Quasiclassical Lie algebras

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Abstract

In this paper we study finite dimensional non-semisimple Lie algebras that can be obtained as Lie algebras of skew-symmetric elements of associative algebras with involution. We call such algebras quasiclassical and characterize them in terms of existence of so-called "$*$-plain" representations. We show that the theory of $*$-plain representations for quasiclassical Lie algebras is almost equivalent to representation theory of associative algebras with involution.

Key Words: non-semisimple Lie algebras, representations, enveloping algebras.

1 Introduction

An antiautomorphism α of an associative algebra is called an *involution* (of the first kind) if $\alpha^2 = 1$. If the involution is fixed we often denote it by the symbol "$*$". Let A be an associative algebra with involution. We denote by $\mathfrak{u}^*(A)$ the vector space of skew-symmetric elements of A, i.e. $\mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$. The space $\mathfrak{u}^*(A)$ is a Lie algebra under the usual bracket multiplication. We denote by $\mathfrak{su}^*(A)$ its commutant $[\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$.

It is well known that classical simple Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$ can be defined as the algebras $\mathfrak{su}^*(A)$ where A is involutory simple (i.e. A has no nontrivial $*$-invariant ideals). It is quite natural to distinguish the class of Lie algebras $\mathfrak{su}^*(A)$ where A is arbitrary. These can be viewed as closest relatives of associative algebras. Is it possible to characterize such Lie algebras L in their internal terms? In this paper we provide such a characterization for L perfect and of finite dimension assuming the ground field $\mathbb{F}$ algebraically closed of characteristic 0. A similar problem is discussed for perfect algebraic groups.

Definition 1.1 Let L be a perfect (i.e. $[L, L] = L$) finite dimensional Lie algebra over $\mathbb{F}$. We say that L is *quasiclassical* if there exists an associative $\mathbb{F}$-algebra A with involution such that $L \cong \mathfrak{su}^*(A)$.

We characterize quasiclassical Lie algebras in terms of the existence of a faithful selfdual (i.e. equivalent to its dual) linear representation whose composition factors satisfy conditions described in the following definition.

Definition 1.2 Let L be a perfect finite dimensional Lie algebra over $\mathbb{F}$. Let V be a finite dimensional selfdual L-module. We say that V is *$*$-plain* if for each nontrivial composition factor W

of V the projection of L in $\mathfrak{gl}(W)$ coincides with one of the classical simple Lie algebras $\mathfrak{sl}(W)$, $\mathfrak{so}(W)$, or $\mathfrak{sp}(W)$ and any two composition factors of V with the same annihilator in L are either isomorphic or dual to each other. If in addition $\dim W \geq 4$ for $\mathfrak{sl}(W)$, $\dim W \geq 7$ for $\mathfrak{so}(W)$, and $\dim W \geq 6$ for $\mathfrak{sp}(W)$, then V is called *strongly ∗-plain*. We say that L is *(strongly) ∗-plain* if L has a faithful (strongly) ∗-plain module.

Observe that V may have trivial (one-dimensional) composition factors. Let W be a nontrivial composition factor of V and let S be a Levi subalgebra of L. Definition 1.2 requires that W is the natural module (or dual) for one of the simple components of S. In particular, simple components of S are only classical, and V is in a sense a simplest S-module. (However, the number of (isomorphic) composition factors in V is not bounded.)

The main result of this paper is the following theorem:

Theorem 1.3 *Let L be a perfect finite dimensional Lie algebra.*

(1) *Assume that L is quasiclassical. Then L is ∗-plain.*

(2) *Assume that L is strongly ∗-plain. Then for each faithful strongly ∗-plain L-module V the enveloping algebra A of L in* $\operatorname{End} V$ *has an involution such that $L = \mathfrak{su}^*(A)$. In particular, L is quasiclassical.*

Thus, a finite dimensional Lie algebra having no "small" quotients is quasiclassical if and only if it is ∗-plain. The fact that V is of simple nature as an S-module does not mean that V has simple nature as an L-module. In fact, if L is strongly plain then in general L can have infinitely many non-isomorphic faithful ∗-plain modules whose restriction to S are isomorphic. Indeed, by Theorem 1.3(2), $L = \mathfrak{su}^*(A)$ and each faithful A-module is strongly ∗-plain for L. If A has infinitely many non-isomorphic faithful modules of the same dimension (which may happen by the Brauer-Thrall conjecture (proved by Nazarova and Roiter [8]), the same is true for faithful ∗-plain L-modules. Therefore, in proving Theorem 1.3 one has to control unlimited number of modules. It is not true that A is uniquely determined by L, even if A has no commutative direct summands. Nevertheless, to large extent L determines A, see Theorem 1.5 below. We emphasize that the conditions for selfdual V to be ∗-plain are described only in terms of the composition factors of V. The essence of Theorem 1.3 is in part (2). A priory, there is no obvious reason to predict that L cannot be smaller than $\mathfrak{su}^*(A)$ (and this may happen if L has small quotients).

We also prove a multiplicative version of Theorem 1.3 by characterizing in similar terms perfect algebraic groups that are isomorphic to the special unitary (or norm) groups of associative algebras with involution. As above, we call such groups *quasiclassical*, and for a perfect algebraic group G we define a *(strongly) ∗-plain* rational G-module exactly as in Definition 1.2. Recall that the *unitary group* of an algebra A with involution is defined as $U^*(A) = \{a \in A \mid a^* = a^{-1}\}$. The *special unitary group* $SU^*(A)$ is the commutator subgroup of $U^*(A)$. Observe that ∗ acts trivially on $\mathbb{F}$ so one does not have to confuse $U^*(A)$ with a classical unitary group.

Theorem 1.4 *Let G be a perfect algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic* 0.

(1) *Assume that G is quasiclassical. Then G is ∗-plain.*

(2) *Assume that G is strongly ∗-plain. Then for each faithful strongly ∗-plain G-module V the algebra A generated by the image of G in* $\operatorname{End} V$ *has an involution such that $G = SU^*(A)$. In particular, G is quasiclassical.*

Let A be an associative algebra with involution. To what extent A is determined by $\mathfrak{u}^*(A)$? This problem was studied by Herstein for simple A (see [6]) and by his successors for prime rings (see [7] for instance). No result is known for A not being semiprime. We investigate this problem for *admissible* finite dimensional algebras A (see Definition 6.2; note that each finite dimensional algebra having no proper ideal of codimension ≤ 36 is admissible). Under this assumption we prove that the Lie algebra $L = \mathfrak{su}^*(A)$ is strongly $*$-plain and generate A (see Theorem 6.3). In particular, L is quasiclassical and A is an enveloping algebra of L. Set

$$\mathrm{Null}(A) = \{a \in A \mid aA = Aa = 0\},$$

i.e., $\mathrm{Null}(A)$ is the two-sided annihilator of A in A. (Note that $\mathrm{Null}(A) = 0$ whenever A has the identity).

Theorem 1.5 *Let A_1 and A_2 be admissible finite dimensional associative algebras with involution. Assume that $\mathfrak{su}^*(A_1) \cong \mathfrak{su}^*(A_2)$. Then there is an admissible algebra A and $*$-invariant ideals $H_1, H_2 \subseteq \mathrm{Null}(A)$ such that for $i = 1, 2$ we have $\mathfrak{su}^*(A) \cong \mathfrak{su}^*(A_i)$ and $A_i \cong A/H_i$. In particular, $A_1/\mathrm{Null}(A_1) \cong A_2/\mathrm{Null}(A_2)$.*

A similar result holds for the special unitary groups of A_i (see Theorem 7.2).

Let L be a Lie algebra and let A be an associative enveloping algebra of L (i.e., A contains L and is generated by L). We say that A is $\mathfrak{P}^*$-*enveloping* for L if A has an involution such that $\mathfrak{su}^*(A) = L$. Theorem 1.5 is a consequence of the following more general result (see also more exact Theorem 6.5).

Theorem 1.6 *Let L be a strongly $*$-plain Lie algebra. Then there exists a (universal) $\mathfrak{P}^*$-enveloping algebra A of L such that each $\mathfrak{P}^*$-enveloping algebra of L is a homomorphic image of A and the corresponding kernel is a $*$-invariant subspace of* $\mathrm{Null}(A)$.

In [2] we studied *quasispecial* (or *plain*) Lie algebras, which are defined as the derived subalgebras of associative algebras. We show in Theorem 6.7 that each plain Lie algebra having no small quotient is $*$-plain. Theorem 6.8 gives a criterion for a $*$-plain Lie algebra to be plain.

2 Some preliminary results on algebras with involution

Recall that a matrix $X \in M_n(\mathbb{F})$ is called *symmetric* (resp., *skew-symmetric*) if $X^t = X$ (resp., $X^t = -X$) where X^t is the matrix transpose to X. Assume that $n = 2l$. Set

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the identity matrix of size $l \times l$ (J is the canonical matrix of a symplectic form). Note that $J^{-1} = J^t = -J$. We say that a matrix $X \in M_n(\mathbb{F})$ is σ-*symmetric* (resp., σ-*skew-symmetric*) if $X^\sigma = X$ (resp., $X^\sigma = -X$) where $X^\sigma = -JX^tJ$. One checks that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}, \tag{1}$$

$\sigma^2 = 1$, and $(X_1X_2)^\sigma = X_2^\sigma X_1^\sigma$. In particular, t and σ are involutions of the algebra $M_n(\mathbb{F})$.

It is well known that the classical Lie algebras can be expressed in the following form.

$$\begin{aligned}
\mathfrak{gl}_n(\mathbb{F}) &= [M_n(\mathbb{F})]; \\
\mathfrak{sl}_n(\mathbb{F}) &= [\mathfrak{gl}_n(\mathbb{F}), \mathfrak{gl}_n(\mathbb{F})] = \{X \in [M_n(\mathbb{F})] \mid \operatorname{tr} X = 0\}; \\
\mathfrak{so}_n(\mathbb{F}) &= \{X \in [M_n(\mathbb{F})] \mid X^t = -X\}; \\
\mathfrak{sp}_n(\mathbb{F}) &= \{X \in [M_n(\mathbb{F})] \mid X^\sigma = -X\}.
\end{aligned}$$

An algebra with involution is called *involutory simple* if it has no nontrivial $*$-invariant ideals. One can easily show that any involutory simple finite dimensional $\mathbb{F}$-algebra is either a full matrix algebra or a direct sum of two (anti)isomorphic matrix algebras. The following two classical results describe the structure of involutory simple algebras.

Lemma 2.1 *Let U be a vector space of dimension n over $\mathbb{F}$. Let α be an involution of the algebra $D = \operatorname{End} U$. Then there exists a basis of U such that this involution can be expressed in the form $X \mapsto X^t$ or $X \mapsto X^\sigma$ for $X \in D \cong M_n(\mathbb{F})$. In particular, $\mathfrak{u}^*(D) \cong \mathfrak{so}_n(\mathbb{F})$ or $\mathfrak{sp}_n(\mathbb{F})$ and U is the natural $\mathfrak{u}^*(D)$-module.*

Lemma 2.2 *Let $D_i = \operatorname{End} U_i$, $i = 1, 2$, where U_i is a vector space of dimension n over $\mathbb{F}$. Let α be an involution of the algebra $D_1 \oplus D_2$ such that $D_i^\alpha = D_{3-i}$. Then there exist bases of U_1 and U_2 such that this involution can be expressed in the form $(X_1, X_2) \mapsto (X_2^t, X_1^t)$ for $X_i \in D_i \cong M_n(\mathbb{F})$. In particular, $\mathfrak{u}^*(D_1 \oplus D_2) = \{(X, -X^t) \mid X \in [M_n(\mathbb{F})]\} \cong \mathfrak{gl}_n(\mathbb{F})$, U_1 is the natural $\mathfrak{u}^*(D_1 \oplus D_2)$-module, and U_2 is the module dual to U_1.*

Let D be a finite dimensional semisimple associative algebra with involution α. Let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of D. Clearly, α permutes the simple components of D. Therefore for each $\zeta \in \Omega$ there exists a unique $\zeta^* \in \Omega$ such that $D_\zeta^\alpha = D_{\zeta^*}$. As $(\zeta^*)^* = \zeta$, the set Ω can be expressed as a disjoint union $\Omega_0 \cup \Omega_1 \cup \Omega_1^*$ where $\Omega_0 = \{\zeta \in \Omega \mid \zeta^* = \zeta\}$ and $\Omega_1^* = \{\zeta^* \mid \zeta \in \Omega_1\}$. Let n_ζ be the dimension of the natural D_ζ-module, so $D_\zeta \cong M_{n_\zeta}(\mathbb{F})$. Applying Lemmas 2.1 and 2.2, we get the following lemma.

Lemma 2.3 *We have $\mathfrak{u}^*(D) = \bigoplus_{\zeta \in \Omega_0 \cup \Omega_1} Q_\zeta$ where*

$$Q_\zeta = \begin{cases} \mathfrak{u}^*(D_\zeta) & \cong \quad \mathfrak{so}_{n_\zeta}(\mathbb{F}), \mathfrak{sp}_{n_\zeta}(\mathbb{F}), \quad \text{if } \zeta \in \Omega_0 \\ \mathfrak{u}^*(D_\zeta \oplus D_{\zeta^*}) & \cong \quad \mathfrak{gl}_{n_\zeta}(\mathbb{F}), \quad \text{if } \zeta \in \Omega_1. \end{cases}$$

In particular, $\mathfrak{su}^*(D) = \bigoplus_{\zeta \in \Omega_0 \cup \Omega_1} [Q_\zeta, Q_\zeta]$ is semisimple.

