# Diagonal locally finite Lie algebras and a version of Ado's theorem

**A.A. Baranov** *

*Institute of Mathematics, Academy of Sciences of Belarus, Surganova 11, Minsk, 220072, Belarus.*

## 1 Introduction

A.E. Zalesskii developed in [14] a new approach to studying the (two-sided) ideal lattices of group algebras of locally finite groups via representation theory of finite groups. He established a 1-1 correspondence between ideals of a group algebra and so-called inductive systems (of modules over finite groups). This observation, in particular, enabled him to describe in [13] the ideal lattices of complex group algebras for locally finite groups that are unions of finite alternating groups. It is now known that this method can be applied to locally finite Lie algebras and their universal enveloping algebras. This approach is used in studying the existence of an embedding of a locally finite Lie algebra $L$ into a locally finite associative algebra. When $L$ is assumed to be finite-dimensional, this result follows from Ado's theorem. However, in general, locally finite Lie algebras do not have this embedding property even if simplicity is assumed.

Therefore, it is natural to seek a description of the simple locally finite Lie algebras for which the locally finite analog of Ado's theorem holds. The machinery developed below allows one to solve this problem as indicated in Corollary 5.11 which is one of the main results of this paper. We observe that, according to a discussion in [15], this problem is a natural analog of the following problem (going back to I. Kaplansky (1965)) for locally finite simple groups: describe the groups $G$ such that the only nontrivial proper ideal of the complex group algebra $\mathbf{C}G$ is the augmentation ideal.

Let $L$ be a locally finite Lie algebra. This means that every finite set of elements of $L$ is contained in a finite-dimensional subalgebra. If the latter can be chosen (semi)simple, then $L$ is called *locally (semi)simple*. Observe that locally finite Lie algebras can be regarded as an asymptotic version of finite-dimensional ones, and appear in various applications as the Lie algebras of direct limits of Lie groups. Suppose that $L$ has countable dimension. (This is in fact a purely technical assumption which allows us to make the exposition more transparent.) Then $L$ can be expressed in the form

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$L = \cup_{i\in\mathbf{N}} L_i$ (or $L = \varinjlim L_i$) where $L_i$ are finite-dimensional Lie algebras, and $L_i \subset L_{i+1}$ for all $i \in \mathbf{N}$. Since $L_i \subset L_{i+1}$, each $L_{i+1}$-module $\varphi$ may be considered to be an $L_i$-module. We use the notation $\varphi\downarrow L_i$ to denote this $L_i$-module. Let $\langle\psi\rangle$ denote the set of inequivalent composition factors of a module $\psi$, and $\operatorname{Irr} L_i$ be the set of inequivalent irreducible $L_i$-modules, Let $\Phi_i$ be a finite subset of $\operatorname{Irr} L_i$. We say that $\Phi = \{\Phi_i\}_{i\in\mathbf{N}}$ is an *inductive system* for $L$ if

$$\cup_{\varphi\in\Phi_{i+1}}\langle\varphi\downarrow L_i\rangle = \Phi_i$$

for all $i \in \mathbf{N}$ (cf. Definition 2.9). It can be shown (see, for instance, [15]) that there is a 1-1 correspondence between the inductive systems for a locally semisimple Lie algebra $L$ and the ideals $X$ of its universal enveloping algebra $U(L)$ with $U(L)/X$ locally finite. This translates the problem into the language of representation theory. For instance, if $L$ is simple, then the analog of Ado's theorem for $L$ holds if and only if there exists a nontrivial inductive system for $L$.

One can ask whether the class of locally semisimple Lie algebras contains all simple locally finite Lie algebras. Recently Yu. Bahturin and H. Strade [3] have constructed examples of simple locally finite Lie algebras that are not locally simple. Actually, their arguments in [3] can be used to prove that these Lie algebras are not locally *semi*simple for the zero characteristic case. Thus, for studying the problem above for simple Lie algebras one cannot restrict oneself to locally semisimple Lie algebras. In particular, it is necessary to look for a reasonable extension of the result above on the 1-1 correspondence to a wider class of locally finite Lie algebras. Our experience shows that it should be the class of *locally perfect* Lie algebras. This class is a radical one (Theorem 2.6), and the quotient of a locally finite Lie algebra by its locally perfect radical is locally solvable. Observe that simple locally finite Lie algebras are locally perfect ([2, Theorem 3.2] and our Theorem 2.8). We establish (Theorem 3.9, see also [4]) a 1-1 correspondence between the set of inductive systems for a locally perfect Lie algebra $L$ and the set of *semiprimitive* ideals $X$ of $U(L)$ with $U(L)/X$ locally finite. This result is similar to Theorem 1.25 in [15] for group algebras over a field of positive characteristic. The proof is based on the following interesting fact (Lemma 3.3) which seems to be unknown. Let $L$ be a finite-dimensional perfect Lie algebra (i.e. $[L, L] = L$), and $\Phi$ a finite subset of $\operatorname{Irr} L$. Then codimensions of the annihilators in $U(L)$ of all finite-dimensional $L$-modules $\varphi$ such that $\langle\varphi\rangle = \Phi$ are bounded by some constant depending on $L$ and $\Phi$.

The correspondence established explains our interest to the inductive systems for locally perfect Lie algebras. The problem of an explicit description of inductive systems for a locally simple Lie algebras was investigated by A.G. Zhilinskii [17, 18]. In Section 5 we extend the notion of *diagonality* introduced by A.G. Zhilinskii [17] to locally perfect Lie algebras and prove (Theorem 5.7) that a locally perfect Lie algebra has a *nondegenerate* (see Definition 5.4) inductive system if and only if it is diagonal. In particular, a simple locally finite Lie algebra has a nontrivial inductive system if and only if it is diagonal. This generalizes the result of A.G. Zhilinskii [17] proved for locally simple Lie algebras. It follows from the 1-1 correspondence above that a simple locally finite Lie algebra is diagonal if and only if it can be embedded into a locally finite associative algebra (Corollary 5.11). Observe that diagonal locally finite Lie algebras (and only they) have nice faithful representations such that their images generate locally finite associative algebras. One can consider these representations as a natural analog of finite-dimensional representations of finite-dimensional Lie algebras. Another, somewhat different approach to representation theory of locally finite Lie algebras is given in [1] where highest weight modules are studied.

Let $L$ be a perfect finite-dimensional Lie algebra, $S = S_1 \oplus \ldots \oplus S_n$ be its Levi subalgebra where $S_1, \ldots, S_n$ are simple components of $S$, and $V_i$ denote the standard $S_i$-module. Since $L$ is perfect, for each $i$ there exists the unique irreducible $L$-module $\mathcal{V}_i$ such that $\mathcal{V}_i{\downarrow}S_i = V_i$. The modules $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are called the *standard* $L$-modules. Let $Q$ be another perfect finite-dimensional Lie algebra. An embedding $L \subset Q$ is called *diagonal* if $\langle W{\downarrow}L \rangle \subseteq \{\mathcal{V}_1, \ldots, \mathcal{V}_n, \mathcal{V}_1^*, \ldots, \mathcal{V}_n^*, T\}$ for all standard $Q$-modules $W$, where $T$ is the trivial one-dimensional $L$-module, $\mathcal{V}_i^*$ is the dual module for $\mathcal{V}_i$. For example, an embedding $f : sl(V) \to sl(W)$ is diagonal if and only if there exist bases of $V$ and $W$ such that $f(A) = \mathrm{diag}(A, \ldots, A, -A^t, \ldots, -A^t)$ for all matrices $A \in sl(V)$. Assume that $L = \varinjlim L_i$ where all $L_i$ are perfect and all embeddings $L_i \subset L_{i+1}$ are diagonal. Then one can check that $L$ is diagonal. Observe that the simple locally finite Lie algebras constructed in [3] are inductive limits of diagonal embeddings, so they are diagonal. One can show that all simple diagonal locally finite Lie algebras can be constructed in such way [5].

Section 6 contains some auxiliary lemmas about branching rules for representations. In Section 7 the notion of a Bratteli diagram for a locally perfect Lie algebra is introduced. In Section 8 (Corollary 8.5) we give a diagonality criterion for locally perfect Lie algebras (on the language of Bratteli diagrams). In Section 9 we prove a general "Ado's theorem" for locally perfect Lie algebras with the trivial centers (Theorem 9.4), i.e. we describe all such algebras that can be embedded into locally finite associative algebras.

A motivation for investigating locally finite Lie algebras can also be found in [2] and [16]. We only want to notice here that there exists some parallelism between the theory of diagonal locally finite Lie algebras and that of locally semisimple associative algebras (see [8, 10]) via the notion of Bratteli diagrams. On the other hand, the questions considered are closely linked with the representation theory of groups that are inductive limits of Lie or algebraic groups. Representations of the most natural examples of such groups were studied by G.I. Olshanskii [11], S. Stratila and D. Voiculescu [12], and others.

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*Notation.*

The ground field $F$ is an algebraically closed field of zero characteristic. $\mathbf{N}$ is the set of natural numbers. All Lie algebras considered are of finite dimension or locally finite. $U(L)$ denotes the universal enveloping algebra of a Lie algebra $L$. If $V$ is an $L$-module, then $\mathrm{Ann}_{U(L)} V$ denotes the annihilator of $V$ in $U(L)$.

Let $L$ be a finite-dimensional Lie algebra. Denote by $\mathrm{Irr}\, L$ the set of inequivalent irreducible finite-dimensional $L$-modules. For an $L$-module $V$ let $\langle V \rangle$ denote the set of inequivalent composition factors of $V$. If $\Phi = \{V_i\}_{i\in I}$ is a set of $L$-modules, then $\langle\Phi\rangle = \cup_{i\in I}\langle V_i\rangle$. If $S$ is a subalgebra of $L$, then $V{\downarrow}S$ denotes the restriction of the $L$-module $V$ to $S$. More general, let $\theta : S \to L$ be a homomorphism of Lie algebras, and $V$ be an $L$-module. Then $\theta$ gives an action of $S$ on $V$. The corresponding module is denoted by $V{\downarrow}^{\theta}_{S}$. Sometimes the symbol $\theta$ is omitted if it is clear which homomorphism is considered. If $\Phi = \{V_i\}_{i\in I}$ is a set of $L$-modules, then $\Phi{\downarrow}^{\theta}_{S} = \{V_i{\downarrow}^{\theta}_{S}\}_{i\in I}$. Denote by $\operatorname{Rad} L$ the solvable radical of $L$. Let $S$ be a Levi subalgebra of $L$, $W$ be an $S$-module. We can consider $W$ as an $L$-module, setting $(\operatorname{Rad} L)W = 0$. Denote this module by $W{\uparrow}L$. If $\Psi = \{W_i\}_{i\in I}$ is a set of $S$-modules, then $\Psi{\uparrow}L = \{W_i{\uparrow}L\}_{i\in I}$. $L$ is called *perfect* if $[L, L] = L$.

Let $A$ be an associative algebra. An ideal $X$ of $A$ is called *primitive* if it is the annihilator of an irreducible $A$-module and *semiprimitive* if it is the intersection of primitive ideals. Equivalently, $X$ is semiprimitive if and only if the Jacobson radical $\operatorname{Rad}(A/X)$ is trivial. Let $A^{(-)}$ be the Lie algebra with the basic set $A$ and the multiplication $[a, b] = ab - ba$. We say that $\varepsilon : L \to A$ is an embedding of a Lie algebra $L$ into $A$ if $\varepsilon$ is an injective homomorphism of $L$ into $A^{(-)}$.

# 2 Locally perfect Lie algebras

The basic tool for investigating locally finite Lie algebras is the notion of a local system.

**Definition 2.1** Let $L$ be a locally finite Lie algebra. A set $\{L_i\}_{i\in I}$ of finite-dimensional subalgebras of $L$ is called a *local system* of $L$ if $L = \cup_{i\in I} L_i$ and for each pair $i, j \in I$ there exists $k \in I$ such that $L_i, L_j \subseteq L_k$.

Set $i \le j$ if $L_i \subseteq L_j$. Then $I$ is a *directed* set, i.e. for each pair $i, j \in I$ there exists $k \in I$ such that $i, j \le k$. It is clear that $L$ is an inductive limit of the algebras $L_i$, that is $L = \varinjlim L_i$.

**Definition 2.2** A local system $\{L_i\}_{i\in I}$ of $L$ is called *perfect* if all $L_i$ are perfect.

**Definition 2.3** A locally finite Lie algebra is called *locally solvable* if it has a local system of solvable algebras and it is called *locally perfect* if no nontrivial quotient is locally solvable.

The following lemma explains this terminology.

**Lemma 2.4** *A locally finite Lie algebra is locally perfect if and only if it has a perfect local system.*

*Proof.* Let $L = \varinjlim L_i$ be locally perfect. For every $i \in I$ denote by $L_i^{(\infty)}$ the smallest member of the derived series of $L_i$. It is clear that $L_i^{(\infty)}$ is perfect, $L_i/L_i^{(\infty)}$ is solvable, and $L_i \subseteq L_j$ implies $L_i^{(\infty)} \subseteq L_j^{(\infty)}$. Hence $L^{(\infty)} = \varinjlim L_i^{(\infty)}$ is an ideal of $L$ and $L/L^{(\infty)}$ is locally solvable. This implies $L = L^{(\infty)}$. Therefore, $\{L_i^{(\infty)}\}_{i\in I}$ is a perfect local system of $L$.

Conversely, let $\{L_i\}_{i\in I}$ be a perfect local system of $L$, and $M$ be an ideal of $L$ with $L/M$ locally solvable. Then $M_i = M \cap L_i$ is an ideal of $L_i$ and the quotient $L_i/M_i$ is solvable. Since all $L_i$ are perfect, $M_i = L_i$ for all $i \in I$, forcing $M = L$.

**Remark 2.5** By the proof of Lemma 2.4, if $L = \varinjlim L_i$, then $\{L_i^{(\infty)}\}_{i\in I}$ is a perfect local system of the ideal $L^{(\infty)} = \varinjlim L_i^{(\infty)}$ of $L$ where $L_i^{(\infty)}$ is the smallest member of the derived series of $L_i$. If $L$ is locally perfect, then $L = L^{(\infty)}$.

In what follows, when we write $L = \varinjlim L_i$ for locally perfect $L$, we mean that $\{L_i\}_{i\in I}$ is a perfect local system of $L$. The significance of locally perfect Lie algebras is shown by the following

**Theorem 2.6** *Let $L$ be a locally finite Lie algebra. Then there exists a locally perfect ideal $P(L)$ of $L$ which contains all other locally perfect ideals of $L$. $P(L)$ is called the locally perfect radical of $L$. The quotient algebra $L/P(L)$ is locally solvable.*

*Proof.* Let $L = \varinjlim L_i$. Set $P(L) = \varinjlim L_i^{(\infty)}$ (see Remark 2.5). Then $P(L)$ is a locally perfect ideal of $L$, and $L/P(L)$ is locally solvable. Let $Q$ be a locally perfect ideal of $L$. Then $Q/(Q \cap P(L))$ is locally solvable. This implies $Q = Q \cap P(L)$, forcing $Q \subseteq P(L)$.

**Definition 2.7** Let $L$ be a locally finite Lie algebra. The largest locally solvable ideal $R(L)$ of $L$ is called the *locally solvable radical.* If $R(L) = 0$, then $L$ is called *semisimple.*

It is well known that the solvable radical of a perfect finite-dimensional Lie algebra is nilpotent. Consequently, the locally solvable radical of a locally perfect Lie algebra is locally nilpotent ($R(L) \cap L_i \subset \operatorname{Rad} L_i$).

**Proposition 2.8** *Let $L$ be a simple locally finite Lie algebra. Then $L$ is semisimple and locally perfect.*

*Proof.* We have to show that $R(L) = 0$ and $P(L) = L$ (see Theorem 2.6 and Definition 2.7). If $R(L) = 0$, then $P(L) \neq 0$. Since $L$ is simple, $P(L) = L$. Therefore, it suffices to prove that $R(L) \neq L$. Assume that $R(L) = L$. Then $L = \varinjlim L_i$ where all $L_i$ are solvable. Since $[L, L] \neq 0$, there exist $x, y \in L$ such that $z = [x, y] \neq 0$. Let $M$ be the ideal of $L$ generated by $z$. Since $L$ is simple, $M = L$, so $x, y \in M$. Therefore, there exists $i \in I$ such that $z \in L_i$ and $x, y \in N$ where $N$ is the ideal of $L_i$ generated by $z$. Take $n \in \mathbf{N}$ such that $z \in L_i^{(n-1)}$ and $z \notin L_i^{(n)}$. Then $x, y \in L_i^{(n-1)}$. Therefore, $z = [x, y] \in [L_i^{(n-1)}, L_i^{(n-1)}] = L_i^{(n)}$. This contradiction establishes the proposition.

Note that a result similar to Proposition 2.8 has been obtained earlier by Yu. Bahturin and H. Strade ([2], Corollary 3.2, Theorem 3.2).

The following notion introduced by A.E. Zalesskii is crucial for what follows.

**Definition 2.9** Let $L = \varinjlim L_i$ be a locally finite Lie algebra, $\Phi_i$ a finite subset of $\operatorname{Irr} L_i$. The set $\Phi = \{\Phi_i\}_{i\in I}$ is called an *inductive system* if $\langle \Phi_j {\downarrow} L_i \rangle = \Phi_i$ for each pair $i < j$.

# 3 Inductive systems and ideals

It is not difficult to show that there is a 1-1 correspondence between the inductive systems for a locally semisimple Lie algebra $L$ and the ideals $X$ of its universal enveloping algebra $U(L)$ with $U(L)/X$ locally finite (see [15]). In this section we extend this result to arbitrary locally perfect Lie algebras. Recall that the solvable radical of a perfect finite-dimensional Lie algebra $L$ coincides with the *nilpotent radical* of $L$, i.e. $\operatorname{Rad} L$ annihilates all irreducible $L$-modules (see, for instance, [6]). Therefore, we can state

**Lemma 3.1** *Let $L$ be a perfect finite-dimensional Lie algebra, $S$ a Levi subalgebra of $L$. Then the map $V \mapsto V{\downarrow}S$ is a 1-1 correspondence between irreducible $L$- and $S$-modules, respectively, (the inverse map is given by $W \mapsto W{\uparrow}L$). In particular, $V{\downarrow}S{\uparrow}L = V$ and $W{\uparrow}L{\downarrow}S = W$.*

Sometimes we shall identify irreducible $L$- and $S$-modules.

**Lemma 3.2** *Let $L$ be a finite-dimensional Lie algebra, $V_1$, $V_2$ be finite-dimensional $L$-modules such that $\operatorname{Ann}_{U(L)} V_1 = \operatorname{Ann}_{U(L)} V_2$. Then $\langle V_1\rangle = \langle V_2\rangle$.*

*Proof.* Set $A = U(L)/\operatorname{Ann}_{U(L)} V_1 = U(L)/\operatorname{Ann}_{U(L)} V_2$. Then $A$ is finite-dimensional, and $V_1$, $V_2$ are faithful $A$-modules. But for each faithful $A$-module $V$ the set $\langle V\rangle$ consists of all irreducible $A$-modules. Therefore, $\langle V_1\rangle = \langle V_2\rangle$ as $A$-modules. Hence $\langle V_1\rangle = \langle V_2\rangle$ as $L$-modules.

