# Plain representations of Lie algebras

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## 1 Introduction

In this paper we study representations of finite dimensional Lie algebras. In this case representations are not necessarily completely reducible. As the general problem is known to be of enormous complexity, we restrict ourselves with representations that particularly well behave on Levi subalgebras. We call such representations plain (Definition 1.1). Informally, we show that the theory of plain representations of a given Lie algebra $L$ is equivalent to representation theory of finitely many finite dimensional associative algebras, also non-semisimple. The sense of this is to distinguish representations of Lie algebras that are of complexity comparable with that of representations of associative algebras. Non-plain representations are intrinsicly much more complex than plain ones. We view our work as a step toward understanding this complexity phenomenon.

We restrict ourselves also with perfect Lie algebras $L$, i.e. such that $L = [L, L]$. In main results we assume that $L$ is perfect and $\mathfrak{sl}_2$-free (which means that $L$ has no quotient isomorphic to $\mathfrak{sl}_2$). The ground field $\mathbb{F}$ is always assumed to be algebraically closed and of characteristic 0.

**Definition 1.1** Let $L$ be a perfect finite dimensional Lie algebra over $\mathbb{F}$ and let $V$ be an $L$-module of finite length. Let $W_1, \dots, W_k$ list the nontrivial composition factors of $V$. We say that $V$ is *plain* if for each $i = 1, \dots, k$ the following conditions hold:

(1) $L/\operatorname{Ann}_L(W_i) \cong \mathfrak{sl}(W_i)$;

(2) the dual $L$-module $W_i^*$ does not occur in $W_1, \dots, W_k$ unless $\dim W_i = 2$ where $W_i$ is selfdual.

We say that $L$ is *plain* if $L$ has a faithful plain module.

Observe that $V$ may have trivial (one-dimensional) composition factors.

**Theorem 1.2** *Let $L$ be a perfect $\mathfrak{sl}_2$-free finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then there exist finitely many finite dimensional associative algebras $A_i$ with $i = 1, \dots, n$ and representations $\eta_i : L \to A_i$ such that $[A_i, A_i] = \eta_i(L)$ and the restrictions of all finite dimensional $A_i$-modules to $L$ are plain and exhaust all plain $L$-modules.*

For an $L$-module $V$ let $\mathrm{Irr}(V)$ denote the set of all composition factors of $V$ (disregarding their multiplicities). For a plain $L$-module $V$ the choice of the corresponding algebra $A_i$ in Theorem 1.2 is determined by $\mathrm{Irr}(V)$, and then $n \leq 2^k$ where $k$ is the number of the simple components of $L/\operatorname{Rad} L$ (see Theorem 6.10 and the proof of Theorem 1.2 for more precise information).

Unexpectedly, $\eta_i(L)$ turns out to coincide with $[A_i, A_i]$, the commutator subalgebra of $A_i$ (under the bracket multiplication). Thanks to this, we have been able to characterize plain $\mathfrak{sl}_2$-free Lie algebras as those of shape $[A, A]$ where $A$ is some finite dimensional associative algebra.

**Definition 1.3** Let $L$ be a perfect (i.e. $[L, L] = L$) finite dimensional Lie algebra over $\mathbb{F}$. We say that $L$ is *quasispecial* if there exists an associative $\mathbb{F}$-algebra $A$ such that $L \cong [A, A]$.

**Theorem 1.4** *Let $L$ be a perfect finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $0$. Suppose that $\mathfrak{sl}_2$ is not a quotient algebra of $L$. Then $L$ is quasispecial if and only if there exists a faithful plain $L$-module.*

Quasispecial Lie algebras can be viewed as natural generalization of algebras $\mathfrak{sl}_n$ so they may be of independent interest.

A crucial part of the proof of Theorems 1.2 and 1.4 is the following result which seems to be of independent significance.