Lemma 2.4 *Let A be a finite dimensional associative algebra with involution α and let N be the radical of A. Then $N^\alpha = N$, $\mathfrak{u}^*(N)$ is a nilpotent ideal of $\mathfrak{u}^*(A)$, and $\mathfrak{u}^*(A)/\mathfrak{u}^*(N) \cong \mathfrak{u}^*(A/N)$. Moreover, there exists a Levi subalgebra D of A such that $D^\alpha = D$.*

Proof. Since N is the largest nilpotent ideal of A, we have that $N^\alpha = N$ and $\mathfrak{u}^*(N)$ is a nilpotent ideal of the Lie algebra $\mathfrak{u}^*(A)$. We denote by the same letter α the involution of the quotient ring A/N induced by α. The homomorphism $\mu : r \mapsto r + N$ is a homomorphism of $\mathfrak{u}^*(A)$ into $\mathfrak{u}^*(A/N)$ with the kernel $\mathfrak{u}^*(N)$. We wish to show that $\mu(\mathfrak{u}^*(A)) = \mathfrak{u}^*(A/N)$. This is equivalent to showing that for all $r \in A$ such that $x = r^\alpha + r \in N$ there exists $r' \in r + N$ such that $(r')^\alpha = -r'$. Setting $r' = r - x/2$, we get $(r')^\alpha = -r'$, as required.

We shall prove by induction on k that there is a Levi subalgebra D_k of A such that the image of D_k in A/N^k is α-invariant. Since N is nilpotent, this will imply our lemma. For $k = 1$ one can take any Levi subalgebra D_1 of A. Assume that the image of D_m in A/N^m is α-invariant. Clearly, D_m^α is a Levi subalgebra of A. By the Levi-Malcev theorem, there exists $r \in N$ such that $D_m^\alpha = (1 + r)D_m(1 + r)^{-1}$. Note that the subalgebra $D_m \oplus N^m$ of A is α-invariant, so such r can be chosen in N^m. Set $D_{m+1} = (1 + r/2)D_m(1 + r/2)^{-1}$. Then D_{m+1} is a Levi subalgebra of A. We have to show that the image of D_{m+1} in A/N^{m+1} is α-invariant. Without loss of generality one can assume that $N^{m+1} = 0$, so $(1+r)^{-1} = 1-r$. As $(D_m^\alpha)^\alpha = D_m$, we get $(1-r^\alpha+r)d(1+r^\alpha-r) \in D_m$ for all $d \in D_m$. Therefore $[D_m, r^\alpha - r] \subseteq D_m$. We need to show that $D_{m+1}^\alpha \subseteq D_{m+1}$, i.e.

$$Q = (1 + r/2)^{-1}((1 + r/2)D_m(1 + r/2)^{-1})^\alpha(1 + r/2) \subseteq D_m.$$

We have

$$Q = (1 - r/2 - r^\alpha/2 + r)D_m(1 - r + r^\alpha/2 + r/2) \subseteq D_m + [D_m, r - r^\alpha] \subseteq D_m,$$

as required. $\square$

Let A be an associative algebra. One can view A as a Lie algebra $[A]$ under the standard bracket multiplication. Let L be a Lie algebra and let

$$\iota : L \to [A]$$

be a Lie algebra monomorphism. A pair (ι, A) is called an *enveloping algebra* for L if $\iota(L)$ generates A as an algebra. In view of the universal property, the map ι uniquely extends to an associative algebra homomorphism

$$\bar{\iota} : \mathcal{A}(L) \to A$$

where $\mathcal{A}(L)$ is the augmentation ideal of the universal enveloping algebra $\mathcal{U}(L)$, i.e. the ideal of codimension 1 generated by L. Note that we do not require A to have the identity, so it is more convenient for us to deal with $\mathcal{A}(L)$ instead of $\mathcal{U}(L)$. Two enveloping algebras (ι_1, A_1) and (ι_2, A_2) are isomorphic if there is an algebra isomorphism $\delta : A_1 \to A_2$ such that $\delta\iota_1 = \iota_2$. Observe that each enveloping algebra (ι, A) is uniquely determined (up to isomorphism) by the corresponding kernel $H_A = \mathrm{Ker}(\bar{\iota})$ in $\mathcal{A}(L)$. In particular, there exists a 1–1 correspondence between the enveloping algebras for L and the ideals H in $\mathcal{A}(L)$ such that $H \cap L = 0$. This gives a partial ordering on the set of enveloping algebras of L: we say that $(\iota_1, A_1) \le (\iota_2, A_2)$ if and only if $H_{A_2} \subseteq H_{A_1}$. Given an enveloping algebra (ι, A) for a Lie algebra L, we often identify L with its image $\iota(L)$ in A, and call A the enveloping algebra, keeping in mind that there is a relevant monomorphism ι.

An involution α of an enveloping algebra A of L is called *standard* if $x^\alpha = -x$ for all $x \in L$. Observe that any antiautomorphism of an algebra is determined by its action on a generating set. So we have

Lemma 2.5 *Let A be an associative enveloping algebra for L. Then A has at most one standard involution.*

The following is well-known.

Lemma 2.6 *There exists a unique standard involution α on $\mathcal{U}(L)$ (and $\mathcal{A}(L)$).*

Proof. For each monomial $x_1 \dots x_k \in \mathcal{U}(L)$ with $x_i \in L$ set $(x_1 \dots x_k)^\alpha = (-x_k) \dots (-x_1)$. One checks that α is a standard involution on $\mathcal{U}(L)$ (and $\mathcal{A}(L)$). The uniqueness follows from Lemma 2.5. $\square$

It is not difficult to observe that there exists a correspondence between enveloping algebras with standard involution and the $*$-invariant ideals H of $\mathcal{A}(L)$ with $H \cap L = 0$ via $H \mapsto \mathcal{A}(L)/H$. Let us denote by $\mathrm{Ann}_{\mathcal{U}(L)} V$ the annihilator of an L-module V in $\mathcal{U}(L)$ and by V^* the L-module dual to V.

Lemma 2.7 *Let V be a finite-dimensional L-module. Then $(\mathrm{Ann}_{\mathcal{U}(L)} V)^\alpha = \mathrm{Ann}_{\mathcal{U}(L)} V^*$ and $(\mathrm{Ann}_{\mathcal{A}(L)} V)^\alpha = \mathrm{Ann}_{\mathcal{A}(L)} V^*$.*

Proof. Recall that V^* can be identified with the space of linear functions on V. For $x \in L$ and $f \in V^*$, the action of x on f is given by the formula $(xf)(v) = f(-xv)$ where $v \in V$. Let $u = \sum a_{i_1 \dots i_k} x_{i_1} \dots x_{i_k} \in \mathcal{U}(L)$ where $a_{i_1 \dots i_k} \in \mathbb{F}$. Then $u^\alpha = \sum a_{i_1 \dots i_k}(-x_{i_k}) \dots (-x_{i_1})$ and

$(u^\alpha f)(v) = f(uv)$ for all $f \in V^*$ and all $v \in V$. In particular, $u^\alpha \in \mathrm{Ann}_{\mathcal{U}(L)} V^*$ if and only if $u \in \mathrm{Ann}_{\mathcal{U}(L)} V$. The proof for $\mathcal{A}(L)$ is similar. $\square$

Let V be a vector space and let L be a Lie subalgebra of $\mathrm{End}_{\mathbb{F}} V$. We shall denote by $\mathcal{E}(L)$ the associative subalgebra in $\mathrm{End}_{\mathbb{F}} V$ generated by L. Note that $\mathcal{E}(L)$ is an enveloping algebra for L.

Lemma 2.8 *Let V be a finite dimensional vector space and let L be a Lie subalgebra of $\mathrm{End}_{\mathbb{F}} V$. Assume that the L-modules V and V^* are isomorphic. Then the enveloping algebra $\mathcal{E}(L)$ has a standard involution.*

Proof. Set $N = \mathrm{Ann}_{\mathcal{A}(L)} V$ and $M = \mathrm{Ann}_{\mathcal{A}(L)} V^*$. Since $V \cong V^*$, we have $N = M$. By Lemma 2.7, $N^\alpha = M$. Therefore $N = N^\alpha$, so $\mathcal{E}(L) = \mathcal{A}(L)/N$ inherits the standard involution. $\square$

3 Matrix algebras with involution

Let $M_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices over $\mathbb{F}$ and $\mathbf{1}_n$ be its identity. Let $D \subseteq M_n(\mathbb{F})$ be a semisimple subalgebra containing $\mathbf{1}_n$ and let A be a subalgebra of $M_n(\mathbb{F})$ such that $DAD = A$. In this section we describe the structure of A assuming that the algebra $D + A$ has an involution α such that $D^\alpha = D$ and $A^\alpha = A$.

First we recall the structure of arbitrary subalgebras A of $M_n(\mathbb{F})$ with $DAD = A$ described in [2]. We denote by V the natural module for $M_n(\mathbb{F})$, so $M_n(\mathbb{F}) \cong \mathrm{End}_{\mathbb{F}} V \cong V \otimes_{\mathbb{F}} V^*$. Since D is semisimple,

$$V = V_1 \oplus \ldots \oplus V_t$$

where $V_1, \ldots, V_t$ are irreducible D-modules. Put $T = \{1, \ldots, t\}$. Let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of D, so

$$D = \bigoplus_{\zeta \in \Omega} D_\zeta,$$

and let W_ζ be the natural D_ζ-module. Let us denote by ν the map $T \to \Omega$ such that for each $i \in T$ the D-modules V_i and $W_{\nu(i)}$ are isomorphic. For a pair $(i,j) \in T \times T$ let $\nu(i,j)$ denote the pair $(\nu(i), \nu(j))$.

Set $n_\zeta = \dim W_\zeta$, so $n = n_{\nu(1)} + \ldots + n_{\nu(t)}$. For $\zeta, \xi \in \Omega$ we denote by $W_{\zeta\xi}$ the vector space $W_\zeta \otimes W_\xi^*$. For each $\zeta \in \Omega$ fix a basis B_ζ of W_ζ. Let B_ζ^* be the dual basis of W_ζ^*. With respect to the chosen bases each $W_{\zeta\xi}$ can be identified with the space of $n_\zeta \times n_\xi$-matrices. Set $n' = \max\{n_\zeta \mid \zeta \in \Omega\}$. It is convenient to identify each $W_{\zeta\xi}$ with a subspace of the matrix algebra $M_{n'}(\mathbb{F})$ extending each $n_\zeta \times n_\xi$ matrix $X \in W_{\zeta\xi}$ to an $n' \times n'$ matrix by 0's (so X is located at the left upper corner of the extended matrix). Let $X_1 \in W_{\zeta_1\xi_1}$ and $X_2 \in W_{\zeta_2\xi_2}$. If $\xi_1 = \zeta_2$, define X_1X_2 as the product of the corresponding matrices in $M_{n'}(\mathbb{F})$. If $\xi_1 \neq \zeta_2$, set $X_1X_2 = 0$. Observe that if $\xi_1 = \zeta_2$, then X_1X_2 belongs to $W_{\zeta_1\xi_2}$ and the multiplication just defined agrees with the usual multiplication of elements from $W_{\zeta_1\xi_1}$ and $W_{\zeta_2\xi_2}$ for $\xi_1 = \zeta_2$:

$$(Y_1 \otimes \varphi_1)(Y_2 \otimes \varphi_2) = \varphi_1(Y_2)Y_1 \otimes \varphi_2 \in W_{\zeta_1\xi_2}$$

where $Y_i \in W_{\zeta_i}$ and $\varphi_i \in W_{\xi_i}^*$ for $i = 1, 2$. We get an algebra structure on the vector space

$$W = \bigoplus_{\zeta,\xi \in \Omega} W_{\zeta\xi}. \tag{2}$$

We denote by $\{e_{ij} \mid 1 \le i, j \le n'\}$ the standard basis of $M_{n'}(\mathbb{F})$ consisting of matrix units, so for each pair $\zeta, \xi \in \Omega$ the set $\{e_{ij} \mid 1 \le i \le n_\zeta,\ 1 \le j \le n_\xi\}$ is the basis of $W_{\zeta\xi}$. For $\zeta, \xi \in \Omega$

we denote by $\Lambda(\zeta,\xi)$ the vector space of all $t\times t$ matrices $\Lambda=(\lambda_{ij})\in M_t(\mathbb{F})$ such that $\lambda_{ij}=0$ if $\nu(i,j)\neq(\zeta,\xi)$. Observe that

$$M_t(\mathbb{F})=\bigoplus_{\zeta,\xi\in\Omega}\Lambda(\zeta,\xi). \tag{3}$$

The following lemma is obvious.

Lemma 3.1 ([2, Lemma 2.1]) *Let $\Lambda_1\in\Lambda(\zeta_1,\xi_1)$ and $\Lambda_2\in\Lambda(\zeta_2,\xi_2)$. If $\Lambda_1\Lambda_2\neq 0$, then $\xi_1=\zeta_2$ and $\Lambda_1\Lambda_2\in\Lambda(\zeta_1,\xi_2)$.*

Let us define a vector space $\mathcal{M}_D$ as follows:

$$\mathcal{M}_D=\bigoplus_{\zeta,\xi\in\Omega}W_{\zeta\xi}\otimes_{\mathbb{F}}\Lambda(\zeta,\xi). \tag{4}$$

Using decompositions (2) and (3), the properties of multiplication in W, and Lemma 3.1, one can observe that $\mathcal{M}_D$ is a subalgebra of the algebra $W\otimes_{\mathbb{F}}M_t(\mathbb{F})$.

