Diagonal direct limits of simple Lie algebras

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1 Introduction

The aim of this article is to classify diagonal locally simple Lie algebras of countable dimension over an algebraically closed field of zero characteristic. We also give some remarks on classification of locally simple associative algebras. Recall that an algebra A is called *locally finite* if any finite subset of A is contained in a finite-dimensional subalgebra. If these subalgebras can be chosen simple, A is called *locally simple*. Observe that A is simple in this case.

Let F be an algebraically closed field of zero characteristic, A be a locally simple associative algebra of countable dimension over F. It follows from the definition that there is an increasing sequence of simple subalgebras $M_1 \subset M_2 \subset M_3 \subset \dots$ of A such that $A = \cup_{i=1}^{\infty} M_i$. It is more convenient to write

$$M_1 \to M_2 \to M_3 \to \dots \tag{1}$$

where '$\to$' stands for a natural embedding. Then one can say that $A = \varinjlim M_i$. Since F is algebraically closed, M_i can be identified with the algebra of all $n_i \times n_i$ matrices for some n_i. Moreover, the embedding $M_i \to M_{i+1}$ can be written in the form

$$M \mapsto \mathrm{diag}(M, \dots, M, 0, \dots, 0) \quad (M \in M_i). \tag{2}$$

Therefore in order to describe locally simple associative algebras of countable dimension one needs to classify direct limits of sequences of matrix algebras (1). Elliot [9] did this in terms of systems of idempotents. It has been shown later that Elliot's invariant can be interpreted in terms of the K_0-functor.

Using elementary methods, we give another parametrization of these algebras (see Theorem 4.4).

Let $F = \mathbf{C}$ where $\mathbf{C}$ denotes the field of complex numbers. If one considers matrix algebra M_i as a $\mathbf{C}^*$-algebra under the involution given by transpose, then the embeddings (2) are $\mathbf{C}^*$-algebra morphisms, i.e. algebra homomorphisms compatible with the involution. Dixmier [7] (using $\mathbf{C}^*$-algebra methods) classified the direct limits of sequences (1) in the category of $\mathbf{C}^*$-algebras, i.e. $\mathbf{C}^*$-algebras $\overline{\varinjlim M_i}$ where the overline stands for the closure under the standard norm. Bratteli [6] proved that $A_1 \cong A_2$ if and only if $\overline{A_1} \cong \overline{A_2}$ where A_1 and A_2 are the direct limits of sequences (1) with semisimple M_i. So our approach here gives a new proof of Dixmier's result. Actually, our parametrization is almost identical to Dixmier's one.

Let L_1 and L_2 be finite-dimensional simple classical Lie algebras. An embedding $L_1 \to L_2$ is called *diagonal* if any nontrivial composition factor of the restriction of the standard L_2-module to L_1 is either the standard L_1-module or dual to it. We illustrate this definition by the following example. An embedding $sl(V) \to sl(W)$ is diagonal if and only if one can choose a basis of W such that

$$M \mapsto \operatorname{diag}(M, \dots, M, -M^t, \dots, -M^t, 0, \dots, 0)$$

for any matrix $M \in sl(V)$. A locally simple Lie algebra L is called *diagonal* if it can be represented as a direct limit of diagonal embeddings of finite-dimensional simple Lie algebras. If L is of countable dimension, then this definition can be reformulated in the following way. There exists an increasing sequence of finite-dimensional simple subalgebras $L_1 \subset L_2 \subset \dots$ such that $L = \cup_{i=1}^{\infty} L_i$ and all embeddings $L_i \subset L_{i+1}$ are diagonal.

The notion of diagonality was introduced by Zalesskii and appeared for the first time in a group theory context, see [13]. The first author [3] extended it to a wider class of locally finite Lie algebras for proving a version of Ado's theorem for locally finite Lie algebras. In [3] necessary and sufficient conditions (under some restrictions) for a locally finite Lie algebra to be embeddable into a locally finite associative algebra were found. For a particular case this was done in fact by the second author. He proved in [15] that a locally simple Lie algebra L can be realized as a Lie subalgebra of a locally finite associative algebra if and only if L is diagonal. In this paper we classify all such algebras of countable dimension. It turns out that there $2^{\aleph_0}$ of them. Bahturin and Strade [1] have been the first to observed that the set of these algebras is uncountable.

It will be shown that the classification of direct limits (1) is equivalent to that of direct limits of *one-sided* sequences of Lie algebras of type A. So we restrict ourselve to the Lie algebra case, pointing out some specific characters of the associative case.

Some related results and problems concerning the theory of simple locally finite associative and Lie algebras as well as simple locally finite groups are discussed in [14] were a motivation for investigation on this topic is given. Note also that some simple Lie algebra limits of particular type were classified by Yanson and Zhdanovich [12]; their results are less complete and are stated in another form. It remains to notice that our results does not exhaust the problem of classification of all diagonal simple locally finite Lie algebras (of countable dimension), since there are examples of such algebras which are not locally semisimple [2] (see also [10] for the case of associative algebras).

2 Diagonal embeddings

Let L be the direct limit of a sequence $L_1 \to L_2 \to \dots$ of simple finite-dimensional Lie algebras. Suppose that L has infinite dimension. Removing some members of the sequence, one can assume that all L_i are classical of the same type A, B, C or D. It is convenient for us do not distinguish the types B and D. So we shall say that L_i is of type O (orthogonal) if it is of type either B or D.

Definition 2.1 An embedding $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ of finite-dimensional classical simple Lie algebras is called *diagonal* if

$$V_2{\downarrow}\mathfrak{g}_1 = \underbrace{V_1 \oplus \dots \oplus V_1}_{l} \oplus \underbrace{V_1^* \oplus \dots \oplus V_1^*}_{r} \oplus \underbrace{T_1 \oplus \dots \oplus T_1}_{z}$$

where V_i is the standard $\mathfrak{g}_i$-module ($i = 1, 2$), V_1^* is dual to V_1 ($r = 0$ if $\mathfrak{g}_1$ is not of type A), and T_1 is the trivial one-dimensional $\mathfrak{g}_1$-module. The triple (l, r, z) is called the *signature* of ε.

Definition 2.2 An embedding $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called *natural* if it is diagonal and $l + r = 1$.

One can identify $\mathfrak{g}_i$ with $sl(V_i)$, $so(V_i)$ or $sp(V_i)$ ($i = 1, 2$). Clearly, an embedding $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ is diagonal if and only if one can choose bases of V_1, V_2 such that

$$\varepsilon(A) = \mathrm{diag}(\underbrace{A, \dots, A}_{l}, \underbrace{-A^t, \dots, -A^t}_{r}, 0, \dots, 0) \tag{3}$$

for any matrix $A \in \mathfrak{g}_1$ ($r = 0$ if $\mathfrak{g}_1$ is not of type A). Moreover, ε is natural if and only if $l + r = 1$. The following is well-known.

Proposition 2.3 *Let $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a diagonal embedding with the signature (l, r, z).*

(i) If $\mathfrak{g}_1$ is of type A and $\mathfrak{g}_2$ is not of type A, then $l = r$.

(ii) If $\mathfrak{g}_1$ is of type C (resp., O) and $\mathfrak{g}_2$ is of type O (resp., C), then l is even.

(iii) If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not of type A and l is even, then there exist an algebra $\mathfrak{g}'$ of type A, a natural embedding $\eta : \mathfrak{g}_1 \to \mathfrak{g}'$ and a diagonal embedding $\eta' : \mathfrak{g}' \to \mathfrak{g}_2$ with the signature $(l/2, l/2, z)$ such that $\varepsilon = \eta'\eta$.

(iv) If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are of type A and $l = r$, then there exist an algebra $\mathfrak{g}'$ of type O (resp., C), diagonal embeddings $\eta : \mathfrak{g}_1 \to \mathfrak{g}'$ with the signature $(1, 1, 0)$ and $\eta' : \mathfrak{g}' \to \mathfrak{g}_2$ with the signature $(l, 0, z)$ such that $\varepsilon = \eta'\eta$.

