Complex finitary simple Lie algebras

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Abstract

An algebra is called finitary if it consists of finite-rank transformations of a vector space. We classify finitary simple Lie algebras over an algebraically closed field of zero characteristic. It is shown that any such algebra is isomorphic to one of the following

(1) a special transvection algebra $\mathfrak{t}(V,\Pi)$;

(2) a finitary orthogonal algebra $\mathfrak{fso}(V,q)$;

(3) a finitary symplectic algebra $\mathfrak{fsp}(V,s)$.

Here V is an infinite dimensional K-space; q (respectively, s) is a symmetric (respectively, skew-symmetric) nondegenerate bilinear form on V; and Π is a subspace of the dual V^* whose annihilator in V is trivial: $0 = \{v \in V \mid \Pi v = 0\}$.

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1. Introduction. Let K be an algebraically closed field of zero characteristic and let V be a vector space over K. All linear transformations of V form a Lie algebra $\mathfrak{gl}(V)$. An element $g \in \mathfrak{gl}(V)$ is called *finitary* if $\dim gV < \infty$. The finitary transformations of V form an ideal $\mathfrak{fgl}(V)$ of $\mathfrak{gl}(V)$, and any subalgebra of $\mathfrak{fgl}(V)$ is called *finitary* (or *cofinite*, see [5]). The aim of this paper is to classify finitary simple Lie algebras over K.

Since $\mathfrak{fgl}(V)$ is locally finite-dimensional (or *locally finite*, for brevity), any finitary Lie algebra is also locally finite. Nowadays locally finite groups and Lie algebras are studied rather intensively (see, for instance, a recent survey of Zalesskii [6]). Finitary simple Lie algebras form a natural subclass in the category of so-called *diagonal* locally finite Lie algebras. The latter were introduced in [1] for generalizing the classical Ado theorem to locally finite Lie algebras. The classification of finitary simple Lie algebras of countable dimension was obtained in [2, Theorem 1.3]. It turns out that there are only three ones: $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$. The analogous problem in group theory has been solved by Hall [4]. He classified simple locally finite groups of finitary linear transformations. We use some notation from [4] translating it into the Lie algebra language. First of all we need to introduce certain classes of finitary simple Lie algebras.

2. Special transvection algebras. In the paper we find convenient to use the term "transvection" in a nonstandard way. An element $t \in \mathfrak{gl}(V)$ will be called a *transvection* if t has $t^2 = 0$ with the range tV of dimension 1 (in the standard definition one takes $t - 1$ rather than t). Choose a representative x of this range, $tV = \langle x \rangle_K$. Then $v \to tv = \lambda x$ ($\lambda \in K$) gives a linear functional $\varphi : v \mapsto \lambda$ such that $\varphi x = 0$ (since

$t^2 = 0$). The pair x, φ completely determines t; and, conversely, for any pair $x \in V$ and $\varphi \in V^*$ (the dual) with $\varphi x = 0$, we have a transvection $t = t_{x\varphi}$ given by

$$t_{x\varphi} v = (\varphi v)x.$$

If $V = K^n$ is spanned by the canonical basis $e_1, \ldots, e_n$ with dual basis $e_1^*, \ldots, e_n^*$, then the transvection $t_{e_i e_j^*}$ is just the matrix unit e_{ij}. The following is well known.

Proposition 1 *If* $\dim V$ *is finite, then* $\mathfrak{sl}(V)$ *is generated by its transvections.*

For general V we define the *finitary special linear algebra* $\mathfrak{fsl}(V)$ to be the set of all finitary linear transformations of V with zero traces. Because it is true in finite dimension, we always have $\mathfrak{fsl}(V) = [\mathfrak{fgl}(V), \mathfrak{fgl}(V)]$ and $\mathfrak{fsl}(V)$ is generated by all the transvections $t_{x\varphi}$, with $x \in V$, and $\varphi x = 0$.

If V has finite dimension, then $V \cong V^*$ and $\mathfrak{fsl}(V) = \mathfrak{sl}(V)$. When V has infinite dimension then V^* has uncountably infinite dimension; and, since $\mathfrak{fsl}(V)$ contains all transvections, $\dim \mathfrak{fsl}(V)$ is uncountable. There is another finitary counterpart to the special linear algebra which remains countably dimensional for countably dimensional V, the stable special linear algebra $\mathfrak{sl}_\infty$. This is best introduced in terms of matrices. Every $n \times n$ matrix A_n can be extended to a $(n+1) \times (n+1)$ matrix A_{n+1} by placing A_n in the upper lefthand corner of A_{n+1} and then bordering A_n with 0's in A_{n+1}. This gives us natural embeddings

$$\mathfrak{sl}_2 \to \mathfrak{sl}_3 \to \ldots \to \mathfrak{sl}_n \to \ldots.$$

The union of this algebras is then the *stable special linear algebra* $\mathfrak{sl}_\infty$ and is countably dimensional. This algebra has a natural finitary action on the K-space V spanned by $B = \{e_1, e_2, \ldots, e_n, \ldots\}$, where $B_n = \{e_1, e_2, \ldots, e_n\}$ is the standard basis for the natural module of $\mathfrak{sl}_n$, for each n.