**Theorem 1.5** *Let $L \subset \operatorname{End} V$ be a perfect $\mathfrak{sl}_2$-free Lie algebra with $V$ being a plain $L$-module. Let $A$ be the enveloping algebra of $L$ in $\operatorname{End} V$. Then $L = [A, A]$.*

A priori, there is no hint that $L$ cannot be smaller than $[A, A]$ (and this can happen if $\mathfrak{sl}_2$ is a quotient of $L$). So we think that Theorem 1.5 is quite unexpected.

If $L$ is quasispecial, there can be (in general) infinitely many nonisomorphic plain $L$-modules of the same dimension. (This follows from the second Brauer-Thrall conjecture proved by Nazarova and Roiter [7], if one observes that the restriction to $L = [A, A]$ of any $A$-module is a plain $L$-module.) However, these possibilities are controlled in a sense by Theorem 1.5, so there is a deep relationship between plain representations of quasispecial Lie algebras and representation theory of associative algebras. We emphasize that the conditions for $V$ to be plain are described only in terms of the composition factors of $V$.

We also provide a multiplicative version of Theorem 1.4 by characterizing in similar terms perfect algebraic groups that are isomorphic to the commutator groups of the unit groups of associative algebras (with identity). As above, we call such groups *quasispecial*, and for a perfect algebraic group $G$ we define a *plain* rational $G$-module exactly as in Definition 1.1 (replacing $\mathfrak{sl}$ by $SL$).

**Theorem 1.6** *Let $G$ be a perfect algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic $0$. Suppose that $SL_2(\mathbb{F})$ is not a quotient group of $G$. Then $G$ is quasispecial if and only if there exists a faithful plain $G$-module.*

Let $A$ be an associative algebra (not necessarily containing the identity). Denote by $[A]$ the Lie algebra obtained from $A$ under the usual bracket multiplication. To what extent $A$ is determined by $[A]$? This problem was studied by Herstein for simple rings $A$ (see [5]) and by his successors for prime and semiprime rings (see [6] for instance). No result is known for $A$ not being semiprime. This problem can also be solved in the framework of the above approach for finite dimensional algebras $A$ having no non-zero quotient of dimension $\leq 4$. We show that (under these assumptions) the Lie algebra $L = [A, A]$ is perfect, $\mathfrak{sl}_2$-free, and generate $A$ (see Theorem 6.3). Moreover, it turns out that $A$ is determined by $L$ almost uniquely. To be more precize, let us introduce some notation. Set

$$\mathrm{Null}(A) = \{a \in A \mid aA = Aa = 0\},$$

i.e., $\mathrm{Null}(A)$ is the two-sided annihilator of $A$ in $A$. (Note that $\mathrm{Null}(A) = 0$ whenever $A$ has the identity). Let $\tilde{A} = A/\mathrm{Null}(A)$. We say that $A$ is weakly indecomposable if $\tilde{A}$ is not a direct sum of proper (two-sided) ideals. One can easily show (see Proposition 3.6) that any finite dimensional algebra $A$ with $A^2 = A$ can be uniquely expressed as $A_1 + \ldots + A_n$ where $A_1, \ldots, A_n$ are weakly indecomposable ideals, $A_i^2 = A_i$, and $A_iA_j = A_jA_i = 0$ for $i \neq j$. For an algebra $B$ we write $A \sim B$ if the weak components $B_1, \ldots, B_n$ of $B$ can be reordered such that $\tilde{B}_i \cong \tilde{A}_i$ or $\tilde{B}_i \cong \tilde{A}_i^{op}$ (the opposite algebra). If $A \sim B$ then we say that $B$ is *quasiisomorphic* to $A$. Observe that $A_i$ and the opposite algebra $A_i^{op}$ can be nonisomorphic while the Lie algebras $[A_i]$ and $[A_i^{op}]$ are always isomorphic.