Proof. (i), (ii). By Malcev's result ([11], see also [8, Appendix, Theorem 0.25]), every symplectic (resp., orthogonal) representation of the algebra $\mathfrak{g}_1$ has the form $\phi_1 \oplus \ldots \oplus \phi_s \oplus \psi_1 \oplus \psi_1^* \oplus \ldots \psi_t \oplus \psi_t^*$ where all ϕ_i and ψ_j are irreducible, all ϕ_i are symplectic (resp., orthogonal). Assume that $\mathfrak{g}_2$ is symplectic (the orthogonal case is analogous). Then $V_2{\downarrow}\mathfrak{g}_1$ is a symplectic representation of $\mathfrak{g}_1$. Since V_1 is not symplectic,

$$V_2{\downarrow}\mathfrak{g}_1 = (V_1 \oplus V_1^*) \oplus \ldots \oplus (V_1 \oplus V_1^*) \oplus T_1 \oplus \ldots \oplus T_1.$$

Therefore $l = r$ for $\mathfrak{g}_1$ of type A, or l is even for $\mathfrak{g}_1$ orthogonal.

(iii). One can assume that $\mathfrak{g}_2$ is symplectic (for $\mathfrak{g}_2$ orthogonal the proof is analogous). Let $f(\cdot, \cdot)$ be the corresponding symplectic form on V_2. We prove the following intermediate fact.

(*) There exists a $\mathfrak{g}_1$-submodule M of $V_2{\downarrow}\mathfrak{g}_1$ such that $M \cong V_1$ and $f|_M \equiv 0$.

Indeed, fix any submodule $M_1 \cong V_1$. Assume that $f|_{M_1}$ is nondegenerate. It follows that $\mathfrak{g}_1$ is symplectic, and $f|_{M_1}$ is a $\mathfrak{g}_1$-invariant symplectic form on M_1. We have $V_2 = M_1 \oplus M_1^\perp$ where $M_1^\perp$ is the orthogonal complement to M_1. Since $l \geq 2$, $M_1^\perp$ contains another $\mathfrak{g}_1$-submodule $M_2 \cong V_1$. If $f|_{M_2} \equiv 0$, then we are done. Assume that $f|_{M_2}$ is also nondegenerate. Let $\alpha : M_1 \to M_2$ be an isomorphism of $\mathfrak{g}_1$-modules M_1 and M_2. Then there exists $\lambda \in F$ such that $f(\alpha(v), \alpha(w)) = \lambda f(v, w)$. Set $M = \{\sqrt{-\lambda}v - \alpha(v) \mid v \in M_1\}$. Then M is a $\mathfrak{g}_1$-submodule of $V_2{\downarrow}\mathfrak{g}_1$ isomorphic to V_1. It remains to note that

$$f(\sqrt{-\lambda}v - \alpha(v), \sqrt{-\lambda}w - \alpha(w)) = -\lambda f(v, w) + f(\alpha(v), \alpha(w)) = 0$$

for all $v, w \in M_1$, i.e. $f|_M \equiv 0$. This proves (*).

Fix any submodule $M_1 \cong V_1$ with $f|_{M_1} \equiv 0$. Since V_2 is completely reducible as $\mathfrak{g}_1$-module, there exists a submodule N_1 such that $V_2{\downarrow}\mathfrak{g}_1 = M_1 \oplus N_1$. Let $\dim V_1 = n$. Fix a basis $\{e_1, \ldots, e_n, \ldots, e_s\}$ of V_2 such that $\langle e_1, \ldots, e_n\rangle_F = M_1$ and $\langle e_{n+1}, \ldots, e_s\rangle_F = N_1$. Let $\{f_1, \ldots, f_s\}$ be the dual basis with respect to f, i.e. $f(e_i, f_j) = \delta_{ij}$. One checks that $M_2 = \langle f_1, \ldots, f_n\rangle_F$ is a $\mathfrak{g}_1$-submodule of V_2 isomorphic to $M_1^* \cong V_1$. Moreover, if A_x and B_x are the matrix realizations of the action of an element $x \in \mathfrak{g}_1$ on M_1 and M_2, respectively, then $B_x = -A_x^t$.

Observe that $f|_{M_2} \equiv 0$. Set $W_1 = M_1 \oplus M_2$. Then $f|_{W_1}$ is nondegenerate. So we have $V_2 = W_1 \oplus W_1^{\perp}$. Using induction on l, one can prove that there exists a basis

$$\{e_1^i, \dots, e_n^i; f_1^j, \dots, f_n^j; t_1, \dots, t_z \mid i, j = 1, \dots, l/2\}$$

of V_2 such that $\langle e_1^i, \dots, e_n^i \rangle_F \cong \langle f_1^j, \dots, f_n^j \rangle_F \cong V_1$ and $Ft_k \cong T_1$ as $\mathfrak{g}_1$-modules; $f(e_i^j, f_p^q) = \delta_{ip}\delta_{jq}$, $f(e_i^j, e_p^q) = f(f_i^j, f_p^q) = 0$. Let $\mathfrak{g}'$ be the algebra of all matrices of the form

$$\mathrm{diag}(\underbrace{A, \dots, A}_{l/2}, \underbrace{-A^t, \dots, -A^t}_{l/2}, \underbrace{0, \dots, 0}_{z})$$

where A runs over all $n \times n$-matrices with zero trace. Then $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}_2$ isomorphic to $sl(V_1)$; $\mathfrak{g}_1 \subset \mathfrak{g}'$; the embedding $\mathfrak{g}' \to \mathfrak{g}_2$ has the signature $(l/2, l/2, z)$; and the embedding $\mathfrak{g}_1 \to \mathfrak{g}'$ is natural.

(iv). Choose a basis $\{e_1^i, \dots, e_n^i; f_1^j, \dots, f_n^j; t_1, \dots, t_z \mid i, j = 1, \dots, l\}$ of V_2 such that $\langle e_1^i, \dots, e_n^i \rangle_F \cong \langle f_1^j, \dots, f_n^j \rangle_F \cong V_1$, $Ft_k \cong T_1$ as $\mathfrak{g}_1$-modules, and our embedding has the form (3) with $r = l$. Set $f(e_i^1, f_j^1) = \delta_{ij}$, $f(e_i^1, e_j^1) = f(f_i^1, f_j^1) = 0$, $f(f_j^1, e_i^1) = \delta_{ij}$ (resp., $f(f_j^1, e_i^1) = -\delta_{ij}$). Then f is an orthogonal (resp., symplectic) form on $W_1 = \langle e_1^1, \dots, e_n^1, f_1^1, \dots, f_n^1 \rangle_F$. Observe that f is $\mathfrak{g}_1$-invariant. Let $\mathfrak{g}'$ be the set of all linear transformations of W_1 that leave f invariant. Then we have a diagonal embedding $\eta : \mathfrak{g}_1 \to \mathfrak{g}'$ with the signature $(1, 1, 0)$. Setting $xe_k^i = xe_k^1$ and $xf_k^j = xf_k^1$ ($i, j = 1, \dots, l$, $x \in \mathfrak{g}'$), we obtain an action of $\mathfrak{g}'$ on V_2 which induces a diagonal embedding $\eta' : \mathfrak{g}' \to \mathfrak{g}_2$ with the signature $(l, 0, z)$. It remains to note that $\varepsilon = \eta'\eta$. □

The following is trivial.

Proposition 2.4 *Let $\varepsilon_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\varepsilon_2 : \mathfrak{g}_2 \to \mathfrak{g}_3$ be diagonal embeddings of the algebras of type A with the signatures (l_1, r_1, z_1) and (l_2, r_2, z_2), respectively. Denote by (l, r, z) the signature of $\varepsilon = \varepsilon_2\varepsilon_1$. Then*

$$\begin{aligned} l &= l_1 l_2 + r_1 r_2, \\ r &= r_1 l_2 + l_1 r_2, \\ z &= z_1(l_2 + r_2) + z_2. \end{aligned}$$

Set $s = l + r$, $c = l - r$, $s_i = l_i + r_i$, $c_i = l_i - r_i$ ($i = 1, 2$). Then one can check that $s = s_1 s_2$ and $c = c_1 c_2$. Moreover, it is not difficult to check that $s = s_1 s_2$ and $z = z_1 s_2 + z_2$ for *all* classical $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ (not necessary of type A). As a result, we have

Corollary 2.5 *Let $\mathfrak{g}_1 \to \dots \to \mathfrak{g}_k$ be a chain of diagonal embeddings, (l_i, r_i, z_i) the signature of $\mathfrak{g}_i \to \mathfrak{g}_{i+1}$, (l, r, z) the signature of $\mathfrak{g}_1 \to \mathfrak{g}_k$, $s_i = l_i + r_i$, $c_i = l_i - r_i$, $s = l + r$, $c = l - r$. Then $s = s_1 \dots s_{k-1}$. Moreover, if all $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ have type A, then $c = c_1 \dots c_{k-1}$.*

Lemma 2.6 *Let $\iota : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}'$ be diagonal embeddings of the algebras of the same type (A, C or O) with the signatures (l, r, z) and (p, q, u), respectively. Assume that a triple of non-negative integers (p', q', u') satisfies the following conditions*

$$\begin{aligned} l + r &= (p+q)(p'+q'), & (4)\\ l - r &= (p-q)(p'-q'), & (5)\\ n_2 &= n'(p'+q') + u' \end{aligned}$$

where n' and n_2 are the dimensions of the standard modules for $\mathfrak{g}'$ and $\mathfrak{g}_2$, respectively. Then there exists a diagonal embedding $\varepsilon' : \mathfrak{g}' \to \mathfrak{g}_2$ with the signature (p', q', u') such that $\iota = \varepsilon'\varepsilon$.