**Lemma 3.3** *Let $L$ be a finite-dimensional perfect Lie algebra, $\Phi$ a finite subset of $\operatorname{Irr} L$. Then codimensions of annihilators in $U(L)$ of all finite-dimensional $L$-modules $V$ such that $\langle V\rangle = \Phi$ are bounded by some constant depending on $L$ and $\Phi$.*

*Proof.* Let $L = S \oplus R$, where $S$ is a Levi subalgebra, $R$ is the radical. Let $\{r_i \mid i = 1, \dots, m\}$ be a basis of $R$. Then, by PBW theorem, $U(S)$ is a subalgebra of $U(L)$, and

$$U(L) = \oplus_{k_1,\dots,k_m} U(S) r_1^{k_1} \dots r_m^{k_m}.$$

Denote by $\mathfrak{M}(\Phi)$ the set of all finite-dimensional $L$-modules $V$ such that $\langle V\rangle = \Phi$. Let $V \in \mathfrak{M}(\Phi)$. It is clear that the image of the ideal $U(L)R$ in the quotient algebra $U_V = U(L)/\operatorname{Ann}_{U(L)} V$ is nilpotent. Since the ideal

$$U(S) \cap \operatorname{Ann}_{U(L)} V = \operatorname{Ann}_{U(S)} V = \cap_{\varphi\in\Phi} \operatorname{Ann}_{U(S)} \varphi$$

of $U(S)$ is semiprimitive and does not depend on the choice of $V$, the subalgebra $P = U(S)/U(S) \cap \operatorname{Ann}_{U(L)} V$ of $U_V$ is semisimple and the same for all $V \in \mathfrak{M}(\Phi)$. Hence $R_V = U(L)R/U(L)R \cap \operatorname{Ann}_{U(L)} V$ is the radical of $U_V$. We have $U_V = P \oplus R_V$. Let $\{p_j \mid j = 1, \dots, l\}$ be a basis of $P$, $\bar r_i$ the image of $r_i$ in $U_V$. Then the elements $p_j\bar r_i$, $\bar r_i$ generate $R_V$, so $R_V$ is a finitely generated algebra. Suppose that there exists $n = n(L, \Phi)$ such that $R_V^n = 0$ (or equivalently, $R^n V = 0$) for all $V \in \mathfrak{M}(\Phi)$. Then it is easy to see that the dimensions of all $R_V, V \in \mathfrak{M}(\Phi)$, are bounded by some constant. Therefore, the same is true for all algebras $U_V, V \in \mathfrak{M}(\Phi)$, as required.

So we have to show that there exists $n = n(L, \Phi)$ such that $R^n V = 0$ for all $V \in \mathfrak{M}(\Phi)$. We proceed by induction on $\dim R$, the case $R = 0$ being clear. Assume that $\dim R > 0$. Since $R$ is a nilpotent Lie algebra, it has the nontrivial center $Z(R)$. As $S$ is semisimple, $Z(R)$ is a completely reducible $S$-module (with respect to the adjoint action). Consider the following two cases.

*Case 1.* $Z(R)$ contains an irreducible submodule $Z$ such that $\dim Z > 1$.

We have $\langle V{\downarrow}S\rangle = \langle V\rangle{\downarrow}S = \Phi{\downarrow}S$. Therefore, the set $\Lambda$ of weights of the module $V{\downarrow}S$ does not depend on the choice of $V \in \mathfrak{M}(\Phi)$ and coincides with the set of all weights of the modules from $\Phi{\downarrow}S$. Denote by $\Omega$ the set of weights of the $S$-module $Z$. We have the decomposition of $V$ and $Z$ in the sum of weight spaces: $V = \oplus_{\lambda\in\Lambda} V_\lambda$, $Z = \oplus_{\omega\in\Omega} Z_\omega$. Since $\Lambda$ is finite, there exists $k \in \mathbf{N}$ such that $(k\omega + \lambda) \notin \Lambda$ for all nonzero $\omega \in \Omega$ and all $\lambda \in \Lambda$. Let $\omega \neq 0$, $z_\omega \in Z_\omega$, $v_\lambda \in V_\lambda$. Consider the element $z_\omega^k v_\lambda \in V$. Let $h$ be an element from the Cartan subalgebra of $S$. Then

$$hz_\omega^k v_\lambda = [h, z_\omega] z_\omega^{k-1} v_\lambda + z_\omega [h, z_\omega] z_\omega^{k-2} v_\lambda + \ldots + z_\omega^{k-1} [h, z_\omega] v_\lambda + z_\omega^k h v_\lambda =$$

$$(k\omega(h) + \lambda(h)) z_\omega^k v_\lambda = (k\omega + \lambda)(h) z_\omega^k v_\lambda.$$

Since $\Lambda$ does not contain the weight $k\omega + \lambda$, we have $z_\omega^k v_\lambda = 0$ for all $v_\lambda$. Hence $z_\omega^k \in \operatorname{Ann}_{U(L)} V$. Now consider elements of $Z_0$ (if $Z_0 \neq 0$). Denote by $E_0$ the linear subspace of $Z_0$ generated by all elements of type $[s_\alpha, z_{-\alpha}]$ where $\alpha$ is a root of $S$, $s_\alpha \in S_\alpha$, $z_{-\alpha} \in Z_{-\alpha}$. It is clear that $\oplus_{\omega\neq 0} Z_\omega \oplus E_0$ is an $S$-submodule of $Z$. The irreducibility of $Z$ forces $E_0 = Z_0$. Thus, the elements of type $[s_\alpha, z_{-\alpha}]$ generate $Z_0$ as a linear space. Since $[Z, Z] = 0$ (i.e. $U(Z)$ is commutative), and $z_{-\alpha}^k \in \operatorname{Ann}_{U(L)} V$ (as the weight $\alpha$ is nonzero), we have

$$(\operatorname{ad} s_\alpha)^k z_{-\alpha}^k = k![s_\alpha, z_{-\alpha}]^k + z_{-\alpha} u \in \operatorname{Ann}_{U(L)} V$$

where $u \in Z^{k-1}$. This implies

$$[s_\alpha, z_{-\alpha}]^{k^2} \equiv (-1/k!)^k (z_{-\alpha} u)^k \equiv (-1/k!)^k z_{-\alpha}^k u^k \equiv 0 \pmod{\operatorname{Ann}_{U(L)} V}.$$

Therefore, there exists a basis $\{z_i \mid i = 1, \ldots, p\}$ of the module $Z$ such that $z_i^{k^2} \in \operatorname{Ann}_{U(L)} V$ for $i = 1, \ldots, p$ and all $V \in \mathfrak{M}(\Phi)$. In view of commutativity of $U(Z)$, there exists $m \in \mathbf{N}$ such that $Z^m \in \operatorname{Ann}_{U(L)} V$ for all $V \in \mathfrak{M}(\Phi)$. Consider a chain of $L$-modules

$$V \supseteq ZV \supseteq \ldots \supseteq Z^{m-1} V \supseteq Z^m V = 0$$

Set $L' = L/Z$, $W = \oplus_{\varphi\in\Phi}\varphi$, $Q_j = Z^{j-1}V/Z^j V$, $\bar{Q}_j = Q_j \oplus W$, $j = 1, \ldots, m$. Since $Z$ annihilates $W$ and all $Q_j$, the modules $\bar{Q}_j$ can be considered as $L'$-modules. The Lie algebra $L'$ is perfect and contains $S$ as a Levi subalgebra. We have

$$\langle \bar{Q}_j{\downarrow}L'\rangle{\downarrow}S = \langle \bar{Q}_j{\downarrow}S\rangle = \langle Q_j{\downarrow}S\rangle \cup \langle \Phi{\downarrow}S\rangle = \langle \Phi{\downarrow}S\rangle.$$

In view of Lemma 3.1, one can assume $\langle \bar{Q}_j{\downarrow}L'\rangle = \Phi$, $j = 1, \ldots, m$. It is valid because we use only properties of weights of $\Phi{\downarrow}S$. Note that $R' = R/Z$ is the radical of $L'$. Since $\dim R' < \dim R$, by inductive hypothesis, there exists $n = n(L', \Phi)$, such that

$R'^n\bar{Q}_j = 0$ for all $j$. In particular, $R'^n Q_j = 0$, so $R^n(Z^{j-1}V/Z^jV) = 0$, $j = 1, \dots, m$. This implies $R^{nm}V = 0$ for all $V \in \mathfrak{M}(\Phi)$, as required.

*Case 2.* $Z(R)$ does not contain nontrivial $S$-submodules.

Since $L = S \oplus R$ is perfect, $R = [L, L] \cap R = [R, R] + [S, R]$. Therefore, $R \neq 0$ implies $[S, R] \neq 0$, so the $S$-module $R$ contains an element $r_\omega$ of a nonzero weight $\omega$. Recall that $R$ is nilpotent. Therefore, there exist $a_1, \dots, a_t \in R, k_1, \dots, k_t \in \mathbf{N}$, such that

$$z = (\operatorname{ad} a_1)^{k_1} \dots (\operatorname{ad} a_t)^{k_t} r_\omega$$

is a nonzero element of $Z(R)$, but

$$(\operatorname{ad} a_i)^{k_i+1}(\operatorname{ad} a_{i+1})^{k_{i+1}} \dots (\operatorname{ad} a_t)^{k_t} r_\omega = 0$$

for each $i = 1, \dots, t$. As in *Case* 1, take $m \in \mathbf{N}$ such that $(m\omega + \lambda) \notin \Lambda$ for all $\lambda \in \Lambda$. Then $r_\omega^m \in \operatorname{Ann}_{U(L)} V$. Since

$$(\operatorname{ad} a_1)^{mk_1} \dots (\operatorname{ad} a_t)^{mk_t} r_\omega^m = c_t (\operatorname{ad} a_1)^{mk_1} \dots (\operatorname{ad} a_{t-1})^{mk_{t-1}} ((\operatorname{ad} a_t)^{k_t} r_\omega)^m = \dots =$$

$$c((\operatorname{ad} a_1)^{k_1} \dots (\operatorname{ad} a_t)^{k_t} r_\omega)^m = cz^m$$

where $c_t, c \in F$, we conclude that $z^m \in \operatorname{Ann}_{U(L)} V$. Denote by $Z$ the one-dimensional subspace generated by $z$. Then $Z$ is an ideal of $L$ and $Z^m V = 0$ for all $V \in \mathfrak{M}(\Phi)$. Arguments analogous to those of *Case 1* (the inductive hypothesis is applied to $L' = L/Z$) show that there exists $n$ such that $R^{nm}V = 0$ for all $V \in \mathfrak{M}(\Phi)$, as required.

**Theorem 3.4** *Let $L$ be a perfect finite-dimensional Lie algebra, $\Phi$ a finite subset of* $\operatorname{Irr} L$, *$\mathfrak{F}(\Phi)$ the set of all ideals $X$ of $U(L)$ such that $U(L)/X$ is finite-dimensional and $\langle U(L)/X \rangle = \Phi$. Then $\mathfrak{F}(\Phi)$ is nonempty and has the smallest element $N(\Phi)$ and the largest element $M(\Phi)$ such that $N(\Phi) \subseteq X \subseteq M(\Phi)$ for all $X \in \mathfrak{F}(\Phi)$. The algebra $U(L)/M(\Phi)$ is semisimple, the algebra $M(\Phi)/N(\Phi)$ is nilpotent.*

*Proof.* Assume that $X \in \mathfrak{F}(\Phi)$. Then $X$ is the annihilator of the $U(L)$-module $V = U(L)/X$. Since the set of composition factors of $V$ is $\Phi$, we have $X \subseteq M(\Phi)$ where $M(\Phi)$ is the annihilator of the completely reducible $U(L)$-module $\oplus_{\varphi \in \Phi} \varphi$. It is clear that $U(L)/M(\Phi)$ is semisimple, and $(M(\Phi)/X)^k = 0$ where $k$ is the number of composition factors of $V$. Therefore, $M(\Phi)$ is the largest element of $\mathfrak{F}(\Phi)$. By Lemma 3.3, codimensions of all ideals from $\mathfrak{F}(\Phi)$ are bounded by some constant. Therefore, every descending chain of ideals from $\mathfrak{F}(\Phi)$ stabilizes. On the other hand, the intersection of any two ideals from $\mathfrak{F}(\Phi)$ belongs to $\mathfrak{F}(\Phi)$ again (the annihilator of the sum of modules is the intersection of their annihilators). Therefore, $\mathfrak{F}(\Phi)$ has the smallest element $N(\Phi)$. Since $N(\Phi) \in \mathfrak{F}(\Phi)$, we get $(M(\Phi)/N(\Phi))^k = 0$ for some $k$.

It is convenient to assume that $\mathfrak{F}(\emptyset) = \{U(L)\}$. Observe that $\mathfrak{F}(\Phi)$ is the set of all ideals $X$ of $U(L)$ such that $N(\Phi) \subseteq X \subseteq M(\Phi)$. So we obtain

**Corollary 3.5** *The set $\mathfrak{F}(\Phi)$ is a finite sublattice of the lattice of all ideals of $U(L)$.*

If $L$ is semisimple, then for each $X \in \mathfrak{F}(\Phi)$ the $L$-module $U(L)/X$ is completely reducible. Therefore, $U(L)/X$-module $U(L)/X$ is completely reducible. Hence the algebra $U(L)/X$ is semisimple. But $\mathfrak{F}(\Phi)$ contains only one semiprimitive ideal $M(\Phi)$. Therefore, we get

**Corollary 3.6** *If $L$ is semisimple, then $|\mathfrak{F}(\Phi)| = 1$ (i.e. $N(\Phi) = M(\Phi)$).*

**Lemma 3.7** *Let $L_1 \subseteq L_2$ be perfect finite-dimensional Lie algebras, $U(L_1) \subseteq U(L_2)$; $\Phi_1$, $\Phi_2$ finite subsets of $\operatorname{Irr} L_1$, $\operatorname{Irr} L_2$, respectively, such that $\langle \Phi_2 \downarrow L_1 \rangle = \Phi_1$. Then $X \in \mathfrak{F}(\Phi_2)$ implies $X \cap U(L_1) \in \mathfrak{F}(\Phi_1)$.*

*Proof.* Since $\operatorname{Ann}_{U(L_1)}(U(L_2)/X) = X \cap U(L_1)$, by Lemma 3.2,

$$\langle U(L_1)/X \cap U(L_1) \rangle = \langle U(L_2)/X \downarrow L_1 \rangle = \langle \Phi_2 \downarrow L_1 \rangle = \Phi_1.$$

Hence $X \cap U(L_1) \in \mathfrak{F}(\Phi_1)$.

**Lemma 3.8** *Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $X$ be an ideal of $U(L)$ with $U(L)/X$ locally finite. Then the set*

$$\Phi(X) = \{\langle U(L_i)/X \cap U(L_i) \rangle\}_{i \in I}$$

*is an inductive system for $L$.*

*Proof.* Since $U(L)/X$ is locally finite, $U(L_i)/X \cap U(L_i)$ is finite-dimensional, so the set $\Phi_i(X) = \langle U(L_i)/X \cap U(L_i) \rangle$ is finite. Further, if $i \leq j$ then $\operatorname{Ann}_{U(L_i)}(U(L_j)/X \cap U(L_j)) = X \cap U(L_i)$. Therefore, by Lemma 3.2,

$$\langle \Phi_j(X) \downarrow L_i \rangle = \langle \langle U(L_j)/X \cap U(L_j) \rangle \downarrow L_i \rangle = \langle U(L_j)/X \cap U(L_j) \downarrow L_i \rangle =$$

$$\langle U(L_i)/X \cap U(L_i) \rangle = \Phi_i(X),$$

so $\Phi(X)$ is an inductive system for $L$.

Denote by $\mathfrak{IS}$, $\mathfrak{LF}$ the sets of inductive systems for a locally perfect Lie algebra $L$, and ideals $X$ of its universal enveloping algebra $U(L)$ with $U(L)/X$ locally finite, respectively. Define the map $f : \mathfrak{LF} \to \mathfrak{IS}$, setting $f(X) = \Phi(X)$ where $X \in \mathfrak{LF}$, $\Phi(X)$ as in Lemma 3.8. Denote by $\mathfrak{LF}(\Phi)$ the inverse image of an inductive system $\Phi$.

**Theorem 3.9** *Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, and the map $f : \mathfrak{LF} \to \mathfrak{IS}$ be as above. Then for each inductive system $\Phi$ the set $\mathfrak{LF}(\Phi)$ is nonempty and has the smallest element $N(\Phi)$ and the largest element $M(\Phi)$ such that $N(\Phi) \subseteq X \subseteq M(\Phi)$ for all $X \in \mathfrak{LF}(\Phi)$. The algebra $U(L)/M(\Phi)$ is semisimple, the algebra $M(\Phi)/N(\Phi)$ is locally nilpotent. Moreover, the map $f$ produces a 1-1 correspondence between the semiprimitive ideals from $\mathfrak{LF}$ and the inductive systems for $L$ (the inverse map is given by $\Phi \mapsto M(\Phi)$).*

*Proof.* Let $\Phi = \{\Phi_i\}_{i\in I}$ be an inductive system. For every $i \in I$ denote by $\mathfrak{F}(\Phi_i)$ the set of ideals $X$ of $U(L_i)$ such that $U(L_i)/X$ is finite-dimensional and $\langle U(L_i)/X\rangle = \Phi_i$. Then by Theorem 3.4, $\mathfrak{F}(\Phi_i)$ is nonempty and has the smallest element $N(\Phi_i)$. Let $i \leq j$. Then $L_i \subseteq L_j$, $U(L_i) \subseteq U(L_j)$ and by definition of an inductive system, $\langle \Phi_j{\downarrow}L_i\rangle = \Phi_i$. By Lemma 3.7, $N(\Phi_j)\cap U(L_i) \in \mathfrak{F}(\Phi_i)$, so $N(\Phi_i) \subseteq N(\Phi_j)$. Since for each $i$ $N(\Phi_i)$ is an ideal of $U(L_i)$, we conclude that $N(\Phi) = \varinjlim N(\Phi_i)$ is an ideal of $U(L)$. Observe that $U(L)/N(\Phi)$ is locally finite. It is clear that for each $i \in I$ there exists $j \geq i$ such that $N(\Phi) \cap U(L_i) = N(\Phi_j) \cap U(L_i) \in \mathfrak{F}(\Phi_i)$. Therefore, $\langle U(L_i)/N(\Phi) \cap U(L_i)\rangle = \Phi_i$, forcing $N(\Phi) \in \mathfrak{LF}(\Phi)$. Denote by $M(\Phi)$ the inverse image of the Jacobson radical of $U(L)/N(\Phi)$ in $U(L)$. Recall that the Jacobson radical of a locally finite algebra is locally nilpotent. Therefore, the algebra $(M(\Phi) \cap U(L_i))/(N(\Phi) \cap U(L_i))$ is nilpotent for each $i \in I$. Since $N(\Phi) \cap U(L_i) \in \mathfrak{F}(\Phi_i)$, we have $M(\Phi) \cap U(L_i) \in \mathfrak{F}(\Phi_i)$, so $M(\Phi) \in \mathfrak{LF}(\Phi)$. Let $X \in \mathfrak{LF}(\Phi)$. By definition, $\langle U(L_i)/X \cap U(L_i)\rangle = \Phi_i$. Hence $X \cap U(L_i) \in \mathfrak{F}(\Phi_i)$. By Theorem 3.4, $N(\Phi_i) \subseteq X \cap U(L_i)$, and $X \cap U(L_i)/N(\Phi_i)$ is nilpotent. Consequently, $N(\Phi) \subseteq X$, and $X/N(\Phi)$ is locally nilpotent, forcing $N(\Phi) \subseteq X \subseteq M(\Phi)$. This completes the proof.

