsto \varphi_u$ produces an isomorphism between the left Δ-spaces V and Π_Φ (V can be considered as a left vector space setting $\delta v = v\bar{\delta}$ for $\delta \in \Delta$ and $v \in V$), so we shall denote the algebra $\mathfrak{F}(V,\Pi_\Phi)$ by $\mathfrak{F}_\Phi(V,V)$. For each $x \in \mathfrak{F}_\Phi(V,V)$ there exists a unique $x^* \in \mathfrak{F}_\Phi(V,V)$ such that $\Phi(xv,w) = \Phi(v,x^*w)$ for all $v,w \in V$. The map $x \mapsto x^*$ is called the *adjoint map with respect to* Φ.

Theorem 5.10 ([11, §4.12]) *Let Φ be a nondegenerate Hermitian or skew-Hermitian form on a vector space V. Then the adjoint map $x \mapsto x^*$ with respect to Φ is an involution of the ring $\mathfrak{F}_\Phi(V,V)$. Conversely, let $\mathcal{A}$ be a simple ring with minimal left ideals and an involution. Then there exists a vector space V with a nondegenerate Hermitian or skew-Hermitian form Φ such that $\mathcal{A}$ can be realized as the ring $\mathfrak{F}_\Phi(V,V)$ with involution given by the adjoint map with respect to Φ.*

Let $\mathcal{A}$ be a ring with involution α. Then one can easily check that the set $\mathfrak{h}^\alpha(\mathcal{A}) = \{a \in \mathcal{A} \mid a^\alpha = -a\}$ is a Lie ring under the usual commutator. Let now we have a vector space V with a Hermitian or skew-Hermitian form Φ. Then we define

$$\begin{aligned} \mathfrak{fh}(V,\Phi) &= \{x \in \mathfrak{F}(V) \mid \Phi(xv,w)+\Phi(v,xw) = 0 \text{ for all } v,w \in V\} \\ \mathfrak{fsh}(V,\Phi) &= [\mathfrak{fh}(V,\Phi),\mathfrak{fh}(V,\Phi)]. \end{aligned}$$

The following is obvious.

Proposition 5.11 *Let V be a vector space with a nondegenerate Hermitian or skew-Hermitian form Φ and α be the involution of $\mathfrak{F}_\Phi(V,V)$ given by the adjoint map with respect to Φ. Then $\mathfrak{h}^\alpha(\mathfrak{F}_\Phi(V,V)) = \mathfrak{fh}(V,\Phi)$.*

6 Finitary simple Lie algebras

In this section we introduce certain classes of finitary simple Lie algebras. Throughout below $\Gamma \subseteq F$ are arbitrary fields with $[\Gamma : F] = 1, 2$; Δ is a finite dimensional central division Γ-algebra; and V is a right vector space over Δ.

Finitary special linear algebras. Let $\Gamma = F$ and let Π be a total subspace of V^*. The *finitary general linear algebra* $\mathfrak{fgl}(V, \Pi)$ is the F-algebra $\mathfrak{F}(V, \Pi)$ under the usual Lie commutator. If $\Pi = V^*$, then we write $\mathfrak{fgl}(V)$ instead of $\mathfrak{fgl}(V, V^*)$. For each $x \in \mathfrak{F}(V)$ one can define its trace $\operatorname{tr} x \in \Delta$ as the trace of x on the finite dimensional Δ-subspace xV. The *finitary special linear algebra* $\mathfrak{fsl}(V, \Pi)$ is the set of all transformations $x \in \mathfrak{fgl}(V, \Pi)$ with $\operatorname{tr} x \in [\Delta, \Delta]$. Clearly, both $\mathfrak{fgl}(V, \Pi)$ and $\mathfrak{fsl}(V, \Pi)$ are finitary Lie algebras over F.

Assume that $k = \dim V$ is finite. Then $\Pi = V^*$, $\mathfrak{fgl}(V, \Pi) \cong \mathfrak{gl}_k(\Delta) = M_k(\Delta)$, and $\mathfrak{fsl}(V, \Pi) \cong \mathfrak{sl}_k(\Delta) = [M_k(\Delta), M_k(\Delta)]$ is a classical Lie algebra over F of type $A^I(\Delta)$. Since $\mathfrak{F}(V, \Pi) = \varinjlim F(V_\tau, \Pi_\tau)$ where V_τ and Π_τ run over all compatible pairs of subspaces in V and Π, we have $\mathfrak{fsl}(V, \Pi) = \varinjlim \mathfrak{sl}(V_\tau, \Pi_\tau)$ with each $\mathfrak{sl}(V_\tau, \Pi_\tau)$ isomorphic to $\mathfrak{sl}_{k_\tau}(\Delta)$, $k_\tau = \dim V_\tau$. Moreover, in view of Remark 5.9, the algebras $\mathfrak{sl}(V_\tau, \Pi_\tau)$ form a natural local system (we shall call it *standard*). We have $\mathfrak{fsl}(V, \Pi) \otimes_F \bar{F} = \varinjlim L(V_\tau, \Pi_\tau)$ with

$$L(V_\tau, \Pi_\tau) = \mathfrak{sl}(V_\tau, \Pi_\tau) \otimes_F \bar{F} \cong \mathfrak{sl}_{n_\tau}(\bar{F}).$$

Recall that $\mathfrak{sl}_{n_\tau}(\bar{F})$ is simple provided $\operatorname{char} F = 0$ or $\operatorname{char} F = p$ and $p \not| n_\tau$. If $p | n_\tau$, then $\mathfrak{sl}_{n_\tau}(\bar{F})$ contains an ideal $I_{n_\tau} = \bar{F} \cdot \operatorname{diag}(1, \dots, 1)$ and $\mathfrak{sl}_{n_\tau}(\bar{F})/I_{n_\tau}$ is simple except in the case $p = n_\tau = 2$ [10, Theorem 7]. Assume that $L(V_\tau, \Pi_\tau) \subset L(V_\rho, \Pi_\rho)$. Since this embedding is natural, we have $n_\tau < n_\rho$ and $I_{n_\tau} \cap I_{n_\rho} = 0$. Therefore $\mathfrak{fsl}(V, \Pi) \otimes_F \bar{F}$ is simple. It follows that $\mathfrak{fsl}(V, \Pi)$ is a central simple Lie algebra of type $A^I(\Delta)$. Note also that $\mathfrak{fsl}(V, \Pi) = [\mathfrak{fgl}(V, \Pi), \mathfrak{fgl}(V, \Pi)]$, since this is true in the case of finite dimension. We summarize the obtained results in the following proposition.