Proof. Rewrite (4) and (5) in the form

$$\begin{aligned} l &= pp' + qq',\\ r &= pq' + qp'. \end{aligned}$$

Let V_1, V_2, V' be the standard modules for $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}'$, respectively. Fix a basis of V_1. One can choose a basis of V' such that the homomorphism ε have the following form.

$$\varepsilon(A) = \mathrm{diag}(\underbrace{A, \ldots, A}_{p}, \underbrace{-A^t, \ldots, -A^t}_{q}, 0, \ldots, 0)$$

for any matrix $A \in \mathfrak{g}_1$. Similarly, one can choose a basis of V_2 such that

$$\iota(A) = \mathrm{diag}(\underbrace{\varepsilon(A), \ldots, \varepsilon(A)}_{p'}, \underbrace{-\varepsilon(A)^t, \ldots, -\varepsilon(A)^t}_{q'}, 0, \ldots, 0)$$

where

$$-\varepsilon(A)^t = \mathrm{diag}(\underbrace{-A^t, \ldots, -A^t}_{p}, \underbrace{A, \ldots, A}_{q}, 0, \ldots, 0)$$

This is valid since the number of matrices A (resp., $-A^t$) in $\iota(A)$ is equal to $l = pp' + qq'$ (resp., $r = pq' + qp'$). Define a homomorphism $\varepsilon' : \mathfrak{g}' \to \mathfrak{g}_2$, setting

$$\varepsilon'(B) = \mathrm{diag}(\underbrace{B, \ldots, B}_{p'}, \underbrace{-B^t, \ldots, -B^t}_{q'}, 0, \ldots, 0)$$

where $B \in \mathfrak{g}'$. Then ε' has the signature (p', q', u'), and $\varepsilon'\varepsilon = \iota$, as required. □

The following lemma can be found in [15] (see also [4, Lemma 5.2]).

Lemma 2.7 *Let $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{s}$ be classical simple Lie algebras. Assume that* $\mathrm{rk}\,\mathfrak{h} > 10$ *and the embedding $\mathfrak{h} \to \mathfrak{s}$ is diagonal. Then the embeddings $\mathfrak{h} \to \mathfrak{g}$ and $\mathfrak{g} \to \mathfrak{s}$ are also diagonal.*

It is not difficult to see that any embedding of associative finite-dimensional simple algebras is "diagonal". That is, it can be written in matrix form (2). So one can easily prove the "associative" analogs of 2.4 – 2.7.

3 Equivalence of sequences and Steinitz numbers

Let $(\mathfrak{g}_i)_I$ and $(\mathfrak{g}'_j)_J$ be sequences of algebras ($I \cong J \cong \mathbf{N}$); $\mathfrak{g}_i \to \mathfrak{g}_{i+1}$ ($i = 1, 2, \ldots$) and $\mathfrak{g}'_j \to \mathfrak{g}'_{j+1}$ ($j = 1, 2, \ldots$) be arbitrary embeddings. Put $\mathfrak{g} = \varinjlim \mathfrak{g}_i$, $\mathfrak{g}' = \varinjlim \mathfrak{g}'_j$. First of all we formulate conditions under which the algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.

Proposition 3.1 $\varinjlim \mathfrak{g}_i \cong \varinjlim \mathfrak{g}'_j$ *if and only if there exist subsequences* $i_1 < i_2 < \ldots$ *of* I, $j_1 < j_2 < \ldots$ *of* J, *and embeddings* $\varepsilon_k : \mathfrak{g}_{i_k} \to \mathfrak{g}'_{j_k}$, $\varepsilon'_k : \mathfrak{g}'_{j_k} \to \mathfrak{g}_{i_{k+1}}$ *(*$k = 1, 2, \ldots$*) such that the following diagram is commutative.*

$$\begin{array}{ccccc} \mathfrak{g}_{i_1} & \longrightarrow & \mathfrak{g}_{i_2} & \longrightarrow & \cdots \\ \downarrow\varepsilon_1 & \nearrow\varepsilon'_1 & \downarrow\varepsilon_2 & \nearrow\varepsilon'_2 & \\ \mathfrak{g}'_{j_1} & \longrightarrow & \mathfrak{g}'_{j_2} & \longrightarrow & \cdots \end{array}$$

Proof. Set $\mathfrak{g} = \varinjlim \mathfrak{g}_i$, $\mathfrak{g}' = \varinjlim \mathfrak{g}'_j$. Assume that there exists an isomorphism $\varepsilon : \mathfrak{g} \to \mathfrak{g}'$. Fix any $i_1 \in I$. Then there exists $j_1 \in J$ such that $\varepsilon(\mathfrak{g}_{i_1}) \subseteq \mathfrak{g}'_{j_1}$. Similarly there exists $i_2 \in I$ such that $\varepsilon^{-1}(\mathfrak{g}'_{j_1}) \subseteq \mathfrak{g}_{i_2}$, and so on. Clearly the corresponding diagram for the subsequences $i_1 < i_2 < \ldots$, $j_1 < j_2 < \ldots$, and embeddings $\varepsilon_k = \varepsilon|\mathfrak{g}_{i_k}$, $\varepsilon'_k = \varepsilon^{-1}|\mathfrak{g}'_{j_k}$, $k = 1, 2, \ldots$, is commutative. Therefore $(\mathfrak{g}_i)_I$ and $(\mathfrak{g}'_j)_J$ are equivalent. The converse statement is obvious. □

Let $(p_1, p_2, \ldots)$ be the increasing sequence of all prime numbers. The set of all mappings from $\{p_1, p_2, \ldots\}$ into the set $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is called the set of *Steinitz numbers*. If a Steinitz number takes a value α_1 at p_1, α_2 at p_2,..., this element will be denoted by $p_1^{\alpha_1} p_2^{\alpha_2} \ldots$. It will be convenient to consider Steinitz numbers as "generalized integers". The set of positive integers $\mathbf{N}$ can be identified in an evident way with a subset of Steinitz numbers. If $\Pi = p_1^{\alpha_1} p_2^{\alpha_2} \ldots$ and $\Pi' = p_1^{\alpha'_1} p_2^{\alpha'_2} \ldots$ are two Steinitz numbers, we shall set $\Pi\Pi' = p_1^{\alpha_1+\alpha'_1} p_2^{\alpha_2+\alpha'_2} \ldots$. We say that Π divides Π' if and only if $\alpha_1 \le \alpha'_1, \alpha_2 \le \alpha'_2 \ldots$. Let $q \in \mathbf{Q}$. We write $\Pi = q\Pi'$ (or $q \in \frac{\Pi}{\Pi'}$) if there exists $n \in \mathbf{N}$ such that $nq \in \mathbf{N}$ and $n\Pi = nq\Pi'$. If there exists non-zero $q \in \mathbf{Q}$ such that $\Pi = q\Pi'$, then we say that Π and Π' are $\mathbf{Q}$-*equivalent* and denote this relation by $\Pi \overset{\mathbf{Q}}{\sim} \Pi'$. Let $\mathcal{S} = (s_1, s_2, \ldots)$ be a sequence of positive integers. Denote by $\mathrm{Stz}(\mathcal{S})$ the Steinitz number $s_1 s_2 s_3 \ldots$.