We assume that the standard basis B in V is such that $B\cap V_i$ is a basis in V_i for each $i\in T$ and $\upsilon_i(B\cap V_i)=B_{\nu(i)}$ where υ_i is a D-module isomorphism from V_i to $W_{\nu(i)}$. In particular, for each $d\in D$ the matrices $d|_{V_i}$ and $d|_{W_{\nu(i)}}$ coincide. The decomposition

$$\operatorname{End}_{\mathbb{F}}V=V\otimes V^*=\bigoplus_{i,j=1}^{t}V_i\otimes V_j^*$$

induce a block structure on the elements of $M_n(\mathbb{F})$, so we view $M_n(\mathbb{F})$ as the set of $t\times t$ block matrices. For $i,j\in T$ let π_{ij} denote the projection of a matrix $X\in M_n(\mathbb{F})$ to its (i,j)-block submatrix. We identify $\pi_{ij}(X)$ with the corresponding matrix in $W_{\nu(i)\nu(j)}$. Let $\{\varepsilon_{ij}\mid 1\le i,j\le t\}$ be the standard basis of $M_t(\mathbb{F})$ (consisting of matrix units). Observe that $\{\varepsilon_{ij}\mid\nu(i,j)=(\zeta,\xi)\}$ is a basis of $\Lambda(\zeta,\xi)$. We have a natural vector space isomorphism $\varphi:M_n(\mathbb{F})\to\mathcal{M}_D$ defined by

$$\varphi(X)=\sum_{i,j\in T}\pi_{ij}(X)\otimes\varepsilon_{ij}.$$

Moreover, we have the following

Proposition 3.2 ([2, Proposition 2.2]) *The map φ is an algebra isomorphism of $M_n(\mathbb{F})$ onto $\mathcal{M}_D$.*

We identify the algebras $\mathcal{M}_D$ and $M_n(\mathbb{F})$. By Lemma 3.1, $\Lambda(\zeta,\zeta)$ is a subalgebra of $M_t(\mathbb{F})$. Set $\varepsilon_\zeta=\sum_{\nu(i)=\zeta}\varepsilon_{ii}$. Then for all $\zeta,\xi\in\Omega$ and all $\Lambda\in\Lambda(\zeta,\xi)$ we have $\varepsilon_\zeta\Lambda=\Lambda\varepsilon_\xi=\Lambda$. In particular, ε_ζ is the identity of the algebra $\Lambda(\zeta,\zeta)$, and $\varepsilon=\sum_{\zeta\in\Omega}\varepsilon_\zeta=\sum_{i=1}^{t}\varepsilon_{ii}$ is the identity of $M_t(\mathbb{F})$. Clearly,

$$D_\zeta=W_{\zeta\zeta}\otimes\varepsilon_\zeta \tag{5}$$

for $\zeta\in\Omega$. We shall often identify the algebras D_ζ and $W_{\zeta\zeta}$.

Right and left multiplications by elements of D induce a structure of an $D\otimes D^{op}$-module on $M_n(\mathbb{F})$. We have

$$D\otimes D^{op}=\bigoplus_{\zeta,\xi\in\Omega}D_{\zeta\xi}$$

where $D_{\zeta\xi}=D_\zeta\otimes D_\xi^{op}$ are the simple components of $D\otimes D^{op}$. The following is obvious.

Lemma 3.3 ([2, Lemma 2.3]) *The formula* (4) *is the decomposition of the* $D \otimes D^{op}$*-module* $M_n(\mathbb{F}) \cong \mathcal{M}_D$ *into its homogeneous components.*

Remark 3.4 Identifying each $D_{\zeta\xi}$ with the algebra $W_{\zeta\zeta} \otimes W_{\xi\xi}^{op}$ and any $D_{\zeta\xi}$-module $M = W_{\zeta\xi} \otimes \Lambda$ ($\Lambda \in \Lambda(\zeta, \xi)$) with $W_{\zeta\xi}$ we can express the action of $D_{\zeta\xi}$ on M in the following form: $(X_{\zeta\zeta} \otimes X_{\xi\xi})X_{\zeta\xi} = X_{\zeta\zeta}X_{\zeta\xi}X_{\xi\xi}$ where X_{ij} is any matrix from $W_{ij} = W_i \otimes W_j^*$ for $i, j \in \{\zeta, \xi\}$.

Proposition 3.5 ([2, Proposition 2.5]) *Let A be an associative subalgebra of $M_n(\mathbb{F})$ such that $DAD = A$. Then*

$$A = \bigoplus_{\zeta,\xi\in\Omega} W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$$

where $\Lambda_A(\zeta, \xi)$ is a subspace of $\Lambda(\zeta, \xi)$.

Remark 3.6 If D is a maximal semisimple subalgebra of A, then the algebra $\Lambda_A = \bigoplus_{\zeta,\xi\in\Omega} \Lambda_A(\zeta, \xi)$ is isomorphic to the *basic* algebra of A (see [9, §6.6]).

Since $D^\alpha = D$, the involution α permutes the simple components of D, so for each $\zeta \in \Omega$ there exists $\zeta^* \in \Omega$ such that $D_\zeta^\alpha = D_{\zeta^*}$. Note that $(\zeta^*)^* = \zeta$.

If we identify D_ζ with $W_{\zeta\zeta}$ (see Remark 3.4), then the map $\alpha : D_\zeta \to D_{\zeta^*}$ yields an antiisomorphism $\alpha_{\zeta\zeta} : W_{\zeta\zeta} \to W_{\zeta^*\zeta^*}$. Moreover, $\alpha_{\zeta^*\zeta^*}\alpha_{\zeta\zeta} = 1$. In particular, if $\zeta^* = \zeta$, then $\alpha_{\zeta\zeta}$ is an involution. By Lemmas 2.1 and 2.2, one can assume that the bases B_ζ of the modules W_ζ are such that for each $\zeta \in \Omega$, we have $X^{\alpha_{\zeta\zeta}} = X^{\tau_{\zeta\zeta}}$ for all matrices $X \in W_{\zeta\zeta}$ where $\tau_{\zeta\zeta} = t$ if $\zeta \neq \zeta^*$, and $\tau_{\zeta\zeta} = t, \sigma$ if $\zeta = \zeta^*$. Set

$$\begin{aligned} \Omega_A &= \{\zeta \in \Omega \mid \zeta^* \neq \zeta\}; \\ \Omega_B &= \{\zeta \in \Omega \mid \zeta^* = \zeta,\ \tau_{\zeta\zeta} = t\}; \\ \Omega_C &= \{\zeta \in \Omega \mid \zeta^* = \zeta,\ \tau_{\zeta\zeta} = \sigma\}. \end{aligned}$$

Then $\Omega = \Omega_A \cup \Omega_B \cup \Omega_C$ (disjoint union). Observe that the map $\alpha : D_\zeta \to D_{\zeta^*}$ can be written in the following form

$$(X \otimes \varepsilon_\zeta)^\alpha = X^{\tau_{\zeta\zeta}} \otimes \varepsilon_{\zeta^*} \text{ for all } X \in W_{\zeta\zeta} \tag{6}$$

(see (5)). Recall that $X^\sigma = -JX^tJ$. Set

$$J_\zeta = \begin{cases} J, & \text{if } \zeta \in \Omega_C; \\ I, & \text{if } \zeta \notin \Omega_C. \end{cases}$$

Note that $J_{\zeta^*} = J_\zeta$ and $J_\zeta^{-1} = J_\zeta^t$. Let us define the map $\tau_{\zeta\xi} : W_{\zeta\xi} \to W_{\xi^*\zeta^*}$ by

$$X^{\tau_{\zeta\xi}} = J_\xi^{-1} X^t J_\zeta. \tag{7}$$

Clearly, for $\xi = \zeta$ this map coincides with the map $\tau_{\zeta\zeta}$ defined above.

Recall that $W_{\zeta\xi}$ is a natural module for $D_{\zeta\xi}$ (see Remark 3.4). We have

$$D_{\xi^*\zeta^*} = D_{\xi^*} \otimes D_{\zeta^*}^{op} = D_\xi^\alpha \otimes (D_\zeta^\alpha)^{op}.$$

Since $D_\xi^\alpha \cong D_\xi^{op}$ and $(D_\zeta^\alpha)^{op} \cong D_\zeta$, the map $\chi_{\zeta\xi} : d_1 \otimes d_2 \mapsto d_2^\alpha \otimes d_1^\alpha$ ($d_1 \in D_\zeta$, $d_2 \in D_\xi$) is a canonical isomorphism of the algebras $D_{\zeta\xi}$ and $D_{\xi^*\zeta^*}$, so $W_{\xi^*\zeta^*}$ can be viewed as a (natural) $D_{\zeta\xi}$-module.

Lemma 3.7 *The map $\tau_{\zeta\xi}$ is an isomorphism of $D_{\zeta\xi}$-modules.*

Proof. We identify D_ζ with $W_{\zeta\zeta}$. Using Remark 3.4 and (6), we get

$$\chi_{\zeta\xi}(X_{\zeta\zeta} \otimes X_{\xi\xi})X_{\zeta\xi}^{\tau_{\zeta\xi}} = X_{\xi\xi}^{\tau_{\xi\xi}} X_{\zeta\xi}^{\tau_{\zeta\xi}} X_{\zeta\zeta}^{\tau_{\zeta\zeta}} = J_\xi^{-1} X_{\xi\xi}^t X_{\zeta\xi}^t X_{\zeta\zeta}^t J_\zeta = ((X_{\zeta\zeta} \otimes X_{\xi\xi})X_{\zeta\xi})^{\tau_{\zeta\xi}}$$

for all $X_{\zeta\zeta} \in W_{\zeta\zeta}$, $X_{\zeta\xi} \in W_{\zeta\xi}$, $X_{\xi\xi} \in W_{\xi\xi}$, as required. □

We shall denote by τ the linear transformation of the space $W = \oplus_{\zeta,\xi\in\Omega} W_{\zeta\xi}$ such that $\tau|_{W_{\zeta\xi}} = \tau_{\zeta\xi}$.

Proposition 3.8 (1) *τ is an antiautomorphism of the algebra W.*

(2) *For $\zeta, \xi \in \Omega$ and $X \in W_{\zeta\xi}$ we have*

$$X^{\tau^2} = \begin{cases} X, & \text{if } \zeta, \xi \in \Omega_C \text{ or } \zeta, \xi \notin \Omega_C; \\ -X, & \text{otherwise.} \end{cases}$$

Proof. (1) For all $X_{\zeta\xi} \in W_{\zeta\xi}$ and $X_{\xi\eta} \in W_{\xi\eta}$ we have

$$(X_{\zeta\xi} X_{\xi\eta})^\tau = J_\eta^{-1} X_{\xi\eta}^t X_{\zeta\xi}^t J_\zeta = X_{\xi\eta}^\tau X_{\zeta\xi}^\tau,$$

so τ is an antiautomorphism. The following equality proves (2):

$$X^{\tau^2} = J_{\zeta^*}^{-1}(J_\xi^{-1} X^t J_\zeta)^t J_{\xi^*} = (J_{\zeta^*}^{-1})^2 X (J_\xi)^2 = \pm X.$$

□

Recall that by Proposition 3.5, any subalgebra A of $M_n(\mathbb{F})$ such that $DAD = A$ can be expressed in the form $A = \bigoplus_{\zeta,\xi\in\Omega} W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$ where $\Lambda_A(\zeta, \xi) \subseteq \Lambda(\zeta, \xi)$. Let Λ_A be the algebra $\oplus_{\zeta,\xi\in\Omega} \Lambda_A(\zeta, \xi)$ (cf. Remark 3.6). The following proposition shows that Λ_A has an antiautomorphism which induce the involution α on A.

Proposition 3.9 *Let A be a subalgebra of $M_n(\mathbb{F})$ such that $DAD = A$. Assume that the algebra $D + A$ has an involution α such that $D^\alpha = D$ and $A^\alpha = A$. Then there exists a (unique) map $\rho : \Lambda_A \to \Lambda_A$ such that the following holds.*

(1) *For all $\zeta, \xi \in \Omega$, $\Lambda \in \Lambda_A(\zeta, \xi)$, and $X \in W_{\zeta\xi}$ we have*

$$(X \otimes \Lambda)^\alpha = X^\tau \otimes \Lambda^\rho.$$

(2) *$\rho(\Lambda_A(\zeta, \xi)) = \Lambda_A(\xi^*, \zeta^*)$.*

(3) *ρ is an antiautomorphism of the algebra Λ_A.*

(4) *For $\zeta, \xi \in \Omega$ and $\Lambda \in \Lambda_A(\zeta, \xi)$ we have*

$$\Lambda^{\rho^2} = \begin{cases} \Lambda, & \text{if } \zeta, \xi \in \Omega_C \text{ or } \zeta, \xi \notin \Omega_C; \\ -\Lambda, & \text{otherwise.} \end{cases}$$

Proof. (1) We have

$$(W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi))^\alpha = (D_\zeta A D_\xi)^\alpha = D_{\xi^*} A D_{\zeta^*} = W_{\xi^*\zeta^*} \otimes \Lambda_A(\xi^*, \zeta^*).$$

Let $0 \neq \Lambda \in \Lambda_A(\zeta, \xi)$. Since $(W_{\zeta\xi} \otimes \Lambda)^\alpha$ is an irreducible $D \otimes D^{op}$-submodule of $M_n(\mathbb{F})$ lying in $W_{\xi^*\zeta^*} \otimes \Lambda_A(\xi^*, \zeta^*)$, there exist $\Lambda' \in \Lambda_A(\xi^*, \zeta^*)$ and a map $\theta : W_{\zeta\xi} \to W_{\xi^*\zeta^*}$ such that $(X \otimes \Lambda)^\alpha = X^\theta \otimes \Lambda'$ for all $X \in W_{\zeta\xi}$. Let $d = d_1 \otimes d_2 \in D_{\zeta\xi} = D_\zeta \otimes D_\xi^{op}$. Then

$$(d(X \otimes \Lambda))^\alpha = (d_1(X \otimes \Lambda)d_2)^\alpha = d_2^\alpha (X^\theta \otimes \Lambda') d_1^\alpha = \chi_{\zeta\xi}(d)(X^\theta \otimes \Lambda').$$

It follows that θ is an isomorphism of irreducible $D_{\zeta\xi}$-modules. By Lemma 3.7, $\theta = \lambda\tau_{\zeta\xi}$ for $\lambda \in \mathbb{F}$. Set $\Lambda^\rho = \lambda\Lambda'$. Then $(X \otimes \Lambda)^\alpha = X^\tau \otimes \Lambda^\rho$ for all $X \in W_{\zeta\xi}$. Note that ρ is linear since α and τ are.

(2) This is obvious.