Proposition 6.1 *Let F be a field, Δ be a finite dimensional central division algebra over F, and let V be an infinite dimensional vector space over Δ. Let Π be a total subspace of V^*. Then*

(i) $\mathfrak{fsl}(V, \Pi) = [\mathfrak{F}(V, \Pi), \mathfrak{F}(V, \Pi)]$;

(ii) $\mathfrak{fsl}(V, \Pi)$ *is a central finitary simple Lie algebra over F of type $A^I(\Delta)$.*

When V has infinite dimension then V^* has uncountably infinite dimension; and since $\mathfrak{fsl}(V) = \mathfrak{fsl}(V, V^*)$ contains all transformations $t_{x\varphi} : v \mapsto x(\varphi v)$ with $\varphi v = 0$, the dimension of $\mathfrak{fsl}(V)$ is uncountably infinite. There is another finitary counterpart to the finitary special linear algebra which remains to be countably dimensional for countably dimensional V; the stable special linear algebra $\mathfrak{sl}_\infty(\Delta)$. This is best introduced in terms of matrices. Every $n \times n$ matrix M can be extended to $(n+1) \times (n+1)$ matrix M' by placing M in the upper lefthand corner of M' and then bordering M with 0's. This gives us natural embeddings

$$\mathfrak{sl}_2(\Delta) \to \mathfrak{sl}_3(\Delta) \to \dots \to \mathfrak{sl}_n(\Delta) \to \dots$$

the union of these algebras is then the *stable special linear algebra* $\mathfrak{sl}_\infty(\Delta)$ and is countably dimensional. Let V be a countably dimensional space with a basis $E = \{e_1, e_2, \dots\}$ and Π be a subspace of V^* spanned by $E^* = \{e_1^*, e_2^*, \dots\}$, the dual of the basis E. Then, clearly, $\mathfrak{fsl}(V, \Pi) \cong \mathfrak{sl}_\infty(\Delta)$.

Proposition 6.2 *The algebra* $\mathfrak{fsl}(V,\Pi)$ *has countable dimension if and only if it is isomorphic to* $\mathfrak{sl}_\infty(\Delta)$.

Proof. Set $\mathcal{L} = \mathfrak{fsl}(V,\Pi)$. Let $\{\mathfrak{sl}(V_\tau,\Pi_\tau)\}$ be the standard local system of $\mathcal{L}$. Since $\dim\mathcal{L}$ is countable, one can choose a chain of subalgebras $\mathfrak{sl}(V_1,\Pi_1) \subset \mathfrak{sl}(V_2,\Pi_2) \subset \dots$ such that $\mathcal{L} = \cup_{i=1}^{\infty}\mathfrak{sl}(V_i,\Pi_i)$. Observe that we have $V_1 \subset V_2 \subset \dots$. Therefore $V = \cup_{i=1}^{\infty}V_i$ has countable dimension. Now it is clear that one can choose a basis $E = \{e_1, e_2, \dots\}$ of V and a basis $E^* = \{e_1^*, e_2^*, \dots\}$ of Π such that E^* is the dual basis of E. Therefore $\mathcal{L} \cong \mathfrak{sl}_\infty(\Delta)$, as required. The converse statement is obvious. □

Finitary special unitary algebras. Assume that Δ has an involution $\sigma : \delta \mapsto \bar{\delta}$ of the second kind, that is, σ acts nontrivially on the center Γ of Δ. Then Γ is a quadratic field extension of the subfield F of σ-invariant elements of Γ. Let Φ be a unitary form on V. The *finitary unitary algebra* $\mathfrak{fu}(V,\Phi)$ is the set of all transformations $x \in \mathfrak{fgl}(V)$ such that

$$\Phi(xv, w) + \Phi(v, xw) = 0 \quad \text{for all } v, w \in V.$$

The *finitary special unitary algebra* $\mathfrak{fsu}(V,\Phi)$ is the set of all $x \in \mathfrak{fu}(V,\Phi)$ with $\operatorname{tr} x \in [\Delta,\Delta]$. We also have $\mathfrak{fsu}(V,\Phi) = [\mathfrak{fu}(V,\Phi), \mathfrak{fu}(V,\Phi)]$ because this is true for finite dimensional V. Both $\mathfrak{fu}(V,\Phi)$ and $\mathfrak{fsu}(V,\Phi)$ are finitary Lie algebras over F. Assume that $\dim V$ is infinite. Let $\{V_\tau\}$ be the set of all finite dimensional subspaces of V with nondegenerate restrictions $\Phi_\tau = \Phi|_{V_\tau}$. For each V_τ we have $V = V_\tau \oplus V_\tau^\perp$ where $V_\tau^\perp$ is the orthogonal complement to V_τ with respect to Φ. Denote by L_τ the algebra of all elements $x \in \mathfrak{fsu}(V,\Phi)$ with $xV_\tau^\perp = 0$. Clearly, $L_\tau \cong \mathfrak{su}(V_\tau,\Phi_\tau)$.

Let L be a finite dimensional subalgebra in $\mathfrak{fsu}(V,\Phi)$. Note that $\dim LV$ is finite. Since Φ is nondegenerate, there exists V_τ containing LV. Since $L \subset \mathfrak{fsu}(V,\Phi)$, L leaves $V_\tau^\perp$ invariant. As $LV \subseteq V_\tau$, we have $LV_\tau^\perp = 0$, so $L \subseteq L_\tau$. Therefore $\{L_\tau\}$ is a local system of $\mathcal{L}$. We shall call this local system *standard*. Note that $L_{\tau_1} \subseteq L_{\tau_2}$ if and only if $V_{\tau_1} \subseteq V_{\tau_2}$. Moreover, if $V_{\tau_1} \subseteq V_{\tau_2}$, then the corresponding embedding of $L_{\tau_1} \cong \mathfrak{su}(V_{\tau_1},\Phi_{\tau_1})$ into $L_{\tau_2} \cong \mathfrak{su}(V_{\tau_2},\Phi_{\tau_2})$ is natural. As in the case of finitary special linear algebras the algebras $\mathfrak{su}(V_\tau,\Phi_\tau)$ are quasisimple [10, Theorem 7] (central simple if $\operatorname{char} F = 0$), so $\mathfrak{fsu}(V,\Phi)$ is central simple. We obtain the following result.