Proposition 3.2 *Let* $\mathcal{S} = (s_i)_I$ *and* $\mathcal{S}' = (s'_j)_J$ *be sequences of positive integers. Then* $q \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ *if and only if for each* $i \in I$ *and* $k \in J$ *there exist* $j = j(i) \in J$ *and* $l = l(k) \in I$ *such that* $s_1 \ldots s_i$ *divides* $qs'_1 \ldots s'_j$ *(over* $\mathbf{Z}$*) and* $qs'_1 \ldots s'_k$ *divides* $s_1 \ldots s_l$ *(over* $\mathbf{Z}$*).*

Proof. Let $q \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$. This is equivalent to say that there exists $n \in \mathbf{N}$ such that $nq \in \mathbf{N}$ and $n\,\mathrm{Stz}(\mathcal{S}) = nq\,\mathrm{Stz}(\mathcal{S}')$. It is not difficult to see that the latter property is equivalent to the following. For each $i \in I$ and $k \in J$ there exist $j = j(i) \in J$ and $l = l(k) \in I$ such that $ns_1 \dots s_i$ divides $nqs'_1 \dots s'_j$ (over $\mathbf{Z}$) and $nqs'_1 \dots s'_k$ divides $ns_1 \dots s_l$ (over $\mathbf{Z}$). So the lemma follows. □

4 The classification of algebras of the same type

Let $(\mathfrak{g}_i)_I$ be a sequence of simple Lie algebras ($I \cong \mathbf{N}$), $\mathfrak{g}_i \to \mathfrak{g}_{i+1}$ ($i = 1, 2, \dots$) diagonal embeddings. Set $\mathfrak{g} = \varinjlim \mathfrak{g}_i$. Removing some members of the sequence one can assume that all $\mathfrak{g}_i$ are of the same type X ($= A, C$ or O) simultaneously. In this case we say that $\mathfrak{g}$ is of type X. The aim of this section is to classify all such algebras of the same type. Throughout this section (l_i, r_i, z_i) will denote the signature of $\mathfrak{g}_i \to \mathfrak{g}_{i+1}$, and n_i will denote the dimension of the standard $\mathfrak{g}_i$-module. We can and shall assume (for type A algebras) that $l_i \geq r_i$ for all i. Indeed, if we replace the base $\mathbf{B} = (\alpha_1, \dots, \alpha_n)$ of the weight lattice for $\mathfrak{g}_i$ by the base $\mathbf{B}' = (\alpha'_1, \dots, \alpha'_n)$ with $\alpha'_k = \alpha_{n+1-k}$ ($k = 1, \dots, n$), then the standard $\mathfrak{g}_i$-module with respect to $\mathbf{B}$ becomes the dual to standard with respect to $\mathbf{B}'$. It is convenient to add to the sequence an "algebra" of dimension 1. Formally we can and shall assume that $n_1 = 1$, $l_1 = n_2$, $r_1 = z_1 = 0$. Denote by $\mathcal{T}$ the triple sequence $\{(l_i, r_i, z_i)\}_{i \in I}$. One can write $\mathfrak{g} = X(\mathcal{T})$ where $X = A, C$ or O.

Set $s_i = l_i + r_i$, $c_i = l_i - r_i$ ($i = 1, 2, \dots$), $\mathcal{S} = (s_i)_{i \in I}$, $\mathcal{C} = (c_i)_{i \in I}$, $s_i^k = s_i \dots s_{k-1}$, $c_i^k = c_i \dots c_{k-1}$.

Put $\delta_i = s_1^i / n_i$. Then

$$\delta_{i+1} = \frac{s_1^{i+1}}{n_{i+1}} = \frac{s_1^i s_i}{n_i s_i + z_i} = \frac{s_1^i}{n_i + (z_i/s_i)} \leq \delta_i. \tag{6}$$

The limit $\delta = \lim_{i \to \infty} \delta_i$ is called the *density index* of $\mathcal{T}$ and is denoted by $\delta(\mathcal{T})$. Since $\delta_2 = s_1/n_2 = 1$, we have $0 \leq \delta \leq 1$. If $\delta = 0$, then the triple sequence is called *sparse*. If there exists i such that for all $j > i$ we have $\delta_j = \delta_i \neq 0$, then the triple sequence is called *pure*. In view of (6) this is equivalent to the following. There exists i such that for all $j \geq i$ we have $z_j = 0$. We say that the triple sequence is *dense* if and only if $0 < \delta < \delta_i$ for all i.

If there exists i such that $c_j = s_j$ (equivalently, $r_j = 0$) for all $j \geq i$, then $\mathcal{T}$ is called *one-sided*. Otherwise, it is called *two-sided*. If for each i there exists $j > i$ such that $c_j = 0$ (equivalently, $l_j = r_j$), then $\mathcal{T}$ is called (two-sided) *symmetric*. Otherwise it is called *non-symmetric*. In the latter case we may and shall assume that $c_i > 0$ for all $i \in \mathbf{N}$. Set $\sigma_i = \frac{c_1 \dots c_i}{s_1 \dots s_i}$.

The limit $\sigma = \lim_{i\to\infty} \sigma_i$ is called the *symmetry index* of $\mathcal{T}$ and is denoted by $\sigma(\mathcal{T})$. Observe that $0 \le \sigma \le 1$. Two-sided non-symmetric triple sequences with $\sigma = 0$ are called *weakly non-symmetric*, and those with $\sigma \neq 0$ are called *strongly non-symmetric*.

Thus all triple sequences can be partitioned into three classes with respect to density and into four classes with respect to symmetry.

Density types

(D1) Sparse ($\delta = 0$).
(D2) Dense ($\delta_i > \delta > 0$ for all i).
(D3) Pure ($\delta_i = \delta > 0$ for some i).

Symmetry types

(S1) One-sided ($r_j = 0$ for all $j \gg 1$).
(S2) Two-sided symmetric ($l_j = r_j$ for an infinite set of j).
(S3) Two-sided weakly non-symmetric ($r_j > 0$ for an infinite set of j, $l_k > r_k$ for all $k \gg 1$, and $\sigma = 0$).
(S4) Two-sided strongly non-symmetric ($r_j > 0$ for an infinite set of j, $l_k > r_k$ for all $k \gg 1$, and $\sigma \neq 0$).

Theorem 4.1 *Let* $\mathcal{T} = \{(l_i, r_i, z_i)\}$ *and* $\mathcal{T}' = \{(l'_i, r'_i, z'_i)\}$. *Set* $\delta = \delta(\mathcal{T})$, $\sigma = \sigma(\mathcal{T})$, $\delta' = \delta(\mathcal{T}')$, $\sigma' = \sigma(\mathcal{T}')$. *Then* $X(\mathcal{T}) \cong X(\mathcal{T}')$ *(*$X = A, C$ *or* O*) if and only if the following conditions hold.*

($\mathcal{A}_1$) *The triple sequences* $\mathcal{T}$ *and* $\mathcal{T}'$ *have the same density type.*

($\mathcal{A}_2$) $\mathrm{Stz}(\mathcal{S}) \overset{\mathbf{Q}}{\sim} \mathrm{Stz}(\mathcal{S}')$.

($\mathcal{A}_3$) $\frac{\delta}{\delta'} \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ *for dense and pure triple sequences (types* (D2) *and* (D3)*).*

($\mathcal{B}_1$) *The triple sequences* $\mathcal{T}$ *and* $\mathcal{T}'$ *have the same symmetry type.*

($\mathcal{B}_2$) $\mathrm{Stz}(\mathcal{C}) \overset{\mathbf{Q}}{\sim} \mathrm{Stz}(\mathcal{C}')$ *for two-sided non-symmetric triple sequences (types* (S3) *and* (S4)*).*

($\mathcal{B}_3$) *There exists* $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ *such that* $\alpha\frac{\sigma}{\sigma'} \in \frac{\mathrm{Stz}(\mathcal{C})}{\mathrm{Stz}(\mathcal{C}')}$ *for two-sided strongly non-symmetric triple sequences (type* (S4)*). Moreover,* $\alpha = \frac{\delta}{\delta'}$ *if in addition the triple sequences are dense or pure (types* (D2) *and* (D3)*).*

Proof of necessity. We shall prove the following more general statement. If $X(\mathcal{T}) \cong X'(\mathcal{T}')$ (we do not demand that $X = X'$), then $\mathcal{T}$ and $\mathcal{T}'$ satisfy the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$. Moreover, if $X = X' = A$, then the conditions $(\mathcal{B}_1), (\mathcal{B}_2), (\mathcal{B}_3)$ hold. Let $(\mathfrak{g}_i)_I$ and $(\mathfrak{g}'_j)_J$ ($I \cong J \cong \mathbf{N}$) be sequences of simple Lie algebras of types X and X', corresponding to the triple sequences $\mathcal{T}$ and $\mathcal{T}'$, respectively. We have $\mathfrak{g} \cong \mathfrak{g}'$ where $\mathfrak{g} = \varinjlim \mathfrak{g}_i$, $\mathfrak{g}' = \varinjlim \mathfrak{g}'_i$. By Proposition 3.1, there exist subsequences $i_1 < i_2 < \dots$ of I, $j_1 < j_2 < \dots$ of J, and embeddings $\varepsilon_k : \mathfrak{g}_{i_k} \to \mathfrak{g}'_{j_k}$, $\varepsilon'_k : \mathfrak{g}'_{j_k} \to \mathfrak{g}_{i_{k+1}}$ $(k = 1, 2, \dots)$ such that the following diagram is commutative.