(3) Let $\Lambda_{\zeta\xi} \in \Lambda_A(\zeta, \xi)$ and $\Lambda_{\xi\eta} \in \Lambda_A(\xi, \eta)$. Then for all $X_{\zeta\xi} \in W_{\zeta\xi}$ and $X_{\xi\eta} \in W_{\xi\eta}$ we have

$$(X_{\zeta\xi}X_{\xi\eta})^\tau \otimes (\Lambda_{\zeta\xi}\Lambda_{\xi\eta})^\rho = (X_{\xi\eta} \otimes \Lambda_{\xi\eta})^\alpha (X_{\zeta\xi} \otimes \Lambda_{\zeta\xi})^\alpha = X^\tau_{\xi\eta}X^\tau_{\zeta\xi} \otimes \Lambda^\rho_{\xi\eta}\Lambda^\rho_{\zeta\xi}.$$

Since $(X_{\zeta\xi}X_{\xi\eta})^\tau = X^\tau_{\xi\eta}X^\tau_{\zeta\xi}$ (see Proposition 3.8(1)), we have $(\Lambda_{\zeta\xi}\Lambda_{\xi\eta})^\rho = \Lambda^\rho_{\xi\eta}\Lambda^\rho_{\zeta\xi}$, so ρ is an antiautomorphism of Λ_A.

(4) This follows from Proposition 3.8(2). □

4 $*$-normal submodules

Let V be the natural module for $M_n(\mathbb{F})$ and let L be a Lie subalgebra of $M_n(\mathbb{F})$ such that the L-modules V and V^* are isomorphic. Let S be a semisimple subalgebra of L and $A = \mathcal{E}(L)$ be the enveloping algebra of L in $M_n(\mathbb{F})$. Since $V \cong V^*$, by Lemma 2.8, A has a standard involution α and $L \subseteq \mathfrak{u}^*(A)$. The aim of this section is to describe irreducible S-submodules in $\mathfrak{u}^*(A)$ under the adjoint action (under some additional assumptions).

Below $S_1, \dots, S_k$ are the simple components of S, so $S = S_1 \oplus \dots \oplus S_k$. As V is completely reducible, $V = V_1 \oplus \dots \oplus V_t$ where $V_1, \dots, V_t$ are irreducible S-modules. Let $\{W_\zeta \mid \zeta \in \Omega\}$ be a set of representatives of the isomorphism classes of irreducible S-submodules of V. Set $\Omega' = \{\zeta \in \Omega \mid W_\zeta \text{ is nontrivial}\}$. If $\Omega \neq \Omega'$, then the set $\Omega \backslash \Omega'$ consists of a single element, which will be denoted by 0. In other words, we use the notation W_0 for the trivial one-dimensional S-module. Let $\mathcal{E}(S)$ be the enveloping algebra of S in $M_n(\mathbb{F})$, i.e. the associative subalgebra generated by S. Set $D = \mathcal{E}(S) + \mathbb{F}\mathbf{1}_n$. Clearly, D is semisimple, and the simple components of D are in a bijective correspondence with isomorphism classes of irreducible S-submodules of V. So

$$D = \bigoplus_{\zeta\in\Omega} D_\zeta$$

with $D_\zeta \cong \operatorname{End} W_\zeta$. The enveloping algebra $\mathcal{E}(S)$ is a subalgebra of D and

$$\mathcal{E}(S) = \bigoplus_{\zeta\in\Omega'} D_\zeta.$$

The Lie algebra $[M_n(\mathbb{F})]$ is an S-module under the adjoint action. Let $\varphi : M_n(\mathbb{F}) \to \mathcal{M}_D$ be the canonical isomorphism (see Proposition 3.2). Observe that the adjoint action of S on $\mathcal{M}_D$ induced by φ agrees with the natural action of S on the direct summands $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$ of $\mathcal{M}_D$ (see (4)): we just view $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$ as the tensor product of the S-module $W_{\zeta\xi} = W_\zeta \otimes W^*_\xi$ and the trivial S-module $\Lambda(\zeta, \xi)$.

Since $s^\alpha = -s$ for all $s \in S$, $\mathcal{E}(S)$ (and D, if we set $\mathbf{1}^\alpha_n = \mathbf{1}_n$)) is α-invariant. We shall denote by the same symbol α the involution of $\mathcal{E}(S)$ and D inherited from the algebra A. As in Section 3 there is a bijection $* : \Omega \to \Omega$ defined as follows: $\zeta^* = \xi$ if and only if $D^\alpha_\zeta = D_\xi$.

Lemma 4.1 *The S-modules W^*_ζ and W_{ζ^*} are isomorphic.*

Proof. Clearly, two irreducible S-modules are isomorphic if and only if their annihilators in $\mathcal{U}(S)$ coincide. By Lemma 2.7, we have

$$\operatorname{Ann}_{\mathcal{U}(S)} W^*_\zeta = (\operatorname{Ann}_{\mathcal{U}(S)} W_\zeta)^\alpha = \operatorname{Ann}_{\mathcal{U}(S)} W_{\zeta^*},$$

as required. □

Let $\tau_{\zeta\xi}$ be an isomorphism of the $D_{\zeta\xi}$-modules $W_{\zeta\xi}$ and $W_{\xi^*\zeta^*}$ introduced in Section 3.

Lemma 4.2 *The map $\tau_{\zeta\xi}$ is an isomorphism of the S-modules $W_{\zeta\xi}$ and $W_{\xi^*\zeta^*}$.*

Proof. Let s_ζ and $s_{\zeta\xi} = s_\zeta \otimes s_\xi$ denote the images of $s \in S$ in D_ζ and $D_{\zeta\xi}$, respectively. Since $s^\alpha = -s$ in D, we have $s_\zeta^\alpha = -s_{\zeta^*}$. Therefore

$$\chi_{\zeta\xi}(s_{\zeta\xi}) = \chi_{\zeta\xi}(s_\zeta \otimes s_\xi) = s_\xi^\alpha \otimes s_\zeta^\alpha = (-s_{\xi^*}) \otimes (-s_{\zeta^*}) = s_{\xi^*\zeta^*}.$$

Hence $\chi_{\zeta\xi}$ gives an isomorphism of the enveloping algebras of the images of S in $\operatorname{End} W_{\zeta\xi} = D_{\zeta\xi}$ and $\operatorname{End} W_{\xi^*\zeta^*} = D_{\xi^*\zeta^*}$. It remains to apply Lemma 3.7. □

Recall that by Proposition 3.9 there are maps τ and ρ such that for all $\zeta, \xi \in \Omega$, we have $(X \otimes \Lambda)^\alpha = X^\tau \otimes \Lambda^\rho$ for all $\Lambda \in \Lambda_A(\zeta, \xi)$ and $X \in W_{\zeta\xi}$.

Definition 4.3 A nonzero S-submodule M of A is called $*$-*normal* if there exist $\zeta, \xi \in \Omega$, a matrix $\Lambda_M \in \Lambda_A(\zeta, \xi)$, and an S-submodule W_M of $W_{\zeta\xi}$ such that

$$M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_M\}.$$

If M is $*$-normal, then we denote by $\bar{M}$ the module $W_{\zeta\xi} \otimes \Lambda_M + W_{\xi^*\zeta^*} \otimes \Lambda_M^\rho \subseteq A$. Note that $\bar{M}^\alpha = \bar{M}$ and $M \subseteq \mathfrak{u}^*(\bar{M}) = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_{\zeta\xi}\}$.

If M is $*$-normal, then the relevant pair (ζ, ξ) is called the *type* of M. Note that the type of M is determined "up to $*$", i.e. the pairs (ζ, ξ) and (ξ^*, ζ^*) denote the same type.

Recall that $D = \mathcal{E}(S) + \mathbb{F}\mathbf{1}_n$. Therefore Lemma 3.3 implies the following.

Lemma 4.4 *Let M be an $*$-normal S-submodule of A. Then $(\mathcal{E}(S) + \mathbb{F}\mathbf{1}_n)M(\mathcal{E}(S) + \mathbb{F}\mathbf{1}_n) = \bar{M}$.*

Recall the map $* : \Omega \to \Omega$ defined as follows: $\zeta^* = \xi$ if and only if $D_\zeta^\alpha = D_\xi$ (or equivalently, by Lemma 4.1, $W_\zeta^* \cong W_\xi$). We denote by $\bar{\Omega}$ (resp. $\bar{\Omega}'$) the quotient of Ω (resp. Ω') by the equivalence relation $\zeta \sim \zeta^*$. For $\zeta \in \Omega$ we denote by $\bar{\zeta}$ its image in $\bar{\Omega}$.

Proposition 4.5 *Assume that there is a bijection between the set of simple components of S and the set $\bar{\Omega}'$ (i.e. the set $\{1, \dots, k\}$ is identified with $\bar{\Omega}'$) such that for each $\zeta \in \Omega'$ the following conditions hold:*

(I) $\operatorname{Ann}_S W_\zeta = \bigoplus_{m \neq \bar{\zeta}} S_m$*; in particular, W_ζ can be considered as an $S_{\bar{\zeta}}$-module;*

(II) *the S-modules $W_{\zeta\zeta}$ and $W_{\zeta\zeta^*}$ do not contain submodules isomorphic to W_ζ or W_{ζ^*};*

(III) *each nontrivial composition factor of the S-module $W_{\zeta\zeta} \oplus W_{\zeta\zeta^*} \oplus W_{\zeta^*\zeta}$ for $\zeta \neq \zeta^*$ and of $W_{\zeta\zeta}$ for $\zeta = \zeta^*$ appears with multiplicity* 1.

Then each nontrivial irreducible S-submodule M of $\mathfrak{u}^*(A)$ is $*$-normal. More exactly, there exist $\zeta, \xi \in \Omega$, $\Lambda_M \in \Lambda(\zeta, \xi)$, and S-submodule $W_M \cong M$ of $W_{\zeta\xi}$ such that $M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_M\}$. Moreover, if $\xi = \zeta^*$, then $\Lambda_M^\rho = \pm\Lambda_M$, $X^\tau = \mp X$ for $X \in W_M$, and $M = \{X \otimes \Lambda_M \mid X \in W_M\}$.

Lemma 4.6 *Assume that the conditions of Proposition* 4.5 *hold. Then for all $\zeta, \xi \in \Omega$ with $\bar{\zeta} \neq \bar{\xi}$ the S-module $W_{\zeta\xi}$ is irreducible.*

Proof. It suffices to note that $W_{\zeta\xi} = W_\zeta \otimes W_\xi^*$ is an irreducible $S_{\bar{\zeta}} \oplus S_{\bar{\xi}}$-module whenever $\bar{\zeta} \neq \bar{\xi}$. □

Lemma 4.7 *Assume that the conditions of Proposition* 4.5 *hold. Let M_i be a nontrivial irreducible submodule of $W_{\zeta_i\xi_i}$ $(i = 1, 2)$ and let $\theta : M_1 \to M_2$ be an isomorphism of S-modules. Then either $(\zeta_1, \xi_1) = (\zeta_2, \xi_2)$ and $\theta = \lambda \cdot 1$ $(\lambda \in \mathbb{F})$, or $(\zeta_1, \xi_1) = (\xi_2^*, \zeta_2^*)$ and $\theta = \lambda \cdot \tau$ $(\lambda \in \mathbb{F})$.*

Proof. Actually, M_i is a faithful $Q_i = S_{\bar{\zeta}_i} + S_{\bar{\xi}_i}$-module (we assume $S_{\bar{0}} = 0$ here). Therefore $M_1 \cong M_2$ implies that $Q_1 = Q_2 = Q$, so $(\bar{\zeta}_1, \bar{\xi}_1)$ coincides with $(\bar{\zeta}_2, \bar{\xi}_2)$ or $(\bar{\xi}_2, \bar{\zeta}_2)$ (we use the assumption (II) here). Assume $\bar{\zeta}_1 \neq \bar{\xi}_1$. Then by Lemma 4.6, $M_i = W_{\zeta_i\xi_i}$ is irreducible for $i = 1, 2$. Therefore either $(\zeta_1, \xi_1) = (\zeta_2, \xi_2)$ and $\theta = \lambda \cdot 1$ or $(\zeta_1, \xi_1) = (\xi_2^*, \zeta_2^*)$ and, by Lemma 4.2, $\theta = \lambda \cdot \tau$, as required.

Therefore we can suppose that $\bar{\zeta}_1 = \bar{\xi}_1 = \bar{\zeta}_2 = \bar{\xi}_2$. It follows from assumption (III) that either $(\zeta_1, \xi_1) = (\zeta_2, \xi_2)$ and $\theta = \lambda \cdot 1$ or $\zeta_1 = \xi_1 = \zeta_2^* = \xi_2^* = \zeta$. Consider the second case. By Lemma 4.2, we have an isomorphism $\tau_{\zeta\zeta} : W_{\zeta\zeta} \to W_{\zeta^*\zeta^*}$. The module $\tau_{\zeta^*\zeta^*}(\theta(M_1))$ is a submodule of $W_{\zeta\zeta}$ isomorphic to M_1, so coincides with M_1 by assumption (III). Since M_1 is irreducible, $\tau_{\zeta^*\zeta^*}\theta = \lambda \cdot 1$ for some $\lambda \in \mathbb{F}$. Applying $\tau_{\zeta\zeta}$ to both sides and using Proposition 3.8(2), we get $\theta = \lambda \cdot \tau$, as required. □

Proof of Proposition 4.5. Let us denote by $M_{\zeta\xi}$ the projection of M to $W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$. Clearly, there exist $\zeta, \xi \in \Omega$ such that $M_{\zeta\xi} \neq 0$. Then $M_{\zeta\xi} \cong M$, so there exist a submodule $W_M \subseteq W_{\zeta\xi}$ and a matrix $\Lambda_M \in \Lambda_A(\zeta, \xi)$ such that $W_M \cong M$ and $M_{\zeta\xi} = W_M \otimes \Lambda_M$. Assume $\xi \neq \zeta^*$. Since $r^\alpha = -r$ for all $r \in M$ and $(X \otimes \Lambda_M)^\alpha = X^\tau \otimes \Lambda_M^\rho$ for all $X \in W_{\zeta\xi}$, the projection of M to $W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi) + W_{\xi^*\zeta^*} \otimes \Lambda_A(\xi^*, \zeta^*)$ is $\{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_M\}$. It remains to note that by Lemma 4.7, all other projections are zeros.