Proposition 6.3 *Let F be a field and let Δ be a finite dimensional division algebra over F with involution σ of the second kind such that F coincides with the set of central σ-invariant elements of Δ. Let V be an infinite dimensional vector space over Δ and let Φ be a nondegenerate unitary form on V. Then*

(i) $\mathfrak{fsu}(V,\Phi) = [\mathfrak{fu}(V,\Phi), \mathfrak{fu}(V,\Phi)]$;

(ii) $\mathfrak{fsu}(V,\Phi)$ *is a central finitary simple Lie algebra over F of type $A^{II}(\Delta)$.*

Finitary orthogonal and finitary symplectic algebras. Let $\Gamma = F$ and let Ψ be a nondegenerate orthogonal form on V. The *finitary orthogonal algebra* $\mathfrak{fo}(V,\Psi)$ is the set of all transformations $x \in \mathfrak{fgl}(V)$ such that

$$\Psi(xv, w) + \Psi(v, xw) = 0 \quad \text{for all } v, w \in V.$$

If we have a nondegenerate symplectic form Θ on V, then we obtain the *finitary symplectic algebra* $\mathfrak{fsp}(V,\Theta)$. If $\operatorname{char} F \neq 2$ and V is finite dimensional, then $\mathfrak{fo}(V,\Psi) = \mathfrak{o}(V,\Psi)$ and $\mathfrak{fsp}(V,\Theta) =$

$\mathfrak{sp}(V,\Theta)$ are classical central simple (except some small dimensions) Lie algebras of types $BD(\Delta)$ and $C(\Delta)$, respectively [10, Lemma 7]. Both $\mathfrak{fo}(V,\Psi)$ and $\mathfrak{fsp}(V,\Theta)$ are finitary Lie algebras over F. As in the case of unitary algebras, it is not difficult to show that $\mathfrak{fo}(V,\Psi)$ and $\mathfrak{fsp}(V,\Theta)$ are direct limits of finite dimensional central simple subalgebras L_τ of the same type ($BD(\Delta)$ or $C(\Delta)$) naturally embedded each into another. So we have the following proposition.

Proposition 6.4 *Let F be a field of characteristic $\neq 2$. Let Δ be a finite dimensional central division algebra over F and let V be an infinite dimensional vector space over Δ. Let Ψ and Θ be nondegenerate orthogonal and symplectic forms on V. Then $\mathfrak{fo}(V,\Psi)$ and $\mathfrak{fsp}(V,\Theta)$ are central finitary simple Lie algebras over F (of types $BD(\Delta)$ or $C(\Delta)$).*

We conclude this section by the following result.

Proposition 6.5 *Let V be a vector space with a nondegenerate Hermitian or skew-Hermitian form Φ. Then the algebra $\mathfrak{fsh}(V,\Phi)$ is countably dimensional if and only if V is countably dimensional.*

Proof. Let $\mathcal{L} = \mathfrak{fsh}(V,\Phi)$ and let $\{L_\tau\}$ be the standard local system of $\mathcal{L}$. Assume that $\mathcal{L}$ is countably dimensional. Then one can choose a chain of subalgebras $L_{\tau_1} \subset L_{\tau_2} \subset \dots$ such that $\mathcal{L} = \cup_{i=1}^\infty L_{\tau_i}$. We have $V_{\tau_1} \subset V_{\tau_2} \subset \dots$, so $V = \cup_{i=1}^\infty V_{\tau_i}$ has countable dimension. The converse statement is obvious. □

7 The classification

In this section we prove the main theorem.

Lemma 7.1 *Let $\mathcal{L} \subseteq \mathfrak{fgl}(V)$ be a finitary simple Lie algebra. Then V has a faithful composition factor.*

Proof. It is well known (see, for instance, [2, 3.1]) that simple locally finite Lie algebras are not locally solvable, so $\mathcal{L}$ has a finite dimensional semisimple subalgebra S (one can take a Levi subalgebra of any non-solvable finite dimensional subalgebra). Fix a nontrivial irreducible submodule U of V. Let W be the $\mathcal{L}$-submodule of V generated by U. By Zorn's lemma, W has a maximal submodule. It remains to note that the relevant factor module is nontrivial, as required. □

Let $\Gamma \supseteq F$ be the centroid of $\mathcal{L}$. Since $\mathcal{L}$ is simple, by [12, Theorem 10.1.1], Γ is a field. Denote by $\mathcal{L}^\Gamma$ the algebra $\mathcal{L}$ considered over Γ. Clearly, $\mathcal{L}^\Gamma$ is simple.

Lemma 7.2 *Let V be a vector space over a field F. Let $\mathcal{L} \subset \mathfrak{fgl}(V)$ be a finitary simple Lie algebra and let $\Gamma \supseteq F$ be the centroid of $\mathcal{L}$. Assume that V is irreducible. Then one can define an action of Γ on V compatible with that of $\mathcal{L}$, i.e. $(\gamma x)v = x(\gamma v) = \gamma(xv)$ for all $\gamma \in \Gamma$, $x \in \mathcal{L}$, and $v \in V$. Moreover, the corresponding Γ-space V^Γ is a finitary irreducible $\mathcal{L}^\Gamma$-module.*

Proof. Denote by $A(\mathcal{L})$ the subalgebra in $U(\mathcal{L})$ generated by $\mathcal{L}$ (this is an ideal of codimension 1 in $U(\mathcal{L})$). Clearly, $A(\mathcal{L})$ inherits from $\mathcal{L}$ the structure of a Γ-module. Let $\gamma \in \Gamma$ and $v \in V$. Then the map $\lambda_\gamma : uv \mapsto (\gamma u)v$ with u running over $A(\mathcal{L})$ is an endomorphism of V. Moreover, $\gamma \mapsto \lambda_\gamma$ is a homomorphism of the field Γ into $\operatorname{End}_F V$. So one can define an action of Γ on V via

$\gamma v = \lambda_\gamma v$. Note that this action commutes with the action of $\mathcal{L}$. Clearly, the relevant Γ-module V^Γ is an irreducible finitary $\mathcal{L}^\Gamma$-module. □

Observe that for all $x \in \mathcal{L}$ and $\gamma \in \Gamma$ we have $\gamma(xV) = x(\gamma V) = xV$. Therefore $[\Gamma : F]$ divides $\operatorname{rk} x = \dim xV$. So we have the following corollary.

Corollary 7.3 *Let $\mathcal{L} \subset \mathfrak{fgl}(V)$ be an irreducible finitary simple Lie algebra and let Γ be the centroid of $\mathcal{L}$. Then $[\Gamma : F]$ is finite and divides rank of any $x \in \mathcal{L}$.*

Proof of Theorem 1.1. Let $\mathcal{L} \subset \mathfrak{fgl}(V)$ be a finitary simple Lie algebra. By Lemmas 7.1 and 7.2, one can assume that $\mathcal{L}$ is central and V is irreducible. Then by Theorem 4.4, there exists a natural local system $\mathfrak{N}$ for $\mathcal{L}$ and for each $L \in \mathfrak{N}$, LV is a finite direct sum of isomorphic natural L-modules. Fixing an appropriate base in the root system of $\bar{L}$, one can assume that LV is a sum of standard L-modules. Denote by I and I_L the annihilators of V in $A(\mathcal{L})$ and $A(L)$, respectively. Set $E = A(\mathcal{L})/I$ and $E_L = A(L)/I_L$. We have $I_L = I \cap A(L)$ and $E = \varinjlim E_L$. Obviously, I_L coincides with the annihilator of the standard $\bar{L}$-module in $A(L)$. Therefore we can use Lemma 2.3. By this lemma, E_L is simple for each $L \in \mathfrak{N}$, so E is simple. Moreover, since E is finitary, by Proposition 5.3 and Lemma 5.4, E is a simple ring with minimal left ideals.