**Theorem 1.7** *Let $A_1$ and $A_2$ be finite dimensional associative algebras. Assume that $[A_1, A_1] \cong [A_2, A_2]$ and each $A_1, A_2$ has no nontrivial homomorphism into $M_2(\mathbb{F})$. Then $A_1 \sim A_2$. Moreover, if $A_1$ or $A_2$ is weakly indecomposable, then there is an algebra $A$ and ideals $H_1, H_2 \subseteq$* $\mathrm{Null}(A)$ *such that for $i = 1, 2$ we have $[A, A] \cong [A_i, A_i]$ and $A_i \cong A/H_i$ or $A_i \cong (A/H_i)^{op}$.*

A similar result holds for the multiplicative groups of $A_i$ (see Theorem 6.14).

Let $L$ be a Lie algebra and let $A$ be an associative enveloping algebra of $L$ (i.e., $A$ contains $L$ and is generated by $L$). We say that $A$ is $\mathfrak{P}$-enveloping for $L$ if $[A, A] = L$. Does there exist a universal $\mathfrak{P}$-enveloping algebra for $L$? In general, the answer is "no", even for $\mathfrak{sl}_3$. This is because a direct sum of plain $L$-modules is not necessarily a plain module. However the set $\mathfrak{V}$ of plain $L$-modules is a disjoint union of finitely many subsets $\mathfrak{V}_1, \ldots, \mathfrak{V}_n$, which are closed under the direct sum operation. We show in Theorem 6.10 that if $L$ is plain and $\mathfrak{sl}_2$-free, then there exist $n$ quasiisomorphic $\mathfrak{P}$-enveloping algebras $U_1, \ldots, U_n$ of $L$ such that each plain $L$-module $V \in \mathfrak{V}$ extends to $U_i$-module. Thus, the theory of plain representations of $L$ reduces to the representation theory of algebras $U_1, \ldots, U_n$. If $L$ is weakly indecomposable, we prove the following more precise result.

**Theorem 1.8** *Let $L$ be a weakly indecomposable $sl_2$-free plain Lie algebra. Then there exists an associative algebra $A$ with the following properties:*

(1) *$A$ is a weakly indecomposable $\mathfrak{P}$-enveloping algebra of $L$;*

(2) *each $\mathfrak{P}$-enveloping algebra of $L$ is a homomorphic image of $A$ or $A^{op}$ with the kernel lying in* $\mathrm{Null}(A)$ *or* $\mathrm{Null}(A^{op})$*, respectively.*

Sections 2-4 contain some auxiliary material. The basic machinery is developed in Section 5 and Section 6 contains proofs of the main results.

# 2 Matrix algebras

Let $M_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices over $\mathbb{F}$ and $\mathbf{1}_n$ be its identity. Let $D \subseteq M_n(\mathbb{F})$ be a semisimple subalgebra containing $\mathbf{1}_n$. In this section we describe subalgebras $A$ of $M_n(\mathbb{F})$ such that $DAD = A$. Note that if $\mathbf{1}_n \in A$, then the condition $DAD = A$ is equivalent to $D \subseteq A$.

We denote by $V$ the natural module for $M_n(\mathbb{F})$, so $M_n(\mathbb{F}) \cong \operatorname{End}_{\mathbb{F}} V \cong V \otimes_{\mathbb{F}} V^*$. Since $D$ is semisimple,

$$V = V_1 \oplus \cdots \oplus V_t$$

where $V_1, \ldots, V_t$ are irreducible $D$-modules. Put $T = \{1, \ldots, t\}$. Let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of $D$, so

$$D = \bigoplus_{\zeta \in \Omega} D_\zeta,$$

and let $W_\zeta$ be the natural $D_\zeta$-module. Let us denote by $\nu$ the map $T \to \Omega$ such that for each $i \in T$ the $D$-modules $V_i$ and $W_{\nu(i)}$ are isomorphic. For a pair $(i, j) \in T \times T$ let $\nu(i, j)$ denote the pair $(\nu(i), \nu(j))$.