$$\begin{array}{ccccccccccccc} \mathfrak{g}_{i_1} & \longrightarrow & \dots & \longrightarrow & \mathfrak{g}_{i_k} & \longrightarrow & \mathfrak{g}_{i_{k+1}} & \longrightarrow & \dots & \longrightarrow & \mathfrak{g}_{i_m} & \longrightarrow & \dots \\ \downarrow\varepsilon_1 & \nearrow\varepsilon'_1 & & \nearrow & \downarrow\varepsilon_k & \nearrow\varepsilon'_k & \downarrow\varepsilon_{k+1} & \nearrow & & \nearrow & \downarrow\varepsilon_m & \nearrow & \\ \mathfrak{g}'_{j_1} & \longrightarrow & \dots & \longrightarrow & \mathfrak{g}'_{j_k} & \longrightarrow & \mathfrak{g}'_{j_{k+1}} & \longrightarrow & \dots & \longrightarrow & \mathfrak{g}'_{j_m} & \longrightarrow & \dots \end{array} \tag{7}$$

One can assume that $\operatorname{rk}\mathfrak{g}_{i_1} > 10$ and $\operatorname{rk}\mathfrak{g}'_{j_1} > 10$. Since the embeddings $\mathfrak{g}_{i_k} \to \mathfrak{g}_{i_{k+1}}$ and $\mathfrak{g}'_{j_k} \to \mathfrak{g}'_{j_{k+1}}$ $(k = 1, 2, \dots)$ are diagonal, by Lemma 2.7, all embeddings ε_k, ε'_k $(k = 1, 2, \dots)$ are also diagonal. Let (p_k, q_k, u_k) (resp., (p'_k, q'_k, u'_k)) be the signature of ε_k (resp., ε'_k). Set $n_i = \dim\mathfrak{g}_i$, $s_i = l_i + r_i$, $c_i = l_i - r_i$, $\delta_i = s^i_1/n_i$, $\delta = \lim_{i\to\infty}\delta_i$ (resp., $n'_j = \dim\mathfrak{g}'_j$, ...). We have

$$n'_{j_m} = (p_m + q_m)n_{i_m} + u_m = (p_m + q_m)s^{i_m}_1\delta^{-1}_{i_m} + u_m = (p_m + q_m)s^{i_k}_1 s^{i_m}_{i_k}\delta^{-1}_{i_m} + u_m. \tag{8}$$

On the other hand,

$$n'_{j_m} = s'^{j_m}_1(\delta'_{j_m})^{-1} = s'^{j_k}_1 s'^{j_m}_{j_k}(\delta'_{j_m})^{-1}. \tag{9}$$

In view of commutativity of the diagram and by Corollary 2.5 we have

$$s^{i_m}_{i_k}(p_m + q_m) = (p_k + q_k)s'^{j_m}_{j_k}. \tag{10}$$

Dividing (8) and (9) by $s'^{j_m}_{j_k}$, we get $(p_k + q_k)s^{i_k}_1\delta^{-1}_{i_m} + u_m/s'^{j_m}_{j_k} = s'^{j_k}_1(\delta'_{j_m})^{-1}$. So

$$(p_k + q_k)s^{i_k}_1\delta'_{j_m} \le s'^{j_k}_1\delta_{i_m}. \tag{11}$$

Taking $m \to \infty$, we obtain $(p_k + q_k)s^{i_k}_1\delta' \le s'^{j_k}_1\delta$. Similarly, we get $(p'_k + q'_k)s'^{j_k}_1\delta \le s^{i_{k+1}}_1\delta'$. By Corollary 2.5, we have $(p_k + q_k)(p'_k + q'_k) = s^{i_{k+1}}_{i_k}$. Hence

$$(p_k + q_k)s^{i_k}_1\delta' \le s'^{j_k}_1\delta \le (p'_k + q'_k)^{-1}s^{i_{k+1}}_1\delta' = (p_k + q_k)s^{i_k}_1\delta'.$$

Therefore

$$(p_k + q_k)s^{i_k}_1\delta' = s'^{j_k}_1\delta, \tag{12}$$

$$(p'_k + q'_k)s'^{j_k}_1\delta = s^{i_{k+1}}_1\delta'. \tag{13}$$

Clearly $\delta = 0$ if and only if $\delta' = 0$. Therefore $\mathcal{T}$ is sparse if and only if $\mathcal{T}'$ is so. If the triple sequence $\mathcal{T}$ is pure, then $\delta = \delta_{i_m}$ for some m. Substracting (12) from (11), we get

$$0 \le (p_k + q_k)s^{i_k}_1(\delta'_{j_m} - \delta') \le s'^{j_k}_1(\delta_{i_m} - \delta) = 0$$

Therefore $\delta'_{j_m} = \delta'$, so $\mathcal{T}'$ is also pure. By symmetry, $\mathcal{T}$ is pure if and only if $\mathcal{T}'$ is pure. So $(\mathcal{A}_1)$ holds.

By (10), $s_{i_k}^{i_m}$ divides $(p_k+q_k)s'^{j_m}_{j_k}$ for all $m>k$. On the other hand, in view of commutativity of the diagram we have

$$s_{i_k}^{i_{m+1}} = (p_k+q_k)s'^{j_m}_{j_k}(p'_m+q'_m), \tag{14}$$

so $(p_k+q_k)s'^{j_m}_{j_k}$ divides $s_{i_k}^{i_{m+1}}$. Therefore by Proposition 3.2,

$$\operatorname{Stz}(\mathcal{S}_{i_k}) = (p_k+q_k)\operatorname{Stz}(\mathcal{S}'_{j_k}), \tag{15}$$

where $\mathcal{S}_{i_k} = (s_{i_k}, s_{i_k+1}, \ldots)$, $\mathcal{S}'_{j_k} = (s'_{j_k}, s'_{j_k+1}, \ldots)$. It follows that $\operatorname{Stz}(\mathcal{S}) \overset{\mathbf{Q}}{\sim} \operatorname{Stz}(\mathcal{S}')$, so $(\mathcal{A}_2)$ holds.

Finally, if δ and δ' are nonzero (dense or pure sequences), then by (12) and (13), $s_1^{i_k}$ divides $(\delta/\delta')s'^{j_k}_1$ and $(\delta/\delta')s'^{j_k}_1$ divides $s_1^{i_{k+1}}$ for any k. Therefore by Proposition 3.2, $\operatorname{Stz}(\mathcal{S}) = (\delta/\delta')\operatorname{Stz}(\mathcal{S}')$, and $(\mathcal{A}_3)$ holds.

Assume now that $X = X' = A$. By Corollary 2.5, one can write down equalities for "differences" similar to (10) and (14).

$$c_{i_k}^{i_m}(p_m-q_m) = (p_k-q_k)c'^{j_m}_{j_k}. \tag{16}$$

$$c_{i_k}^{i_{m+1}} = (p_k-q_k)c'^{j_m}_{j_k}(p'_m-q'_m), \tag{17}$$

If $\mathcal{T}'$ is symmetric, then by definition, for each k there exists m such that $c'^{j_m}_{j_k} = 0$. It follows from (17) that $c_{i_k}^{i_{m+1}} = 0$, so $\mathcal{T}$ is symmetric. Therefore, $\mathcal{T}$ is symmetric if and only if $\mathcal{T}'$ is so. Assume that $\mathcal{T}$ is non-symmetric. Recall that in this case one can suppose that all c_i and c'_j are nonzero. Dividing (17) by (14), we get

$$\frac{c_{i_k}^{i_{m+1}}}{s_{i_k}^{i_{m+1}}} = \frac{(p_k-q_k)}{(p_k+q_k)}\frac{c'^{j_m}_{j_k}}{s'^{j_m}_{j_k}}\frac{(p'_m-q'_m)}{(p'_m+q'_m)}, \tag{18}$$

or equivalently,

$$\sigma_1^{i_{m+1}}\cdot\frac{s_1^{i_k}}{c_1^{i_k}} = \sigma'^{j_m}_1\cdot\frac{(p_k-q_k)}{(p_k+q_k)}\frac{s'^{j_k}_1}{c'^{j_k}_1}\frac{(p'_m-q'_m)}{(p'_m+q'_m)}, \tag{19}$$