Assume now that $\xi = \zeta^*$. Then $W_{\zeta\xi} = W_{\xi^*\zeta^*}$ and $\Lambda_A(\zeta, \xi) = \Lambda_A(\xi^*, \zeta^*)$. Therefore $M = M_{\zeta\xi} = W_M \otimes \Lambda_M$. Since $M \subseteq \mathfrak{u}^*(A)$, $X \otimes \Lambda_M = -(X \otimes \Lambda_M)^\alpha = -X^\tau \otimes \Lambda_M^\rho$ for all $X \in W_M$. Hence as above $M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_M\}$. Moreover, there exists $\lambda \in \mathbb{F}$ such that $\Lambda_M^\rho = \lambda\Lambda_M$. Since $\rho^2 = 1$ (see Proposition 3.9(4)), we have $\lambda = \pm 1$, so $\Lambda_M^\rho = \pm\Lambda_M$ and $X^\tau = \mp X$, as required.
□

5 Enveloping algebras of $*$-plain Lie algebras

Let $L \subset M_n(\mathbb{F})$ be a perfect Lie algebra. Then $L = S \oplus R$ where S is a maximal semisimple subalgebra and R is the radical of L. Let V be the natural $M_n(\mathbb{F})$-module, We adopt the notation of Section 4. As above, we denote by $S_1, \dots, S_k$ the simple components of S and by $A = \mathcal{E}(L)$ the enveloping algebra of L in $M_n(\mathbb{F})$. Throughout below we assume that the L-modules V and V^* are isomorphic. This implies that the enveloping algebra $\mathcal{E}(L)$ has the standard involution α, and there is a map $* : \Omega \to \Omega$ defined as follows: $\zeta^* = \xi$ if and only if $W_\zeta^* \cong W_\xi$. The involution α extends to $D = \mathcal{E}(S) + \mathbb{F}\mathbf{1}_n$. Let $\{W_\zeta \mid \zeta \in \Omega\}$ be a set of representatives of the isomorphism classes of all irreducible L-submodules of V, and let $\Omega' = \{\zeta \in \Omega \mid W_\zeta \text{ is nontrivial}\}$. We denote by $\bar{\Omega}$ (resp. $\bar{\Omega}'$) the quotient of Ω (resp. Ω') by the equivalence relation $\zeta \sim \zeta^*$. For $\zeta \in \Omega$ we denote by $\bar{\zeta}$ its image in $\bar{\Omega}$ and by n_ζ (or $n_{\bar{\zeta}}$) the dimension of W_ζ. In this section we assume that L is perfect and V is a strongly $*$-plain L-module. In other words, we assume that the set $\bar{\Omega}'$ is identified with the set $\{1, \dots, k\}$ (which labels the simple components of S) such that

- $\operatorname{Ann}_S W_\zeta = \bigoplus_{m \neq \bar{\zeta}} S_m$ for $\zeta \in \Omega'$;
- for each $\zeta \in \Omega'$, $S_{\bar{\zeta}}$ is isomorphic to one of the following algebras

$$\begin{aligned} &\mathfrak{sl}(W_\zeta) && (n_\zeta \geq 4); \\ &\mathfrak{so}(W_\zeta) && (n_\zeta \geq 7); \\ &\mathfrak{sp}(W_\zeta) && (n_\zeta \geq 6). \end{aligned}$$

In particular, W_ζ ($\zeta \in \Omega'$) is the natural $S_{\bar\zeta}$-module or dual to it. Then one can easily verify that the conditions (I)–(III) of Proposition 4.5 hold. Therefore each nontrivial irreducible S-submodule of L is $*$-normal. The aim of this section is to show that $L = \mathfrak{su}^*(A)$ (see Proposition 5.10).

Remark 5.1 (1) We exclude $\mathfrak{sl}_3$ from the list because $U \otimes U$ contains a submodule isomorphic to U^* where U is the standard $\mathfrak{sl}_3$-module. This contradicts assumption (II) of Proposition 4.5.

(2) The algebras $\mathfrak{sl}_2$ and $\mathfrak{sp}_4$ should be excluded as otherwise L may be smaller than $\mathfrak{su}^*(A)$.

(3) The algebra $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is not simple.

(4) If we admit the algebras $\mathfrak{so}_5$ and $\mathfrak{so}_6$, the main result of this section Proposition 5.10 (as well as Theorem 1.3) remains to be valid, but Theorem 1.5 not (in view of the isomorphism $\mathfrak{so}_6 \cong \mathfrak{sl}_4$).

We denote by ${}_LL$ the adjoint module for L and by ${}_SL$ its restriction to S. Let R_0 be the sum of all trivial submodules of ${}_SL$ and let L_1 be the sum of all nontrivial irreducible submodules. Since ${}_SL$ is completely reducible, ${}_SL = L_1 \oplus R_0$. Note that $R_0 \subset R$. Set

$$W'_{\zeta\xi} = \begin{cases} W_{\zeta\xi}, & \text{if } \zeta \neq \xi, \\ \{X \mid X \in W_{\zeta\xi},\ \operatorname{tr} X = 0\}, & \text{if } \zeta = \xi. \end{cases}$$

Clearly, $W'_{\zeta\xi}$ is a submodule of $W_{\zeta\xi}$. Using Lemma 2.1 and 2.2 and equation (6), we get

$$S_{\bar\zeta} = \{X \otimes \varepsilon_\zeta - X^\tau \otimes \varepsilon_{\zeta^*} \mid X \in W'_{\zeta\zeta}\}$$

where $\tau = \sigma$ if $\zeta \in \Omega_C$, and $\tau = t$ otherwise. For $\zeta \in \Omega$ and $\xi = \zeta, \zeta^*$ let us denote by $W^+_{\zeta\xi}$ (resp. $W^-_{\zeta\xi}$) the set of τ-symmetric (resp. τ-skew-symmetric) matrices of $W'_{\zeta\xi}$ ($\tau = \sigma$ if $\zeta \in \Omega_C$, and $\tau = t$ otherwise). Set

$$\begin{aligned} Z_A &= \{\zeta \in \Omega' \mid S_{\bar\zeta} \cong \mathfrak{sl}_{n_\zeta}(\mathbb{F})\}; \\ Z_B &= \{\zeta \in \Omega' \mid S_{\bar\zeta} \cong \mathfrak{so}_{n_\zeta}(\mathbb{F})\}; \\ Z_C &= \{\zeta \in \Omega' \mid S_{\bar\zeta} \cong \mathfrak{sp}_{n_\zeta}(\mathbb{F})\}. \end{aligned}$$

The following can be checked by the direct calculations.

Lemma 5.2 *Let $\zeta, \xi \in \Omega$. Then*

$$W_{\zeta\xi} = \begin{cases} W_{\zeta\xi}, & \text{if } \bar\zeta \neq \bar\xi \text{ or } \zeta = \xi = 0; \\ W'_{\zeta\xi} \oplus \mathbb{F}I, & \text{if } \zeta = \xi \in \Omega_A; \\ W^+_{\zeta\xi} \oplus W^-_{\zeta\xi}, & \text{if } \zeta = \xi^* \in \Omega_A; \\ W^+_{\zeta\xi} \oplus W^-_{\zeta\xi} \oplus \mathbb{F}I, & \text{if } \zeta = \xi \in \Omega_B \cup \Omega_C \end{cases}$$

is the decomposition of the S-module $W_{\zeta\xi} = W_\zeta \otimes W^*_\xi$ into pairwise nonisomorphic irreducible submodules.

Recall that we denote by A the algebra $\mathcal{E}(L)$, so $L \subseteq \mathfrak{u}^*(A)$.

Lemma 5.3 *Let M be a nontrivial irreducible S-submodule of L. Then there exist $\zeta, \xi \in \Omega$ and $\Lambda_M \in \Lambda_A(\zeta, \xi)$ such that $M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda^\rho_M \mid X \in W'_{\zeta\xi}\}$. Moreover, if $\zeta = \xi^*$, then $M = W^\pm_{\zeta\xi} \otimes \Lambda_M$ and $\Lambda^\rho_M = \mp\Lambda_M$.*

Proof. By Proposition 4.5, there exist $\zeta, \xi \in \Omega$, $\Lambda_M \in \Lambda(\zeta, \xi)$, and an S-submodule $W_M \cong M$ of $W_{\zeta\xi}$ such that $M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_M\}$. Moreover, if $\xi = \zeta^*$, then $\Lambda_M^\rho = \pm\Lambda_M$, $X^\tau = \mp X$ for $X \in W_M$, and $M = \{X \otimes \Lambda_M \mid X \in W_M\}$. If $\bar{\zeta} \neq \bar{\xi}$ or $\zeta = \xi \in \Omega_A$, then by Lemma 5.2, $W_M = W'_{\zeta\xi}$, and we are done. Assume that $\xi = \zeta^*$. Then by Lemma 5.2, $W_M = W^{\pm}_{\zeta\xi}$, so $M = W^{\pm}_{\zeta\xi} \otimes \Lambda_M$. As $X^\tau = \pm X$ for $X \in W^{\pm}_{\zeta\xi}$, we have $\Lambda_M^\rho = \mp\Lambda_M$. Hence $M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W'_{\zeta\xi}\}$, as required. $\square$

Lemma 5.4 *Let M be as in Lemma 5.3. Then*

$$\mathfrak{u}^*(\bar{M}) = \begin{cases} M, & \text{if } \zeta \neq \xi; \\ M \oplus \mathbb{F}(I \otimes \Lambda_M - I \otimes \Lambda_M^\rho), & \text{if } \zeta = \xi \in \Omega_A; \\ M, & \text{if } \zeta = \xi \in \Omega_B \cup \Omega_C \text{ and } M \cong W^-_{\zeta\zeta}; \\ M \oplus \mathbb{F}(I \otimes \Lambda_M), & \text{if } \zeta = \xi \in \Omega_B \cup \Omega_C \text{ and } M \cong W^+_{\zeta\zeta}. \end{cases}$$

Proof. We have $\mathfrak{u}^*(\bar{M}) = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_{\zeta\xi}\}$. If $\zeta \neq \xi$, then $W'_{\zeta\xi} = W_{\zeta\xi}$, so $\mathfrak{u}^*(\bar{M}) = M$, as required. Assume that $\zeta = \xi$. Then $W_{\zeta\zeta} = W'_{\zeta\zeta} \oplus \mathbb{F}I$. By (7), $I^\tau = I$, so $\mathfrak{u}^*(\bar{M}) = M \oplus \mathbb{F}(I \otimes \Lambda_M - I \otimes \Lambda_M^\rho)$ if $\zeta = \xi \in \Omega_A$. Assume that $\zeta = \xi \in \Omega_B \cup \Omega_C$. Then by Lemma 5.3, $M = W^{\pm}_{\zeta\xi} \otimes \Lambda_M$ and $\Lambda_M^\rho = \mp\Lambda_M$. Therefore $\mathfrak{u}^*(\bar{M}) = M$ if $M \cong W^-_{\zeta\zeta}$, and $\mathfrak{u}^*(\bar{M}) = M \oplus \mathbb{F}(I \otimes \Lambda_M)$ if $M \cong W^+_{\zeta\zeta}$. $\square$

Let $\mathfrak{M}$ be the set of all nontrivial irreducible submodules of ${}_SL$. Put

$$\bar{L}_1 = \sum_{M \in \mathfrak{M}} \bar{M}.$$

Lemma 5.5 *Let $x \in L$ and $\zeta, \xi \in \Omega$. Assume that x has a nonzero projection $x_{\zeta\xi} = X_1 \otimes \Lambda_1 + \ldots + X_m \otimes \Lambda_m$ to $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$ where $X_1, \ldots, X_m \in W_{\zeta\xi}$ are linearly independent. For $\zeta = \xi$ assume additionally that the matrices $I, X_1, \ldots, X_m$ are linearly independent. Then $W_{\zeta\xi} \otimes \Lambda_i \subseteq \bar{L}_1$ for each $i = 1, \ldots, m$.*

Proof. Let M be the S-submodule of L generated by x. Then $M = T \oplus M'$ where T is a trivial S-module and $M' = M_1 \oplus \ldots \oplus M_l$ is a sum of nontrivial irreducible S-modules M_j. By Proposition 4.5, each M_i is normal, so set $\bar{M}' = \bar{M}_1 + \ldots + \bar{M}_l \subseteq \bar{L}_1$. Let $T_{\zeta\xi}$ and $\bar{M}'_{\zeta\xi}$ be the projections of T and $\bar{M}'$, respectively, to $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$. It follows from definition that $\bar{M}'_{\zeta\xi} = W_{\zeta\xi} \otimes \Lambda_M(\zeta, \xi) \subseteq \bar{L}_1$ where $\Lambda_M(\zeta, \xi)$ is a subspace of $\Lambda(\zeta, \xi)$. Clearly it suffices to show that $\Lambda_i \in \Lambda_M(\zeta, \xi)$ for each $i = 1, \ldots, m$. Assume that this is not the case. Reordering the indices one can assume that $\{1, \ldots, m'\}$ is a maximal subset of $\{1, \ldots, m\}$ such that $\Lambda_1, \ldots, \Lambda_{m'}$ are linearly independent modulo $\Lambda_M(\zeta, \xi)$. Then the image of $x_{\zeta\xi}$ in the quotient $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)/W_{\zeta\xi} \otimes \Lambda_M(\zeta, \xi)$ is $X'_1 \otimes \Lambda'_1 + \ldots + X'_{m'} \otimes \Lambda'_{m'}$ where $X'_i = X_i + \sum_{j=m'+1}^m \lambda_i^j X_j$ and Λ'_i is the image of Λ_i in $\Lambda(\zeta, \xi)/\Lambda_M(\zeta, \xi)$. By Lemma 5.2, $T_{\zeta\xi}$ is either 0 (for $\zeta \neq \xi$) or lies in $\mathbb{F}I \otimes \Lambda(\zeta, \xi)$ (for $\zeta = \xi$). As $X'_1, \ldots, X'_m$ (resp. $I, X'_1, \ldots, X'_m$) are linearly independent for $\zeta \neq \xi$ (resp. $\zeta = \xi$), we get that $x_{\zeta\xi} \notin M^0_{\zeta\xi} + \bar{M}'_{\zeta\xi}$, which contradicts the assumption. The lemma follows. $\square$

Definition 5.6 A $*$-normal submodule M is called degenerate if $M = \{X \otimes \Lambda_M - X^\tau \otimes \Lambda_M^\rho \mid X \in W_{0\xi}\}$ for some $\xi \in \Omega'$ and $\Lambda_M \in \Lambda_A(0, \xi)$. For such M we denote $\bar{M}^l = W_{0\xi} \otimes \Lambda_M$ and $\bar{M}^r = W_{\xi^*0} \otimes \Lambda_M^\rho$, so $\bar{M} = \bar{M}^l \oplus \bar{M}^r$.