Set $n_\zeta = \dim W_\zeta$, so $n = n_{\nu(1)} + \cdots + n_{\nu(t)}$. For $\zeta, \xi \in \Omega$ we denote by $W_{\zeta\xi}$ the vector space $W_\zeta \otimes W_\xi^*$. For each $\zeta \in \Omega$ fix a base $B_\zeta$ of $W_\zeta$. Let $B_\zeta^*$ be the dual base of $W_\zeta^*$. With respect to the chosen bases each $W_{\zeta\xi}$ can be identified with the space of $n_\zeta \times n_\xi$-matrices. Set $n' = \max\{n_\zeta \mid \zeta \in \Omega\}$. It is convenient to identify each $W_{\zeta\xi}$ with a subspace of the matrix algebra $M_{n'}(\mathbb{F})$ extending each $n_\zeta \times n_\xi$ matrix $X \in W_{\zeta\xi}$ to an $n' \times n'$ matrix by 0's (so $X$ is located at the left upper corner of the extended matrix). Let $X_1 \in W_{\zeta_1\xi_1}$ and $X_2 \in W_{\zeta_2\xi_2}$. If $\xi_1 = \zeta_2$, define $X_1X_2$ as the product of the corresponding matrices in $M_{n'}(\mathbb{F})$. If $\xi_1 \neq \zeta_2$, set $X_1X_2 = 0$. Observe that if $\xi_1 = \zeta_2$, then $X_1X_2$ belongs to $W_{\zeta_1\xi_2}$ and the multiplication just defined agrees with the usual multiplication of elements from $W_{\zeta_1\xi_1}$ and $W_{\zeta_2\xi_2}$ for $\xi_1 = \zeta_2$:

$$(Y_1 \otimes \varphi_1)(Y_2 \otimes \varphi_2) = \varphi_1(Y_2)Y_1 \otimes \varphi_2 \in W_{\zeta_1\xi_2}$$

where $Y_i \in W_{\zeta_i}$ and $\varphi_i \in W_{\xi_i}^*$ for $i = 1, 2$. We get an algebra structure on the vector space

$$W = \bigoplus_{\zeta,\xi \in \Omega} W_{\zeta\xi}. \tag{1}$$

We denote by $\{e_{ij} \mid 1 \leq i, j \leq n'\}$ the standard basis of $M_{n'}(\mathbb{F})$ consisting of matrix units, so for each pair $\zeta, \xi \in \Omega$ the set $\{e_{ij} \mid 1 \leq i \leq n_\zeta,\ 1 \leq j \leq n_\xi\}$ is the basis of $W_{\zeta\xi}$. For $\zeta, \xi \in \Omega$ we denote by $\Lambda(\zeta, \xi)$ the vector space of all $t \times t$ matrices $\Lambda = (\lambda_{ij}) \in M_t(\mathbb{F})$ such that $\lambda_{ij} = 0$ if $\nu(i, j) \neq (\zeta, \xi)$. Observe that

$$M_t(\mathbb{F}) = \bigoplus_{\zeta,\xi \in \Omega} \Lambda(\zeta, \xi). \tag{2}$$

**Lemma 2.1** *Let $\Lambda_1 \in \Lambda(\zeta_1, \xi_1)$ and $\Lambda_2 \in \Lambda(\zeta_2, \xi_2)$. If $\Lambda_1\Lambda_2 \neq 0$, then $\xi_1 = \zeta_2$ and $\Lambda_1\Lambda_2 \in \Lambda(\zeta_1, \xi_2)$.*

*Proof.* Let $\Lambda_1 = (\lambda_{ij}^1)$ and $\Lambda_2 = (\lambda_{ij}^2)$. Then $(\Lambda_1\Lambda_2)_{ij} = \sum_{s=1}^t \lambda_{is}^1 \lambda_{sj}^2$. If $\xi_1 \neq \zeta_2$, then all the summands are zeros for all $i, j$, so $\Lambda_1\Lambda_2 = 0$. If $\nu(i) \neq \zeta_1$ or $\nu(j) \neq \xi_2$, then all the summands are zeros as well. Therefore $\Lambda_1\Lambda_2 \in \Lambda(\zeta_1, \xi_2)$. □