Taking $m\to\infty$, we get

$$\sigma\cdot\frac{s_1^{i_k}}{c_1^{i_k}} \le \sigma'\cdot\frac{(p_k-q_k)}{(p_k+q_k)}\frac{s'^{j_k}_1}{c'^{j_k}_1}, \tag{20}$$

Similarly, dividing (16) by (10) and taking $m\to\infty$, we get

$$\sigma\cdot\frac{s_1^{i_k}}{c_1^{i_k}} \ge \sigma'\cdot\frac{(p_k-q_k)}{(p_k+q_k)}\frac{s'^{j_k}_1}{c'^{j_k}_1}, \tag{21}$$

Combining with (20), we obtain

$$\sigma \cdot \frac{s_1^{i_k}}{c_1^{i_k}} = \sigma' \cdot \frac{(p_k - q_k)}{(p_k + q_k)} \frac{s_1'^{j_k}}{c_1'^{j_k}}, \tag{22}$$

It follows that $\sigma = 0$ if and only if $\sigma' = 0$. That is, $\mathcal{T}$ is weakly non-symmetric if and only if $\mathcal{T}'$ is so. Assume that $\mathcal{T}'$ is one-sided. Then $\sigma' = \sigma_1'^{j_m}$ for some m. Substracting (22) from (19), we have $0 \le (\sigma_1^{i_{m+1}} - \sigma) s_1^{i_k}/c_1^{i_k} \le 0$. Therefore $\sigma_1^{i_{m+1}} = \sigma$, i.e. $\mathcal{T}$ is one-sided. So $(\mathcal{B}_1)$ holds.

Similarly to (15), one can get

$$\mathrm{Stz}(\mathcal{C}_{i_k}) = (p_k - q_k)\,\mathrm{Stz}(\mathcal{C}'_{j_k}), \tag{23}$$

It follows that $\mathrm{Stz}(\mathcal{C}) \overset{\mathbf{Q}}{\sim} \mathrm{Stz}(\mathcal{C}')$, so $(\mathcal{B}_2)$ holds.

Assume now that $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric. That is, $\sigma \neq 0$, $\sigma' \neq 0$. Set $\alpha = (p_k + q_k) s_1^{i_k}/s_1'^{j_k}$. Then (22) can be rewritten in the form

$$\frac{\sigma}{\sigma'} \alpha c_1'^{j_k} = (p_k - q_k) c_1^{i_k} \tag{24}$$

Observe that $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$. Indeed, using (15), we have

$$\alpha\,\mathrm{Stz}(\mathcal{S}') = (p_k + q_k) s_1^{i_k}\,\mathrm{Stz}(\mathcal{S}'_{j_k}) = s_1^{i_k}\,\mathrm{Stz}(\mathcal{S}_{i_k}) = \mathrm{Stz}(\mathcal{S}).$$

Moreover, if $\mathcal{T}$ and $\mathcal{T}'$ are dense or pure, then by (12), $\alpha = \delta/\delta'$. It follows from (24) and (23) that

$$\frac{\sigma}{\sigma'} \alpha\,\mathrm{Stz}(\mathcal{C}') = (p_k - q_k) c_1^{i_k}\,\mathrm{Stz}(\mathcal{C}'_{j_k}) = c_1^{i_k}\,\mathrm{Stz}(\mathcal{C}_{i_k}) = \mathrm{Stz}(\mathcal{C}).$$

Therefore, $\frac{\sigma}{\sigma'}\alpha \in \frac{\mathrm{Stz}(\mathcal{C})}{\mathrm{Stz}(\mathcal{C}')}$. This proves $(\mathcal{B}_3)$. □

To prove the sufficiency in Theorem 4.1, we need the following lemma.

Lemma 4.2 *Let $\mathcal{T}$ and $\mathcal{T}'$ satisfy the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$, $(\mathcal{B}_1), (\mathcal{B}_2)$, $(\mathcal{B}_3)$ of the theorem. Fix $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ ($\alpha = \delta/\delta'$ if $\mathcal{T}$ and $\mathcal{T}'$ are dense or pure), $\beta \in \frac{\mathrm{Stz}(\mathcal{C})}{\mathrm{Stz}(\mathcal{C}')}$ for the case of two-sided non-symmetric triple sequences ($\beta/\alpha = \sigma/\sigma'$ if $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric). Let i, j, a, b be integers such that*

(a) $\alpha s_1'^{j} = a s_1^{i}$,
(b) $\beta c_1'^{j} = b c_1^{i}$ (for two-sided non-symmetric $\mathcal{T}$ and $\mathcal{T}'$).

Then there exists $k > i$ such that $a' = s_i^k/a$ and $b' = c_i^k/b$ are integers of the same parity (a' is even and $c_i^k = 0$ for the case of symmetric $\mathcal{T}$ and $\mathcal{T}'$), $a' \ge b'$ and $n_k \ge a' n_j'$.

Proof. If otherwise is not specified we assume that $\mathcal{T}$ and $\mathcal{T}'$ are two-sided non-symmetric. The case of one-sided and symmetric sequences can be settled by removing from the proof the arguments with c_i, β, b.

Since $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ and $\alpha s_1'^j = a s_1^i$, we have

$$\mathrm{Stz}(\mathcal{S}_i) = \alpha (s_1^i)^{-1} s_1'^j \,\mathrm{Stz}(\mathcal{S}_j') = a\,\mathrm{Stz}(\mathcal{S}_j'). \tag{25}$$

Similarly, we get

$$\mathrm{Stz}(\mathcal{C}_i) = b\,\mathrm{Stz}(\mathcal{C}_j'). \tag{26}$$

Therefore there exists $k_1 > i$ such that $a' = s_i^k/a$ and $b' = c_i^k/b$ are integers for all $k \geq k_1$. Since for each m the integers $s_m' = l_m' + r_m'$ and $c_m' = l_m' - r_m'$ have the same parity, 2 divides $\mathrm{Stz}(\mathcal{S}_j')$ if and only if 2 divides $\mathrm{Stz}(\mathcal{C}_j')$ (for symmetric sequences 2 divides $\mathrm{Stz}(\mathcal{S}_j')$ always). Therefore by (25) and (26), there exists $k_2 \geq k_1$ such that the integers a' and b' have the same parity (a' is even and $c_i^k = 0$ for the case of symmetric $\mathcal{T}$ and $\mathcal{T}'$) for all $k \geq k_2$. Set $\gamma_k = b'/a'$. In view of (a) and (b), we have

$$\gamma_k = \frac{c_i^k}{b} \cdot \frac{a}{s_i^k} = \frac{c_1^i c_i^k}{\beta c_1'^j} \cdot \frac{\alpha s_1'^j}{s_1^i s_i^k} = \frac{\alpha}{\beta} \cdot \frac{\sigma_1^k}{\sigma_1'^j}.$$

If $\mathcal{T}$ and $\mathcal{T}'$ are weakly non-symmetric, then $\sigma_1^k \to 0$ as $k \to \infty$, so $\gamma_k \to 0$. If $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric, then by assumption $\beta/\alpha = \sigma/\sigma'$, so

$$\gamma_k \to \frac{\alpha}{\beta} \cdot \frac{\sigma}{\sigma_1'^j} = \frac{\sigma'}{\sigma_1'^j} < 1$$

as $k \to \infty$. In both cases there exists $k_3 \geq k_2$ such that $\gamma_k \leq 1$ (i.e. $a' \geq b'$) for all $k \geq k_3$.