We can unify and generalize our two infinite dimensional versions of the special linear algebra, $\mathfrak{fsl}(V)$ and $\mathfrak{sl}_\infty$, by first realizing that both are generated by transvections. The algebra $\mathfrak{fsl}(V)$ is generated by all transvections, while Proposition 1 and our construction show that $\mathfrak{sl}_\infty$ is generated by the various elementary matrix transvections e_{ij}.

Let U be a K-subspace of V and Π a K-subspace of the dual V^*. Then the special transvection algebra $\mathfrak{t}(U, \Pi)$ is defined as

$$\mathfrak{t}(U, \Pi) = \langle t_{x\varphi} \mid x \in U, \quad \varphi \in \Pi, \quad \varphi x = 0 \rangle,$$

the subalgebra of $\mathfrak{gl}(V)$ generated by all the transvections $t_{x\varphi}$ where the eligible pairs x, φ are restricted to U and Π. The algebra $\mathfrak{t}(U, \Pi)$ is a Lie algebra analog of the special transvection group $T(U, \Pi)$ introduced by Cameron and Hall [3]. Clearly, such an algebra is finitary. In fact it is a subalgebra of $\mathfrak{fsl}(V) = \mathfrak{t}(V, V^*)$. On the other hand, $\mathfrak{sl}_\infty = \mathfrak{t}(V, \Pi)$, where Π is a subspace of V^* spanned by $B^* = \{e_1^*, e_2^*, \ldots, e_n^*, \ldots\}$, the dual of the basis $B = \{e_1, e_2, \ldots, e_n, \ldots\}$. Assume that the annihilator $\mathrm{Ann}_V \Pi = \{v \in V \mid \Pi v = 0\}$ of Π in V is trivial. If $\dim V$ is finite, then the only possible such choice for Π is the complete dual V^*; and $\mathfrak{t}(V, \Pi) = \mathfrak{sl}(V) \cong \mathfrak{sl}_n$. Assume now that $\dim V$ is infinite. Let $t_{x_1\varphi_1}, \ldots, t_{x_n\varphi_n}$ be a finite set of transvections. Set $V_1 = \langle x_1, \ldots, x_n \rangle_K$, $\Pi_1 = \langle \varphi_1, \ldots, \varphi_n \rangle_K$. Fix a basis $\{\psi_1, \ldots, \psi_k\}$ of Π_1. One can choose elements $e_1, \ldots, e_k \in V$ such that $\psi_i e_j = \delta_{ij}$. Moreover, there exists a basis $\{e_1, \ldots, e_k, \ldots, e_m\}$ of $V_2 = \langle e_1, \ldots, e_k \rangle_K + V_1$ such that $\psi_i e_j = \delta_{ij}$ for $i \leq k$ and $j \leq m$. Since $\mathrm{Ann}_V \Pi = 0$, there exist $\psi_{k+1}, \ldots, \psi_m \in \Pi$ such that $\psi_i e_j = \delta_{ij}$ for $1 \leq i, j \leq m$. Set $\Pi_2 = \langle \psi_1, \ldots, \psi_m \rangle_K$. Then we have $t_{x_1\varphi_1}, \ldots, t_{x_n\varphi_n} \in \mathfrak{t}(V_2, \Pi_2) \cong \mathfrak{sl}(V_2)$. We say that finite-dimensional subspaces $V' \subset V$ and $\Pi' \subset V^*$ are compatible if $\dim V' = \dim \Pi'$ and $\mathrm{Ann}_{V'} \Pi' = 0$. Clearly, if V' and Π' are compatible, then there exist bases $\{e_1, \ldots, e_m\}$ of V' and $\{\psi_1, \ldots, \psi_m\}$ of Π' such that $\psi_i e_j = \delta_{ij}$ for $1 \leq i, j \leq m$, so $\mathfrak{t}(V', \Pi') \cong \mathfrak{sl}(V')$. By the arguments above, $\mathfrak{t}(V, \Pi)$ is the direct limit of finite-dimensional subalgebras $\mathfrak{t}(V', \Pi') \cong \mathfrak{sl}(V')$ where V' and Π' run over all compatible pairs of subspaces in V and Π. Note that $\mathfrak{t}(V', \Pi') \subseteq \mathfrak{t}(V'', \Pi'')$ if and only if $V' \subseteq V''$ and $\Pi' \subseteq \Pi''$. Moreover, if $V' \subseteq V''$ and $\Pi' \subseteq \Pi''$, then the corresponding embedding $\mathfrak{sl}(V')$ into $\mathfrak{sl}(V'')$ is natural. We say that an embedding $L_1 \to L_2$ of finite-dimensional classical simple Lie algebras is *natural* if the restriction of the standard L_2-module to L_1 involves a unique nontrivial composition factor, and the latter is isomorphic to the standard L_1-module or dual to it. So we have