Note that $\mathfrak{LF}(\Phi)$ is the set of all ideals $X$ of $U(L)$ such that $N(\Phi) \subseteq X \subseteq M(\Phi)$. So the set $\mathfrak{LF}(\Phi)$ is a sublattice of the lattice of all ideals of $U(L)$. By Lemma 3.7, $X \cap U(L_i) \in \mathfrak{F}(\Phi_i)$ for $i \leq j$ and $X \in \mathfrak{F}(\Phi_j)$, so we have a morphism of finite lattices $f_{ji} : \mathfrak{F}(\Phi_j) \to \mathfrak{F}(\Phi_i)$. Therefore we have

**Corollary 3.10** *The lattice $\mathfrak{LF}(\Phi)$ is a projective limit of the finite lattices $\mathfrak{F}(\Phi_i)$.*

If all $L_i$ are semisimple, then by Corollary 3.6, all $\mathfrak{F}(\Phi_i)$ are one-element sets. So we obtain already known result:

**Corollary 3.11** *If $L = \varinjlim L_i$ where all $L_i$ are semisimple (i.e. $L$ is locally semisimple), then $|\mathfrak{LF}(\Phi)| = 1$ (i.e. $N(\Phi) = M(\Phi)$) for all inductive systems $\Phi$.*

# 4 Abstract Levi subalgebras

**Definition 4.1** A subalgebra $S$ of a locally finite Lie algebra $L = \varinjlim L_i$ is called a *Levi subalgebra* associated with the local system $\{L_i\}_{i\in I}$, if $S = \varinjlim S_i$ where $S_i$ is a Levi subalgebra of $L_i$ and $S_i \subseteq S_j$ for each pair $i \leq j$.

Note that a Levi subalgebra is locally semisimple.

**Lemma 4.2** *If a locally perfect Lie algebra $L$ has a Levi subalgebra $S$, then $S$ is associated with a perfect local system of $L$.*

*Proof.* Assume that $S$ is associated with a local system $\{L_i\}_{i\in I}$. Since $L$ is locally perfect, by Remark 2.5, $\{L_i^{(\infty)}\}_{i\in I}$ is a perfect local system of $L$. Since $S_i = S \cap L_i$ is semisimple, we obtain $S_i \subseteq L_i^{(\infty)}$. Hence $S_i$ is a Levi subalgebra of $L_i^{(\infty)}$, so $S$ is associated with $\{L_i^{(\infty)}\}_{i\in I}$.

It is not clear, whether all locally finite Lie algebras have Levi subalgebras. However, we have

**Lemma 4.3** *Locally finite Lie algebras of countable dimensions have Levi subalgebras.*

*Proof.* Assume that $L$ is a locally finite Lie algebra of countable dimension. Then $L = \cup_{i\in\mathbf{N}} L_i$ where $L_i \subset L_{i+1}$ for $i \in \mathbf{N}$. Since every semisimple subalgebra of a finite-dimensional Lie algebra lies in a Levi subalgebra, for each $i \in \mathbf{N}$ there exists a Levi subalgebra $S_i$ of $L_i$ such that $S_{i-1} \subseteq S_i$, so $S = \cup_{i\in\mathbf{N}} S_i$ is a Levi subalgebra of $L$.

It is not difficult to show (using Lemma 3.1) that the description of inductive systems for a locally perfect Lie algebra is equivalent to the similar question for its Levi subalgebra. So, by Lemma 4.3, the case of Lie algebras of countable dimensions is clear. To handle the general case, we need to introduce the notion of an abstract Levi subalgebra.

**Definition 4.4** Let $L = \varinjlim L_i$ be a locally finite Lie algebra. For every $i \in I$ select a Levi subalgebra $S_i$ of $L_i$. The set $\mathfrak{S} = \{S_i\}_{i\in I}$ is called an *abstract Levi subalgebra* of $L$ associated with the local system $\{L_i\}_{i\in I}$.

Now we are going to introduce the notion of an inductive system for an abstract Levi subalgebra. For this we need

**Lemma 4.5** *Let $L_1 \subseteq L_2$ be finite-dimensional Lie algebras; $S_1$, $S_2$ Levi subalgebras of $L_1$, $L_2$, respectively. Then there exists an automorphism $\theta$ of $L_2$ (called special) such that $\theta(S_1) \subseteq S_2$, $\theta(l) = l + r(l)$ for all $l \in L_2$ where $r(l)$ are elements in the nilpotent radical of $L_2$. Moreover, the monomorphism $S_1 \to S_2$ induced by $\theta$ does not depend on the choice of such $\theta$.*

*Proof.* The existence of such an automorphism follows from the Levi-Malcev theorem. For any two such automorphisms $\theta_1$, $\theta_2$ and any $s \in S_1$ we have $\theta_1(s) = s + r_1(s) \in S_2$, $\theta_2(s) = s + r_2(s) \in S_2$, so $r_1(s) - r_2(s) \in S_2$. Since $S_2 \cap \operatorname{Rad} L_2 = 0$, we have $r_1(s) = r_2(s)$ and $\theta_1(s) = \theta_2(s)$ for all $s \in S_2$.

**Definition 4.6** Let $\mathfrak{S} = \{S_i\}_{i\in I}$ be an abstract Levi subalgebra of $L = \varinjlim L_i$, $\Psi_i$ be a finite subset of $\operatorname{Irr} S_i$. The set $\Psi = \{\Psi_i\}_{i\in I}$ is called an *inductive system* for $\mathfrak{S}$, if $\langle \Psi_j{\downarrow}^{\theta_{ij}}_{S_i} \rangle = \Psi_i$ for each pair $i \le j$ where $\theta_{ij} : S_i \to S_j$ is the monomorphism described by Lemma 4.5.

**Proposition 4.7** *Let $L_1$, $L_2$, $S_1$, $S_2$, $\theta$ be as in Lemma 4.5. Let also $L_1$, $L_2$ be perfect, $V$ be an irreducible $L_2$-module, $W = V{\downarrow}S_2$. Then $\langle V{\downarrow}L_1 \rangle = \langle W{\downarrow}^{\theta}_{S_1} \rangle{\uparrow}L_1$ and $\langle V{\downarrow}L_1 \rangle{\downarrow}S_1 = \langle W{\downarrow}^{\theta}_{S_1} \rangle$.*

*Proof.* Since $V$ is irreducible, $(\operatorname{Rad} L_2)V = 0$, so $\theta(s)v = (s + r(s))v = sv$ for all $s \in S_1$, $v \in V$. Hence $W{\downarrow}^{\theta}_{S_1} = V{\downarrow}^{\theta}_{S_1} = V{\downarrow}S_1$, and, applying Lemma 3.1, we obtain the required equalities.

**Corollary 4.8** *Let $\mathfrak{S} = \{S_i\}_{i\in I}$ be an abstract Levi subalgebra of $L = \varinjlim L_i$. Then for each triple $i \le j \le k$ and each irreducible $S_k$-module $W$ we have $\langle W{\downarrow}^{\theta_{ik}}_{S_i} \rangle = \langle \langle W{\downarrow}^{\theta_{jk}}_{S_j} \rangle{\downarrow}^{\theta_{ij}}_{S_i} \rangle$.*

*Proof.* Set $V = W\uparrow L_k$. It is clear that $\langle V\downarrow L_i\rangle = \langle\langle V\downarrow L_j\rangle\downarrow L_i\rangle$ . By Proposition 4.7,

$$\langle V\downarrow L_i\rangle\downarrow S_i = \langle W\downarrow^{\theta_{ik}}_{S_i}\rangle,$$

$$\langle\langle V\downarrow L_j\rangle\downarrow L_i\rangle\downarrow S_i = \langle\langle\langle V\downarrow L_j\rangle\downarrow S_j\rangle\downarrow^{\theta_{ij}}_{S_i}\rangle = \langle\langle W\downarrow^{\theta_{jk}}_{S_j}\rangle\downarrow^{\theta_{ij}}_{S_i}\rangle,$$

so the statement follows.

In view of Proposition 4.7 and Lemma 3.1, we can state

**Theorem 4.9** *Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $\mathfrak{S} = \{S_i\}_{i\in I}$ be an abstract Levi subalgebra of $L$. Then the map $\Phi = \{\Phi_i\}_{i\in I} \mapsto \Phi\downarrow\mathfrak{S} = \{\Phi_i\downarrow S_i\}_{i\in I}$ is a 1-1 correspondence between the inductive systems for $L$ and those for $\mathfrak{S}$ (the inverse map is given by $\Psi = \{\Psi_i\}_{i\in I} \mapsto \Psi\uparrow L = \{\Psi_i\uparrow L_i\}_{i\in I}$).*

Since we deal only with systems of modules over an abstract Levi subalgebra $\mathfrak{S} = \{S_i\}_{i\in I}$, the Proposition 4.7 and the Corollary 4.8 enable us to assume that $S_i$ is a "subalgebra" of $S_j$ for $i < j$, so one handles abstract Levi subalgebras in the same manner as ordinary ones (Definition 4.1). In particular, we often omit the symbol $\theta_{ij}$ in $\langle W\downarrow^{\theta_{ij}}_{S_i}\rangle$ where $W$ is an $S_j$-module. By Proposition 4.7, the branching rules for an abstract Levi subalgebra $\mathfrak{S} = \{S_i\}_{i\in I}$ do not depend on the choice of $S_i$. So in what follows we do not distinguish abstract Levi subalgebras associated with the same local system.

# 5 Diagonal Lie algebras

Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $\mathfrak{S} = \{S_i\}_{i\in I}$ an abstract Levi subalgebra associated with $\{L_i\}_{i\in I}$. Let $S_i^1, S_i^2, \ldots, S_i^{n_i}$ be the simple components of $S_i$, i.e. $S_i = S_i^1 \oplus S_i^2 \oplus \ldots \oplus S_i^{n_i}$. Recall that each irreducible $S_i$-module $M$ can be written in the (canonical) form $M = M_1 \otimes \ldots \otimes M_{n_i}$ where $M_k$ is an irreducible $S_i$-module such that $M_k\downarrow S_i^k$ is irreducible and $\langle M_k\downarrow S_i^l\rangle = \{T_i^l\}$ for $l \neq k$ were $T_i^l$ is the trivial one-dimensional $S_i^l$-module. Denote by $V_i^k$ the standard module for classical $S_i^k$ and the module of the minimal dimension for exceptional $S_i^k$. We shall identify this module with the corresponding irreducible $S_i$-module (such that all $S_i^l$ act trivially for $l \neq k$). Put $\mathfrak{V}_i = \{V_i^1, V_i^2, \ldots, V_i^{n_i}\}$, $\mathfrak{T}_i = \cup_{j\geq i}\langle\mathfrak{V}_j\downarrow S_i\rangle$, and $\mathfrak{T}_i^k = \cup_{j\geq i}\langle\mathfrak{V}_j\downarrow S_i^k\rangle = \langle\mathfrak{T}_i\downarrow S_i^k\rangle$.

**Definition 5.1** An abstract Levi subalgebra $\mathfrak{S} = \{S_i\}_{i\in I}$ is called *diagonal* if the set $\mathfrak{T}_i$ is finite for each $i \in I$ (or equivalently, $\mathfrak{T}_i^k$ is finite for each $i \in I$ and each $k = 1, \ldots, n_i$), otherwise $\mathfrak{S}$ is called *nondiagonal.*

**Definition 5.2** A locally perfect Lie algebra $L = \varinjlim L_i$ is called *diagonal (nondiagonal)* (with respect to the local system $\{L_i\}_{i\in I}$) if an abstract Levi subalgebra $\mathfrak{S}$ associated with the local system $\{L_i\}_{i\in I}$ is diagonal (nondiagonal).

Observe, that the notion of diagonality in Definition 5.2 does not depend on the choice of an abstract Levi subalgebra $\mathfrak{S}$ of $L$ associated with the given local system (see the remarks at the end of Section 4). But the Definition 5.2 permits $L$ to be diagonal and nondiagonal simultaneously (with respect to the distinct local systems). However, we will show below (Corollary 5.9) that if a locally perfect Lie algebra is diagonal with respect to some perfect local system, then it is diagonal with respect to each one. This implies immediately that nondiagonal Lie algebras are exactly those which are not diagonal. So one can introduce

**Definition 5.3** A locally finite Lie algebra is called *diagonal (nondiagonal)* if its locally perfect radical is diagonal (nondiagonal).

**Definition 5.4** An inductive system $\Psi = \{\Psi_i\}_{i\in I}$ for an abstract Levi subalgebra $\mathfrak{S} = \{S_i\}_{i\in I}$ is called *nondegenerate* if $\langle \Psi_i \downarrow S_i^k \rangle \neq \{T_i^k\}$ for each $i \in I$ and each $k = 1, \ldots, n_i$, otherwise $\Psi$ is called *degenerate*. An inductive system $\Phi = \{\Phi_i\}_{i\in I}$ for a locally perfect Lie algebra $L = \varinjlim L_i$ is called *nondegenerate (degenerate)* if $\Phi \downarrow \mathfrak{S} = \{\Phi_i \downarrow S_i\}_{i\in I}$ is a nondegenerate (degenerate) inductive system for an abstract Levi subalgebra $\mathfrak{S}$ of $L$ associated with the local system $\{L_i\}_{i\in I}$.

**Lemma 5.5** *An abstract Levi subalgebra $\mathfrak{S} = \{S_i\}_{i\in I}$ is diagonal if and only if there exists a nondegenerate inductive system for $\mathfrak{S}$.*

*Proof of necessity.* Since $\mathfrak{S}$ is diagonal, the set $\mathfrak{T}_i = \cup_{j\geq i} \langle \mathfrak{V}_j \downarrow S_i \rangle$ is finite for each $i \in I$. Hence for each $i \in I$ there exists $p = p(i) > i$ such that for every $q \geq p$ we have $\cup_{j\geq q} \langle \mathfrak{V}_j \downarrow S_i \rangle = \cup_{j\geq p} \langle \mathfrak{V}_j \downarrow S_i \rangle$. Put $\Psi_i = \cup_{j\geq p} \langle \mathfrak{V}_j \downarrow S_i \rangle$. Observe that $\Psi_i \subseteq \mathfrak{T}_i$, so $\Psi_i$ is finite. Show that $\Psi = \{\Psi_i\}_{i\in I}$ is a nondegenerate inductive system for $\mathfrak{S}$. Since $i < p = p(i)$, we have the monomorphism $\theta_{ip} : S_i \to S_p$, so $\langle \mathfrak{V}_p \downarrow S_i^k \rangle \neq \{T_i^k\}$ for $k = 1, \ldots, n_i$. This implies $\langle \Psi_i \downarrow S_i^k \rangle \neq \{T_i^k\}$ for $k = 1, \ldots, n_i$, so $\Psi$ is nondegenerate. Further, let $i < j$. Find $q \in I$ such that $q \geq p(i), p(j)$. Then

$$\langle \Psi_j \downarrow S_i \rangle = \langle \cup_{l\geq q} \langle \mathfrak{V}_l \downarrow S_j \rangle \downarrow S_i \rangle = \cup_{l\geq q} \langle \mathfrak{V}_l \downarrow S_i \rangle = \Psi_i,$$

so $\Psi$ is an inductive system for $\mathfrak{S}$.

The sufficiency in Lemma 5.5 we shall prove in Section 7.

**Lemma 5.6** *Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $\Phi = \{\Phi_i\}_{i\in I}$ a nondegenerate inductive system for $L$. Then for each ideal $X$ of $U(L)$ such that $X \in \mathfrak{LF}(\Phi)$ (see Theorem 3.9), $X \cap L$ is a locally solvable ideal of $L$. Conversely, if $X$ is an ideal of $U(L)$ such that $U(L)/X$ is locally finite, and $X \cap L$ is a locally solvable ideal of $L$, then the inductive system $\Phi(X) = \{\Phi_i(X)\}_{i\in I}$ is nondegenerate, where $\Phi_i(X) = \langle U(L_i)/X \cap U(L_i) \rangle$.*

*Proof.* Assume that $X \cap L$ is not a locally solvable ideal of $L$. Then there exists $i \in I$ such that $X \cap L_i$ is not a solvable ideal of $L_i$. Hence $X \cap S_i \neq 0$ for a Levi subalgebra $S_i$ of $L_i$, so $S_i^k \subset X$ for some $k$. Therefore,

$$\langle \Phi_i \downarrow S_i^k \rangle = \langle U(L_i)/X \cap U(L_i) \downarrow S_i^k \rangle = \{T_i^k\}.$$

It follows that $\Phi \downarrow \mathfrak{S}$ is degenerate, where $\mathfrak{S} = \{S_i\}_{i\in I}$ is an abstract Levi subalgebra of $L$, so $\Phi$ is degenerate too.

Conversely, assume that $X \cap L$ is a locally solvable ideal of $L$. Then $X \cap S_i = 0$ for all $i \in I$. Therefore,

$$\langle \Phi_i(X) \downarrow S_i^k \rangle = \langle U(L_i)/X \cap U(L_i) \downarrow S_i^k \rangle \neq \{T_i^k\}$$

for each $i \in I$, $k = 1, \dots, n_i$, so $\Phi(X)$ is nondegenerate.

Let $\Phi$ be a degenerate inductive system for $L$, $X \in \mathfrak{LF}(\Phi)$. Then by Lemma 5.6, $X \cap L$ is not locally solvable ideal of $L$. Assume that $L$ is simple. Then $L \subset X$, so $X$ is the *augmentation* ideal of $U(L)$ (i.e. the ideal generated by $L$). Hence $\Phi = \Phi(X)$ is the trivial inductive system for $L$, i.e. $\Phi_i = \{T_i\}$, $i \in I$. So we conclude that all nontrivial inductive systems for a simple locally finite Lie algebra are nondegenerate.

Now we can prove the main result of this section.

**Theorem 5.7** *Let $L$ be a locally perfect Lie algebra, $\{L_i\}_{i \in I}$ a perfect local system of $L$. Then the following conditions are equivalent.*

*(a) $L$ is diagonal (with respect to $\{L_i\}_{i \in I}$).*

*(b) There exists a nondegenerate inductive system for $L$ (with respect to $\{L_i\}_{i \in I}$).*

*(c) There exists an ideal $X$ of $U(L)$ such that $U(L)/X$ is locally finite and $L \cap X$ is a locally solvable ideal of $L$.*

*(d) There exists a locally solvable ideal $R$ of $L$ such that $L/R$ can be embedded into a locally finite associative algebra.*

*Proof.* The equivalence (a)⇔(b) follows from the Definition 5.2, Definition 5.4 and Lemma 5.5. The equivalence (b)⇔(c) follows from Lemma 5.6. The equivalence (c)⇔(d) is obvious.

**Remark 5.8** It is not difficult to see that one can assume the ideal $X$ in (c) to be semiprimitive and the associative algebra in (d) to be semisimple.

The following statement eliminates the shortcoming in Definition 5.2.

**Corollary 5.9** *If a locally perfect Lie algebra is diagonal with respect to some local system, then it is diagonal with respect to each one.*

*Proof.* This follows from the equivalence (a)⇔(c) in the Theorem 5.7.