Assume that $\mathcal{L}$ is of type A^I. Then $L = [E_L, E_L]$ for each L, so $\mathcal{L} = [E, E]$. By Theorem 5.6, E is isomorphic to $\mathfrak{F}(U, \Pi)$ for some vector space U over a skew field Δ and total subspace Π of U^*. Observe that Δ is a finite dimensional division F-algebra isomorphic to $\Delta(\mathcal{L})$. Indeed, take any subalgebra A of $\mathfrak{F}(U, \Pi)$ isomorphic to $M_k(\Delta)$ for sufficiently large k. Δ is isomorphic to a subalgebra of A, so acts by finitary transformations on U. Since Δ is a division algebra, this implies that Δ is finite dimensional. Now note that $\mathcal{L}$ contains a subalgebra isomorphic to $\mathfrak{sl}_k(\Delta) = [A, A]$. Therefore by Lemma 2.8, $\Delta = \Delta(\mathcal{L})$. By Proposition 6.1(i),

$$\mathcal{L} = [E, E] \cong [\mathfrak{F}(U, \Pi), \mathfrak{F}(U, \Pi)] = \mathfrak{fsl}(U, \Pi).$$

It follows that $\mathcal{L}$ and $\mathfrak{fsl}(U, \Pi)$ are isomorphic as rings. It remains to note that F is the centroid of $\mathcal{L}$ and, consequently, of $\mathfrak{fsl}(U, \Pi)$. Therefore $\mathcal{L}$ and $\mathfrak{fsl}(U, \Pi)$ are isomorphic as F-algebras.

Assume that $\mathcal{L}$ is not of type A^I. Denote by α the standard involution of $A(\mathcal{L})$. Observe that the restriction of α to $A(L)$ is the standard involution of $A(L)$. Therefore $I_L^\alpha = I_L$, $I^\alpha = I$, and all E_L and E inherit the involution α. Since $L = \mathfrak{h}^\alpha(E_L)$ (or $L = [\mathfrak{h}^\alpha(E_L), \mathfrak{h}^\alpha(E_L)]$ for $\mathcal{L}$ of type A^{II}) for all $L \in \mathfrak{N}$, we have $\mathcal{L} = \mathfrak{h}^\alpha(E)$ (or $\mathcal{L} = [\mathfrak{h}^\alpha(E), \mathfrak{h}^\alpha(E)]$ for $\mathcal{L}$ of type A^{II}). By Theorem 5.10, there exists a vector space U over a skew field Δ (as above $\Delta = \Delta(\mathcal{L})$) with a nondegenerate Hermitian or skew-Hermitian form Φ such that E can be realized as the ring $\mathfrak{F}_\Phi(U, U)$ with involution given by the adjoint map with respect to Φ. Now, in view of Proposition 5.11, we have $\mathcal{L} = \mathfrak{fh}(U, \Phi)$ (or $\mathcal{L} = \mathfrak{fsh}(U, \Phi)$ if $\mathcal{L}$ is of type A^{II}). Assume that the form Φ is skew-Hermitian of the second kind, i.e. the relevant involution $\delta \mapsto \bar{\delta}$ of Δ acts nontrivially on the center of Δ. Then one can find a nonzero central element $\delta \in \Delta$ such that $\bar{\delta} = -\delta$. Set $\Phi' = \delta\Phi$. Then one can check that $\mathfrak{fh}(U, \Phi) = \mathfrak{fh}(U, \Phi')$ and Φ' is a Hermitian form on U. So we can exclude skew-Hermitian forms of the second kind from our consideration. Any other form is either unitary, or orthogonal, or symplectic. It follows that $\mathcal{L}$ is isomorphic to either $\mathfrak{fsu}(U, \Phi)$, or $\mathfrak{fo}(U, \Phi)$, or $\mathfrak{fsp}(U, \Phi)$, respectively. The theorem follows. □

The classification of all finitary simple Lie algebras (not necessary central) can be obtained using the following.

Proposition 7.4 *Let K be a field and F be a finite field extension of K. Let $\mathcal{L}$ be one of the central finitary simple Lie F-algebras described in Theorem 1.1. Denote by $\mathcal{L}^K$ the algebra $\mathcal{L}$ considered over the field K. Then $\mathcal{L}^K$ is finitary and simple. Moreover, any finitary simple Lie algebra over K can be obtained in such a way.*

Proof. The simplicity of $\mathcal{L}^K$ follows from [12, Theorem 10.1.3]. Now if M is a finitary simple Lie algebra over K, then by Corollary 7.3, the centroid Γ of M is a finite extension of K, so the proposition follows. □

8 Finitary modules

The aim of this section is to describe finitary modules for central finitary simple Lie algebras.

Let V be a vector space. Let Π be a total subspace of V^*. Observe that Π can be viewed as a *right* vector space over the opposite skew field Δ^{op}. Moreover, V, viewed as a left vector space over Δ^{op}, can be identified with a subspace of Π^* setting $e\varphi = \varphi e$ for all $e \in V$ and $\varphi \in \Pi$. One can check that V is total. So one can define the ring $\mathfrak{F}(\Pi, V)$. This is exactly the ring of all transformations of the right Δ^{op}-space Π of the form $\varphi \mapsto \sum_{i=1}^{n} \varphi_i(e_i\varphi)$ where $e_i \in V$ and $\varphi_i \in \Pi$ for $i = 1, \dots, n$.

Lemma 8.1 *Let V be a vector space, Π be a total subspace of V^*, and Φ be a nondegenerate Hermitian or skew-Hermitian form on V. Then*

(1) the enveloping algebra of $\mathfrak{fsl}(V, \Pi)$ in $\operatorname{End} V$ is $\mathfrak{F}(V, \Pi)$;

(2) the enveloping algebra of $\mathfrak{fsl}(V, \Pi)$ in $\operatorname{End} \Pi$ is $\mathfrak{F}(\Pi, V)$;

(3) the enveloping algebra of $\mathfrak{fsh}(V, \Phi)$ in $\operatorname{End} V$ is $\mathfrak{F}_\Phi(V, V)$.