Proposition 2 *Let V be an infinite dimensional vector space and let $\Pi \subseteq V^*$ be such that $\mathrm{Ann}_V \Pi = 0$. Then $\mathfrak{t}(V, \Pi)$ is isomorphic to a direct limit of natural embeddings of finite-dimensional special linear algebras. In particular, $\mathfrak{t}(V, \Pi)$ is simple.*

3. Finitary orthogonal and finitary symplectic algebras. We now consider a K-space V endowed with a nondegenerate bilinear *symmetric* form $q : V \times V \to K$. One says that an element $g \in \mathfrak{gl}(V)$ leaves q invariant if

$$q(gu, v) + q(u, gv) = 0 \quad \text{for all } u, v \in V.$$

The *special orthogonal algebra* (or, simply, *orthogonal algebra*) $\mathfrak{so}(V, q)$ is the set of all elements $g \in \mathfrak{gl}(V)$ that leave q invariant. The *finitary orthogonal algebra* $\mathfrak{fso}(V, q)$ is defined as $\mathfrak{fso}(V, q) = \mathfrak{so}(V, q) \cap \mathfrak{fgl}(V)$. If we have a nondegenerate bilinear *skew-symmetric* form s on V, then we obtain the *symplectic algebra* $\mathfrak{sp}(V, s)$ and the *finitary symplectic algebra* $\mathfrak{fsp}(V, s) = \mathfrak{sp}(V, s) \cap \mathfrak{fgl}(V)$. If $\dim V = n$ is finite, then $\mathfrak{fso}(V, q) = \mathfrak{so}(V, q) \cong \mathfrak{so}_n$ and $\mathfrak{fsp}(V, s) = \mathfrak{sp}(V, s) \cong \mathfrak{sp}_n$ (in the latter case n must be even). It is worth mentioning that if the dimension of V is infinite but countable, there are unique nondegenerate symmetric and skew-symmetric forms up to equivalence. In this case $\mathfrak{fso}(V, q) \cong \mathfrak{so}_\infty$ and $\mathfrak{fsp}(V, s) \cong \mathfrak{sp}_\infty$, where $\mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$ are the *stable orthogonal algebra* and the *stable symplectic algebra*, respectively, defined as the direct limits of the natural embeddings of the corresponding classical algebras

$$\mathfrak{so}_2 \to \mathfrak{so}_3 \to \ldots \to \mathfrak{so}_n \to \ldots,$$

$$\mathfrak{sp}_2 \to \mathfrak{sp}_4 \to \ldots \to \mathfrak{sp}_{2n} \to \ldots.$$

On the other hand, for a given V of uncountable dimension, there are many fundamentally different nondegenerate symmetric and skew-symmetric forms and so different algebras (see, for instance, [3]).

Assume now that $\dim V$ is infinite. Let L be a finite-dimensional subalgebra in $\mathfrak{fso}(V,q)$. Set $V_1 = LV$. Note that $\dim V_1$ is finite. Since q is nondegenerate, there exists a finite-dimensional subspace V_2 of V containing V_1 such that the restriction $q' = q|_{V_2}$ is nondegenerate. Therefore one can decompose V as $V = V_2 \oplus V_2^{\perp}$. Since $L \subset \mathfrak{so}(V,q)$, L leaves $V_2^{\perp}$ invariant. As $LV \subseteq V_2$, we have $LV_2^{\perp} = 0$. For a finite-dimensional subspace U of V with $q|_U$ nondegenerate, denote by $L(U)$ the algebra of all elements $g \in \mathfrak{fso}(V,q)$ such that $gU^{\perp} = 0$. We have

$$L \subseteq L(V_2) \cong \mathfrak{so}(V_2, q') \cong \mathfrak{so}_n$$

where $n = \dim V_2$. Therefore $\mathfrak{fso}(V,q)$ is the direct limit of finite-dimensional subalgebras $L(U) \cong \mathfrak{so}(U, q|_U)$, where U runs over all finite-dimensional subspaces of V such that $q|_U$ is nondegenerate. Note that $L(U) \subseteq L(U')$ if and only if $U \subseteq U'$. Moreover, if $U \subseteq U'$, then the corresponding embedding $\mathfrak{so}(U, q|_U)$ into $\mathfrak{so}(U', q|_{U'})$ is natural. Arguing similarly for the symplectic case, we obtain