If $L$ is semisimple, then it does not contain locally solvable ideals. Therefore, we have

**Corollary 5.10** *A semisimple locally perfect Lie algebra can be embedded into a locally finite associative algebra if and only if it is diagonal.*

From Corollary 5.10 and Proposition 2.8 we immediately obtain

**Corollary 5.11** *A simple locally finite Lie algebra can be embedded into a locally finite associative algebra if and only if it is diagonal.*

# 6 Branching

We need to introduce a bit of notation. Throughout this section, $\mathfrak{h}$ and $\mathfrak{g}$ denote finite-dimensional simple Lie algebras of ranks $m$ and $n$, respectively ($\operatorname{rk}\mathfrak{h} = m$, $\operatorname{rk}\mathfrak{g} = n$). It is convenient for us to denote by $\mu$, $\nu$ (resp., $\lambda$, $\eta$ ) weights of $\mathfrak{h}$ (resp., $\mathfrak{g}$) and simultaneously irreducible $\mathfrak{h}$-modules (resp., $\mathfrak{g}$-modules) with corresponding highest weights (if these weights are dominant). Denote by $\omega_1, \dots, \omega_m$ the fundamental weights of $\mathfrak{h}$, by $\pi_1, \dots, \pi_n$ those of $\mathfrak{g}$, by $\alpha_1, \dots, \alpha_m$ the simple roots of $\mathfrak{h}$ (we label simple roots and fundamental weights as in [7]). By $\omega_0$, $\pi_0$ we mean the zero weights (or the trivial one-dimensional modules) of the corresponding algebras. Define two linear functions $\delta$ and $\sigma$ on weights of $\mathfrak{h}$, writing down their values on the fundamental weights. In the following list $\sigma(\omega_i) = p_i$, $\delta(\omega_i) = q_i$, $1 \le i \le m$, is abbreviated by $\sigma = (p_1, \dots, p_m)$, $\delta = (q_1, \dots, q_m)$. Put

$$\begin{array}{lll}
\sigma = (1,1,\dots,1,1,1), & \delta = (1,2,\dots,k,k,\dots,2,1) & (A_{2k}); \\
\sigma = (1,1,\dots,1,1,1), & \delta = (1,2,\dots,k+1,\dots,2,1) & (A_{2k+1}); \\
\sigma = (1,1,\dots,1,1,1), & \delta = (1,2,\dots,m-2,m-1,m) & (C_m,\ m \ge 2); \\
\sigma = (1,1,\dots,1,1,\frac{1}{2}), & \delta = (1,2,\dots,m-2,m-1,[\frac{m}{2}]) & (B_m,\ m \ge 3); \\
\sigma = (1,1,\dots,1,\frac{1}{2},\frac{1}{2}), & \delta = (1,2,\dots,2k-2,k-1,k) & (D_{2k},\ k \ge 2); \\
\sigma = (1,1,\dots,1,\frac{1}{2},\frac{1}{2}), & \delta = (1,2,\dots,2k-1,k,k) & (D_{2k+1},\ k \ge 2); \\
\sigma = \delta = (2,2,3,4,3,2) & & (E_6); \\
\sigma = \delta = (2,2,3,4,3,2,1) & & (E_7); \\
\sigma = \delta = (4,5,7,10,8,6,4,2) & & (E_8); \\
\sigma = \delta = (2,3,2,1) & & (F_4); \\
\sigma = \delta = (1,2) & & (G_2).
\end{array}$$

It is not difficult to check that $\delta(\alpha_p) \ge 0$ and $\sigma(\alpha_p) \ge 0$ for all $p = 1, \dots, m$. Note that for classical $\mathfrak{h}$ we have $\delta(\alpha_p) > 0$ only in the following cases:

$$\begin{array}{ll}
\delta(\alpha_k) = \delta(\alpha_{k+1}) = 1 & (A_{2k}); \\
\delta(\alpha_{k+1}) = 2 & (A_{2k+1}); \\
\delta(\alpha_m) = 2 & (C_m); \\
\delta(\alpha_{2k}) = 1 & (B_{2k},\ B_{2k+1}); \\
\delta(\alpha_{2k}) = 2 & (D_{2k}); \\
\delta(\alpha_{2k}) = \delta(\alpha_{2k+1}) = 1 & (D_{2k+1}).
\end{array}$$

Notice that the function similar to $\delta$ was introduced by A.G. Zhilinskii in [17]. He also proved there Lemma 6.7 stated below. But in order to ensure the completeness of the exposition, we give here his arguments (with some modification).

Let a permutation $\tau$ of the fundamental weights of $\mathfrak{h}$ corresponds to the nontrivial automorphism of the Dynkin diagram for $\mathfrak{h}$ of type $A_m$, $D_{2k+1}$ or $E_6$, and $\tau = (1)$ for other $\mathfrak{h}$. For any weight $\mu = \sum a_i\omega_i$ of $\mathfrak{h}$ set $\hat{\mu} = \sum a_{\tau(i)}\omega_i$. Recall that $\hat{\mu} = -w_0(\mu)$ where $w_0$ is the longest element of the corresponding Weyl group. Note that for dominant $\mu$ the module $\hat{\mu}$ is dual to $\mu$, i.e. $\mu^* = \hat{\mu}$.

The definition of $\delta$ and $\sigma$ yields the following properties immediately.

**Proposition 6.1** *Let $\mu$ be a weight of $\mathfrak{h}$.*

(a) $\delta(\mu) \geq 0$ *and* $\sigma(\mu) \geq 0$ *for dominant* $\mu$. *Moreover, if* $\mu$ *is nonzero dominant, then* $\delta(\mu) \geq 1$ *and* $\sigma(\mu) \geq \frac{1}{2}$.

(b) $\sigma(\mu) \leq \delta(\mu) \leq m\sigma(\mu)$ *for dominant* $\mu$.

(c) $\delta(\mu) = \delta(\hat{\mu})$ *and* $\sigma(\mu) = \sigma(\hat{\mu})$.

To avoid repetitions, sometimes we shall use the symbol $\chi$ to denote both $\delta$ and $\sigma$. Let $V$ be an $\mathfrak{h}$-module with the set of weights $M$. Set $\chi_{\mathfrak{h}}(V) = \sup\{\chi(\mu)\}_{\mu\in M}$. It is well known that $\hat{M} = \{\hat{\mu} \mid \mu \in M\}$ is the set of weights of the dual module $V^*$. Therefore, in view of Proposition 6.1 (c), we have $\chi_{\mathfrak{h}}(V^*) = \chi_{\mathfrak{h}}(V)$. If $\Phi = \{V_i\}_{i\in I}$ is a set of $\mathfrak{h}$-modules, set $\chi_{\mathfrak{h}}(\Phi) = \sup\{\chi_{\mathfrak{h}}(V_i)\}_{i\in I}$. It is clear that $\chi_{\mathfrak{h}}(V) = \chi_{\mathfrak{h}}(\langle V\rangle)$. Let $\mu$ and $\nu$ be weights of $\mathfrak{h}$ such that $\mu - \nu$ is a sum of positive roots. Since $\chi(\alpha) \geq 0$ for all simple roots $\alpha$, we have $\chi(\mu) \geq \chi(\nu)$. Consequently, if $\mu$ is an irreducible $\mathfrak{h}$-module, then $\chi_{\mathfrak{h}}(\mu) = \chi(\mu)$. Let $\theta$ be a monomorphism of $\mathfrak{h}$ into $\mathfrak{g}$ (in what follows we write simply $\mathfrak{h} \to \mathfrak{g}$), $V$ a $\mathfrak{g}$-module. Set $\chi_{\mathfrak{h}}(V) = \chi_{\mathfrak{h}}(V{\downarrow}\mathfrak{h})$. So we have defined the function $\chi_{\mathfrak{h}}$ on the dominant weights of $\mathfrak{g}$ (by means of $\chi_{\mathfrak{h}}(\lambda) = \chi_{\mathfrak{h}}(\lambda{\downarrow}\mathfrak{h})$). Note that $\chi_{\mathfrak{h}}(\hat{\lambda}) = \chi_{\mathfrak{h}}(\lambda)$ (since the modules $\hat{\lambda}$ and $\lambda^*$ are isomorphic). Show that this function is additive on the set of dominant weights of $\mathfrak{g}$. Let $\Lambda$, $H$ and $\Sigma$ be the sets of weights of irreducible $\mathfrak{g}$-modules $\lambda$, $\eta$, and $\lambda+\eta$, respectively. Then, according to [7, Section 8.7.4], $\Lambda + H = \Sigma$. We can assume that a Cartan subalgebra of $\mathfrak{h}$ is contained in that of $\mathfrak{g}$. Therefore, we have a homomorphism $\epsilon$ of the group of weights of $\mathfrak{g}$ onto that of $\mathfrak{h}$. Consequently, $\epsilon(\Lambda) + \epsilon(H) = \epsilon(\Sigma)$ where $\epsilon(\Lambda)$, $\epsilon(H)$ and $\epsilon(\Sigma)$ are the sets of weights of the $\mathfrak{h}$-modules $\lambda{\downarrow}\mathfrak{h}$, $\eta{\downarrow}\mathfrak{h}$, and $(\lambda+\eta){\downarrow}\mathfrak{h}$, respectively. Hence

$$\chi_{\mathfrak{h}}(\lambda+\eta) = \sup\{\chi(\gamma)\}_{\gamma\in\epsilon(\Sigma)} = \sup\{\chi(\mu)\}_{\mu\in\epsilon(\Lambda)} + \sup\{\chi(\nu)\}_{\nu\in\epsilon(H)} = \chi_{\mathfrak{h}}(\lambda) + \chi_{\mathfrak{h}}(\eta).$$

So we have proved that the functions $\delta_{\mathfrak{h}}$ and $\sigma_{\mathfrak{h}}$ are additive on dominant weights of $\mathfrak{g}$.

Let $\dim \pi_1 = N$, $\{v_1, \ldots, v_N\}$ be a basis of the $\mathfrak{g}$-module $\pi_1$ consisting of weight vectors, $\lambda_k$ be the weight of $v_k$ with respect to $\mathfrak{g}$, $\mu_k = \epsilon(\lambda_k)$ be the weight of $v_k$ with respect to $\mathfrak{h}$, $\lambda_{N+1-k} = -\lambda_k$ for $\mathfrak{g}$ of type $B_n$, $C_n$, $D_n$ (this forces, in particular, $\mu_{N+1-k} = -\mu_k$). Recall that each module having a weight $\mu$ has also a weight $-\hat{\mu}$ with the same multiplicity. Since $\chi(-\hat{\mu}) = -\chi(\mu)$, we can suppose that $\chi(\mu_{N+1-k}) = -\chi(\mu_k)$. Moreover, we shall always assume, unless otherwise stated, that $\mu_1$ is dominant and $\chi(\mu_k) \geq \chi(\mu_{k+1})$, $1 \leq k \leq N-1$ (it is not necessary that this ordering is the same both for $\delta$ and $\sigma$). In particular, $\chi_{\mathfrak{h}}(\pi_1) = \chi(\mu_1)$. Observe that $\chi(\mu_q) \geq 0$ for $q \leq N/2$. If $\mathfrak{g}$ is of type $C_n$, then, according to [7], the component $\pi_q$ of the $\mathfrak{g}$-module $\wedge^q\pi_1$ contains all vectors of type $v_{i_1} \wedge \ldots \wedge v_{i_q}$ such that $i_r + i_s \neq N+1$ (here $q \leq N/2$) for all $1 \leq r, s \leq q$. Hence

$$\chi_{\mathfrak{h}}(\pi_q) \geq \sup\{\chi(\mu_{i_1} + \ldots + \mu_{i_q}) \mid i_1 < ... < i_q, i_r + i_s \neq N+1\} =$$

$$\chi(\mu_1 + \ldots + \mu_q) = \sup\{\chi(\mu_{i_1} + \ldots + \mu_{i_q}) \mid i_1 < ... < i_q\} = \chi_{\mathfrak{h}}(\wedge^q\pi_1).$$

Since $\pi_q$ is a submodule of $\wedge^q\pi_1$, we have $\chi_{\mathfrak{h}}(\pi_q) = \chi_{\mathfrak{h}}(\wedge^q\pi_1)$ for $\mathfrak{g}$ of type $C_n$. It is well known that $\pi_q = \wedge^q\pi_1$ and $\hat{\pi}_q = \pi_{n+1-q}$, $0 \leq q \leq n$, for $\mathfrak{h}$ of type $A_n$. Hence $\chi_{\mathfrak{h}}(\pi_q) = \chi_{\mathfrak{h}}(\pi_{n+1-q}) = \chi_{\mathfrak{h}}(\wedge^q\pi_1)$. If $\mathfrak{g}$ is of type $B_n$ and $1 \leq q \leq n-1$, or $\mathfrak{g}$ is of type $D_n$ and $1 \leq q \leq n-2$, then $\pi_q = \wedge^q\pi_1$. Set $\bar{q} = q$ for $\mathfrak{g}$ of types $B_n$, $C_n$ and $D_n$, and $\bar{q} = \min(q, n+1-q)$ for $\mathfrak{g}$ of type $A_n$. We have proved the following

**Lemma 6.2** *Let $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$. Then $\delta_{\mathfrak{h}}(\pi_q) = \delta_{\mathfrak{h}}(\wedge^{\bar{q}}\pi_1)$ and $\sigma_{\mathfrak{h}}(\pi_q) = \sigma_{\mathfrak{h}}(\wedge^{\bar{q}}\pi_1)$ for $q = 1, \ldots, n$, except the following cases: $\mathfrak{g}$ is of type $B_n$ ($q = n$) and $\mathfrak{g}$ is of type $D_n$ ($q = n-1, n$).*

Recall that $\frac{1}{2}(\pm\lambda_1 \pm \ldots \pm \lambda_n)$ are the weights of $\pi_n$ for $\mathfrak{g}$ of type $B_n$, and of $\pi_{n-1} \oplus \pi_n$ for $\mathfrak{g}$ of type $D_n$. In the second case the weights with odd number of minuses are of $\pi_{n-1}$, and those with even number of minuses are of $\pi_n$. Therefore, we immediately obtain

**Lemma 6.3** *Let $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$. Then $\chi_{\mathfrak{h}}(\pi_q) = \chi(\mu_1) + \ldots + \chi(\mu_{\bar{q}})$ for $q = 1, \ldots, n$, except the following cases: $\mathfrak{g}$ is of type $B_n$ ($q = n$), and $\mathfrak{g}$ is of type $D_n$ ($q = n-1, n$). In the exceptional cases*

$$\chi_{\mathfrak{h}}(\pi_n) = \frac{1}{2}(\chi(\mu_1) + \ldots + \chi(\mu_n)), \quad \chi_{\mathfrak{h}}(\pi_{n-1}) = \frac{1}{2}(\chi(\mu_1) + \ldots + \chi(\mu_{n-1}) - \chi(\mu_n)).$$

Since $\chi$ is additive and $\chi(\mu_1) = \chi_{\mathfrak{h}}(\pi_1)$, by Lemma 6.3, we obtain

**Lemma 6.4** *Let $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$, and let $\lambda$ be a nontrivial irreducible $\mathfrak{g}$-module. Then $\chi_{\mathfrak{h}}(\lambda) \geq \chi_{\mathfrak{h}}(\pi_1)$, except possibly in the following cases: $\mathfrak{g}$ is of type $B_n$ ($\lambda = \pi_n$), and $\mathfrak{g}$ is of type $D_n$ ($\lambda = \pi_{n-1}, \pi_n$). In the exceptional cases $\chi_{\mathfrak{h}}(\lambda) \geq \frac{1}{2}\chi_{\mathfrak{h}}(\pi_1)$.*

**Lemma 6.5** *Let $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$, and let $V$ be a $\mathfrak{g}$-module. Then $\sigma_{\mathfrak{h}}(V) \geq \sigma_{\mathfrak{h}}(\pi_1)\sigma_{\mathfrak{g}}(V)$. In particular, $\sigma_{\mathfrak{h}}(V) \geq \frac{1}{2}\sigma_{\mathfrak{g}}(V)$.*

*Proof.* Take $\lambda \in \langle V \rangle$ such that $\sigma_{\mathfrak{g}}(\lambda) = \sigma_{\mathfrak{g}}(V)$. By Lemma 6.4 and the definition of $\sigma$, $\sigma_{\mathfrak{h}}(\pi_q) \geq \sigma_{\mathfrak{h}}(\pi_1)\sigma_{\mathfrak{g}}(\pi_q)$ for $q = 1, \ldots, n$. Hence in view of additivity of $\sigma$ we obtain

$$\sigma_{\mathfrak{h}}(V) \geq \sigma_{\mathfrak{h}}(\lambda) \geq \sigma_{\mathfrak{h}}(\pi_1)\sigma_{\mathfrak{g}}(\lambda) = \sigma_{\mathfrak{h}}(\pi_1)\sigma_{\mathfrak{g}}(V).$$

**Definition 6.6** Let $\mathfrak{h}$ and $\mathfrak{g}$ be classical. An embedding $\mathfrak{h} \to \mathfrak{g}$ is called *diagonal* if $\langle \pi_1 \downarrow \mathfrak{h} \rangle \subseteq \{\omega_0, \omega_1, \hat{\omega}_1\}$ (this forces $\delta_{\mathfrak{h}}(\pi_1) = 1$), otherwise an embedding $\mathfrak{h} \to \mathfrak{g}$ is called *nondiagonal.*

Observe that if $\delta_{\mathfrak{h}}(\pi_1) = 1$ and $\operatorname{rk} \mathfrak{h} > 4$, then the embedding $\mathfrak{h} \to \mathfrak{g}$ is diagonal.

**Lemma 6.7** *Let $\operatorname{rk} \mathfrak{h} > 10$, $\mathfrak{h} \to \mathfrak{g}$. Then $\delta_{\mathfrak{h}}(\lambda) \geq \delta_{\mathfrak{h}}(\pi_1)$ for all irreducible $\mathfrak{g}$-modules $\lambda \neq \pi_0$. Moreover, $\delta_{\mathfrak{h}}(\lambda) > \delta_{\mathfrak{h}}(\pi_1)$ for $\lambda \notin \{\pi_0, \pi_1, \hat{\pi}_1\}$.*

*Proof.* It suffices to show that $\delta_{\mathfrak{h}}(\pi_q) > \delta_{\mathfrak{h}}(\pi_1)$ for $\pi_q \notin \{\pi_0, \pi_1, \hat{\pi}_1\}$. Assume that $\delta_{\mathfrak{h}}(\pi_1) > 4$. Recall that $\delta(\alpha) \leq 2$ for all simple roots $\alpha$ of $\mathfrak{h}$. Therefore, for $\mathfrak{g}$ of type $B_n$ ($q = n$) and for $\mathfrak{g}$ of type $D_n$ ($q = n-1, n$) we have

$$\delta_{\mathfrak{h}}(\pi_q) \geq \frac{1}{2}(\delta(\mu_1) + \delta(\mu_2) + \delta(\mu_3)) \geq \frac{1}{2}(3\delta(\mu_1) - 4) > \delta(\mu_1) = \delta_{\mathfrak{h}}(\pi_1).$$

In the other cases:

$$\delta_{\mathfrak{h}}(\pi_q) \geq \delta(\mu_1) + \delta(\mu_2) \geq 2\delta(\mu_1) - 2 > \delta(\mu_1) = \delta_{\mathfrak{h}}(\pi_1).$$

If $\delta_{\mathfrak{h}}(\pi_1) \leq 4$, then for $m > 10$ we have

$$\mu_1 \in \{c_1\omega_1 + \ldots + c_4\omega_4 + d_1\hat{\omega}_1 + \ldots + d_4\hat{\omega}_4 \mid c_1 + \ldots + 4c_4 + d_1 + \ldots + 4d_4 \leq 4\}.$$

Therefore, $\delta(\mu_1) = \delta(\mu_2) = \delta(\mu_3)$, so either $\delta_{\mathfrak{h}}(\pi_q) \geq 2\delta(\mu_1) > \delta_{\mathfrak{h}}(\pi_1)$ or $\delta_{\mathfrak{h}}(\pi_q) \geq \frac{3}{2}\delta(\mu_1) > \delta_{\mathfrak{h}}(\pi_1)$. This completes the proof.