Proof. Let $\mathcal{L} \cong \mathfrak{fsl}(V, \Pi)$ or $\mathfrak{fsh}(V, \Phi)$ and let $\{L_\tau\}$ be the standard local system of $\mathcal{L}$. Recall that in the case (1) and (2) $L_\tau = \mathfrak{sl}(V_\tau, \Pi_\tau)$ where V_τ and Π_τ are compatible subspaces in V and Π. By [12, Lemma 10.4.4], the enveloping algebras of L_τ in V and Π are $F(V_\tau, \Pi_\tau)$ and $F(\Pi_\tau, V_\tau)$, respectively, so (1) and (2) follows. For the case (3) we have $L_\tau = \mathfrak{sh}(V_\tau, \Phi_\tau)$ with $\Phi_\tau = \Phi|_{V_\tau}$. Let $V_\tau^\perp$ be the orthogonal complement to V_τ, i.e. $V = V_\tau \oplus V_\tau^\perp$. One can extend Φ_τ to V setting $\Phi_\tau(V_\tau^\perp, V) = 0$. Let Π_τ be the set of all linear functions $\varphi \in V^*$ with $\varphi(V_\tau^\perp) = 0$. Note that Π_τ is the image of V_τ in V^* under the canonical map $u \mapsto \varphi_u$ with $\varphi_u(v) = \Phi(v, u)$. We have

$$L_\tau = \mathfrak{sh}(V_\tau, \Phi_\tau) \subset \mathfrak{F}(V_\tau, \Pi_\tau) \cong M_{k_\tau}(\Delta)$$

where $k_\tau = \dim V_\tau$. By [12, Lemma 10.4.4], the enveloping algebra of L_τ is isomorphic to the full matrix algebra $M_{k_\tau}(\Delta)$, so coincide with $\mathfrak{F}(V_\tau, \Pi_\tau)$. Therefore the enveloping algebra of $\mathfrak{fsh}(V, \Phi)$ in $\operatorname{End} V$ is $\mathfrak{F}_\Phi(V, V)$. □

Theorem 8.2 *Let V be a vector space, Π be a total subspace of V^*, and Φ be a nondegenerate Hermitian or skew-Hermitian form on V. Let $\mathcal{L} = \mathfrak{fsl}(V, \Pi)$ or $\mathfrak{fsh}(V, \Phi)$ and let W be a finitary $\mathcal{L}$-module, i.e. $\mathcal{L} \subset \mathfrak{fgl}(W)$. Then $W = W^0 \oplus W^1 \oplus W^2$ where $\mathcal{L}W^0 = 0$, the module W^1 is a finite direct sum of copies of V, and either $W^2 = 0$ or $\mathcal{L} = \mathfrak{fsl}(V, \Pi)$ and W^2 is a finite direct sum of copies of Π.*

Proof. Let $\{L_\tau\}$ be the standard local system of $\mathcal{L}$. In view of Theorem 4.4, one can assume that for all τ, $L_\tau W$ is a finite direct sum of isomorphic natural L_τ-modules. Let $A(\mathcal{L})$ and $A(L_\tau)$ be the augmentation ideals in $U(\mathcal{L})$ and $U(L_\tau)$, respectively. We have $A(\mathcal{L}) = \varinjlim A(L_\tau)$. Set $I = \mathrm{Ann}_{A(\mathcal{L})} W$, $I_\tau = \mathrm{Ann}_{A(L_\tau)} W$, $E = A(\mathcal{L})/I$, and $E_\tau = A(L_\tau)/I_\tau$. Clearly, $I_\tau = I \cap A(L_\tau)$; E and E_τ are the enveloping algebras of $\mathcal{L}$ and L_τ in $\mathrm{End}\, W$; and $E = \varinjlim E_\tau$. Assume that for each τ, $L_\tau W$ is a sum of standard L_τ-modules. Then $I_\tau = \mathrm{Ann}_{A(L_\tau)} V$, so $I = \mathrm{Ann}_{A(\mathcal{L})} V$. Therefore the enveloping algebras of $\mathcal{L}$ in $\mathrm{End}\, W$ and $\mathrm{End}\, V$ are isomorphic. Hence by Lemma 8.1, $E \cong \mathfrak{F}(V, \Pi)$ for $\mathcal{L} = \mathfrak{fsl}(V, \Pi)$ and $E \cong \mathfrak{F}_\Phi(V, V)$ for $\mathcal{L} = \mathfrak{fsh}(V, \Phi)$. Denote by E^0 the algebra E considered as a ring. In both cases E^0 is primitive simple with minimal left ideals. Since $E^0 W = W$, by [11, Theorem 4.14.1], the E^0-module W is a direct sum of submodules isomorphic to V. By [11, Proposition 1.9.2] (see also similar Lemma 7.2 for the case of Lie algebras), the E^0 module V has a unique structure of a E-module. It follows that W is a sum of E-modules isomorphic to V. Since $E \subseteq \mathfrak{F}(W)$, this sum is finite. If $\mathcal{L} = \mathfrak{fsl}(V, \Pi)$ and for all τ, $L_\tau W$ is a direct sum of modules dual to standard, then by Lemma 8.1, $E = \mathfrak{F}(\Pi, V)$, so W is a finite direct sum of modules isomorphic to Π. The theorem follows. □

9 Real finitary simple Lie algebras

The aim of this section is to prove Theorems 1.3 and 1.4, which describe real finitary simple Lie algebras.

Proof of Theorem 1.3. By Frobenius theorem, there are only three finite dimensional division algebras over $\mathbb{R}$: $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, and the algebra of quaternions $\mathbb{H}$. The algebra $\mathbb{R}$ has no nontrivial involutions. The algebras $\mathbb{C}$ and $\mathbb{H}$ have *standard* involutions $\delta = a+bi \mapsto \bar{\delta} = a-bi$ and $\delta = a + bi + cj + dk \mapsto \bar{\delta} = a - bi - cj - dk$, respectively. (Here $\{1, i\}$ and $\{1, i, j, k\}$ are the standard bases of $\mathbb{C}$ and $\mathbb{H}$, respectively). We shall denote both these involutions by σ. Now by Theorem 1.1, any real finitary simple Lie algebra is isomorphic to one of the following algebras: $\mathfrak{fsl}(V, \Pi)$ (type $A^I(\Delta)$), $\mathfrak{fsu}(V, \Phi^\alpha)$ (type $A^{II}(\Delta)$), $\mathfrak{fo}(V, \Psi^\alpha)$ (type $BD(\Delta)$ or $C(\Delta)$), $\mathfrak{fsp}(V, \Theta^\alpha)$ (type $BD(\Delta)$ or $C(\Delta)$), where V is a vector space over $\Delta = \mathbb{R}, \mathbb{C}, \mathbb{H}$; α is an involution of the $\mathbb{R}$-algebra Δ; Φ^α, Ψ^α, and Θ^α are nondegenerate forms; and Π is a total subspace of V^*. Consider the following cases.