Proposition 3 *Let V be an infinite dimensional vector space with a nondegenerate bilinear symmetric form q (resp., skew-symmetric form s). Then $\mathfrak{fso}(V,q)$ (resp., $\mathfrak{fsp}(V,s)$) is isomorphic to a direct limit of natural embeddings of finite-dimensional orthogonal (resp., symplectic) algebras. In particular, $\mathfrak{fso}(V,q)$ and $\mathfrak{fsp}(V,s)$ are simple.*

4. The classification. A set $\{L_i\}_{i\in I}$ of finite-dimensional subalgebras of a locally finite Lie algebra L is called a *local system* if $L = \cup_{i\in I} L_i$ and for any pair $i, j \in I$ there exists $k \in I$ such that $L_i, L_j \subseteq L_k$. Set $i \leq j$ if $L_i \subseteq L_j$. Then I is a *directed* set, i.e. for any pair $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$. It is clear that L is the direct limit of the algebras L_i, that is, $L = \varinjlim L_i$. If the set I has the smallest element (say 1), then the local system $\{L_i\}_{i\in I}$ is called *conical*. This means that $L_1 \subseteq L_i$ for all $i \in I$.

Proposition 4 ([2, Theorem 5.1 and Theorem 5.6]) *Let L be a finitary simple Lie algebra. Then there exists a conical local system of L consisting of classical finite-dimensional simple Lie algebras of the same type A, B, or C, such that the corresponding embeddings of each finite-dimensional algebra into another are natural.*

We now in position to prove the main result.

Theorem 1 *Each infinite dimensional simple Lie algebra over K which has a faithful representation as a finitary Lie algebra is isomorphic to one of the following:*

(1) a special transvection algebra $\mathfrak{t}(V,\Pi)$;

(2) a finitary orthogonal algebra $\mathfrak{fso}(V,q)$;

(3) a finitary symplectic algebra $\mathfrak{fsp}(V,s)$.

Here V is an infinite dimensional K-space; q (respectively, s) is a symmetric (respectively, skew-symmetric) nondegenerate bilinear form on V; and Π is a subspace of the dual V^* whose annihilator in V is trivial: $0 = \{v \in V \mid \Pi v = 0\}$.

P r o o f. Let L be a finitary simple Lie algebra. By Proposition 4, there exists a local system $\{L_i\}_{i\in I}$ of L such that I has the smallest element (say 1), all L_i are classical simple of the same type (A, B, or C), and all embeddings $L_i \subset L_j$ for $i < j$ are natural. Denote by V_i the standard L_i-module. Assume for a moment that all L_i have type A. Observe that if we replace the base $B = \{\alpha_1, \dots, \alpha_n\}$ of the root system of L_i by the base $B' = \{\alpha'_1, \dots, \alpha'_n\}$ with $\alpha'_k = \alpha_{n+1-k}$, the dual module V_i^* becomes standard with respect to B'. Hence one can assume that for each j the nontrivial composition factor of the restriction $V_j{\downarrow}L_1$ is isomorphic to V_1. Therefore (as for types B and C) for each pair $i < j$ the nontrivial composition factor of the restriction $V_j{\downarrow}L_i$ is isomorphic to V_i. Now we return to the general situation. It follows that there exists a unique (up to a scalar multiplier) injective homomorphism of L_i-modules $\rho_{ij} : V_i \to V_j$. We construct the set of nonzero scalars $\{\lambda_{ij}\}_{i<j}$ in the following manner. Put $\lambda_{1j} = 1$ for all j; the λ_{ij} with $1 < i < j$ are determined from the equality $\lambda_{ij}\rho_{ij} \circ \rho_{1i} = \rho_{1j}$. We define the homomorphism $\bar\rho_{ij} : V_i \to V_j$ by $\bar\rho_{ij} = \lambda_{ij}\rho_{ij}$, so we have $\bar\rho_{ij} \circ \bar\rho_{1i} = \bar\rho_{1j}$ for all $1 < i < j$. One can easily check that $\bar\rho_{jk} \circ \bar\rho_{ij} = \bar\rho_{ik}$ for all $i < j < k$. Indeed, by construction, $\bar\rho_{jk} \circ \bar\rho_{ij} = \lambda\bar\rho_{ik}$ for some $\lambda \in K$. Therefore