Let $\mathfrak{h} \to \mathfrak{g}$. Denote by $\tau(\mathfrak{h}, \mathfrak{g})$ the number of nontrivial composition factors of the module $\pi_1{\downarrow}\mathfrak{h}$.

**Lemma 6.8** *Let $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$, $c \geq 1$, $\tau(\mathfrak{h}, \mathfrak{g}) \geq 2c$. Let also $V$ be a nontrivial $\mathfrak{g}$-module and $\delta_{\mathfrak{g}}(V) > 4c^2$. Then $\delta_{\mathfrak{h}}(V) > c$.*

*Proof.* Take $\lambda = \sum b_j\pi_j \in \langle V\rangle$ such that $\delta_{\mathfrak{g}}(\lambda) = \delta_{\mathfrak{g}}(V)$. Note that

$$\delta_{\mathfrak{h}}(V) \geq \delta_{\mathfrak{h}}(\lambda) = \sum b_j\delta_{\mathfrak{h}}(\pi_j) \geq \sum b_j.$$

So if $\sum b_j > c$, then we are done. Assume that $\sum b_j \leq c$. Since $\delta_{\mathfrak{g}}(\lambda) > 3c^2 \geq c(2c+2)$, we have $b_q > 0$ for some $q$ such that $\bar{q} > 2c + 2$. Therefore,

$$\delta_{\mathfrak{h}}(V) \geq \delta_{\mathfrak{h}}(\lambda) \geq \delta_{\mathfrak{h}}(\pi_q) \geq \frac{1}{2}(\delta(\nu_1) + \ldots + \delta(\nu_{\bar{q}-2})) \geq \frac{1}{2}(\bar{q} - 2) > c$$

where $\nu_1, \ldots, \nu_{\bar{q}-2}$ are the weights of primitive vectors (with nonzero weights) of $\pi_1{\downarrow}\mathfrak{h}$.

**Lemma 6.9** *Let $\mathfrak{h}$ be classical, $\mu = \sum a_i\omega_i$ be an irreducible $\mathfrak{h}$-module. Assume that $a_p \neq 0$ for some $p$ ($p \neq m$ for $\mathfrak{h}$ of type $A_m$). Then the module $\mu$ contains $p$ distinct weights $\nu_1, \ldots, \nu_p$ such that $\sigma(\nu_i) = \sigma(\mu)$, $1 \leq i \leq p$.*

*Proof.* Set $\nu_1 = \mu$, $\beta_i = \alpha_i + \alpha_{i+1} + \ldots + \alpha_p$, $2 \leq i \leq p$ (for $\mathfrak{h}$ of type $D_m$ ($p = m$) set $\beta_{p-1} = \alpha_{p-2} + \alpha_p$). It is well known that $\beta_i$ are roots of $\mathfrak{h}$. Put $\langle\beta, \alpha\rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ where $( \, , \, )$ is the standard nondegenerate symmetric bilinear form on the weight system associated with the Killing form. Since

$$\langle\mu, \beta_i\rangle = \frac{2(\mu, \alpha_i + \ldots + \alpha_{p-1}) + 2(\mu, \alpha_p)}{(\beta_i, \beta_i)} \geq \langle\mu, \alpha_p\rangle\frac{(\alpha_p, \alpha_p)}{(\beta_i, \beta_i)} \geq \frac{1}{3}a_p > 0,$$

the module $\mu$ contains the weights $\nu_i = \mu - \beta_i$. As $\sigma(\beta_i) = 0$, we get $\sigma(\nu_i) = \sigma(\mu)$, $1 \leq i \leq p$.

**Lemma 6.10** *Let $\mathfrak{h}$ and $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$. Let also $V$ be a $\mathfrak{g}$-module, $\delta_{\mathfrak{h}}(\pi_1) \geq k^2$, $\delta_{\mathfrak{g}}(V) \geq \frac{k}{2}$. Then $\sigma_{\mathfrak{h}}(V) \geq \frac{k}{4}$.*

*Proof.* Let $\mu = \sum a_i\omega_i$ be an irreducible $\mathfrak{h}$-module from $\langle\pi_1{\downarrow}\mathfrak{h}\rangle$ such that $\delta_{\mathfrak{h}}(\mu) = \delta_{\mathfrak{h}}(\pi_1)$. By Lemma 6.4, $\sigma_{\mathfrak{h}}(\lambda) \geq \frac{1}{2}\sigma_{\mathfrak{h}}(\pi_1) \geq \frac{1}{2}\sigma_{\mathfrak{h}}(\mu)$. Therefore, if $\sigma_{\mathfrak{h}}(\mu) \geq \frac{k}{2}$, then we are done. Assume that $\sigma_{\mathfrak{h}}(\mu) < \frac{k}{2}$. Then at most $k - 1$ coefficients $a_i$ are nonzero. Since

$$\delta_{\mathfrak{h}}(\mu) = \delta_{\mathfrak{h}}(\pi_1) \geq k^2 > (k-1)(k+1),$$

there exists $p > k+1$ ($k+1 < p < m-k-1$ for $\mathfrak{h}$ of type $A_m$) such that $a_p \neq 0$. Hence by Lemma 6.9, the module $\mu$ contains at least $k$ distinct weights $\nu_1, \ldots, \nu_k$ such that $\sigma(\nu_i) = \sigma(\mu)$, $1 \leq i \leq k$. Since $k \leq p-2 \leq m-2 \leq n-2$ ($k \leq m/2 \leq n/2$ for $\mathfrak{h}$ of type $A_m$), we have

$$\sigma_{\mathfrak{h}}(\pi_q) = \sigma_{\mathfrak{h}}(\pi_{\bar{q}}) \geq \sigma_{\mathfrak{h}}(\nu_1) + \ldots + \sigma_{\mathfrak{h}}(\nu_{\bar{q}}) = \bar{q}\sigma_{\mathfrak{h}}(\pi_1), \quad \bar{q} \leq k;$$

$$\sigma_{\mathfrak{h}}(\pi_q) = \sigma_{\mathfrak{h}}(\pi_{\bar{q}}) \geq \frac{1}{2}(\sigma_{\mathfrak{h}}(\nu_1) + \ldots + \sigma_{\mathfrak{h}}(\nu_k)) = \frac{k}{2}\sigma_{\mathfrak{h}}(\pi_1), \quad \bar{q} > k.$$

Take $\lambda = \sum b_j \pi_j \in \langle V \rangle$ such that $\delta_{\mathfrak{g}}(\lambda) = \delta_{\mathfrak{g}}(V)$. Assume that $b_q > 0$ for some $q$ such that $\bar{q} > k$. Then $\sigma_{\mathfrak{h}}(V) \geq \sigma_{\mathfrak{h}}(\lambda) \geq \sigma_{\mathfrak{h}}(\pi_q) \geq \frac{k}{2}\sigma_{\mathfrak{h}}(\pi_1) \geq \frac{k}{4}$, as required. Hence we can assume that $b_j = 0$ for $\bar{j} > k$. This yields

$$\sigma_{\mathfrak{h}}(V) \geq \sigma_{\mathfrak{h}}(\lambda) = \sum b_j \sigma_{\mathfrak{h}}(\pi_j) \geq \sum b_j \bar{j} \sigma_{\mathfrak{h}}(\pi_1) = \delta_{\mathfrak{g}}(\lambda)\sigma_{\mathfrak{h}}(\pi_1) \geq \frac{k}{2} \cdot \frac{1}{2} = \frac{k}{4}.$$

So the lemma follows.

The following lemma seems to be true for all classical $\mathfrak{h}$ and $\mathfrak{g}$. However, to simplify the proof we assume that $\operatorname{rk}\mathfrak{h} > 12$.

**Lemma 6.11** *Let* $\operatorname{rk}\mathfrak{h} > 12$, $\mathfrak{h} \to \mathfrak{g}$, *and* $V$ *be a* $\mathfrak{g}$*-module. Then*

$$\delta_{\mathfrak{h}}(V) \leq \delta_{\mathfrak{h}}(\pi_1)\delta_{\mathfrak{g}}(V).$$

*Proof.* It suffices to prove that $\delta_{\mathfrak{h}}(\lambda) \leq \delta_{\mathfrak{h}}(\pi_1)\delta_{\mathfrak{g}}(\lambda)$ for all irreducible $\mathfrak{g}$-modules $\lambda$. In view of additivity of $\delta$ we need only show that $\delta_{\mathfrak{h}}(\pi_q) \leq \delta_{\mathfrak{h}}(\pi_1)\delta_{\mathfrak{g}}(\pi_q)$ for $q = 1, \ldots, n$. By Lemma 6.3,

$$\delta_{\mathfrak{h}}(\pi_q) = \delta(\mu_1) + \ldots + \delta(\mu_{\bar{q}}) \leq \bar{q}\delta(\mu_1) = \delta_{\mathfrak{g}}(\pi_q)\delta_{\mathfrak{h}}(\pi_1),$$

as desired, except possibly the cases $\mathfrak{g}$ of types $B_n$ ($q = n$) and $D_n$ ($q = n-1, n$). If $\mathfrak{g}$ is of type $D_{2k+1}$, then $\hat{\pi}_n = \pi_{n-1}$. Hence

$$\delta_{\mathfrak{h}}(\pi_n) = \delta_{\mathfrak{h}}(\pi_{n-1}) = \frac{1}{2}(\delta(\mu_1) + \ldots + \delta(\mu_{n-1}) - \delta(\mu_n)) \leq \frac{n-1}{2}\delta_{\mathfrak{h}}(\pi_1),$$

as required. If $\mathfrak{g}$ is of type $D_{2k}$ or $B_{2k}$ and $q = n$, then

$$\delta_{\mathfrak{h}}(\pi_n) = \frac{1}{2}(\delta(\mu_1) + \ldots + \delta(\mu_n)) \leq \frac{n}{2}\delta(\mu_1) = \delta_{\mathfrak{g}}(\pi_n)\delta_{\mathfrak{h}}(\pi_1),$$

as required. It remains to consider two cases:

(I) $\mathfrak{g}$ is of type $B_{2k+1}$ ($q = n$);

(II) $\mathfrak{g}$ is of type $D_{2k}$ ($q = n-1$).

Put $\Delta_1 = \delta(\mu_1) + \ldots + \delta(\mu_n)$ and $\Delta_2 = \delta(\mu_1) + \ldots + \delta(\mu_{n-1}) - \delta(\mu_n)$. We have to prove that $\Delta_1 \leq (n-1)\delta(\mu_1)$ in (I), and $\Delta_2 \leq (n-2)\delta(\mu_1)$ in (II). The choice of the ordering of $\mu_i$ shows that there exists $p$ such that $\delta(\mu_i) \neq 0$ for $1 \leq i \leq p$, $N-p+1 \leq i \leq N$ ($N = 2n+1 = 4k+3$ in (I), and $N = 2n = 4k$ in (II)), and $\delta(\mu_i) = 0$ otherwise. (Observe that $\delta(\mu_{n+1}) = 0$ in (I).) Therefore, if the module $\pi_1{\downarrow}\mathfrak{h}$ contains more than one weight $\mu$ such that $\delta(\mu)=0$ (more than two such weights for (II)), then $\delta(\mu_n)=0$ ($\delta(\mu_{n-1})=\delta(\mu_n)=0$ for (II)), and we are done. Assume this is false. Consider the following cases.

*Case* 1. $\mathfrak{h}$ is of type $A_{2l}$, $B_m$ or $D_{2l+1}$. Then $\delta(\alpha)=0,1$ for all simple roots $\alpha$ of $\mathfrak{h}$. Let $\Lambda$ be the set of weights of an irreducible $\mathfrak{h}$-module $\mu$. It is clear that

$$\{\delta(\nu)\mid \nu\in\Lambda\}=\{-\delta(\mu),-\delta(\mu)+1,\dots,\delta(\mu)-1,\delta(\mu)\}.$$

Therefore, each composition factor of $\pi_1{\downarrow}\mathfrak{h}$ contains a weight $\nu$ such that $\delta(\nu)=0$. Hence by the remark above, $\pi_1$ is an irreducible $\mathfrak{h}$-module in (I), or the length of the module $\pi_1{\downarrow}\mathfrak{h}$ is at most 2 in (II). Consider the following subcases.

(a) $\delta(\mu_1)=1$. Then $\mathfrak{h}\to\mathfrak{g}$ is a diagonal embedding, i.e. $\langle\pi_1{\downarrow}\mathfrak{h}\rangle\subseteq\{\omega_0,\omega_1,\hat\omega_1\}$. In (I) we have $\pi_1{\downarrow}\mathfrak{h}=\omega_1$ or $\hat\omega_1$, so $\mathfrak{h}=\mathfrak{g}$, and we are done. Consider the case (II). First assume that $\mathfrak{h}$ is of type $A_{2l}$. Then $\{\omega_1,\hat\omega_1\}\subseteq\langle\pi_1{\downarrow}\mathfrak{h}\rangle$. Since the length of $\pi_1{\downarrow}\mathfrak{h}$ is at most 2, we have $\pi_1{\downarrow}\mathfrak{h}=\omega_1\oplus\hat\omega_1$. But this is impossible as $\dim\omega_1\oplus\hat\omega_1=4l+2\neq 4k=\dim\pi_1$. Assume that $\mathfrak{h}$ has type $D_{2l+1}$. Then the module $\omega_1$ contains two weights $\mu$ ($\mu=\pm(\omega_{m-1}-\omega_m)$) such that $\delta(\mu)=0$. Therefore, the length of $\pi_1{\downarrow}\mathfrak{h}$ is at most one, i.e. $\pi_1$ is an irreducible $\mathfrak{h}$-module with respect to the natural embedding $\mathfrak{h}\to\mathfrak{g}$. But this is impossible since $2l+1\neq 2k$. Assume that $\mathfrak{h}$ has type $B_m$. Since $\dim\omega_1=2m+1$ and $\dim\pi_1=4k$, the multiplicity of the module $\omega_1$ in $\pi_1{\downarrow}\mathfrak{h}$ is one, forcing $\pi_1{\downarrow}\mathfrak{h}=\omega_1\oplus\omega_0$. Therefore, we have the natural embedding $B_{2k-1}\to D_{2k}$. Recall that $\pm\alpha_m$ and 0 are weights of the module $\omega_1$. Since $m=2k-1$ is odd, $\delta(\pm\alpha_m)=0$. Hence this case is impossible.

(b) $\delta(\mu_1)=2$. Then $\nu\in\langle\pi_1{\downarrow}\mathfrak{h}\rangle$ where $\nu$ is one of the modules $2\omega_1$, $\omega_1+\hat\omega_1$, $\omega_2$. It suffices to find two weights $\nu_1$, $\nu_2$, of $\nu$ such that $\delta(\nu_1)=\delta(\nu_2)=1$. Indeed, this forces

$$\Delta_1\le(n-2)\delta(\mu_1)+\delta(\nu_1)+\delta(\nu_2)=(n-2)\delta(\mu_1)+1+1=(n-1)\delta(\mu_1)$$

in (I), and

$$\Delta_2\le(n-3)\delta(\mu_1)+\delta(\nu_1)+\delta(\nu_2)+\delta(\mu_{2k})=(n-2)\delta(\mu_1)$$

in (II), as required. Let $\mu$ be a weight of $\omega_1$ such that $\delta(\mu)=0$. Set $\nu_1=\omega_1+\mu$, $\nu_2=\omega_1-\alpha_1+\mu$, for $\nu=2\omega_1$, $\omega_2$, and $\nu_1=\omega_1+\hat\mu$, $\nu_2=\omega_1-\alpha_1+\hat\mu$, for $\nu=\omega_1+\hat\omega_1$. It is clear that $\nu_1$, $\nu_2$, are weights of $\nu$, and $\delta(\nu_1)=\delta(\nu_2)=1$, as desired.

(c) $\delta(\mu_1)\ge 3$. Then the module $\pi_1{\downarrow}\mathfrak{h}$ contains weights $\nu_1$, $\nu_2$ such that $\delta(\nu_1)=2$, $\delta(\nu_2)=1$. Therefore, we have

$$\Delta_1\le(n-2)\delta(\mu_1)+\delta(\nu_1)+\delta(\nu_2)\le(n-1)\delta(\mu_1),$$

$$\Delta_2\le(n-3)\delta(\mu_1)+\delta(\nu_1)+\delta(\nu_2)+\delta(\mu_{2k})\le(n-2)\delta(\mu_1)$$

in (I), (II), respectively, as required.

*Case 2.* $\mathfrak{h}$ is of type $A_{2l+1}$, $C_m$ or $D_{2l}$. Then $\delta(\alpha)=0,2$ for all simple roots $\alpha$ of $\mathfrak{h}$. Consider the following subcases.

(a) $\delta(\mu_1)=1$. The modules $\omega_1$ and $\hat\omega_1$ do not contain weights $\mu$ such that $\delta(\mu)=0$. Denote by $z$ the number of trivial composition factors of $\pi_1{\downarrow}\mathfrak{h}$, and by $r$ the number of nontrivial those (they are isomorphic to $\omega_1$ or $\hat{\omega}_1$). By the remark above, $z \le 1$ for (I), or $z \le 2$ for (II). Consider the case (I). By Malcev's theorem ([9], Appendix, Theorem 0.25), every orthogonal representation of $\mathfrak{h}$ can be written in the form $\varphi_1 \oplus \ldots \oplus \varphi_s \oplus \psi_1 \oplus \hat{\psi}_1 \oplus \ldots \oplus \psi_t \oplus \hat{\psi}_t$ where $\varphi_i$, $\psi_i$ are irreducible, $\varphi_i$ are orthogonal. Observe that $\pi_1$ is an orthogonal representation of $\mathfrak{h}$. If $\mathfrak{h}$ is of type $A_{2l+1}$ or $C_m$, then $\omega_1$ and $\hat{\omega}_1$ are not orthogonal. Therefore, $r$ is even. Since $\dim \omega_1 = \dim \hat{\omega}_1$ is even, and $\dim \pi_1 = 4k+3$, we conclude $z \ge 3$. If $\mathfrak{h}$ is of type $D_{2l}$, then $\dim \omega_1 = 4l$, so as above $z \ge 3$. This contradicts the remark above ($z \le 1$). Now consider the case (II). If $\mathfrak{h}$ is of type $A_{2l+1}$ or $C_m$, then as above $r$ is even. Since the dimension of $\omega_1$ is even too, and $\dim \pi_1 = 4k$, we have $4|z$. If $\mathfrak{h}$ is of type $D_{2l}$, then as above, $4|z$. Since $z \le 2$, we conclude $z = 0$. Hence $\delta(\mu_1) = \ldots = \delta(\mu_{2k}) = 1$. Therefore, $\Delta_2 = (n-2)\delta(\mu_1)$, as required.