Lemma 5.7 *Let M_1 and M_2 be nontrivial irreducible submodules of L. Then $\bar{M}_1\bar{M}_2 \subseteq \bar{L}_1$, except possibly in the case when M_1 and M_2 are degenerate. In the exceptional case $\bar{M}_1^r\bar{M}_2^r = \bar{M}_1^l\bar{M}_2^l = 0$, $\bar{M}_1^r\bar{M}_2^l \subset \bar{L}_1$, and $\bar{M}_1^l\bar{M}_2^r \subset W_{00} \otimes \Lambda(0, 0)$. Moreover, $\mathfrak{u}^*(\bar{M}_1^l\bar{M}_2^r + \bar{M}_2^l\bar{M}_1^r) \subset R_0 + \mathfrak{u}^*(\bar{L}_1)$.*

Proof. By Proposition 4.5, M_1 and M_2 are $*$-normal, so

$$M_i = \{X \otimes \Lambda_i + X^\tau \otimes (-\Lambda_i^\rho) \mid X \in W'_{\zeta_i\xi_i}\} \quad (i=1,2)$$

for some $\zeta_i, \xi_i \in \Omega$ and $\Lambda_i \in \Lambda_A(\zeta_i, \xi_i)$. In view of symmetry between ζ_1 and ξ_1, ζ_2 and ξ_2, it suffices to show that $\Lambda_{12} \neq 0$ implies that $W_{\zeta_1\xi_2} \otimes \Lambda_{12} \subseteq \bar{L}_1$ (except possibly in the case when M_1 and M_2 are degenerate). So let $\Lambda_{12} \neq 0$. Then $\xi_1 = \zeta_2$. We shall use Lemma 5.5, so we need to find $x \in L$ with nonzero projection to $W_{\zeta_1\xi_2} \otimes \Lambda_A(\zeta_1, \xi_2)$. Let $x_i = X_i \otimes \Lambda_i + X_i^\tau \otimes (-\Lambda_i^\rho) \in M_i$ with $X_i \in W'_{\zeta_i\xi_i}$. Then

$$x = [x_1, x_2] = \sum_{\mu,\nu \in \{1,\tau\}} X_1^\mu X_2^\nu \otimes \Lambda_1^\mu \Lambda_2^\nu - \sum_{\mu,\nu \in \{1,\tau\}} X_2^\mu X_1^\nu \otimes \Lambda_2^\mu \Lambda_1^\nu \in L$$

where $X_i^1 = X_i$, $\Lambda_i^1 = \Lambda_i$, and $\Lambda_i^\tau = -\Lambda_i^\rho$. Recall that $(X_iX_j)^\tau = X_j^\tau X_i^\tau$ and $(\Lambda_i\Lambda_j)^\rho = \Lambda_j^\rho \Lambda_i^\rho$. Therefore

$$\begin{aligned} x &= (X_{12} \otimes \Lambda_{12} - X_{12}^\tau \otimes \Lambda_{12}^\rho) - (X_{21} \otimes \Lambda_{21} - X_{21}^\tau \otimes \Lambda_{21}^\rho) \\ &\quad -(Y_{12} \otimes \Gamma_{12} - Y_{12}^\tau \otimes \Gamma_{12}^\rho) + (Y_{21} \otimes \Gamma_{21} - Y_{21}^\tau \otimes \Gamma_{21}^\rho) \end{aligned}$$

where $X_{ij} = X_iX_j$, $\Lambda_{ij} = \Lambda_i\Lambda_j$, $Y_{12} = X_1X_2^\tau$, $Y_{21} = X_2^\tau X_1$, $\Gamma_{12} = \Lambda_1\Lambda_2^\rho$, and $\Gamma_{21} = \Lambda_2^\rho\Lambda_1$.

Consider the following cases.

Case 1: $\bar{\zeta}_1 = \bar{\xi}_1 = \bar{\zeta}_2 = \bar{\xi}_2 = l$. Then $W_{\zeta_1\xi_1}$ and $W_{\zeta_2\xi_2}$ are the sets of $n_l \times n_l$-matrices with $n_l \geq 3$ ($n_l \geq 6$ if $l \in \bar{\Omega}_C$), and either $\tau = t$ (for $l \in \bar{\Omega}_A \cup \bar{\Omega}_B$) or $\tau = \sigma$ (for $l \in \bar{\Omega}_C$). Set $X_1 = e_{12}$ and $X_2 = e_{23}$. Then either $X_2^\tau = X_2^t = e_{32}$, or $l \in \bar{\Omega}_C$ and $X_2^\tau = X_2^\sigma = e_{m+3,m+2}$ where $m = n_l/2 \geq 3$ (see (1)). One checks that $X_{21} = Y_{12} = Y_{21} = 0$, $X_{12} = e_{13}$, and $X_{12}^\tau = e_{13}^\tau = e_{31}$ or $e_{m+3,m+1}$. It follows that $x = e_{13} \otimes \Lambda_{12} - e_{13}^\tau \otimes \Lambda_{12}^\rho \in L$. Since I, e_{13}, and e_{13}^τ are linearly independent, by Lemma 5.5, $W_{\zeta_1\xi_2} \otimes \Lambda_{12} \subseteq \bar{L}_1$.

Case 2: $\bar{\zeta}_1 \neq \bar{\xi}_1 = \bar{\zeta}_2 = \bar{\xi}_2 = l$. Note that $l \neq 0$. Therefore one can set $X_1 = e_{11}$ and $X_2 = e_{12}$. Then $X_2^\tau = e_{21}, e_{m+2,m+1}$. As $\Lambda_{21} = \Gamma_{21} = 0$ and $Y_{12} = 0$, we have $x = e_{12} \otimes \Lambda_{12} - e_{12}^\tau \otimes \Lambda_{12}^\rho$ and can apply Lemma 5.5.

Case 3: $\bar{\zeta}_1 = \bar{\xi}_1 = \bar{\zeta}_2 \neq \bar{\xi}_2$. The arguments are as in Case 2, setting $X_1 = e_{21}$ and $X_2 = e_{11}$.

Case 4: $\bar{\zeta}_1 \neq \bar{\xi}_1 = \bar{\zeta}_2 \neq \bar{\xi}_2$ and $\bar{\zeta}_1 \neq \bar{\xi}_2$. Then $\Lambda_{21} = \Gamma_{12} = \Gamma_{21} = 0$. Set $X_1 = X_2 = e_{11}$. Note that $\Lambda_{12} \in \Lambda(\zeta_1, \xi_2)$ and $\Lambda_{12}^\rho \in \Lambda(\xi_2^*, \zeta_1^*)$. Since $\bar{\zeta}_1 \neq \bar{\xi}_2$, the projection of x to $W_{\zeta_1\xi_2} \otimes \Lambda_A(\zeta_1, \xi_2)$ is $e_{11} \otimes \Lambda_{12}$ and we can apply Lemma 5.5.

Case 5: $\bar{\xi}_1 = \bar{\zeta}_2 \neq \bar{\zeta}_1 = \bar{\xi}_2 \neq 0$. Set $X_1 = e_{11}$ and $X_2 = e_{12}$. Then $\Gamma_{12} = \Gamma_{21} = 0$, $X_{21} = 0$, and we have $x = e_{12} \otimes \Lambda_{12} - e_{12}^\tau \otimes \Lambda_{12}^\rho$, as required.

Case 6: $\bar{\xi}_1 = \bar{\zeta}_2 \neq 0$ and $\bar{\zeta}_1 = \bar{\xi}_2 = 0$ (exceptional case). Then $\Gamma_{12} = \Gamma_{21} = 0$ and we have

$$x = (X_{12} \otimes \Lambda_{12} - X_{12}^\tau \otimes \Lambda_{12}^\rho) - (X_{21} \otimes \Lambda_{21} - X_{21}^\tau \otimes \Lambda_{21}^\rho).$$

Setting $X_1 = e_{11}$ and $X_2 = e_{21}$, and applying Lemma 5.5, we see that the second summand always belongs to $\mathfrak{u}^*(\bar{L}_1)$. Setting $X_1 = X_2 = e_{11}$, we conclude that

$$P = \mathbb{F}(I \otimes \Lambda_{12} - I \otimes \Lambda_{12}^\rho) \subset \mathfrak{u}^*(\bar{L}_1) + R_0.$$

It remains to note that $\mathfrak{u}^*(\bar{M}_1^l \bar{M}_2^r + \bar{M}_2^l \bar{M}_1^r) = P$. The lemma follows. □

Set

$$\mathcal{E}_0 = \sum \bar{M}_1^l \bar{M}_2^r$$

where M_1 and M_2 run over all pairs of degenerate submodules of L_1.

Lemma 5.8 *Let $M_1, \ldots, M_d$ be nontrivial irreducible submodules of L. Then $\bar{M}_1 \ldots \bar{M}_d \subseteq \bar{L}_1 + \mathcal{E}_0$.*

Proof. Proceed by induction on d, the case $d = 1$ being trivial. The case $d = 2$ immediately follows from Lemma 5.7. Let $d \geq 3$. Assume that the module M_2 is nondegenerate. Then by Lemma 5.7, $\bar{M}_1 \bar{M}_2$ is a sum of $\bar{Q}_j$ for some nontrivial irreducible submodules Q_j of L, so the lemma follows by inductive hypothesis. Therefore one can assume that M_2 is degenerate. Then $\bar{M}_2 = \bar{M}_2^l + \bar{M}_2^r$ and by Lemma 5.7, $M_1 M_2^l \subseteq \bar{L}_1$ and $\bar{M}_2^r M_3 \subseteq \bar{L}_1$, so we can apply the induction. The lemma follows. □

For completeness we include the proof of the following lemma from [2].

Lemma 5.9 *Let L be a finite dimensional perfect Lie algebra, S a Levi subalgebra of L. Then L is generated as a Lie algebra by nontrivial irreducible S-submodules of L.*

Proof. Express L in the form $L = S \oplus R$ where $R = \operatorname{Rad} L$. Pick irreducible S-submodules $M_1, \ldots, M_k$ of R such that $R = M_1 \oplus \ldots \oplus M_k \oplus [R, R]$. Since L is perfect, all M_i are nontrivial and R is nilpotent. As R is nilpotent, the vector space $M_1 + \ldots + M_k$ generates R as an algebra (see [4, ch. I, §4, exercise 4]). Therefore, the simple components of S and $M_1, \ldots, M_k$ generate L. □

Proposition 5.10 *We have*

(i) $\mathcal{E}(L) = \bar{L}_1 + \mathcal{E}_0$;

(ii) $\mathfrak{u}^*(\mathcal{E}(L)) = \mathfrak{u}^*(\bar{L}_1) + R_0$;

(iii) $\mathfrak{su}^*(\mathcal{E}(L)) = L$.

Proof. (i). Set $B = \bar{L}_1 + \mathcal{E}_0$. By Lemma 4.4, for each nontrivial irreducible submodule M of L we have $\bar{M} \subset \mathcal{E}(L)$. Therefore $B \subset \mathcal{E}(L)$. By Lemma 5.9, L is generated by nontrivial irreducible S-submodules, hence $\mathcal{E}(L)$ does. Thus by Lemma 5.8, $\mathcal{E}(L) \subseteq B$, so $\mathcal{E}(L) = B$.

(ii). By Lemma 5.7, $\mathfrak{u}^*(\mathcal{E}_0) \subseteq \mathfrak{u}^*(\bar{L}_1) + R_0$. Thus

$$\mathfrak{u}^*(\mathcal{E}(L)) = \mathfrak{u}^*(\bar{L}_1) + \mathfrak{u}^*(\mathcal{E}_0) \subseteq \mathfrak{u}^*(\bar{L}_1) + R_0 \subseteq \mathfrak{u}^*(\mathcal{E}(L)).$$

Therefore $\mathfrak{u}^*(\mathcal{E}(L)) = \mathfrak{u}^*(\bar{L}_1) + R_0$.

(iii). Since L is perfect, it suffices to show that the following inclusions hold:

$$[\mathfrak{u}^*(\bar{L}_1), \mathfrak{u}^*(\bar{L}_1)] \subseteq L, \tag{8}$$

$$[\mathfrak{u}^*(\bar{L}_1), R_0] \subseteq L. \tag{9}$$

Assume that (8) holds. We are going to prove (9), i.e. to show that $[\mathfrak{u}^*(\bar{M}), R_0] \subseteq L$ for each nontrivial irreducible submodule M of L. Let (ζ, ξ) be the type of M. If $\zeta \neq \xi$, then by Lemma 5.4, $\mathfrak{u}^*(\bar{M}) = M$ and $[M, R_0] \subseteq L$, as required. So one can suppose that $\zeta = \xi \neq 0$. Note that any element $x \in \mathcal{E}_0$ can be expressed in the form $x = X \otimes \Lambda$ where $X \in \mathbb{F}$ and $\Lambda \in \Lambda(0, 0)$. Hence $x\bar{M} = \bar{M}x = 0$, so $[\bar{M}, \mathcal{E}_0] = 0$. Therefore we have

$$[\mathfrak{u}^*(\bar{M}), R_0] \subseteq [\mathfrak{u}^*(\bar{M}), \mathfrak{u}^*(\bar{L}_1) + \mathcal{E}_0] \subseteq [\mathfrak{u}^*(\bar{M}), \mathfrak{u}^*(\bar{L}_1)] \subseteq L,$$

as required.