Set $\nu_k = n_k/a' - n_j'$. We have to show that $\nu_k \geq 0$ for sufficiently large k. One has

$$\nu_k = \frac{n_k}{a'} - n_j' = \frac{s_1^k}{a'\delta_k} - \frac{s_1'^j}{\delta_j'} = \frac{a s_1^i}{\delta_k} - \frac{s_1'^j}{\delta_j'} = s_1'^j \left(\frac{\alpha}{\delta_k} - \frac{1}{\delta_j'}\right).$$

(The last equality follows from (a).) If $\mathcal{T}$ and $\mathcal{T}'$ are sparse, then $\delta_k \to 0$ as $k \to \infty$, so $\nu_k \to +\infty$. Therefore there exists $k_4 \geq k_3$ such that $\nu_k \geq 0$ for all $k \geq k_4$. Let $\mathcal{T}$ and $\mathcal{T}'$ be dense. Then $\alpha = \delta/\delta'$ and $\delta_j' > \delta'$. Therefore

$$\nu_k = s_1'^j \left(\frac{\delta}{\delta_k} \cdot \frac{1}{\delta'} - \frac{1}{\delta_j'}\right) \to s_1'^j \left(\frac{1}{\delta'} - \frac{1}{\delta_j'}\right) > 0,$$

as $k \to \infty$. Hence there exists $k_4 \geq k_3$ such that $\nu_k \geq 0$ for all $k \geq k_4$. Let $\mathcal{T}$ and $\mathcal{T}'$ be pure. Then there exists $k_4 \geq k_3$ such that $\delta = \delta_k$ for all $k \geq k_4$. Therefore

$$\nu_k = s_1'^j \left(\frac{1}{\delta'} - \frac{1}{\delta_j'}\right) \geq 0,$$

for all $k \geq k_4$. So each $k \geq k_4$ satisfies the assumptions of the theorem. $\square$

Proof of sufficiency in Theorem 4.1. According to Proposition 3.1 we have to construct sequences $i_1 < i_2 < \ldots$, $j_1 < j_2 < \ldots$, and embeddings $\varepsilon_k : \mathfrak{g}_{i_k} \to \mathfrak{g}'_{j_k}$, $\varepsilon'_k : \mathfrak{g}'_{j_k} \to \mathfrak{g}_{i_{k+1}}$ $(k = 1, 2, \ldots)$ such that the diagram (7) is commutative. Fix $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ ($\alpha = \delta/\delta'$ if $\mathcal{T}$ and $\mathcal{T}'$ are dense or pure) and $\beta \in \frac{\mathrm{Stz}(\mathcal{C})}{\mathrm{Stz}(\mathcal{C}')}$ for the case of two-sided non-symmetric triple sequences ($\beta/\alpha = \sigma/\sigma'$ if $\mathcal{T}$ and $\mathcal{T}'$ are strongly non-symmetric). Fix also $j_0 \in J$. Since $\mathrm{Stz}(\mathcal{S}') = \alpha^{-1}\,\mathrm{Stz}(\mathcal{S})$ and $\mathrm{Stz}(\mathcal{C}') = \beta^{-1}\,\mathrm{Stz}(\mathcal{C})$, by Proposition 3.2, there exists $i_1 \in I$ such that

(a_0) $\alpha^{-1} s_1^{i_1} = a_0 s_1'^{j_0}$,

(b_0) $\beta^{-1} c_1^{i_1} = b_0 c_1'^{j_0}$ (for two-sided non-symmetric $\mathcal{T}$ and $\mathcal{T}'$)

where $a_0, b_0 \in \mathbf{N}$. Applying Lemma 4.2 (interchanging $\mathcal{T}$ and $\mathcal{T}'$), we find j_1 such that $a_1 = s_{j_0}'^{j_1}/a_0$ and $b_1 = c_{j_0}'^{j_1}/b_0$ are integers of the same parity (a_1 is even if $\mathcal{T}$ and $\mathcal{T}'$ are symmetric), $a_1 \geq b_1$ and $n'_{j_1} \geq a_1 n_{i_1}$. Set $p_1 = (a_1 + b_1)/2$, $q_1 = (a_1 - b_1)/2$, $u_1 = n'_{j_1} - a_1 n_{i_1}$ ($p_1 = q_1 = a_1/2$ for symmetric sequences). Fix any diagonal embedding $\varepsilon_1 : \mathfrak{g}_{i_1} \to \mathfrak{g}'_{j_1}$ with the signature (p_1, q_1, u_1). We have

(a_1) $\alpha s_1'^{j_1} = a_1 s_1^{i_1}$,

(b_1) $\beta c_1'^{j_1} = b_1 c_1^{i_1}$.

Proceed by induction. Assume that sequences $i_1 < \ldots < i_k$, $j_1 < \ldots < j_k$ and embeddings $\varepsilon_1, \varepsilon'_1, \ldots, \varepsilon_k$ have been constructed, and the following conditions hold.

(a_k) $\alpha s_1'^{j_k} = a_k s_1^{i_k}$,

(b_k) $\beta c_1'^{j_k} = b_k c_1^{i_k}$

where $a_k = p_k + q_k$, $b_k = p_k - q_k$. Construct an embedding ε'_k. By Lemma 4.2, there exists $i_{k+1} > i_k$ such that $a'_k = s_{i_k}^{i_{k+1}}/a_k$ and $b'_k = c_{i_k}^{i_{k+1}}/b_k$ are integers of the same parity (a'_k is even if $\mathcal{T}$ and $\mathcal{T}'$ are symmetric), $a'_k \geq b'_k$ and $n_{i_{k+1}} \geq a'_k n'_{j_k}$. Set $p'_k = (a'_k + b'_k)/2$, $q'_k = (a'_k - b'_k)/2$, $u'_k = n_{i_{k+1}} - a'_k n'_{j_k}$ ($p'_k = q'_k = a'_k/2$ for symmetric sequences). Since

$$\begin{aligned}(p_k + q_k)(p'_k + q'_k) &= a_k a'_k = s_{i_k}^{i_{k+1}},\\ (p_k - q_k)(p'_k - q'_k) &= b_k b'_k = c_{i_k}^{i_{k+1}},\end{aligned}$$

and $u'_k \geq 0$, by Lemma 2.6 there exists a diagonal embedding $\varepsilon'_k : \mathfrak{g}'_{j_k} \to \mathfrak{g}_{i_{k+1}}$ such that $\iota_k = \varepsilon'_k \varepsilon_k$ where ι_k denotes the embedding $\mathfrak{g}_{i_k} \to \mathfrak{g}_{i_{k+1}}$. Observe that

(a'_k) $\alpha^{-1} s_1^{i_{k+1}} = a'_k s_1'^{j_k}$,

(b'_k) $\beta^{-1} c_1^{i_{k+1}} = b'_k s_1'^{j_k}$.

Therefore Lemma 4.2 can be applied once more (interchanging $\mathcal{T}$ and $\mathcal{T}'$). So the result follows by induction. $\square$

Remark 4.3 It is not difficult to see that for pure triple sequences one can always assume that all $z_i = 0$ (removing a finite number of first members of the sequences and adding the one-dimensional algebra again). In this situation $\delta = 1$, so the condition $(\mathcal{A}_3)$ can be rewritten in the form $\mathrm{Stz}(\mathcal{S}) = \mathrm{Stz}(\mathcal{S}')$.

Consider the case of associative algebras. Recall that any locally simple associative algebra M can be represented as the direct limit of an increasing sequence of matrix algebras: $M = \varinjlim M_i$ $(i \in \mathbf{N})$. Let $\mathcal{T} = ((l_i.r_i, z_i))_{i\in\mathbf{N}}$ be the corresponding triple sequence (by convention, $r_i = 0$ for all i). One can write $M = M(\mathcal{T})$. Clearly, the classification of the algebras $M(\mathcal{T})$ is equivalent to that of direct limits of one-sided sequences of Lie algebras of type A. Thus, we have

Theorem 4.4 *$M(\mathcal{T}) \cong M(\mathcal{T}')$ if and only if the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ hold.*

Let now the ground field be the field of complex numbers $\mathbf{C}$. It has been note in the introduction that our result yields the classification of $\mathbf{C}^*$-algebras which are inductive limits of sequences of complex matrix algebras. This classification is similar to Dixmier's one [7]. He partitioned these algebras into three classes:

(i) matroid separable $\mathbf{C}^*$-algebras with unit (the limits of pure sequences),

(ii) matroid separable finite $\mathbf{C}^*$-algebras without unit (the limits of dense sequences),

(iii) matroid separable infinite $\mathbf{C}^*$-algebras (the limits of sparse sequences).

Dixmier's parametrization is almost identical to our one. He uses "generalized" integers n (which corresponds to our Steinitz numbers $\mathrm{Stz}(\mathcal{S})$) and the real parameter $\theta \in [1, +\infty]$ instead of our δ. One can easily check that $\theta = \delta^{-1}$.

5 Isomorphisms of algebras of different types and general parametrization

In this section we find conditions under which $X(\mathcal{T}) \cong X'(\mathcal{T}')$ where $\mathcal{T}$ and $\mathcal{T}'$ are triple sequences and $X \neq X'$. We also give a general parametrization of countable locally simple Lie algebras.