(b) $\delta(\mu_1) = 2, 3, 4, 5$. Take $\nu \in \langle \pi_1 {\downarrow} \mathfrak{h} \rangle$ such that $\delta_{\mathfrak{h}}(\nu) = \delta(\mu_1)$. Since $\operatorname{rk} \mathfrak{h} > 12$, $\nu$ has the form $\nu = c_1\omega_1 + \ldots + c_5\omega_5 + d_1\hat{\omega}_1 + \ldots + d_5\hat{\omega}_5$ with $c_1 + \ldots + 5c_5 + d_1 + \ldots + 5d_5 = \delta(\mu_1)$. It suffices to find three weights $\nu_1$, $\nu_2$, $\nu_3$ of $\nu$ such that $\delta(\nu_1) = \delta(\nu_2) = \delta(\nu_3) = 0$ (then we are done, by the remark above) or $\delta(\nu_1) = \delta(\nu_2) = \delta(\nu_3) = 1$ (then $\delta(\mu_1) \ge 3$ and $\Delta_1, \Delta_2 \le (n-3)\delta(\mu_1) + 1 + 1 + 1 \le (n-2)\delta(\mu_1)$, as required). Set $r = c_1 + \ldots + 5c_5$, $l = d_1 + \ldots + 5d_5$. Then the set of weights of the module $\wedge^r \omega_1 \otimes \wedge^l \hat{\omega}_1$ lies in that of $\nu$. Observe that $r + l = \delta(\mu_1) \le 5$. Set $\rho_0 = \omega_1, \rho_i = \omega_1 - \alpha_1 - \ldots - \alpha_i$, $i = 1, \ldots, 6$. Then $\rho_i$, $-\hat{\rho}_i$, $i = 0, \ldots, 6$, are weights of $\omega_1$, and $\hat{\rho}_i$, $-\rho_i$, $i = 0, \ldots, 6$, are weights of $\hat{\omega}_1$. Observe that $\delta(\rho_i) = \delta(\hat{\rho}_i) = 1$, $\delta(-\hat{\rho}_i) = \delta(-\rho_i) = -1$. Set $\nu_j = \tau_{j-1} + \ldots + \tau_{j+r+l-2}$, $j = 1, 2, 3$, where $\tau_k \in \{\rho_k, -\hat{\rho}_k, \hat{\rho}_k, -\rho_k\}$. It is clear that we can choose $\tau_k$ in such a way that $\nu_j$ are weights of $\wedge^r \omega_1 \otimes \wedge^l \hat{\omega}_1$ ( consequently, of $\nu$) and $\delta(\nu_j) = 0$ or 1. So we complete this subcase.

(c) $\delta(\mu_1) \ge 6$. Then the module $\pi_1 {\downarrow} \mathfrak{h}$ contains weights $\mu_r$, $\mu_s$, $\mu_t$ such that $\delta(\mu_r) = 4$, $\delta(\mu_s) = 2$, $\delta(\mu_t) = 0$ for even $\delta(\mu_1)$, and $\delta(\mu_r) = 5$, $\delta(\mu_s) = 3$, $\delta(\mu_t) = 1$ for odd $\delta(\mu_1)$. In the first case we have

$$\Delta_1 \le (n-2)\delta(\mu_1) + \delta(\mu_r) + \delta(\mu_s) = (n-2)\delta(\mu_1) + 4 + 2 \le (n-1)\delta(\mu_1),$$

$$\Delta_2 \le (n-3)\delta(\mu_1) + \delta(\mu_r) + \delta(\mu_s) + \delta(\mu_t) \le (n-2)\delta(\mu_1)$$

in (I), (II), respectively. In the second case

$$\Delta_1 \le (n-2)\delta(\mu_1) + \delta(\mu_s) + \delta(\mu_t) = (n-2)\delta(\mu_1) + 3 + 1 < (n-1)\delta(\mu_1)$$

for (I). Consider (II). If $t \ne 2k$ then $\Delta_2 \le (n-3)\delta(\mu_1) + 3 + 1 < (n-2)\delta(\mu_1)$. If $t = 2k$ then $\Delta_2 \le (n-3)\delta(\mu_1) + 5 + 3 - 1 < (n-2)\delta(\mu_1)$, so the lemma follows.

**Proposition 6.12** *Let $\mathfrak{h}$ and $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$ be a diagonal embedding, and $V$ be a $\mathfrak{g}$-module.*

- *(a) If $\operatorname{rk} \mathfrak{h} > 12$, then $\delta_{\mathfrak{h}}(V) \le \delta_{\mathfrak{g}}(V)$.*
- *(b) $\sigma_{\mathfrak{h}}(V) \ge \sigma_{\mathfrak{g}}(V)$.*
- *(c) If $\tau(\mathfrak{h}, \mathfrak{g}) = 1$, then $\sigma_{\mathfrak{h}}(V) = \sigma_{\mathfrak{g}}(V)$.*

*Proof.* (a) follows from Lemma 6.11.
(b). Since $\mathfrak{h} \to \mathfrak{g}$ is diagonal, we have $\sigma_{\mathfrak{h}}(\pi_1) = 1$. Therefore, by Lemma 6.5, $\sigma_{\mathfrak{h}}(V) \geq \sigma_{\mathfrak{g}}(V)$.
(c). In view of (b) it suffices to show that $\sigma_{\mathfrak{h}}(V) \leq \sigma_{\mathfrak{g}}(V)$. Since $\tau(\mathfrak{h}, \mathfrak{g}) = 1$, we have $\pi_1{\downarrow}\mathfrak{h} = \nu \oplus \omega_0 \oplus \ldots \oplus \omega_0$ where $\nu = \omega_1, \hat{\omega}_1$. Recall that $\sigma(\alpha_1) = \sigma(\hat{\alpha}_1) = 1$. Therefore, $\sigma(\mu_1) = 1$, $\sigma(\mu_2) = \ldots = \sigma(\mu_n) = 0$. Hence by Lemma 6.3, we have $\sigma_{\mathfrak{h}}(\pi_q) = \sigma_{\mathfrak{g}}(\pi_q)$, $1 \leq q \leq n$. Take $\lambda \in \langle V \rangle$ such that $\sigma_{\mathfrak{h}}(\lambda) = \sigma_{\mathfrak{h}}(V)$. In view of additivity of $\sigma$, $\sigma_{\mathfrak{h}}(V) = \sigma_{\mathfrak{h}}(\lambda) = \sigma_{\mathfrak{g}}(\lambda) \leq \sigma_{\mathfrak{g}}(V)$.

Let $\mu$ be a nontrivial irreducible $\mathfrak{h}$-module. Estimate the dimension of $\mu$. By Weyl's formula [7],

$$\dim \mu = \prod_{\alpha > 0} \frac{\langle \mu + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

where $\rho$ is the sum of the fundamental weights of $\mathfrak{h}$. Denote by $a = a(\mathfrak{h})$ the number of positive roots of $\mathfrak{h}$. Choose $b = b(\mathfrak{h})$ such that $\langle \mu + \rho, \alpha \rangle < b\delta(\mu)$ for all dominant nonzero $\mu$ and all positive roots $\alpha$. Since $\langle \rho, \alpha \rangle$ does not depend on $\mu$, we have $\dim \mu < c\delta(\mu)^a$ for some $c = c(\mathfrak{h})$. Set $x = 2c$, $y = a + 1$. Note that $y \geq 2$.

**Lemma 6.13** *Let $\mathfrak{g}$ be classical, $\mathfrak{h} \to \mathfrak{g}$, $x = x(\mathfrak{h})$ and $y = y(\mathfrak{h})$ be as above, and $V$ be a nontrivial $\mathfrak{g}$-module. Then $\delta_{\mathfrak{h}}(V) < x\delta_{\mathfrak{h}}(\pi_1)^y \tau(\mathfrak{h}, \mathfrak{g})\sigma_{\mathfrak{g}}(V)$.*

*Proof.* By the remark above, $\dim \mu < \frac{x}{2}\delta_{\mathfrak{h}}(\pi_1)^{y-1}$ for all $\mu \in \langle \pi_1{\downarrow}\mathfrak{h} \rangle$. Therefore, $\pi_1{\downarrow}\mathfrak{h}$ contains less than $\frac{x}{2}\delta_{\mathfrak{h}}(\pi_1)^{y-1}\tau(\mathfrak{h}, \mathfrak{g})$ base vectors of nonzero weights. Hence by Lemma 6.3,

$$\delta_{\mathfrak{h}}(\pi_q) \leq \delta(\mu_1) + \ldots + \delta(\mu_q) < \frac{x}{2}\delta_{\mathfrak{h}}(\pi_1)^{y-1}\tau(\mathfrak{h}, \mathfrak{g})\delta_{\mathfrak{h}}(\pi_1) = \frac{x}{2}\delta_{\mathfrak{h}}(\pi_1)^y\tau(\mathfrak{h}, \mathfrak{g})$$

for $q = 1, \ldots, n$. Take $\lambda = \sum b_j\pi_j \in \langle V \rangle$ such that $\delta_{\mathfrak{h}}(\lambda) = \delta_{\mathfrak{h}}(V)$. Then

$$\delta_{\mathfrak{h}}(V) = \delta_{\mathfrak{h}}(\lambda) = \sum b_j\delta_{\mathfrak{h}}(\pi_j) < (\sum b_j)\frac{x}{2}\delta_{\mathfrak{h}}(\pi_1)^y\tau(\mathfrak{h}, \mathfrak{g}) \leq x\delta_{\mathfrak{h}}(\pi_1)^y\tau(\mathfrak{h}, \mathfrak{g})\sigma_{\mathfrak{g}}(V),$$

as required.

Set $d = \sup\{\delta_{\mathfrak{h}}(\pi_j) \mid \mathfrak{h} \to \mathfrak{g},\ n = \operatorname{rk}\mathfrak{g} \leq 12,\ 1 \leq j \leq n\}$.

**Lemma 6.14** *Let $\operatorname{rk}\mathfrak{g} \leq 12$, $\mathfrak{h} \to \mathfrak{g}$, and let $V$ be a $\mathfrak{g}$-module. Then $\delta_{\mathfrak{h}}(V) \leq d\delta_{\mathfrak{g}}(V)$ and $\delta_{\mathfrak{h}}(V) \leq 2d\sigma_{\mathfrak{g}}(V)$.*

*Proof.* Take $\lambda = \sum b_j\pi_j \in \langle V \rangle$ such that $\delta_{\mathfrak{h}}(\lambda) = \delta_{\mathfrak{h}}(V)$. Then $\delta_{\mathfrak{h}}(V) = \delta_{\mathfrak{h}}(\lambda) = \sum b_j\delta_{\mathfrak{h}}(\pi_j) \leq d\sum b_j$. Since $\sum b_j \leq 2\sigma_{\mathfrak{g}}(\lambda) \leq 2\sigma_{\mathfrak{g}}(V)$ and $\sum b_j \leq \delta_{\mathfrak{g}}(\lambda) \leq \delta_{\mathfrak{g}}(V)$, we obtain the required bounds.

Denote by $\rho_0$ the trivial one-dimensional module for a simple Lie algebra $\mathfrak{s}$.

**Lemma 6.15** *Let $\mathfrak{g}$ be of type $B_n$ or $D_n$, $\operatorname{rk}\mathfrak{h} \geq 4$, $\mathfrak{h} \oplus \mathfrak{s} \to \mathfrak{g}$. Suppose that $\langle M{\downarrow}\mathfrak{h} \rangle \neq \{\omega_0\}$ and $\langle M{\downarrow}\mathfrak{s} \rangle \neq \{\rho_0\}$ for some $M \in \langle \pi_1{\downarrow}\mathfrak{h} \oplus \mathfrak{s} \rangle$. Then $\langle N{\downarrow}\mathfrak{h} \rangle \neq \{\omega_0\}$ and $\langle N{\downarrow}\mathfrak{s} \rangle \neq \{\rho_0\}$ for some $N \in \langle \pi{\downarrow}\mathfrak{h} \oplus \mathfrak{s} \rangle$ where $\pi = \pi_n$ for $\mathfrak{g}$ of type $B_n$ and $\pi = \pi_{n-1}, \pi_n$ for $\mathfrak{g}$ of type $D_n$.*

*Proof.* Let $\{v_1, \dots, v_N\}$ be a weight basis of the module $\pi_1$, $\lambda_i$ be the weight of $v_i$ with respect to $\mathfrak{g}$, $\mu_i$ be the weight of $v_i$ with respect to $\mathfrak{h} \oplus \mathfrak{s}$. We can assume that $\lambda_i = -\lambda_{N+1-i}$, $\mu_i = -\mu_{N+1-i}$. Recall that we can write each weight $\mu$ of $\mathfrak{h} \oplus \mathfrak{s}$ in the form $\mu = (\mu', \mu'')$ where $\mu'$, $\mu''$ are weights of $\mathfrak{h}, \mathfrak{s}$, respectively. It is clear that if $\langle V{\downarrow}\mathfrak{h}\rangle \neq \{\omega_0\}$ and $\langle V{\downarrow}\mathfrak{s}\rangle \neq \{\rho_0\}$ for some simple $\mathfrak{h} \oplus \mathfrak{s}$-module $V$, then $V$ contains a weight $\mu = (\mu', \mu'')$ such that $\mu'$, $\mu''$ are nonzero dominant weights. Conversely, if a simple $\mathfrak{h} \oplus \mathfrak{s}$-module $V$ contains a weight $\mu = (\mu', \mu'')$ with nonzero $\mu'$, $\mu''$, then $\langle V{\downarrow}\mathfrak{h}\rangle \neq \{\omega_0\}$ and $\langle V{\downarrow}\mathfrak{s}\rangle \neq \{\rho_0\}$. Therefore, it suffices to find a weight $\mu = (\mu', \mu'')$ such that $\mu'$, $\mu''$ are nonzero. Since $\operatorname{rk}\mathfrak{h} \geq 4$, the module $M{\downarrow}\mathfrak{h}$ contains five distinct weights $\nu_1, \dots, \nu_5$. Denote by $\rho$ the highest weight of $M{\downarrow}\mathfrak{s}$. Observe that $\rho \neq 0$. Then $M{\downarrow}\mathfrak{h} \oplus \mathfrak{s}$ contains the weights $(\nu_1, \rho), \dots, (\nu_5, \rho)$. We can assume that $(\nu_i, \rho) = \mu_i$, $i = 1, \dots, 5$. Set $\xi = \mu_6 + \ldots + \mu_n$,

$$\xi_1 = \frac{1}{2}(\mu_1 - \mu_2 - \mu_3 + \mu_4 + \mu_5 + \xi), \qquad \xi_2 = \frac{1}{2}(-\mu_1 + \mu_2 - \mu_3 + \mu_4 + \mu_5 + \xi),$$

$$\xi_3 = \frac{1}{2}(\mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5 + \xi), \qquad \xi_4 = \frac{1}{2}(-\mu_1 + \mu_2 - \mu_3 - \mu_4 - \mu_5 + \xi).$$

Then $\xi_1, \dots, \xi_4$ are weights of $\pi_n{\downarrow}\mathfrak{h} \oplus \mathfrak{s}$. Observe that $\xi_1' - \xi_2' = \xi_3' - \xi_4' = \nu_1 - \nu_2 \neq 0$, $\xi_1'' = \xi_2'' = (\xi'' + \rho)/2$, $\xi_3'' = \xi_4'' = (\xi'' - 3\rho)/2$. Therefore, either $\xi_1'' = \xi_2'' \neq 0$ or $\xi_3'' = \xi_4'' \neq 0$. Since $\xi_1' \neq \xi_2'$ and $\xi_3' \neq \xi_4'$, we conclude that $\xi_i' \neq 0$ and $\xi_i'' \neq 0$ for some $i$, as desired. Set

$$\zeta_1 = \frac{1}{2}(\mu_1 - \mu_2 + \mu_3 + \mu_4 + \mu_5 + \xi), \qquad \zeta_2 = \frac{1}{2}(-\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \xi),$$

$$\zeta_3 = \frac{1}{2}(\mu_1 - \mu_2 - \mu_3 - \mu_4 + \mu_5 + \xi), \qquad \zeta_4 = \frac{1}{2}(-\mu_1 + \mu_2 - \mu_3 - \mu_4 + \mu_5 + \xi).$$

Then $\zeta_1, \dots, \zeta_4$ are weights of $\pi_{n-1}{\downarrow}\mathfrak{h} \oplus \mathfrak{s}$. Observe that $\zeta_1' - \zeta_2' = \zeta_3' - \zeta_4' = \nu_1 - \nu_2 \neq 0$, $\zeta_1'' = \zeta_2'' = (\xi'' + 3\rho)/2$, $\zeta_3'' = \zeta_4'' = (\xi'' - \rho)/2$. As above we conclude that $\zeta_i' \neq 0$ and $\zeta_i'' \neq 0$ for some $i$. So the lemma follows.

# 7 Bratteli diagrams

The notion of Bratteli diagram was introduced for investigation of locally semisimple associative algebras (see [8, 10]). In this paper we adapt it to handle locally finite Lie algebras.

We use notations of Sections 4, 5, 6. Let $L$ be a locally finite Lie algebra of countable dimension. Then $L$ can be expressed in the form $L = \varinjlim L_i$ where $L_i \subset L_{i+1}$, $i \in \mathbf{N}$. Let $\mathfrak{S} = \{S_i\}_{i \in \mathbf{N}}$ be an abstract Levi subalgebra associated with the local system $\{L_i\}_{i \in \mathbf{N}}$. Let $S_i = S_i^1 \oplus \ldots \oplus S_i^{n_i}$ where $S_i^k$ are simple components of $S_i$. A *Bratteli diagram* of $L$ associated with the local system $\{L_i\}_{i \in \mathbf{N}}$ is an $\mathbf{N}$-graded graph $\mathfrak{B}$ defined as follows. The *nodes* at the level $i$ of $\mathfrak{B}$ are in one-to-one correspondence with the simple components of $S_i$, and they are labelled by the corresponding components. Two nodes $S_i^k$ and $S_{i+1}^l$ of the neighboring levels are joined by an *edge* (denoted by $(S_i^k, S_{i+1}^l)$) if $\langle V_{i+1}^l{\downarrow}S_i^k\rangle \neq \{T_i^k\}$. By *path* we mean an increasing (finite or infinite) sequence of nodes $\Gamma = (S_i^k, S_{i+1}^l, S_{i+2}^m, \ldots)$ such that all the neighboring nodes are joined by edges. Denote by $\mathcal{P}(S_i^k)$ the set of paths beginning in $S_i^k$, by $\mathcal{P}(S_i^k, S_j^l)$ the set of paths beginning in $S_i^k$ and ending in $S_j^l$. A node $S_j^l$ is called $S_i^k$-*accessible* if $\mathcal{P}(S_i^k, S_j^l) \neq \emptyset$. A node $S_j^l$ is called *critical* if $\operatorname{rk} S_j^l > 12$ and there exists $M \in \langle V_j^l \downarrow S_{j-1} \rangle$ such that $\langle M \downarrow S_{j-1}^p \rangle \neq \{T_{j-1}^p\}$ and $\langle M \downarrow S_{j-1}^q \rangle \neq \{T_{j-1}^q\}$ for two distinct nodes on the level $j-1$. Let $\Gamma \in \mathcal{P}(S_i^k)$. A node $S_j^l \in \Gamma$ is called $\Gamma$-*critical of degree* $g$ if $\operatorname{rk} S_j^l > 12$ and there exists $M \in \langle V_j^l \downarrow S_{j-1} \rangle$ such that $\langle M \downarrow S_{j-1}^{p_i} \rangle \neq \{T_{j-1}^{p_i}\}$ for $g \geq 2$ distinct $S_i^k$-accessible nodes $S_{j-1}^{p_i}$, $i = 1, \ldots, g$, with $S_{j-1}^{p_1} \in \Gamma$, and such $g$ is maximal. An edge $(S_i^k, S_{i+1}^l)$ is called *nonstandard* if $\langle V_{i+1}^l \downarrow S_i^k \rangle \not\subseteq \{T_i^k, V_i^k, V_i^{k^*}\}$ and $\operatorname{rk} S_{i+1}^l > 12$, and *standard* otherwise. Let $j > i$, $V$ be an $S_j$-module. Set $\delta_i^k(V) = \delta(V \downarrow S_i^k)$, $\sigma_i^k(V) = \sigma(V \downarrow S_i^k)$. By $\delta$-*rank* and $\sigma$-*rank* of an edge $(S_i^k, S_{i+1}^l)$ we mean the numbers $\delta_i^k(V_{i+1}^l)$ and $\sigma_i^k(V_{i+1}^l)$, respectively. As above $\tau(S_i^k, S_j^l)$ is the number of nontrivial composition factors of $V_j^l \downarrow S_i^k$.