$\Delta = \mathbb{R}$. Then we have the algebras $\mathfrak{fsl}(V, \Pi)$ (type $A^I(\mathbb{R})$), $\mathfrak{fo}(V, \Psi)$ (type $BD(\mathbb{R})$), and $\mathfrak{fsp}(V, \Theta)$ (type $C(\mathbb{R})$).

$\Delta = \mathbb{C}$. Since the $\mathbb{R}$-algebra $\mathbb{C}$ is not central and has only standard involution (which is of the second kind), we have only $A^{II}(\mathbb{C})$-type algebras: $\mathfrak{fsu}(V, \Phi^\sigma)$.

$\Delta = \mathbb{H}$. Assume that α is a nonstandard involution of $\mathbb{H}$. Then $\beta : \delta \mapsto \alpha(\bar{\delta})$ is an automorphism of the algebra $\mathbb{H}$. Since $\mathbb{H}$ is central simple, each automorphism of $\mathbb{H}$ is internal, i.e. there is $q \in \mathbb{H}$ such that $\alpha(\bar{\delta}) = q\delta q^{-1}$ for all $\delta \in \mathbb{H}$, or equivalently, $\alpha(\delta) = q\bar{\delta}q^{-1}$. Since $\alpha^2 = 1$, one easily checks that $\bar{q} = -q$. Let now Φ^α be an orthogonal (resp. symplectic) form on a vector space V over $\mathbb{H}$ with respect to the involution α. Then $\Phi = \Phi^\alpha q$ is a symplectic (resp. orthogonal) form on V with respect to the standard involution σ of $\mathbb{H}$. Indeed, we have for all $v, w \in V$

$$\Phi(v, w) = \Phi^\alpha(v, w)q = \pm\alpha(\Phi^\alpha(w, v))q = \pm q\overline{\Phi^\alpha(w, v)} = \mp\overline{\Phi(w, v)}.$$

Therefore we do not need to consider nonstandard involutions, so we get the following algebras: $\mathfrak{fsl}(V,\Pi)$ (type $A^I(\mathbb{H})$), $\mathfrak{fo}(V,\Psi^\sigma)$ (type $C(\mathbb{H})$), and $\mathfrak{fsp}(V,\Theta^\sigma)$ (type $D(\mathbb{H})$). The theorem follows. □

Proof of Theorem 1.4. Let $\mathcal{L}$ be a central real finitary simple Lie algebra of (infinite) countable dimension. We identify $\mathcal{L}$ with one of the algebras in Theorem 1.3. Since $\mathcal{L}$ has countable dimension, by Propositions 6.2 and 6.5, the underlying space V also has countable dimension. Moreover, $\mathfrak{sl}_\infty(\mathbb{R})$ and $\mathfrak{sl}_\infty(\mathbb{H})$ are unique countably dimensional algebras of types $A^I(\mathbb{R})$ and $A^I(\mathbb{H})$, respectively. So one can assume that $\mathcal{L}$ preserves a nondegenerate Hermitian or skew-Hermitian form Φ. We shall consider the following cases.

(a) $\Delta = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$; and Φ is Hermitian (unitary or orthogonal).
(b) $\Delta = \mathbb{R}$ and Φ is skew-Hermitian (symplectic).
(c) $\Delta = \mathbb{H}$ and Φ is skew-Hermitian (symplectic).

Fix a chain $V_1 \subset V_2 \subset \cdots \subset V_n \subset \ldots$ of subspaces in V such that $\Phi|_{V_n}$ is nondegenerate and $V = \cup_{i=1}^\infty V_i$. Set $W_0 = V_1$. Let W_i be the orthogonal complement to V_i in V_{i+1}, that is, $V_{i+1} = V_i + W_i$ and $\Phi(V_i, W_i) = 0$. Then $V = \oplus_{i=0}^\infty W_i$. Note that $\Phi|_{W_i}$ is nondegenerate for all $i \geq 0$. Let now W be a finite dimensional vector space over $\Delta = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ with a nondegenerate Hermitian or skew-Hermitian form Φ with respect to the standard involution. By classical results (see also [12, §10.7]), there exists a basis $\{w_1, \ldots, w_n\}$ of W such that in the cases (a), (b), and (c) above we have, respectively,
(a) $\Phi(w_i, w_j) = \pm\delta_{ij}$, $1 \leq i, j \leq n$;
(b) $\Phi(w_i, w_j) = 0$ for all i and j, except $\Phi(w_{2i-1}, w_{2i}) = 1$ for $1 \leq i \leq n/2$ (n is even);
(c) $\Phi(w_i, w_j) = q\delta_{ij}$, $1 \leq i, j \leq n$, where q is any fixed element in $\mathbb{H}$ with $\bar{q} = -q$.
Moreover, the numbers of -1 and 1 in the case (a) is an invariant. Choosing such a basis in each W_i and reordering these basic vectors, one can get a basis $E = \{e_1, e_2, \ldots, e_n, \ldots\}$ of V such that the matrix $(\Phi(e_i, e_j))$ of the form Φ in the case (a) coincides with $\pm I_k$, $k = 0, 1, 2, \ldots, \infty$, (see Introduction); in the case (b) coincides with J; and in the case (c) coincides with qI. Since the forms Φ and $-\Phi$ yield the same algebras, one can assume that the matrix of Φ in the case (a) is I_k for some k. Observe that for $\Delta = \mathbb{H}$ all skew-Hermitian forms Φ are equivalent. Therefore in the case (c) there exists a basis of V such that the matrix of Φ is J (since the form given by J is skew-Hermitian). We redenote this basis by E.

We identify the algebra $M_\infty(\Delta)$ with the corresponding set of finitary transformations of V (with respect to the basis E). Set $U_n = \langle e_1, \ldots, e_{2n}\rangle_\Delta$ and $U_n^\perp = \langle e_{2n+1}, e_{2n+2}, \ldots\rangle$. We have $V = U_n \oplus U_n^\perp$ for all $n \geq 1$. Let now x be a finitary transformation of V preserving the form Φ, that is,

$$\Phi(xv, w) + \Phi(v, xw) = 0 \quad \text{for all } v, w \in V. \tag{1}$$

Then there exists n such that $xV \subseteq U_n$. Since x preserves Φ, $xU_n^\perp \subseteq U_n^\perp$. As $U_n \cap U_n^\perp = 0$, we have $xU_n^\perp = 0$. It follows that $x \in M_\infty(\Delta)$. Let X and $B = I_k, J$ be the matrices of x and Φ with respect to the basis E. Then the condition (1) can be written in the matrix form

$$X^*B + BX = 0$$

where $X^* = X^t$ for $\Delta = \mathbb{R}$, and $X^* = \bar{X}^t$ for $\Delta = \mathbb{C}, \mathbb{H}$. It follows that any real finitary central simple Lie algebra of countable dimension is isomorphic to one of the algebras (1)–(7) in the theorem.