Let us define a vector space $\mathcal{M}_D$ as follows:

$$\mathcal{M}_D = \bigoplus_{\zeta,\xi \in \Omega} W_{\zeta\xi} \otimes_{\mathbb{F}} \Lambda(\zeta, \xi). \tag{3}$$

Using decompositions (1) and (2), the properties of multiplication in $W$, and Lemma 2.1, one can observe that $\mathcal{M}_D$ is a subalgebra of the algebra $W \otimes_{\mathbb{F}} M_t(\mathbb{F})$.

We assume that the standard basis $B$ in $V$ is such that $B \cap V_i$ is a basis in $V_i$ for each $i \in T$ and $\upsilon_i(B \cap V_i) = B_{\nu(i)}$ where $\upsilon_i$ is a $D$-module isomorphism from $V_i$ to $W_{\nu(i)}$. In particular, for each $d \in D$ the matrices $d|_{V_i}$ and $d|_{W_{\nu(i)}}$ coincide. The decomposition

$$\mathrm{End}_{\mathbb{F}} V = V \otimes V^* = \bigoplus_{i,j=1}^{t} V_i \otimes V_j^*$$

induce a block structure on the elements of $M_n(\mathbb{F})$, so we view $M_n(\mathbb{F})$ as the set of $t \times t$ block matrices. For $i, j \in T$ let $\pi_{ij}$ denote the projection of a matrix $X \in M_n(\mathbb{F})$ to its $(i, j)$-block submatrix. We identify $\pi_{ij}(X)$ with the corresponding matrix in $W_{\nu(i)\nu(j)}$. Let $\{\varepsilon_{ij} \mid 1 \leq i, j \leq t\}$ be the standard basis of $M_t(\mathbb{F})$ (consisting of matrix units). Observe that $\{\varepsilon_{ij} \mid \nu(i, j) = (\zeta, \xi)\}$ is a basis of $\Lambda(\zeta, \xi)$. We have a natural vector space isomorphism $\varphi : M_n(\mathbb{F}) \to \mathcal{M}_D$ defined by

$$\varphi(X) = \sum_{i,j \in T} \pi_{ij}(X) \otimes \varepsilon_{ij}.$$

**Proposition 2.2** *The map $\varphi$ is an algebra isomorphism of $M_n(\mathbb{F})$ onto $\mathcal{M}_D$.*

*Proof.* This can be easily verified using the standard bases of $M_n(\mathbb{F})$ and $\mathcal{M}_D$. $\square$

We identify the algebras $\mathcal{M}_D$ and $M_n(\mathbb{F})$. By Lemma 2.1, $\Lambda(\zeta, \zeta)$ is a subalgebra of $M_t(\mathbb{F})$. Set $\varepsilon_\zeta = \sum_{\nu(i)=\zeta} \varepsilon_{ii}$. Then for all $\zeta, \xi \in \Omega$ and all $\Lambda \in \Lambda(\zeta, \xi)$ we have $\varepsilon_\zeta \Lambda = \Lambda \varepsilon_\xi = \Lambda$. In particular, $\varepsilon_\zeta$ is the identity of the algebra $\Lambda(\zeta, \zeta)$, and $\varepsilon = \sum_{\zeta \in \Omega} \varepsilon_\zeta = \sum_{i=1}^{t} \varepsilon_{ii}$ is the identity of $M_t(\mathbb{F})$. Clearly,

$$D_\zeta = W_{\zeta\zeta} \otimes \varepsilon_\zeta \tag{4}$$

for $\zeta \in \Omega$. We shall often identify the algebras $D_\zeta$ and $W_{\zeta\zeta}$.