It remains to prove (8), i.e. to show that $[\mathfrak{u}^*(\bar{M_1}), \mathfrak{u}^*(\bar{M_2})] \subseteq L$ for all nontrivial irreducible M_1 and M_2. Let (ζ_i, ξ_i) be the type of M_i. If $\zeta_i \neq \xi_i$ for each i, then $\mathfrak{u}^*(\bar{M_i}) = M_i$, and we are done. So one can assume that $\zeta_1 = \xi_1 = \zeta \neq 0$. By Lemma 5.7,

$$[\mathfrak{u}^*(\bar{M_1}), \mathfrak{u}^*(\bar{M_2})] \subseteq \mathfrak{u}^*(\bar{M_1}\bar{M_2} + \bar{M_2}\bar{M_1}) = \mathfrak{u}^*(\bar{Q}_1 + \ldots + \bar{Q}_d)$$

for some nontrivial irreducible S-submodules Q_j of L. Suppose that $\zeta_2 \neq \zeta$, or $\xi_2 \neq \zeta$. Then by Lemma 5.4, $\mathfrak{u}^*(\bar{Q_i}) = Q_i \subseteq L$, so $[\mathfrak{u}^*(\bar{M_1}), \mathfrak{u}^*(\bar{M_2})] \subseteq L$. Hence we assume that $\zeta_1 = \xi_1 = \zeta_2 = \xi_2 = \zeta$. Recall that $M_i = \{X \otimes \Lambda_i + X^\tau \otimes (-\Lambda_i^\rho) \mid X \in W'_{\zeta\zeta}\}$ $(i = 1, 2)$ for some $\Lambda_i \in \Lambda_A(\zeta, \zeta)$. Set

$$I_i = I \otimes \Lambda_i - I \otimes \Lambda_i^\rho.$$

where I is the identity matrix. Note that $I_i \in \mathfrak{u}^*(\bar{M_i})$. If $\zeta \in \Omega_B \cup \Omega_C$, then by Lemma 5.3, $M_i \cong W^{\pm}_{\zeta\zeta}$ and $\Lambda_i^\rho = \mp\Lambda_i$, so either $I_i = 0$ (for $M_i \cong W^-_{\zeta\zeta}$) or $I_i = 2I \otimes \Lambda_i$ (for $M_i \cong W^+_{\zeta\zeta}$). By Lemma 5.4, we have $\mathfrak{u}^*(\bar{M_i}) = M_i + \mathbb{F}I_i$, so

$$[\mathfrak{u}^*(\bar{M_1}), \mathfrak{u}^*(\bar{M_2})] = [M_1, M_2] + [M_1, I_2] + [M_2, I_1] + \mathbb{F}[I_1, I_2].$$

Clearly $[M_1, M_2] \subseteq L$. We now show that $P = [M_1, I_2] \subseteq L$. One can assume that $I_2 \neq 0$ and $P \neq 0$. Since S acts trivially on I_2, the S-module P is a homomorphic image of M_1 (the homomorphism is given by the map $x \mapsto [x, I_2]$, $x \in M_1$). As M_1 is irreducible and $P \neq 0$, we have $M_1 \cong P$. By Lemma 5.7, P is a submodule of $\bar{M}_1\bar{M}_2 + \bar{M}_2\bar{M}_1 \subseteq \bar{L}_1$. As $I_2^\alpha = -I_2$ and $M_1 \subset \mathfrak{u}^*(\bar{L}_1)$, we have $P \subseteq \mathfrak{u}^*(\bar{L}_1)$. Since P is nontrivial and irreducible, $P \subseteq L$, as required. Similarly, $[M_2, I_1] \subseteq L$.

It remains to check that $[I_1, I_2] \subset L$. Recall that $n_\zeta = \dim W_\zeta$. Set $m = n_\zeta$ if $\zeta \notin \Omega_C$, and $m = n_\zeta/2$, otherwise. By our assumptions, $m \geq 3$. Fix any nonzero $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that $\sum \alpha_i = 0$ and $\sum \alpha_i^{-1} = 0$ (as $\mathbb{F}$ is algebraically closed, one can take all mth roots of unity). Set $X_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_m)$ and $X_2 = \mathrm{diag}(\alpha_1^{-1}, \ldots, \alpha_m^{-1})$ if $\zeta \notin \Omega_C$; and $X_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_m, \alpha_1, \ldots, \alpha_m)$ and $X_2 = \mathrm{diag}(\alpha_1^{-1}, \ldots, \alpha_m^{-1}, \alpha_1^{-1}, \ldots, \alpha_m^{-1})$ otherwise. Then $X_1^\tau = X_1$, $X_2^\tau = X_2$, and $X_1X_2 = X_2X_1 = I$. Since X_1 and X_2 have zero traces, by Lemma 5.3, the element $x_i = X_i \otimes \Lambda_i - X_i \otimes \Lambda_i^\rho$ lies in M_i for $i = 1, 2$. It remains to note that $[I_1, I_2] = [x_1, x_2] \in L$. □

6 Main results

Let A be a finite dimensional associative algebra. We say that A is *perfect* if $A^2 = A$. Set $\bar{A} = A$ if A has the identity $\mathbf{1}_A$ and $\bar{A} = A \oplus \mathbb{F}\mathbf{1}_{\bar{A}}$ (the algebra that is obtained from A by external adjointing the identity) otherwise. If A is an enveloping algebra of a Lie algebra L, then the algebra $\bar{A}$ can be viewed as a faithful L-module ${}_L\bar{A}$ (under the regular action). Note that the enveloping algebra of the image of L in the algebra of transformations of the space $\bar{A}$ is isomorphic to A.

Lemma 6.1 *Let A be a finite dimensional associative algebra with involution and L be a Lie subalgebra of $\mathfrak{su}^*(A)$. Then there is a finite dimensional vector space $V(A)$ such that $\bar{A}$ is isomorphic to a subalgebra of* $\mathrm{End}\, V(A)$ *with the same identity, and the L-module $V(A)$ is selfdual.*

Proof. Since $L \subseteq \mathfrak{su}^*(A) \subseteq \mathfrak{su}^*(\bar{A})$, the kernel K of the map $\mathcal{U}(L) \to \bar{A}$ is an α-invariant ideal of $\mathcal{U}(L)$. Therefore by Lemma 2.7,

$$\mathrm{Ann}_{\mathcal{U}(L)}({}_L\bar{A}) = K = K^\alpha = \mathrm{Ann}_{\mathcal{U}(L)}({}_L\bar{A})^*.$$

Consider an L-module $V(A) = {}_L\bar{A} \oplus ({}_L\bar{A})^*$. Then A is the enveloping algebra of L in $\mathrm{End}\, V(A)$. Moreover, $\mathbf{1}_{\bar{A}}$ acts identically on $V(A)$. Clearly, $V(A)$ is selfdual. □

Definition 6.2 Let A be a finite dimensional associative algebra with involution α. For a $*$-invariant ideal M of A set $Q_M = A/M$ and denote by α_M the involution of Q_M induced by α. The algebra A is called *admissible* if A is perfect and for each maximal $*$-invariant ideal M of A one of the following holds (below $d = \sqrt{\dim Q_M}$):

(1) Q_M is not simple and $d' = d/\sqrt{2} \geq 4$, i.e. $\mathfrak{u}^*(Q_M) \cong \mathfrak{gl}_{d'}$;

(2) Q_M is simple, $d \geq 7$, and α_M is orthogonal, i.e. $\mathfrak{u}^*(Q_M) \cong \mathfrak{so}_d$;

(3) Q_M is simple, $d \geq 6$, and α_M is symplectic, i.e. $\mathfrak{u}^*(Q_M) \cong \mathfrak{sp}_d$.

Theorem 6.3 *Let A be an admissible associative algebra with involution. Then $L = \mathfrak{su}^*(A)$ is a perfect Lie algebra, $V(A)$ is a faithful strongly $*$-plain L-module, and L generates A.*

Proof. Let $n = \dim V(A)$. By Lemma 6.1, $\bar{A}$ is a subalgebra of $\operatorname{End} V(A) \cong M_n(\mathbb{F})$ with the same identity. By Lemma 2.4 there exists a Levi subalgebra D of $\bar{A}$ such that $D^\alpha = D$. Let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of D. Since A is admissible, D contains at most one component D_0 of dimension 1. Clearly, $D' = \bigoplus_{\zeta \neq 0} D_\zeta$ is an α-invariant Levi subalgebra of A. It follows from Lemmas 2.3 and 2.4 that $S = \mathfrak{su}^*(D) = \mathfrak{su}^*(D')$ is a Levi subalgebra of $\mathfrak{su}^*(A)$ and the S-module $V(A)$ is $*$-plain. Moreover, since D' is admissible, $V(A)$ is strongly $*$-plain.

It remains to show that L is perfect and generates A. Using Proposition 3.9, we get that

$$A = \bigoplus_{\zeta,\xi \in \Omega} W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$$

and

$$\mathfrak{u}^*(A) = \sum_{\zeta,\xi \in \Omega} \sum_{\Lambda \in \Lambda_A(\zeta,\xi)} M^\Lambda_{\zeta\xi}$$

where $M^\Lambda_{\zeta\xi} = \{X \otimes \Lambda - X^\tau \otimes \Lambda^\rho \mid X \in W_{\zeta\xi}\}$. Set $(M'_{\zeta\xi})^\Lambda = \{X \otimes \Lambda - X^\tau \otimes \Lambda^\rho \mid X \in W'_{\zeta\xi}\}$. Let L' be a Lie subalgebra of $\mathfrak{u}^*(A)$ generated by all $(M'_{\zeta\xi})^\Lambda$ with $(\zeta, \xi) \neq (0,0)$ and $\Lambda \in \Lambda_A(\zeta, \xi)$. One can easily check that L' contains $S = \mathfrak{su}^*(D')$ and is perfect. Let A' be the enveloping algebra of L' in $M_n(\mathbb{F})$. Then by Lemma 4.4, A' is generated by all $W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$ with $(\zeta, \xi) \neq (0,0)$. Therefore A' is an ideal of A. Since A' contains the Levi subalgebra D', A/A' is nilpotent. As A is admissible, $A' = A$. By Proposition 5.10, $L = \mathfrak{su}^*(A) = L'$, so L is perfect and generates A, as required. □

Proof of Theorem 1.3. (1) Let A be a $\mathfrak{P}^*$-enveloping of L, i.e. $\mathfrak{su}^*(A) = L$. Clearly, A is perfect. Arguing as in the first part of the proof of Theorem 6.3, one can easily show that $V(A)$ is a faithful $*$-plain L-module.

(2) This is proved in Proposition 5.10(iii). □

Proposition 6.4 *Let L be a strongly $*$-plain Lie algebra and let A be a $\mathfrak{P}^*$-enveloping of L. Let H be a $*$-invariant ideal of A such that $H \cap L \subseteq Z(L)$. Then $HA = AH = 0$, i.e. $H \subseteq \operatorname{Null}(A)$.*

Proof. Since, $\mathfrak{su}^*(A) = L$, we have $[L, \mathfrak{u}^*(H)] \subseteq L \cap \mathfrak{u}^*(H) = L \cap H \subseteq Z(L)$. As $[L, L] = L$, we have

$$[L, \mathfrak{u}^*(H)] = [[L, L], \mathfrak{u}^*(H)] \subseteq [L, [L, \mathfrak{u}^*(H)] \subseteq [L, Z(L)]] = 0$$

i.e., L commutes with $\mathfrak{u}^*(H)$. Hence A commutes with $\mathfrak{u}^*(H)$. As in the proof of Theorem 6.3 we view A as a subalgebra of $M_n(\mathbb{F})$ for some n. Let D' be a $*$-invariant Levi subalgebra of A and $D = D' + \mathbb{F}\mathbf{1}_n$. Clearly, D commutes with $\mathfrak{u}^*(H)$. As H is an ideal of A, $DHD = H$. Therefore, by Proposition 3.5, $H = \bigoplus_{\zeta,\xi \in \Omega} W_{\zeta\xi} \otimes \Lambda_H(\zeta, \xi)$. Let $S = \mathfrak{su}^*(D)$. Obviously, if $\Lambda_H(\zeta, \xi) \neq 0$ for some $(\zeta,\xi)\neq(0,0)$, then $[S,\mathfrak{u}^*(H)]\neq 0$. Therefore $H=W_{00}\otimes\Lambda_H(0,0)$. Let $\Lambda\in\Lambda_A(\zeta,\xi)$ with $(\zeta,\xi)\neq(0,0)$. Since H is an ideal of A,

$$\Lambda\Lambda_H(0,0)=\Lambda_H(0,0)\Lambda=0.$$

As A is generated by $W_{\zeta\xi}\otimes\Lambda_A(\zeta,\xi)$ with $(\zeta,\xi)\neq(0,0)$, we have $AH=HA=0$, i.e. $H\subseteq\mathrm{Null}(A)$, as required. $\square$

We denote by $\mathfrak{P}^*(L)$ the poset of $\mathfrak{P}^*$-enveloping algebras of L under the ordering "$\leq$" (see Section 2). The following theorem describes $\mathfrak{P}^*(L)$ for strongly $*$-plain Lie algebras L.