Lemma 5.1 *Let $\mathcal{T}$ be a two-sided symmetric triple sequence, $\mathcal{S} = \mathcal{S}(\mathcal{T})$. Then 2^∞ divides* $\mathrm{Stz}(\mathcal{S})$.

Proof. By definition, $l_i = r_i$ (in particular, $s_i = l_i + r_i$ is even) for an infinite set of i. Therefore 2^∞ divides $\mathrm{Stz}(\mathcal{S})$. □

Theorem 5.2 *Let $\mathcal{T}$, $\mathcal{T}'$ be triple sequences.*

(i) $A(\mathcal{T}) \cong O(\mathcal{T}')$ *(resp.,*$A(\mathcal{T}) \cong C(\mathcal{T}')$*) if and only if $\mathcal{T}$ is two-sided symmetric, 2^∞ divides* $\mathrm{Stz}(\mathcal{S}')$ *and the conditions* $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ *of Theorem 4.1 hold.*

(ii) $O(\mathcal{T}) \cong C(\mathcal{T}')$ *if and only if 2^∞ divides both* $\mathrm{Stz}(\mathcal{S})$ *and* $\mathrm{Stz}(\mathcal{S}')$*, and the conditions* $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ *of Theorem 4.1 hold.*

Proof. (i). Set $\mathfrak{g} = A(\mathcal{T})$, $\mathfrak{g}' = O(\mathcal{T}')$. Assume that $\mathfrak{g} \cong \mathfrak{g}'$. The validity of the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ have been verified in the proof of Theorem 4.1. Further, denote by (x_k, y_k, z_k) the signature of $\mathfrak{g}_{i_k} \to \mathfrak{g}_{i_{k+1}}$ (see diagram (7)). In view of commutativity of the diagram we have $x_k = p_k p'_k$, $y_k = q_k p'_k$ where (p_k, q_k, u_k) and $(p'_k, 0, u'_k)$ are the signatures of ε_k and ε'_k, respectively. By Proposition 2.3 (i) $p_k = q_k$, so $x_k = y_k$. Therefore $c_{i_k}^{i_{k+1}} = x_k - y_k = 0$, so $\mathcal{T}$ is two-sided symmetric. By Lemma 5.1, 2^∞ divides $\mathrm{Stz}(\mathcal{S})$. Therefore in view of condition $(\mathcal{A}_2)$, 2^∞ divides $\mathrm{Stz}(\mathcal{S}')$.
Conversely. Let we have algebras $\mathfrak{g} = A(\mathcal{T})$ and $\mathfrak{g}' = O(\mathcal{T}')$ such that $\mathcal{T}$ is two-sided symmetric, 2^∞ divides $\mathrm{Stz}(\mathcal{S}')$ and the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ hold. There exists a sequence of indices $j_1 < j_2 < \ldots$ such that $s'^{j_{k+1}}_{j_k}$ is even for all $k = 1, 2, \ldots$. By Proposition 2.3 (iii), there exists an algebra $\mathfrak{g}''_k$ of type A and diagonal embeddings $\mathfrak{g}'_{j_k} \to \mathfrak{g}''_k$ and $\mathfrak{g}''_k \to \mathfrak{g}'_{j_{k+1}}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g}'_{j_k} & \longrightarrow & \mathfrak{g}'_{j_{k+1}} \\ & \searrow \quad \nearrow & \\ & \mathfrak{g}''_k & \end{array}$$

is commutative. Set $\mathfrak{g}'' = \varinjlim \mathfrak{g}''_k$. Let $\mathcal{T}''$ be the corresponding triple sequence. We have $\mathfrak{g}'' = A(\mathcal{T}'')$. By construction $\mathfrak{g}'' \cong \mathfrak{g}'$. Moreover, by the above arguments (the proof of necessity) $\mathcal{T}''$ is symmetric and the conditions $(\mathcal{A}_1), (\mathcal{A}_2), (\mathcal{A}_3)$ (for $\mathcal{T}'$ and $\mathcal{T}''$) hold. Since the same is true for the pair $\mathcal{T}$, $\mathcal{T}'$, we conclude that the pair $\mathcal{T}$, $\mathcal{T}''$ also satisfies these conditions. Indeed, the validity of $(\mathcal{A}_1)$ is trivial. Further, since $\mathrm{Stz}(\mathcal{S}') \overset{\mathbf{Q}}{\sim} \mathrm{Stz}(\mathcal{S}'')$ and $\mathrm{Stz}(\mathcal{S}) \overset{\mathbf{Q}}{\sim} \mathrm{Stz}(\mathcal{S}')$, we have $\mathrm{Stz}(\mathcal{S}) \overset{\mathbf{Q}}{\sim} \mathrm{Stz}(\mathcal{S}'')$. Finally, if $\frac{\delta'}{\delta''} \in \frac{\mathrm{Stz}(\mathcal{S}')}{\mathrm{Stz}(\mathcal{S}'')}$ and $\frac{\delta}{\delta'} \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$, then

$$\mathrm{Stz}(\mathcal{S}') = \frac{\delta'}{\delta''}\,\mathrm{Stz}(\mathcal{S}'') = \frac{\delta'}{\delta}\,\mathrm{Stz}(\mathcal{S}),$$

so $\frac{\delta}{\delta''} \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}'')}$. Consequently, by Theorem 4.1, $A(\mathcal{T}) \cong A(\mathcal{T}'')$, i.e. $\mathfrak{g} \cong \mathfrak{g}''$. Therefore $\mathfrak{g} \cong \mathfrak{g}'$. The proof for the case $\mathfrak{g}' = C(\mathcal{T}')$ is analogous.
(ii). Let $O(\mathcal{T}) \cong C(\mathcal{T}')$. Using Proposition 2.3 $(ii), (iii)$, it is not difficult to construct an algebra $A(\mathcal{T}'') \cong O(\mathcal{T}) \cong C(\mathcal{T}')$. The claim now follows from Theorem 5.2 (i). To prove the converse statement we construct $A(\mathcal{T}'')$ isomorphic to $O(\mathcal{T})$ and use Theorem 5.2 (i). □

It remains to give concluding remarks on the parametrization. Let $\mathfrak{g}$ be a diagonal locally simple Lie algebra of countable dimension. First of all we choose an increasing sequence $(\mathfrak{g}_i)_{i\in\mathbf{N}}$ of subalgebras of the same type $X = A$, B, or C with $\varinjlim \mathfrak{g}_i = \mathfrak{g}$. Next we construct the corresponding triple sequence $\mathcal{T} = ((l_i, r_i, z_i))_{i\in\mathbf{N}}$, the sequences of "sums" $\mathcal{S} = (l_i + r_i)_{i\in\mathbf{N}}$ and (for $X = A$ only) "differences" $\mathcal{C} = (l_i - r_i)_{i\in\mathbf{N}}$. Finally, we determine the density type D=(D1), (D2) or (D3), the density index $\delta = \delta(\mathcal{T})$, Steinitz number $\Pi_{\mathcal{S}} = \mathrm{Stz}(\mathcal{S})$, and (for $X = A$ only) the symmetry type S=(S1), (S2), (S3), or (S4), the symmetry index $\sigma = \sigma(\mathcal{T})$, Steinitz number $\Pi_{\mathcal{C}} = \mathrm{Stz}(\mathcal{C})$. So one can associate with any algebra $\mathfrak{g}$ a tuple

$$\mathcal{P}(\mathfrak{g}) = (X, \mathrm{D}, \mathrm{S}, \delta, \sigma, \Pi_{\mathcal{S}}, \Pi_{\mathcal{C}})$$

where $X, \mathrm{D}, \mathrm{S}$ describe a type of $\mathfrak{g}$; δ and σ are real numbers $(0 \leq \delta, \sigma \leq 1)$; $\Pi_{\mathcal{S}}$ and $\Pi_{\mathcal{C}}$ are Steinitz numbers. For $X = C, O$ (and $X = A$ with one-sided or symmetric $\mathcal{T}$) we use a shorter variant of the correspondence:

$$\mathfrak{g} \mapsto (X, \mathrm{D}, \delta, \Pi_{\mathcal{S}}).$$

By Theorem 4.1, tuples associated with two nonisomorphic algebras are distinct. The question under which conditions $\mathfrak{g}$ and $\mathfrak{g}'$ with tuples $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{g}')$ are isomorphic have been solved by Theorems 4.1 and 5.2.

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