Right and left multiplications by elements of $D$ induce a structure of an $D \otimes D^{op}$-module on $M_n(\mathbb{F})$. We have

$$D \otimes D^{op} = \bigoplus_{\zeta, \xi \in \Omega} D_{\zeta\xi}$$

where $D_{\zeta\xi} = D_\zeta \otimes D_\xi^{op}$ are the simple components of $D \otimes D^{op}$. The following is obvious.

**Lemma 2.3** *The formula* (3) *is the decomposition of the $D \otimes D^{op}$-module $M_n(\mathbb{F}) \cong \mathcal{M}_D$ into its homogeneous components.*

**Remark 2.4** Identifying each $D_{\zeta\xi}$ with the algebra $W_{\zeta\zeta} \otimes W_{\xi\xi}^{op}$ and any $D_{\zeta\xi}$-module $M = W_{\zeta\xi} \otimes \Lambda$ ($\Lambda \in \Lambda(\zeta, \xi)$) with $W_{\zeta\xi}$ we can represent the action of $D_{\zeta\xi}$ on $M$ in the following form: $(X_{\zeta\zeta} \otimes X_{\xi\xi})X_{\zeta\xi} = X_{\zeta\zeta} X_{\zeta\xi} X_{\xi\xi}$ where $X_{ij}$ is any matrix from $W_{ij} = W_i \otimes W_j^*$ for $i, j \in \{\zeta, \xi\}$.

**Proposition 2.5** *Let $A$ be an associative subalgebra of $M_n(\mathbb{F})$ such that $DAD = A$. Then*

$$A = \bigoplus_{\zeta, \xi \in \Omega} W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$$

*where $\Lambda_A(\zeta, \xi)$ is a subspace of $\Lambda(\zeta, \xi)$.*

*Proof.* It suffices to observe that $A$ is a $D \otimes D^{op}$-submodule of $M_n(\mathbb{F})$ and to apply Lemma 2.3. $\square$

**Remark 2.6** If $D$ is a maximal semisimple subalgebra of $A$, then the algebra $\bigoplus_{\zeta,\xi\in\Omega}\Lambda_A(\zeta,\xi)$ is isomorphic to the *basic* algebra of $A$ (see [8, §6.6]).

**Lemma 2.7** *Let $A$ be an associative subalgebra of $M_n(\mathbb{F})$ and $D = \mathbb{F}\mathbf{1}_n + D'$ where $D'$ is a semisimple subalgebra of $A$. Then for any ideal $B$ of $A$ we have*

$$B = \bigoplus_{\zeta,\xi\in\Omega} W_{\zeta\xi}\otimes\Lambda_B(\zeta,\xi)$$

*where $\Lambda_B(\zeta,\xi)$ is a subspace of $\Lambda(\zeta,\xi)$.*

*Proof.* It suffices to note that $DBD = B$ and to apply Proposition 2.5. □

# 3 Weak decompositions

In this section we study weak decompositions of associative and Lie algebras. Let $A$ be an associative algebra. The algebra $A$ is called *perfect* if $A^2 = A$. Recall that $\mathrm{Null}(A) = \{a \in A \mid aA = Aa = 0\}$. We call $\mathrm{Null}(A)$ the *null-radical* of $A$. The following proposition shows that the null-radical has radical-like properties.

**Proposition 3.1** *Let $A$ be perfect, $N \subseteq \mathrm{Null}(A)$, and $A' = A/N$. Let $N' \subseteq \mathrm{Null}(A')$ and $\bar{N}$ be the complete inverse image of $N'$ in $A$. Then $\bar{N} \subseteq \mathrm{Null}(A)$. In particular, $\mathrm{Null}(A/\mathrm{Null}(A)) = 0$.*

*Proof.* We have $A\bar{N} + \bar{N}A = A^2\bar{N} + \bar{N}A^2 \subseteq AN + NA = 0$, as required. □

**Definition 3.2** Let $A$ and $A'$ be associative algebras and $\pi : A' \to A$ be a surjective homomorphism. The pair $(\pi, A')$ (or, simply, $A'$) is called a *null-extension* of $A$ if $\mathrm{Ker}\,\pi \subseteq \mathrm{Null}(A')$. A null-extension $A'$ is called a *null-covering* of $A$ if $A'$ is perfect. A null-covering $(\rho, U)$ is *universal* if for each null-covering $(\pi, A')$ of $A$ there is a homomorphism $\tau : U \to A'$ such that $\pi\tau = \rho$. An algebra $A$ is called *null-closed* if $A$ is its own universal null-covering.