Theorem 6.5 *Let L be a strongly $*$-plain Lie algebra. Then the class $\mathfrak{P}^*(L)$ contains a universal object $\mathcal{N}(L)$. More precisely, there is an algebra $\mathcal{N}(L)\in\mathfrak{P}^*(L)$ such that for each $A\in\mathfrak{P}^*(L)$ one has $A\leq\mathcal{N}(L)$, the kernel H_A of the corresponding homomorphism $\mathcal{N}(L)\to A$ is a $*$-invariant subspace of* $\mathrm{Null}(\mathcal{N}(L))$*, and $H_A\cap L=0$. In particular, the poset $\mathfrak{P}^*(L)$ is antiisomorphic to the poset of all $*$-invariant subspaces $H\subseteq\mathrm{Null}(\mathcal{N}(L))$ with $H\cap L=0$.*

Proof. In the proof of Theorem 1.3 it has been actually shown that for any strongly $*$-plain Lie algebra L the class of all $\mathfrak{P}^*$-envelopings for L coincides with that of all envelopings in $\mathrm{End}\,V$ where V runs over all strongly $*$-plain modules for L. Denote by $\mathrm{Irr}\,L$ the set of all nontrivial irreducible L-modules (up to equivalence), and by $\mathrm{Irr}(V)$ the set of inequivalent nontrivial composition factors of an L-module V. Since the algebras $\mathfrak{sl}_n$ $(n\geq 4)$, $\mathfrak{so}_n$ $(n\geq 7)$, $\mathfrak{sp}_n$ $(n\geq 6)$ are pairwise non-isomorphic, there is a finite subset $\Phi\subset\mathrm{Irr}\,L$ such that $\mathrm{Irr}(V)=\Phi$ for all strongly $*$-plain L-modules V. Let $\mathcal{N}=\mathcal{N}(L)$ be the universal enveloping of L corresponding to Φ (see [2, Corollary 6.2]) and let W be an L-module such that $\mathcal{N}$ is the enveloping of L in $\mathrm{End}\,W$. Set $M=W\oplus W^*$. Since $\mathrm{Irr}(M)=\Phi$, $\mathcal{N}$ is the enveloping of L in $\mathrm{End}\,M$. As M is strongly $*$-plain, $\mathcal{N}$ is a $\mathfrak{P}^*$-enveloping. Let $A\in\mathfrak{P}^*(L)$ and H_A be the kernel of the corresponding homomorphism $\mathcal{N}(L)\to A$. Then $H_A\cap L=0$. Therefore by Proposition 6.4, $H_A\subseteq\mathrm{Null}(\mathcal{N}(L))$. $\square$

Proof of Theorem 1.5. By Theorem 6.3, L is strongly $*$-plain. Since A_1 and A_2 are $\mathfrak{P}^*$-enveloping algebras for L, the result follows from Theorem 6.5. $\square$

Remark 6.6 The poset $\mathfrak{P}^*(L)$ may not contain the smallest element.

Let L be a perfect finite dimensional Lie algebra over $\mathbb{F}$. Let A be a finite dimensional enveloping algebra of L. We say that A is $\mathfrak{P}$-*enveloping* for L if $L=[A,A]$. The algebra L is called *quasispecial* if it has a $\mathfrak{P}$-enveloping algebra. Let V be an L-module of finite length. Let $W_1,\dots,W_k$ list the nontrivial composition factors of V. We say that V is *plain* if for each $i=1,\dots,k$ we have $L/\mathrm{Ann}_L(W_i)\cong\mathfrak{sl}(W_i)$ and the dual L-module W_i^* does not occur in $W_1,\dots,W_k$ unless $\dim W_i=2$. We say that L is *plain* if L has a faithful plain module. Quasispecial Lie algebras are studied in [2]. In particular, it is shown there that a perfect Lie algebra having no quotient isomorphic to $\mathfrak{sl}_2$ is quasispecial if and only if it is plain. Two theorems below show the relation between quasispecial (plain) and quasiclassical ($*$-plain) Lie algebras.

Theorem 6.7 *Let L be a plain Lie algebra. Then L is $*$-plain. Moreover, if L has no quotient isomorphic to $\mathfrak{sl}_2$ or $\mathfrak{sl}_3$, then L is strongly $*$-plain.*

Proof. Let V be a faithful plain L-module. Then $V\oplus V^*$ is a $*$-plain L-module, so L is $*$-plain. Moreover, L is strongly $*$-plain if L has no quotient isomorphic to $\mathfrak{sl}_2$ or $\mathfrak{sl}_3$. $\square$

Theorem 6.8 *Let L be a strongly $*$-plain Lie algebra. Then L is plain if and only if there exists a $\mathfrak{P}^*$-enveloping algebra A of L such that $A = B \oplus B^*$ where B is an ideal of A. In that case B is a $\mathfrak{P}$-enveloping algebra of L and for each $\mathfrak{P}$-enveloping algebra C of L the algebra $C \oplus C^{\mathrm{op}}$ with involution $* : (c_1, c_2) \mapsto (c_2, c_1)$ is $\mathfrak{P}^*$-enveloping for L.*

Proof. Assume that L is plain. Observe that L is strongly plain. Let C be a $\mathfrak{P}$-enveloping algebra of L. Set $A = C \oplus C^{\mathrm{op}}$. Then the map $* : (c_1, c_2) \mapsto (c_2, c_1)$ is an involution of A and $L \cong \mathfrak{su}^*(A)$. Observe that A is admissible. Therefore by Theorem 6.3, A is a $\mathfrak{P}^*$-enveloping algebra of L.

Conversely, let A be a $\mathfrak{P}^*$-enveloping algebra of L such that $A = B \oplus B^*$ where B is an ideal of A. Then, obviously, $L = \mathfrak{su}^*(A) \cong [B, B] = \tilde{L}$. Moreover, $\tilde{L}$ generates B, so B is a $\mathfrak{P}$-enveloping algebra of L and L is plain. □

7 Quasiclassical algebraic groups

Let A be an associative algebra. Set $\bar{A} = A$ if $\mathbf{1}_A \in A$ and $\bar{A} = A \oplus \mathbb{F}\mathbf{1}_{\bar{A}}$ (the algebra that is obtained from A by external adjointing the identity) otherwise. If $\mathbf{1}_A \in A$, we denote by $U(A)$ the group of invertible elements of A. Since the external adjointing the identity to A does not change the commutator subgroup $U(A)'$ of $U(A)$ we can define $U(A)'$ even if A has no the identity.

Assume that A has an involution. Clearly, the involution of A uniquely extends to an involution of $\bar{A}$. If $\mathbf{1}_A \in A$, we denote by $U^*(A) = \{a \in A \mid aa^* = \mathbf{1}_A\}$ the *unitary group* of A and by $SU^*(A)$ the *special unitary group* of A, which is defined to be the commutator subgroup of $U^*(A)$. As above, the external adjointing the identity to A does not change $SU^*(A)$. This allows us to define $SU^*(A)$ even if A has no identity by setting $SU^*(A) = SU^*(\bar{A})$.

Lemma 7.1 *Let A be a finite dimensional associative $\mathbb{F}$-algebra and let A_0 be the smallest ideal of A such that the quotient A/A_0 is semisimple and each component of A/A_0 has dimension 1. Then $U(A)' = U(A_0)'$. Moreover, if A has an involution, then A_0 is $*$-invariant and $SU^*(A) = SU^*(A_0)$.*

Proof. This can be easily proved fixing a ($*$-invariant) Levi subalgebra of A (see Lemma 2.4). □

We say that an associative algebra A is *reduced* if $A = A_0$, i.e. A has no simple 1-dimensional quotient. By Lemma 7.1, $U(A)' = U(A_0)'$ and $SU^*(A) = SU^*(A_0)$.

Let G be a perfect group, A be a reduced algebra, and

$$\iota : G \to U(A)'$$

be a monomorphism. The pair (ι, A) is called an *enveloping algebra* for G if $\iota(G)$ generates $\bar{A}$ as an algebra. In view of the universal property, the map ι uniquely extends to an associative algebra homomorphism

$$\bar{\iota} : \mathbb{F}G \to \bar{A}.$$

Let $\varphi : G \to \operatorname{End} V$ be a faithful representation of a perfect group G and let A be a subalgebra of $\operatorname{End} V$ generated by $\varphi(G)$. Then one can easily check that $A = A_0 + \mathbb{F}I$ where I is the identity transformation of V. The algebra A_0 is called the enveloping algebra of G in $\operatorname{End} V$. Conversely, let B be an enveloping algebra for a perfect group G. The algebra $\bar{B}$ can be viewed as an G-module ${}_G\bar{B}$ (under the regular action). One can check that the enveloping algebra of the image of G in $\operatorname{End} \bar{B}$ is isomorphic to B.

An enveloping algebra A of a perfect group G is called $\mathfrak{P}^*$-*enveloping* if A has an involution such that

$$G = SU^*(A).$$

We denote by $\mathfrak{P}^*(G)$ the set of $\mathfrak{P}^*$-enveloping algebras for a group G. Let V be a faithful rational finite dimensional G-module isomorphic to its dual module V^*. We say that V is *$*$-plain* if for each nontrivial composition factor W of V the image of G in $GL(W)$ is one of the groups $SL(W)$, $SO(W)$, or $Sp(W)$. Moreover, if in addition $\dim W \geq 4$ for $SL(W)$, $\dim W \geq 7$ for $SO(W)$, and $\dim W \geq 6$ for $Sp(W)$, then V is called *strongly $*$-plain*. We say that G is *(strongly) $*$-plain* if G has a faithful (strongly) $*$-plain module.

Proof of Theorem 1.4. (1) Let A be a reduced algebra with involution such that $G \cong SU^*(A)$. It suffices to show that $V(A) = {}_G\bar{A} \oplus ({}_G\bar{A})^*$ is a faithful $*$-plain G-module. Arguing as in Lemma 6.1, we can view $\bar{A}$ as a subalgebra of $\operatorname{End} V(A) \cong M_n(\mathbb{F})$ with the same identity (here $n = \dim V(A)$). Let D be a $*$-invariant Levi subalgebra of $\bar{A}$ (see Lemma 2.4). Clearly, $U^*(D)$ is a direct product of groups isomorphic to GL, SO, and Sp (cf. Lemma 2.3). Let $N = \operatorname{Rad} \bar{A}$ be the radical of $\bar{A}$. As N is nilpotent and $*$-invariant, $U^*(N)$ is a unipotent normal subgroup of $U^*(\bar{A})$. Clearly, $U^*(\bar{A}) = U^*(D)U^*(N)$. Therefore $G = SU^*(A) = SU^*(\bar{A}) = SU^*(D)R$ where R is the unipotent radical of $SU^*(A)$. It remains to observe that $SU^*(D)$ is a $*$-plain subgroup of $GL(V)$ and $V(A)$ is a $*$-plain $SU^*(D)$-module.
(2) Let $\varphi : G \to GL(V)$ be a faithful strongly $*$-plain representation of G and let $W_1, \dots, W_n$ list the nontrivial composition factors of V. Let $\varphi_i : G \to GL(W_i)$ denote the restriction of φ to W_i and let $\tau_i : GL(W_i) \to PGL(W_i)$ be the natural projection. Then $G_i = G/\operatorname{Ker} \varphi_i$ is isomorphic to $SL(W_i)$ $SO(W_i)$, or $Sp(W_i)$, and $H_i = G_i/\operatorname{Ker} \tau_i \circ \varphi_i$ is isomorphic to $PSL(W_i)$, $PSO(W_i)$, or $PSp(W_i)$. Set $H = \cap_{i=1}^n \operatorname{Ker} \varphi_i$ and $M = \cap_{i=1}^n \operatorname{Ker} \tau_i \circ \varphi_i$. Then H acts trivially on each composition factor of V so H is unipotent, hence H is nilpotent. As $\operatorname{Ker} \tau_i$ is abelian, M/N is abelian so M is solvable. As G/M is a finite subdirect product of simple groups, it is in fact a direct product. Clearly, G/M have no non-trivial finite quotient. Therefore, if X is a finite index subgroup of G then $G = XM$. If $X \neq G$, then G/X is not solvable as G is perfect. Besides, $G/X \cong MX/X \cong M/(M \cap X)$. This is a contradiction as M is solvable. This means that G contains no proper subgroup of finite index. Therefore, G is connected.

Let A be the enveloping algebra of G in $\operatorname{End} V$ and let L be the Lie algebra of G. We identify L with the corresponding subalgebra of $\mathfrak{gl}(V)$. Since G is perfect, L is perfect as well. Observe that L is strongly $*$-plain and V is a $*$-plain L-module. Let B be the enveloping algebra of L in $\operatorname{End} V$. By Lemma 2.8, B admits standard involution and by Theorem 1.3(2), $L = \mathfrak{su}^*(B)$. Since G is connected, by [5, Corollary 2 of Theorem II.12.8], $\mathbb{F}I + B = \mathbb{F}I + A$ where I is the identity of $\operatorname{End} V$. Hence $L = \mathfrak{su}^*(A)$. Set $H = U^*(A + \mathbb{F}I)$. One can easily show that H is a connected algebraic group and the Lie algebra of H is $\mathfrak{u}^*(A + \mathbb{F}I)$. By [3, I.2.3], the commutator subgroup H' is a connected algebraic subgroup of H, and by [3, II.7.8], $L = [\mathfrak{u}^*(A + \mathbb{F}I), \mathfrak{u}^*(A + \mathbb{F}I)] = \mathfrak{su}^*(A)$ is the Lie algebra of H'. Since $H' \supseteq G$ are connected and have the same Lie algebras, by [3, II.7.1], $G = H' = SU^*(A)$, as required. $\square$

One can easily describe the properties of $\mathfrak{P}^*$-enveloping algebras for G in the spirit of Theorem 6.5. Observe that in Theorem 1.5 only reduced associative algebras are considered. We have the following analog of Theorem 1.5 for algebraic groups.

Theorem 7.2 *Let A_1 and A_2 be admissible finite dimensional associative algebras with involution. Assume that $SU^*(A_1) \cong SU^*(A_2)$. Then there is an admissible algebra A and $*$-invariant ideals $H_1, H_2 \subseteq \operatorname{Null}(A)$ such that for $i = 1, 2$ we have $SU^*(A) \cong SU^*(A_i)$ and $A_i \cong A/H_i$. In particular, $A_1/\operatorname{Null}(A_1)$ and $A_2/\operatorname{Null}(A_2)$ are isomorphic as algebras with involution.*

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