For the case of a Lie algebra $L = \varinjlim L_i$ of uncountable dimension the picture is as follows. For every ascending chain of indices $C : i_1 < i_2 < \ldots$ we construct its own Bratteli diagram $\mathfrak{B}_C$ (the Bratteli diagram of the Lie algebra $L_C = \cup_{j \in \mathbf{N}} L_{i_j}$). By Bratteli diagram $\mathfrak{B}$ of $L$ we mean the set of all $\mathfrak{B}_C$. We say that $\mathfrak{B}$ satisfy a property $\Pi$ if all $\mathfrak{B}_C$ satisfy $\Pi$.

The arguments similar to those at the end of Section 4 show that a Bratteli diagram associated with the local system $\{L_i\}_{i \in I}$ does not depend on the choice of an abstract Levi subalgebra associated with $\{L_i\}_{i \in I}$. Therefore, the choice of $\{L_i\}_{i \in I}$ uniquely determines the corresponding Bratteli diagram.

The following lemma is obvious.

**Lemma 7.1** *Let $S_j^l$ be $S_i^k$-accessible. Then* $\operatorname{rk} S_j^l \geq \operatorname{rk} S_i^k$.

Let $i < j$, and $\theta_{ij} : S_i \to S_j$ be the monomorphism described by Lemma 4.5. For every pair $k, l$ define the homomorphism $\theta_{ij}^{kl} : S_i^k \to S_j^l$ by means of $\theta_{ij}^{kl} = \pi_j^l \circ \theta_{ij}$ where $\pi_j^l$ is the projection $S_j \to S_j^l$. So we have the action of $S_i^k$ on $S_j^l$-modules. Let $M$ be an irreducible $S_j^l$-module, $N$ be an irreducible $S_j$-module such that $N \downarrow S_j^l = M$. Then it is clear that $\langle M \downarrow S_i^k \rangle = \langle N \downarrow S_i^k \rangle$. In view of this we shall identify such $S_j^l$- and $S_j$-modules.

**Lemma 7.2** *Let $i < j$, and $M$ be an irreducible $S_j$-module. Then*

$$\chi_i^k(M) = \chi_i^k(M \downarrow S_j^1) + \ldots + \chi_i^k(M \downarrow S_j^{n_j}) \quad \textit{for} \quad \chi_i^k = \delta_i^k, \sigma_i^k.$$

*Proof.* Write $M$ in the canonical form $M = M_1 \otimes \ldots \otimes M_{n_j}$. Since $\chi_i^k$ is additive on the dominant weights of $S_j$, and one can identify the module $M_l$ with a module from $\langle M \downarrow S_j^l \rangle$, we get the required equality.

The next lemma immediately follows from Lemma 7.2.

**Lemma 7.3** *Let $i < j$, $M$ be an irreducible $S_j$-module. Then*

$$\chi_i^k(M \downarrow S_j^l) \leq \chi_i^k(M) \quad \textit{for} \quad \chi_i^k = \delta_i^k, \sigma_i^k.$$

Now we can prove sufficiency in Lemma 5.5.

*Proof of sufficiency in Lemma 5.5.* Assume that $\Psi = \{\Psi_i\}_{i\in I}$ is a nondegenerate inductive system for $\mathfrak{S}$. Put $c_i^k = \delta_i^k(\Psi_i)$. Estimate the value of $\delta_i^k(V_j^l)$ where $j > i$, $1 \le l \le n_j$. If $S_j^l$ is exceptional, then $\delta_i^k(V_j^l) \le d$, where $d$ is a constant which does not depend on $S_i^k$ and $S_j^l$. Assume that $S_j^l$ is classical. Since $\Psi$ is nondegenerate, $\langle \Psi_j \downarrow S_j^l\rangle \ne \{T_j^l\}$. Let $W$ be a nontrivial $S_j^l$-module from $\langle \Psi_j \downarrow S_j^l\rangle$. Then by Lemma 7.3,

$$\delta_i^k(W) \le \delta_i^k(\Psi_j) = \delta_i^k(\Psi_j \downarrow S_i) = \delta_i^k(\Psi_i) = c_i^k.$$

Since $W$ is nontrivial, by Lemma 6.4, $\delta_i^k(V_j^l) \le 2\delta_i^k(W) \le 2c_i^k$. Summarizing, $\delta_i^k(V_j^l) \le \max(d, 2c_i^k)$ for all $j > i$ and $l = 1, \dots, n_j$. This implies $\delta_i^k(\mathfrak{T}_i^k) \le \max(d, 2c_i^k)$. Hence the set $\mathfrak{T}_i^k$ is finite for every $i \in I$ and every $k \in \{1, \dots, n_i\}$. Therefore, by Definition 5.1, $\mathfrak{S}$ is diagonal.

**Lemma 7.4** *Let* $\operatorname{rk} S_i^k > 12$, *and* $\Gamma = (S_i^k, S_{i+1}^m, \dots, S_{j-1}^n, S_j^l) \in \mathcal{P}(S_i^k, S_j^l)$. *Then* $\delta_i^k(V_j^l) \ge \delta_i^k(V_{i+1}^m) + r$ *where* $r$ *is the number of nonstandard edges of the subpath* $\Gamma' = (S_{i+1}^m, \dots, S_{j-1}^n, S_j^l)$.

*Proof.* By Lemmas 7.3 and 6.7,

$$\delta_i^k(V_j^l) = \delta_i^k(V_j^l \downarrow S_{j-1}) \ge \delta_i^k(V_j^l \downarrow S_{j-1}^n) \ge \delta_i^k(V_{j-1}^n) + \epsilon$$

where $\epsilon = 0$ if the edge $(S_{j-1}^n, S_j^l)$ is standard, and $\epsilon = 1$ otherwise. Consequently, $\delta_i^k(V_j^l) \ge \delta_i^k(V_{i+1}^m) + r$ where $r$ is the number of nonstandard edges of the subpath $\Gamma' = (S_{i+1}^m, \dots, S_{j-1}^n, S_j^l)$.

Since either $\delta_i^k(V_{i+1}^m) = 1$ or $\delta_i^k(V_{i+1}^m) \ge 2$, we can state

**Lemma 7.5** *Let* $\operatorname{rk} S_i^k > 12$, $\Gamma \in \mathcal{P}(S_i^k, S_j^l)$. *Then* $\delta_i^k(V_j^l) \ge t+1$ *where* $t$ *is the number of nonstandard edges of* $\Gamma$.

**Lemma 7.6** *Let* $\Gamma = (S_i^k, \dots, S_{j-1}^n, S_j^l) \in \mathcal{P}(S_i^k, S_j^l)$, *and* $M$ *be a nontrivial irreducible* $S_j^l$*-module. Assume that* $\operatorname{rk} S_{j-1}^n > 12$, *and* $S_j^l$ *is* $\Gamma$*-critical. Then* $\delta_i^k(M) \ge \delta_i^k(N) + 1$ *where* $N$ *is a nontrivial irreducible* $S_{j-1}^n$*-module.*

*Proof.* By definition, $\langle V_j^l \downarrow S_{j-1}\rangle$ contains an irreducible $S_{j-1}$-module $P$ such that $\langle P \downarrow S_{j-1}^n\rangle \ne \{T_{j-1}^n\}$ and $\langle P \downarrow S_{j-1}^m\rangle \ne \{T_{j-1}^m\}$ for some $m \ne n$ where $S_{j-1}^m$ is $S_i^k$-accessible. Hence by Lemmas 6.4, 7.2,

$$\delta_i^k(M) \ge \delta_i^k(V_j^l) = \delta_i^k(V_j^l \downarrow S_{j-1}) \ge \delta_i^k(P) \ge \delta_i^k(P \downarrow S_{j-1}^n) + \delta_i^k(P \downarrow S_{j-1}^m) \ge \delta_i^k(P \downarrow S_{j-1}^n) + 1,$$

as required (taking nontrivial $N$ in $\langle P \downarrow S_{j-1}^n\rangle$), except possibly the cases of spinor $M$. But in these exceptional cases, by Lemma 6.15, there exists a module $Q \in \langle M \downarrow S_{j-1}\rangle$ such that $\langle Q \downarrow S_{j-1}^n\rangle \ne \{T_{j-1}^n\}$ and $\langle Q \downarrow S_{j-1}^m\rangle \ne \{T_{j-1}^m\}$. Therefore,

$$\delta_i^k(M) = \delta_i^k(M \downarrow S_{j-1}) \ge \delta_i^k(Q) \ge \delta_i^k(Q \downarrow S_{j-1}^n) + \delta_i^k(Q \downarrow S_{j-1}^m) \ge$$

$$\delta_i^k(Q \downarrow S_{j-1}^n) + 1 \ge \delta_i^k(N) + 1,$$

where $N$ is a nontrivial module from $\langle Q \downarrow S_{j-1}^n\rangle$, as required.

# 8 Diagonality criterion

**Definition 8.1** A path $\Gamma \in \mathcal{P}(S_i^k)$ is called $(c_1, \ldots, c_5)$-*admissible* if

(1) the number of nonstandard edges of $\Gamma$ is at most $c_1$,

(2) the number of $\Gamma$-critical nodes of $\Gamma$ is at most $c_2$,

(3) the degree of each $\Gamma$-critical node of $\Gamma$ is at most $c_3$,

(4) $\sigma$-rank of each edge of $\Gamma$ is at most $c_4$,

(5) if $S_t^m, S_{t+1}^n \in \Gamma$ and $\tau(S_i^k, S_t^m) > c_5$, then $\delta$-rank of the edge $(S_t^m, S_{t+1}^n)$ is at most $c_4$.

Recall that $\tau(S_i^k, S_t^m)$ is the number of nontrivial composition factors of $V_t^m{\downarrow}S_i^k$, and $\mathfrak{T}_i^k = \cup_{j\geq i} \cup_{1\leq l\leq n_j} \langle V_j^l{\downarrow}S_i^k\rangle$.

**Theorem 8.2** *Let* $\delta_i^k(\mathfrak{T}_i^k) = c$. *Set* $c_1 = 512c^2$, $c_2 = c$, $c_3 = c$, $c_4 = 24c^2$, $c_5 = 2c$. *Then all paths in* $\mathcal{P}(S_i^k)$ *are* $(c_1, \ldots, c_5)$*-admissible.*

*Proof.* Let $\Gamma \in \mathcal{P}(S_i^k)$. Assume that $\Gamma$ is not $(c_1, \ldots, c_5)$-admissible. Show that $\delta_i^k(V_j^l) > c$ for some $S_j^l$. This forces $\delta_i^k(\mathfrak{T}_i^k) > c$ which contradicts to hypothesis.
(1) Assume that $\Gamma$ contains more than $512c^2$ nonstandard edges. Then there exist nodes $S_t^m$, $S_u^n$, $S_j^l$ of $\Gamma$ with $t < u < j$ such that $\operatorname{rk} S_t^m > 12$ and the subpaths $\Gamma_1 = (S_t^m, \ldots, S_u^n)$ and $\Gamma_2 = (S_u^n, \ldots, S_j^l)$ of $\Gamma$ contain exactly $256c^2$ nonstandard edges. Hence by Lemma 7.5, $\delta_t^m(V_u^n) > 256c^2$ and $\delta_u^n(V_j^l) > 256c^2$. Therefore, by Lemmas 7.3, 6.10,

$$\sigma_t^m(V_j^l) = \sigma_t^m(V_j^l{\downarrow}S_u) \geq \sigma_t^m(V_j^l{\downarrow}S_u^n) \geq 4c.$$

By Proposition 6.1 (b) and Lemmas 7.3, 6.5,

$$\delta_i^k(V_j^l) \geq \sigma_i^k(V_j^l) \geq \sigma_i^k(V_j^l{\downarrow}S_t^m) \geq \frac{1}{2}\sigma_t^m(V_j^l) \geq 2c > c.$$

(2) Assume that $\Gamma$ contains more than $c$ $\Gamma$-critical nodes. One can suppose that

$$\Gamma = (S_i^k, \ldots, S_{t-1}^m, S_t^n, \ldots, S_{u-1}^p, S_u^q, \ldots, S_{j-1}^r, S_j^l, \ldots)$$

where $S_t^n, \ldots, S_u^q, S_j^l$ are $c$ $\Gamma$-critical nodes, and $\operatorname{rk} S_{t-1}^m > 12$. Hence $c$ times applying Lemma 7.6, we get

$$\delta_i^k(V_j^l) \geq \delta_i^k(M) + 1 \geq \delta_i^k(M{\downarrow}S_u^q) + 1 \geq \delta_i^k(N) + 2 \geq \ldots \geq \delta_i^k(P) + c > c$$

where $M$, $N$, $P$ are nontrivial irreducible $S_{j-1}^r$-, $S_{u-1}^p$-, $S_{t-1}^m$-modules, respectively.
(3) Assume that the degree of a $\Gamma$-critical node $S_j^l$ of $\Gamma$ is more than $c$. Then by definition, there exists $M \in \langle V_j^l{\downarrow}S_{j-1}\rangle$ such that $\langle M{\downarrow}S_{j-1}^{p_i}\rangle \neq \{T_{j-1}^{p_i}\}$ for $c+1$ distinct $S_i^k$-accessible nodes $S_{j-1}^{p_i}$, $i = 1, \ldots, c+1$. Therefore, by Lemma 7.2,

$$\delta_i^k(V_j^l) = \delta_i^k(V_j^l{\downarrow}S_{j-1}) \geq \delta_i^k(M) = \delta_i^k(M{\downarrow}S_{j-1}^{p_1}) + \ldots + \delta_i^k(M{\downarrow}S_{j-1}^{p_{c+1}}) \geq c + 1 > c.$$

(4) Assume that $\sigma$-rank of an edge $(S_t^m, S_{t+1}^n)$ of $\Gamma$ is more than $24c^2$. Then by Proposition 6.1 (b) and Lemmas 7.3, 6.5,

$$\delta_i^k(V_{t+1}^n) \geq \sigma_i^k(V_{t+1}^n) \geq \sigma_i^k(V_{t+1}^n{\downarrow}S_t^m) \geq \frac{1}{2}\sigma_t^m(V_{t+1}^n) > 12c^2 > c.$$

(5) Assume that an edge $(S_t^m, S_{t+1}^n)$ of $\Gamma$ has $\delta$-rank more than $24c^2$ and $\tau(S_i^k, S_t^m) > 2c$. If $\operatorname{rk} S_t^m \leq 12$, then by Proposition 6.1 (b),

$$\sigma_t^m(V_{t+1}^n) \geq \frac{1}{12}\delta_t^m(V_{t+1}^n) > 2c^2.$$

Therefore, by Proposition 6.1 (b) and Lemma 6.5,

$$\delta_i^k(V_{t+1}^n) \geq \sigma_i^k(V_{t+1}^n) \geq \frac{1}{2}\sigma_t^m(V_{t+1}^n) > c^2 > c,$$

as desired. Assume now that $\operatorname{rk} S_t^m > 12$. Let $M$ be an irreducible $S_t^m$-module from $\langle V_{t+1}^n{\downarrow}S_t^m\rangle$ such that $\delta_t^m(M) = \delta_t^m(V_{t+1}^n) > 24c^2$. Since $\tau(S_i^k, S_t^m) > 2c$, by Lemma 6.8, $\delta_i^k(V_{t+1}^n) \geq \delta_i^k(M) > c$. This completes the proof.

Denote by $d$ the maximum of the value $\delta_{\mathfrak{h}}(V)$ for various embeddings $\mathfrak{h} \to \mathfrak{g}$ where $\operatorname{rk}\mathfrak{g} \leq 12$ and $V$ runs over the fundamental modules of $\mathfrak{g}$.