It remains to show that the algebras described in the theorem are pairwise nonisomorphic. In view of Proposition 4.3, it suffices to prove that all algebras of the same series (2), (4), or (6) are pairwise nonisomorphic. So assume that $\mathcal{L} = \mathfrak{o}_\infty(\mathbb{R}, k) \cong \mathfrak{o}_\infty(\mathbb{R}, l)$ for some $k < l$ (the other series can be treated similarly). This means that $\mathcal{L}$ has two different natural local systems $\mathfrak{N}_1 = \{L_n\}$ and $\mathfrak{N}_2 = \{Q_m\}$ where $L_n = \mathfrak{o}_n(\mathbb{R}, k)$, $n > 2k$; $Q_m = \mathfrak{o}_m(\mathbb{R}, l)$ for $l < \infty$; and $Q_m = \mathfrak{o}_m(\mathbb{R}, m/2)$ with m even and greater than 4 for $l = \infty$. Therefore there are integers $m < n < m'$ with $m > 2l$ for $l < \infty$ and $m > 2k$ for $l = \infty$ such that $Q_m \subset L_n \subset Q_{m'}$. Since the embedding $Q_m \subset Q_{m'}$ is natural, by Lemma 2.8, the embedding $Q_m \subset L_n$ is natural. Represent the latter algebra in the form $L_n = \mathfrak{h}(V, \Phi)$ where Φ is a nondegenerate orthogonal form on V with inertia indices $(k, n-k)$ (the numbers of -1 and 1 in the canonical matrix presentation). We have a unique decomposition $V = W \oplus U$ where W is a natural Q_m-module and $Q_m U = 0$. Assume that $\Phi|_W$ is degenerate. Then there exists $w \in W$ and $u \in U$ such that $\Phi(w, u) \neq 0$. Since $Q_m W = W$, there exist $x \in Q_m \subset L_n$ and $w' \in W$ such that $w = xw'$. Then

$$\Phi(xw', u) + \Phi(w', xu) = \Phi(w, u) \neq 0.$$

It follows that $x \in L_n$ does not respect Φ, which contradicts the assumption. Therefore $\Phi|_W$ is nondegenerate. Let $(k_1, m - k_1)$ be the inertia indices of this form. Clearly, $k_1 \leq k$. Observe that $Q_m \subseteq \mathfrak{h}(W, \Phi|_W) = \mathfrak{o}_m(\mathbb{R}, k_1)$. Since the dimensions of these algebras coincide (extending the field to $\mathbb{C}$, we obtain the algebras isomorphic to $\mathfrak{o}_s(\mathbb{C})$ with $s = m(m-1)/2$), we have $Q_m = \mathfrak{h}(W, \Phi|_W)$, i.e. either $\mathfrak{o}_m(\mathbb{R}, l) \cong \mathfrak{o}_m(\mathbb{R}, k_1)$ with $m > 2l > 2k_1$ for $l < \infty$, or $\mathfrak{o}_m(\mathbb{R}, m/2) \cong \mathfrak{o}_m(\mathbb{R}, k_1)$ with $m > 2k_1$ for $l = \infty$. It follows from Jacobson's classification that both these isomorphisms are impossible ($\mathfrak{o}_p(\mathbb{R}, q_1) \cong \mathfrak{o}_p(\mathbb{R}, q_2)$ with $p > 4$ if and only if either $q_1 = q_2$ or $q_1 = p - q_2$). The proposition follows. □

10 Finitary irreducible Lie algebras

The aim of this section is to classify infinite dimensional finitary irreducible Lie algebras over a field of characteristic 0. In fact the main result (Theorem 1.6) holds for any field of characteristic $\neq 2$ provided Theorem 1.1 and Theorem 8.2 (for irreducible finitary modules) are valid in the case of positive odd characteristic.

Let Δ be a finite dimensional division algebra over F and V be a (right) vector space over Δ. An element $t \in \operatorname{End} V$ is called a *transvection* if its rank (the dimension of tV) is 1, i.e. there exists $u \in V$ and $\varphi \in V^*$ such that $tv = u(\varphi v)$ for all $v \in V$. We shall write $t = t_{u\varphi}$. Note that we use the term "transvection" in a nonstandard way (in the standard definition one takes $t - 1$ rather than t). Recall that for a total subspace Π of V^* the ring $\mathfrak{F}(V, \Pi)$ can be defined as

$$\mathfrak{F}(V, \Pi) = \langle t_{u\varphi} \mid u \in V \;\; \varphi \in \Pi \rangle_{\mathbb{Z}}.$$

The following properties of transvections can be checked by the direct calculations.

Lemma 10.1 *For all $v, u \in V$, $\varphi, \psi \in \Pi$, and $\delta \in \Delta$ we have*

(1) $t_{v+u,\varphi} = t_{v\varphi} + t_{u\varphi}$*;*

(2) $t_{u,\varphi+\psi} = t_{u\varphi} + t_{u\psi}$*;*

(3) $t_{u\delta,\varphi} = t_{u,\delta\varphi}$*;*

(4) $t_{u\varphi}t_{v\psi} = t_{u(\varphi v),\psi} = t_{u,(\varphi v)\psi}$. *In particular* $t_{u\varphi}t_{v\psi} = 0$ *if and only if* $\varphi v = 0$.

Note that any element $x \in \mathfrak{F}(V,\Pi)$ can be represented as a sum of $n = \operatorname{rk} x$ transvections.

Lemma 10.2 *Let* $x \in \mathfrak{F}(V,\Pi)$ *and* $n = \operatorname{rk} x$. *Assume that* x *is represented as a sum of* n *transvections* $x = t_{u_1\varphi_1} + \cdots + t_{u_n\varphi_n}$. *Then*

(1) $u_1, \dots, u_n$ *are linearly independent over* Δ*;*

(2) $\varphi_1, \dots, \varphi_n$ *are linearly independent over* Δ*;*

(3) $\varphi_1, \dots, \varphi_n \in \Pi$.