$$\lambda\bar\rho_{1k} = \lambda\bar\rho_{ik}\bar\rho_{1i} = \bar\rho_{jk} \circ \bar\rho_{ij} \circ \bar\rho_{1i} = \bar\rho_{jk} \circ \bar\rho_{1j} = \bar\rho_{1k},$$

so $\lambda = 1$. It follows that one can construct the inductive limit $V = \varinjlim V_i$ with respect to the embeddings $\bar\rho_{ij}$. We shall identify V_i with the corresponding subspace in V, so $V_i \subseteq V_j$ whenever $i \le j$. Clearly, V is a faithful finitary L-module and we can identify L with the subalgebra of $\mathfrak{fgl}(V)$. Consider the following cases.

Case 1: all L_i have type A (special linear). Fix $i \in I$. In view of complete reducibility V can be decomposed as $V = V_i \oplus V_i^0$ where $L_iV_i^0 = 0$. Choose bases $\{e_1, \dots, e_n\}$ of V_i and $\{e_{n+1}, e_{n+2}, \dots\}$ of V_i^0. Define the functions $\varphi_1, \dots, \varphi_n \in V^*$ by $\varphi_i e_j = \delta_{ij}$. Set $\Pi_i = \langle\varphi_1, \dots, \varphi_n\rangle_K \subset V^*$. Obviously, $L_i = \mathfrak{t}(V_i, \Pi_i)$. Let $i \le j$. Then

$$\mathfrak{t}(V_i, \Pi_i) = L_i \subseteq L_j = \mathfrak{t}(V_j, \Pi_j).$$

Therefore $\Pi_i \subseteq \Pi_j$. Set $\Pi = \varinjlim \Pi_i \subseteq V^*$. Clearly, $\mathrm{Ann}_V \Pi = 0$ and $L \subseteq \mathfrak{t}(V, \Pi)$. It remains to show that $L = \mathfrak{t}(V, \Pi)$. It suffices to check that any transvection $t_{v\varphi}$ with $v \in V$ and $\varphi \in \Pi$ belongs to L. Find $i, j \in I$ such that $v \in V_i$ and $\varphi \in \Pi_j$. Take any $k > i, j$. Then $v \in V_k$ and $\varphi \in \Pi_k$. Therefore

$$t_{v\varphi} \in \mathfrak{t}(V_k, \Pi_k) = L_k \subset L,$$

as required.

Case 2: all L_i have type B (orthogonal). Since $L_i \cong \mathfrak{so}(V_i)$, there exists a unique (up to a scalar multiplier) nondegenerate symmetric form q_i on V_i that is invariant under L_i. Let $i < j$. Since $L_i \subset L_j$, L_i leaves invariant q_j on V_j. Therefore $q_j|_{V_i} = \lambda_{ij}q_i$ where $\lambda_{ij} \in K$. Note that $\lambda_{ij} \neq 0$. Indeed, otherwise the restriction $V_j{\downarrow}L_i$ has at least two nontrivial composition factors isomorphic to V_i. Set $\bar{q}_1 = q_1$ and $\bar{q}_j = (1/\lambda_{1j})q_j$ for all $j \neq 1$. Then $\bar{q}_j|_{V_1} = \bar{q}_1$ for all $j \in I$. Therefore for each pair $i < j$ we have $\bar{q}_j|_{V_i} = \bar{q}_i$. Define the form q on V by $q|_{V_i} = \bar{q}_i$. Clearly, q is symmetric and nondegenerate. By construction, $L \subseteq \mathfrak{fso}(V, q)$. Let now $g \in \mathfrak{fso}(V, q)$. Then $gV \subseteq V_i$ for some i. Since g leaves q invariant, $gV_i^{\perp} = 0$ where $V_i^{\perp}$ is the orthogonal complement of V_i in V. Note that we have also $L_i V_i^{\perp} = 0$. Since $L_i \cong \mathfrak{so}(V_i)$, we have $g \in L_i \subset L$. Therefore $L = \mathfrak{fso}(V, q)$, as required.

Case 3: all L_i have type C (symplectic). Arguing as in Case 2, one can show that $L = \mathfrak{fsp}(V, s)$, where s is a nondegenerate skew-symmetric form on V. The theorem follows.

R e m a r k. We distribute finitary simple Lie algebras by types: A (special transvection algebras), B (finitary orthogonal algebras), C (finitary symplectic algebras). It is shown in [2, Proposition 5.3] that algebras of different types are nonisomorphic.

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