**Theorem 8.3** *Suppose that all paths from $\mathcal{P}(S_i^k, S_j^l)$ are $(c_1, \ldots, c_5)$-admissible. Let $x = x(S_i^k)$, $y = y(S_i^k)$ be as in Lemma 6.13. Set $m = \max(c_3, c_4, c_5, d, x)$, $n = 2(c_1 + c_2 + 1)$. Then $\delta_i^k(V_j^l) < m^{y^n}$.*

*Proof.* If $\operatorname{rk} S_j^l \leq 12$, then $\delta_i^k(V_j^l) \leq d \leq m < m^{y^n}$. Assume that $\operatorname{rk} S_j^l > 12$. Proceed by induction on $n$, the case $n = 0$ being clear. Consider $\langle V_j^l{\downarrow}S_{j-1}\rangle$. Let $M \in \langle V_j^l{\downarrow}S_{j-1}\rangle$. Estimate the value $\delta_i^k(M)$. It suffices to show that $\delta_i^k(M) < m^{y^n}$. Consider the following cases.
*Case* 1. $S_j^l$ is $\Gamma$-critical for some $\Gamma \in \mathcal{P}(S_i^k, S_j^l)$. Let $M = M_1 \otimes \ldots \otimes M_{n_{j-1}}$ be the canonical form of $M$. Then by Lemma 7.2,

$$\delta_i^k(M) = \delta_i^k(M_1) + \ldots + \delta_i^k(M_{n_{j-1}}).$$

If $S_{j-1}^p$ is not $S_i^k$-accessible, then $\delta_i^k(M_p) = 0$. Since the degree of $S_j^l$ is at most $c_3 \leq m$, this sum contains at most $m$ nonzero terms. Therefore, it suffices to show that $\delta_i^k(M_p) \leq m^{y^n - 1}$. If $\operatorname{rk} S_{j-1}^p \leq 12$, then by Lemma 6.14,

$$\delta_i^k(M_p) \leq 2d\sigma_{j-1}^p(M_p) \leq 2m^2 < m^{y^n - 1},$$

as required. So one can assume that $\operatorname{rk} S_{j-1}^p \geq 12$. Since all paths from $\mathcal{P}(S_i^k, S_{j-1}^p)$ are $(c_1, c_2 - 1, c_3, c_4, c_5)$-admissible, by inductive hypothesis, $\delta_i^k(V_{j-1}^p) < m^{y^{n-2}}$. If the edge $(S_{j-1}^p, S_j^l)$ is standard, then

$$\delta_i^k(M_p) = \delta_i^k(V_{j-1}^p) < m^{y^{n-2}} < m^{y^n - 1}.$$

So it remains to consider the case of nonstandard $(S^p_{j-1}, S^l_j)$. If $\tau(S^k_i, S^p_{j-1}) > c_5$, then by Definition 8.1 (5), $\delta^p_{j-1}(V^l_j) \le c_4 \le m$. Hence by Lemma 6.11,

$$\delta^k_i(M_p) \le \delta^k_i(V^p_{j-1})\delta^p_{j-1}(M_p) < m^{y^{n-2}} m < m^{y^n-1}.$$

If $\tau(S^k_i, S^p_{j-1}) \le c_5$, then by Lemma 6.13,

$$\delta^k_i(M_p) \le x\delta^k_i(V^p_{j-1})^y \tau(S^k_i, S^p_{j-1})\sigma^p_{j-1}(M_p) < m^3(m^{y^{n-2}})^y < m^{y^n-1},$$

as required. This proves *Case 1.*

*Case* 2. $S^l_j$ is not $\Gamma$-critical for all $\Gamma \in \mathcal{P}(S^k_i, S^l_j)$. Then we have $\delta^k_i(M) = \delta^k_i(M_p)$ for some $p$. The arguments are similar to those of *Case* 1. If $\operatorname{rk} S^p_{j-1} \le 12$, then by Lemma 6.14,

$$\delta^k_i(M_p) \le 2d\sigma^p_{j-1}(M_p) \le 2m^2 < m^{y^n}.$$

Assume that $\operatorname{rk} S^p_{j-1} \ge 12$. If the edge $(S^p_{j-1}, S^l_j)$ is standard, then $\delta^k_i(M_p) = \delta^k_i(V^p_{j-1})$, so we can complete the proof by induction on $t = j - i$. Assume that the edge $(S^p_{j-1}, S^l_j)$ is nonstandard. Then all paths from $\mathcal{P}(S^k_i, S^p_{j-1})$ are $(c_1 - 1, c_2, c_3, c_4, c_5)$-admissible. Hence by inductive hypothesis, $\delta^k_i(V^p_{j-1}) < m^{y^{n-2}}$. If $\tau(S^k_i, S^p_{j-1}) > c_5$, then by Definition 8.1 (5), $\delta^p_{j-1}(V^l_j) \le c_4 \le m$. Hence by Lemma 6.11,

$$\delta^k_i(M_p) \le \delta^k_i(V^p_{j-1})\delta^p_{j-1}(M_p) < m^{y^{n-2}} m < m^{y^n},$$

as required. If $\tau(S^k_i, S^p_{j-1}) \le c_5$, then by Lemma 6.13,

$$\delta^k_i(M_p) \le x\delta^k_i(V^p_{j-1})^y \tau(S^k_i, S^p_{j-1})\sigma^p_{j-1}(M_p) < m^3(m^{y^{n-2}})^y < m^{y^n},$$

as required. This proves *Case 2*, and with it the theorem.

**Definition 8.4** Let $L = \varinjlim L_i$ be a locally finite Lie algebra. The Bratteli diagram $\mathfrak{B}$ associated with the local system $\{L_i\}_{i\in I}$ is called *diagonal* if for each $i \in I$ and each $k \in \{1, \dots, n_i\}$ there exist $c_1, \dots, c_5 \in \mathbf{N}$ such that all paths from $\mathcal{P}(S^k_i)$ are $(c_1, \dots, c_5)$-admissible.

**Corollary 8.5 (Diagonality criterion)** *A locally finite Lie algebra $L$ is diagonal if and only if it has a diagonal Bratteli diagram. Moreover, all Bratteli diagrams of every diagonal locally finite Lie algebra are diagonal.*

*Proof.* Since every abstract Levi subalgebra of $L$ lies in the locally perfect radical $P(L)$ (in particular, Bratteli diagrams of $L$ depends only on $P(L)$), one can assume that $L$ is locally perfect. Suppose that $L$ is diagonal. Then by Definition 5.2 and Corollary 5.9, each abstract Levi subalgebra $\mathfrak{S} = \{S_i\}_{i\in I}$ of $L$ is diagonal, i.e. $\mathfrak{T}^k_i$ is finite for all $i \in I$, $k = 1, \dots, n_i$, or equivalently, $\delta^k_i(\mathfrak{T}^k_i) < c^k_i$ for some $c^k_i \in \mathbf{N}$. Hence by Theorem 8.2, all paths from $\mathcal{P}(S^k_i)$ are $(c_1, \dots, c_5)$-admissible for some $c_1, \dots, c_5 \in \mathbf{N}$. Consequently, by Definition 8.4, the Bratteli diagram associated with $\{L_i\}_{i\in I}$ is diagonal. Conversely, if the Bratteli diagram associated with a local system $\{L_i\}_{i\in I}$ is diagonal, then by Theorem 8.3, for each $i$ and $k$ there exists $c^k_i \in \mathbf{N}$ such that $\delta^k_i(V^l_j) < c^k_i$ for all $j > i$, $1 \le l \le n_j$. Consequently, $\delta^k_i(\mathfrak{T}^k_i) < c^k_i$, so $\mathfrak{T}^k_i$ is finite. Therefore, by Definition 5.1, the corresponding abstract Levi subalgebra $\mathfrak{S}$ is diagonal. Hence $L$ is diagonal.

# 9 Ado's theorem for locally perfect Lie algebras

In this section we establish conditions under which a locally perfect Lie algebra $L$ with the trivial center can be embedded into a locally finite associative algebra. If $L$ is semisimple, then the answer on this question is given by Corollary 5.10. In the general case it remains to find conditions for the locally solvable radical of $L$.

Denote by $\operatorname{Ad} L$ the associative subalgebra of $\operatorname{End}_F L$ generated by all $\operatorname{ad} x$, $x \in L$. Consider $L$ as $L$-module (with respect to the adjoint action). It is clear that $\operatorname{Ad} L = U(L)/\operatorname{Ann}_{U(L)} L$.

**Theorem 9.1** *Let $L$ be a locally finite Lie algebra with the trivial center. The following conditions are equivalent.*

*(1) $L$ is a subalgebra of $A^{(-)}$ where $A$ is a locally finite associative algebra.*

*(2) For each $x \in L$ there exists a polynomial $f_x$ such that $f_x(\operatorname{ad} x) = 0$ (in $\operatorname{End} L$).*

*(3) $\operatorname{Ad} L$ is locally finite.*

*Proof.* (1)⇒ (2). If (1) holds, then for each $x \in L$ there exists a polynomial $g_x$ such that $g_x(x) = 0$ in $A$. Hence $g_x(l_x) = g_x(r_x) = 0$ in $\operatorname{End}_F A$ where $l_x : a \mapsto xa$, $r_x : a \mapsto ax$ are endomorphisms of $A$. Since $l_x r_x = r_x l_x$ and $\operatorname{ad} x = l_x - r_x$, there exists a polynomial $f_x$ such that $f_x(\operatorname{ad} x) = 0$ in $\operatorname{End}_F A$. Hence $f_x(\operatorname{ad} x) = 0$ in $\operatorname{End}_F L$.
(2)⇒(3). Let $B$ be a finitely generated subalgebra of $\operatorname{Ad} L$. We have to prove that $B$ is finite-dimensional. Without loss of generality one can assume that $B$ is generated by elements $\operatorname{ad} x_1, \ldots, \operatorname{ad} x_m$ where $x_1, \ldots, x_m$ are linearly independent elements of $L$. Let $x_1, \ldots, x_m, \ldots, x_n$ be a basis of the subalgebra of $L$ generated by $x_1, \ldots, x_m$, and $C$ be the subalgebra of $\operatorname{Ad} L$ generated by $\operatorname{ad} x_1, \ldots, \operatorname{ad} x_n$. It is clear that $B \subseteq C$. Since

$$[\operatorname{ad} x_i, \operatorname{ad} x_j] = \operatorname{ad}[x_i, x_j] \in \langle \operatorname{ad} x_1, \ldots, \operatorname{ad} x_n \rangle_F,$$

$C$ is a linear subspace of the linear space $V$ generated by the elements of type

$$(\operatorname{ad} x_1)^{k_1} \ldots (\operatorname{ad} x_n)^{k_n}.$$

Since for each $i$ there exists a polynomial $f_i$ such that $f_i(\operatorname{ad} x_i) = 0$, the space $V$ is finite-dimensional. Hence $B$ is finite-dimensional.
(3)⇒(1). Since $L$ has the trivial center, the map $x \mapsto \operatorname{ad} x$ is an embedding of $L$ into $\operatorname{Ad} L^{(-)}$.

Observe that Theorem 9.1 gives another interesting characterization of diagonal Lie algebras.

**Corollary 9.2** *A semisimple locally perfect Lie algebra $L$ is diagonal if and only if for each $x \in L$ there exists a polynomial $f_x$ such that $f_x(\operatorname{ad} x) = 0$ (in $\operatorname{End} L$).*

*Proof.* This follows from Corollary 5.10 and the equivalence (1)⇔(2) in Theorem 9.1.

Let $L = \varinjlim L_i$ be locally perfect, $\mathfrak{S} = \{S_i\}_{i \in I}$ an abstract Levi subalgebra of $L$, $\mathfrak{B}$ the corresponding Bratteli diagram. Recall that $L_i = S_i \oplus R_i$ where $R_i = \operatorname{Rad} L_i$. One can consider $L_i$ and $R_i$ as $L_i$-modules (with respect to the adjoint action).

**Definition 9.3** A node $S_i^k$ of $\mathfrak{B}$ is called *R-regular* if there exist $d_1, d_2, d_3 \in \mathbf{N}$ such that for each $j \geq i$ one has

(1) for each $M \in \langle R_j{\downarrow}S_j\rangle$ the number of $S_i^k$-accessible nodes $S_j^l$ at the level $j$ such that $\langle M{\downarrow}S_j^l\rangle \neq \langle T_j^l\rangle$ is at most $d_1$;

(2) if $S_j^l$ is $S_i^k$-accessible then $\sigma_j^l(R_j) \leq d_2$;

(3) if $\tau(S_i^k, S_j^l) > d_3$ then $\delta_j^l(R_j) \leq d_2$.

**Theorem 9.4 ("Ado's theorem")** *Let $L$ be a locally perfect Lie algebra with the trivial center, $\mathfrak{B}$ a Bratteli diagram of $L$. Then $L$ can be embedded into $A^{(-)}$ where $A$ is a locally finite associative algebra if and only if $L$ is diagonal and all nodes of $\mathfrak{B}$ are R-regular.*

*Proof.* Since the center of $L$ is trivial, the theorem follows from the equivalence (1)⇔(3) of Theorem 9.1 and the equivalence (1)⇔(6) of Lemma 9.5.

**Lemma 9.5** *Let $L = \varinjlim L_i$ be a locally perfect Lie algebra, $\mathfrak{S} = \{S_i\}_{i\in I}$ an abstract Levi subalgebra, $\mathfrak{B}$ the corresponding Bratteli diagram of $L$. Then the following conditions are equivalent.*

*(1)* $\operatorname{Ad} L = U(L)/\operatorname{Ann}_{U(L)} L$ *is locally finite.*

*(2)* $\Phi = \{\Phi_i\}$ *with* $\Phi_i = \langle L{\downarrow}L_i\rangle$ *is an inductive system for $L$.*

*(3) For each $i \in I$ the set $\langle L{\downarrow}L_i\rangle = \cup_{j\geq i}\langle L_j{\downarrow}L_i\rangle$ is finite.*

*(4) For each node $S_i^k$ of $\mathfrak{B}$ there exists $d_i^k \in \mathbf{N}$ such that $\delta_i^k(L_j) \leq d_i^k$ for all $j \geq i$.*

*(5) $L$ is diagonal, and for each node $S_i^k$ of $\mathfrak{B}$ there exists $c_i^k \in \mathbf{N}$ such that $\delta_i^k(R_j) \leq c_i^k$ for all $j \geq i$.*

*(6) $L$ is diagonal, and all nodes of $\mathfrak{B}$ are R-regular.*

*Proof.* (1)⇔(2). Set $X = \operatorname{Ann}_{U(L)} L$. By Theorem 3.9, it suffices to show that $\Phi(X) = \Phi$, i.e.

$$\Phi_i(X) = \langle U(L_i)/X \cap U(L_i)\rangle = \langle L{\downarrow}L_i\rangle = \Phi_i$$

for all $i \in I$ (assuming $\Phi_i(X)$ or $\Phi_i$ to be finite). This follows from Lemma 3.2 and the equality below.

$$\operatorname{Ann}_{U(L_i)}(U(L_i)/X \cap U(L_i)) = X \cap U(L_i) = \operatorname{Ann}_{U(L_i)} L = \operatorname{Ann}_{U(L_i)}(L{\downarrow}L_i)$$

(2)⇔(3) is obvious.
(3)⇔(4). In view of Theorem 3.1, $|\langle L{\downarrow}L_i\rangle| = |\langle L{\downarrow}S_i\rangle| = |\cup_{j\geq i}\langle L_j{\downarrow}S_i\rangle|$. It remains to observe that the set $\cup_{j\geq i}\langle L_j{\downarrow}S_i\rangle$ is finite if and only if for each $k = 1, \ldots, n_i$ there exists $d_i^k \in \mathbf{N}$ such that $\delta_i^k(L_j) \leq d_i^k$ for all $j \geq i$.

(4)⇔(5). Since $L_j = \oplus_{l=1}^{n_j} S_j^l \oplus R_j$, it suffices to show that $L$ is diagonal if and only if for each node $S_i^k$ of $\mathfrak{B}$ there exists $d_i^k \in \mathbf{N}$ such that $\delta_i^k(S_j^l) \leq d_i^k$ for all $j \geq i$, $l = 1, \ldots, n_j$. Since $S_j^l \subset V_j^l \otimes V_j^{l^*}$ as $S_i^k$-modules, we have

$$\delta_i^k(S_j^l) \leq \delta_i^k(V_j^l \otimes V_j^{l^*}) = \delta_i^k(V_j^l) + \delta_i^k(V_j^{l^*}) = 2\delta_i^k(V_j^l).$$

On the other hand, by Lemma 6.4, $\delta_i^k(V_j^l) \leq 2\delta_i^k(S_j^l)$. The claim now follows from Definition 5.1.

(5)⇔(6). Let $i \in I$, $k \in \{1, \ldots, n_i\}$, $j \geq i$. Since $L$ is diagonal, by Corollary 8.5 and Definition 8.4, there exist $c_1, \ldots, c_5 \in \mathbf{N}$ such that all paths from $\mathcal{P}(S_i^k)$ are $(c_1, \ldots, c_5)$-admissible. "Add" a new node $S_{j+1}^x = sl(R_j)$ (with the corresponding edges) at the level $j+1$, setting $V_{j+1}^x{\downarrow}S_j = R_j{\downarrow}S_j$. It is clear that $\delta_i^k(R_j) = \delta_i^k(V_{j+1}^x)$.

Assume that a node $S_i^k$ is $R$-regular (Definition 9.3). Then all paths from $\mathcal{P}(S_i^k, S_{j+1}^x)$ are $(c_1+1, c_2+1, \max(c_3, d_1), \max(c_4, d_2), \max(c_5, d_3))$-admissible. Therefore, by Theorem 8.3, there exists $c_i^k \in \mathbf{N}$ such that $\delta_i^k(R_j) = \delta_i^k(V_{j+1}^x) \leq c_i^k$ for all $j \geq i$.

Conversely, assume that $\delta_i^k(R_j) = \delta_i^k(V_{j+1}^x) \leq c_i^k$ for all $j \geq i$. Then by Theorem 8.2, there exist $c_1, \ldots, c_5 \in \mathbf{N}$ such that all paths from $\mathcal{P}(S_i^k, S_{j+1}^x)$ are $(c_1, \ldots, c_5)$-admissible for all $j \geq i$. Set $d_1 = c_3$, $d_2 = c_4$, $d_3 = c_5$. It is not difficult to see that the node $S_i^k$ is $R$-regular in the sense of Definition 9.3. So the lemma follows.

We conclude this section by a statement which specifies inductive systems.

**Theorem 9.6** *Let $L = \varinjlim L_i$ be a diagonal locally perfect Lie algebra, $\mathfrak{B}$ the Bratteli diagram of $L$ associated with $\{L_i\}_{i \in I}$, $\Phi = \{\Phi_i\}_{i \in I}$ an inductive system for $L$. Then each node $S_i^k$ of $\mathfrak{B}$ is $\Phi$-regular, i.e. there exist $d_1, d_2, d_3 \in \mathbf{N}$ depending on $i$ and $k$ such that for each $j \geq i$ one has*

*(1) for each $M \in \Phi_j{\downarrow}S_j$ the number of $S_i^k$-accessible nodes $S_j^l$ at the level $j$ such that $\langle M{\downarrow}S_j^l \rangle \neq \langle T_j^l \rangle$ is at most $d_1$;*

*(2) if $S_j^l$ is $S_i^k$-accessible then $\sigma_j^l(\Phi_j) \leq d_2$;*

*(3) if $\tau(S_i^k, S_j^l) > d_3$ then $\delta_j^l(\Phi_j) \leq d_2$.*

*Proof.* The arguments are similar to those of the proof (5)⇒(6) in Lemma 9.5. Set $c_i^k = \delta_i^k(\Phi_i)$. Since $\Phi_j{\downarrow}L_i = \Phi_i$, we have $\delta_i^k(\Phi_j) = \delta_i^k(\Phi_i) \leq c_i^k$ for all $j \geq i$. Then by Theorem 8.2, there exist $c_1, \ldots, c_5 \in \mathbf{N}$ such that all paths from $\mathcal{P}(S_i^k, S_{j+1}^x)$ are $(c_1, \ldots, c_5)$-admissible for all $j \geq i$ where $S_{j+1}^x$ is a "new" node of $\mathfrak{B}$ such that $\langle V_{j+1}^x{\downarrow}S_j \rangle = \Phi_j{\downarrow}S_j$. Set $d_1 = c_3$, $d_2 = c_4$, $d_3 = c_5$. It is not difficult to see that the node $S_i^k$ is $\Phi$-regular.

If $L = \varinjlim L_i$ is simple, then the situation is much easier. One can show ([5]) that for each inductive system $\Phi = \{\Phi_i\}_{i \in I}$ there exist $d_1, d_2 \in \mathbf{N}$ such that for all $i \in I$ and all $M \in \Phi_i{\downarrow}S_i$ the number of nodes $S_i^k$ at the level $i$ such that $\langle M{\downarrow}S_i^k \rangle \neq \langle T_i^k \rangle$ is at most $d_1$, and $\sigma_i^k(\Phi_i) \leq d_2$ for all nodes $S_i^k$. Moreover, if $L$ is *non-finitary*, then $\delta_i^k(\Phi_i) \leq d_2$ for all $S_i^k$.

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