Proof. (1) and (2) are obvious. Since $x \in \mathfrak{F}(V,\Pi)$, it can be represented in the form $x = t_{v_1\psi_1} + \cdots + t_{v_m\psi_m}$ with $\psi_1, \dots, \psi_m \in \Pi$. Using Lemma 10.1(1–3), one can assume that $v_1, \dots, v_m$ and $\psi_1, \dots, \psi_m$ are linearly independent. Therefore

$$\langle v_1, \dots, v_m\rangle_\Delta = xV = \langle u_1, \dots, u_n\rangle_\Delta.$$

Representing each v_i as a linear combination of the u_j and using Lemma 10.1, we have that $x = t_{u_1\theta_1} + \cdots + t_{u_n\theta_n}$ with $\theta_1, \dots, \theta_n \in \Pi$. Therefore $t_{u_1,\varphi_1-\theta_1} + \cdots + t_{u_n,\varphi_n-\theta_n} = 0$. This implies $\varphi_i = \theta_i \in \Pi$ for all i. □

Proposition 10.3 *Let* V *be an infinite dimensional vector space over* Δ *and* Π *be a total subspace of* V^*. *Let* x *be a finitary endomorphism of* V *as an* F*-space. Assume that* $[x, \mathfrak{F}(V,\Pi)] \subseteq \mathfrak{F}(V,\Pi)$. *Then* $x \in \mathfrak{F}(V,\Pi)$. *Moreover, if* $[x, \mathfrak{F}(V,\Pi)] = 0$*, then* $x = 0$.

Proof. Assume that $x \neq 0$. First of all we show that x is Δ-linear. Since x is finitary, there exists $u \in V$ such that $xu = 0$. Let $\varphi \in \Pi$. Set

$$\tau = [x, t_{u\varphi}] = xt_{u\varphi} - t_{u\varphi}x = -t_{u\varphi}x.$$

By assumption, τ is Δ-linear. Therefore for all $v \in V$ and $\delta \in \Delta$ we have

$$0 = \tau(v\delta) - (\tau v)\delta = -t_{u\varphi}x(v\delta) + t_{u\varphi}(xv)\delta = u\varphi((xv)\delta - x(v\delta)).$$

Since φ can be taken arbitrary in Π and Π is total, $(xv)\delta = x(v\delta)$, as desired.

Since x is finitary and Δ-linear, x can be represented in the form $x = t_{u_1\varphi_1} + \cdots + t_{u_n\varphi_n}$ where all $u_1, \dots, u_n$ and $\varphi_1, \dots, \varphi_n$ are linearly independent over Δ. It suffices to show that $\varphi_i \in \Pi$ for $i = 1, \dots, n$. Without loss of generality we can restrict ourselves to the case $i = 1$. Take $u \in V$ and $\varphi \in \Pi$ such that $\varphi_1 u = \cdots = \varphi_n u = 0$, $\varphi u_1 = 1$, and $\varphi u_2 = \cdots = \varphi u_n = 0$. Then it is not difficult to check that $[x, t_{u\varphi}] = -t_{u\varphi}t_{u_1\varphi_1} = -t_{u\varphi_1}$. Now by Lemma 10.1(3), $t_{u\varphi_1} \in \mathfrak{F}(V,\Pi)$ if and only if $\varphi_1 \in \Pi$, as required. Observe that $[x, t_{u\varphi}] \neq 0$. Therefore $[x, \mathfrak{F}(V,\Pi)] = 0$ implies $x = 0$. The proposition follows. □

Proof of Theorem 1.6. By [14, 15], $\mathcal{L}$ has a unique minimal ideal $\mathcal{M}$. Moreover, $\mathcal{M}$ is simple and $\mathcal{M} = [[\mathcal{L}^{(4)}, \mathcal{L}]^{(1)}, \mathcal{L}]$ ($\mathcal{M} = [\mathcal{L}, \mathcal{L}]$ if $\operatorname{char} F = 0$). Since $\mathcal{M}$ is an ideal of $\mathcal{L}$, by [13], V is an irreducible $\mathcal{M}$-module. Let Δ_1 be the centralizer of the $\mathcal{M}$-module V. We have $\Delta_1 \subseteq \Delta$. Let Γ be the centroid of $\mathcal{M}$. Denote by $\mathcal{M}^\Gamma$ the Lie ring $\mathcal{M}$ considered as a Γ-algebra. The algebra $\mathcal{M}^\Gamma$ is central simple. By Lemma 7.2, V respects the action of Γ, so $\Gamma \subseteq \Delta_1$. By Theorem 8.2,

V is the standard $\mathcal{M}^\Gamma$-module. Therefore $E_\mathcal{M} = E_{\mathcal{M}^\Gamma} = \mathfrak{F}(V_1, \Pi)$ where $E_\mathcal{M}$ and $E_{\mathcal{M}^\Gamma}$ are the enveloping rings of $\mathcal{M}$ and $\mathcal{M}^\Gamma$, respectively, V_1 is the abelian group V considered as a Δ_1-space, and Π is a total subspace of V_1^*. Since $\mathcal{M}$ is an ideal, $[x, E_\mathcal{M}] \subseteq E_\mathcal{M}$ for all $x \in \mathcal{L}$. Therefore by Proposition 10.3, $\mathcal{L} \subseteq E_\mathcal{M}$. It follows that $\Delta_1 = \Delta$ and $E_\mathcal{L} = E_\mathcal{M} = \mathfrak{F}(V, \Pi)$. If $\mathcal{M}$ is of type A^I, then $\mathcal{M} = \mathfrak{fsl}(V, \Pi)$, so we get (1). Assume now that $\mathcal{M}$ preserves a Hermitian or skew-Hermitian form Φ, i.e. $\mathcal{M} = \mathfrak{fsh}(V, \Phi) = \mathfrak{sh}^\alpha(E_\mathcal{M})$ where α is the relevant involution of $E_\mathcal{M}$. Let $x \in \mathcal{L}$ and $y \in \mathcal{M}$. Since y and $z = [x, y]$ lie in $\mathcal{M}$, we have $y^\alpha = -y$ and $z^\alpha = -z$. Therefore

$$[x + x^\alpha, y] = xy + x^\alpha y - yx - yx^\alpha = (xy - yx) + (xy - yx)^\alpha = z + z^\alpha = 0.$$

It follows that $[x + x^\alpha, \mathcal{M}] = 0$, so $[x + x^\alpha, E_\mathcal{M}] = 0$. Now by Proposition 10.3, $x + x^\alpha = 0$, so $x = -x^\alpha$. Therefore $\mathcal{L} \subseteq \mathfrak{fh}(V, \Phi) = \mathfrak{h}^\alpha(E_\mathcal{M})$ and we get (2), (3), or (4). If $\mathcal{M} = [\mathcal{L}, \mathcal{L}]$ is central over F, then the codimensions of $\mathfrak{fsl}(V, \Pi)$ in $\mathfrak{fgl}(V, \Pi)$ and of $\mathfrak{fsu}(V, \Phi^\sigma)$ in $\mathfrak{fu}(V, \Phi^\sigma)$ equal to 1, since this is true in the finite dimensional case. Therefore either $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ is simple or $\mathcal{L} = \mathfrak{fgl}(V, \Pi)$ or $\mathfrak{fu}(V, \Phi^\sigma)$. The theorem follows. □

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