Standard arguments show that an algebra $A$ admits a universal null-covering if and only if $A$ is perfect. In that case the universal null-covering is unique and can be constructed as follows. Denote by $M$ the vector space $A \otimes_A A$ and by $U$ the vector space $A \oplus M$. The space $U$ can be viewed as an associative algebra if we set $AM = MA = 0$, $M^2 = 0$, and $ab = a \cdot b + a \otimes_A b$ for $a, b \in A$ where $a \cdot b$ is the product in $A$. Then the algebra $U^2$ is the universal null-covering of $A$. We state the following standard properties of null-extensions.

**Proposition 3.3** (1) *The universal null-covering of a perfect algebra is null-closed.*

(2) *Every null-extension of a null-closed algebra splits.*

(3) *If $A$ is null-closed and $N$ is a subspace of* $\mathrm{Null}(A)$, *then $A$ is the universal covering of $A/N$.*

*Proof.* (1) This follows from Proposition 3.1.

(2) Let $(\pi, B)$ be a null-extension of $A$. Set $\tau = \pi|_{B^2}$. Observe that $B^2$ is perfect, so $(\tau, B^2)$ is a null-covering of $A$. Now (1) implies that $\tau$ is a bijection and $B = B^2 \oplus \mathrm{Ker}\,\pi$.

(3) This is obvious. □

**Lemma 3.4** *Let $A$ be a perfect algebra and let $A_1, \ldots, A_n$ be perfect ideals of $A$ such that $A = A_1 + \cdots + A_n$. Then the following conditions are equivalent.*

(W1) $A_i \cap A_j \subseteq \mathrm{Null}(A)$ *for all* $i \neq j$.

(W2) $A_i A_j = 0$ *for all* $i \neq j$.

*Proof.* (W1)$\Rightarrow$(W2): $A_i A_j = A_i A_j A_j \subseteq (A_i \cap A_j)A_j = 0$.

(W2)$\Rightarrow$(W1): For each $k$ we have $(A_i \cap A_j)A_k \subseteq (A_i A_k) \cap (A_j A_k) = 0$. Therefore $(A_i \cap A_j)A = 0$. Similarly, $A(A_i \cap A_j) = 0$. $\square$

**Definition 3.5** A perfect algebra $A$ is called *weakly decomposable* if there is a nontrivial (weak) decomposition as in Lemma 3.4 (which will be denoted by $A = A_1 \uplus \cdots \uplus A_n$) such that (W1) or (W2) holds. Otherwise $A$ is called *weakly indecomposable.* If each algebra $A_i$ for $i = 1, \ldots, n$ is weakly indecomposable, then the decomposition is called *complete*.

The following is standard.

**Proposition 3.6** *Each finite dimensional perfect algebra has a unique weak decomposition $A = A_1 \uplus \cdots \uplus A_n$ into weakly indecomposable ideals $A_i$. Moreover, if* $\mathrm{Null}(A) = 0$ *(resp., $A$ is null-closed), then the sum is direct and* $\mathrm{Null}(A_i) = 0$ *(resp., $A_i$ is null-closed) for each $i = 1, \ldots, n$.*

Let $A$ be a perfect subalgebra of $M_n(\mathbb{F})$ and $D'$ be a Levi subalgebra of $A$. Set $D = D' + \mathbb{F}\mathbf{1}_n$. Clearly, $D$ is semisimple and $D'$ is an ideal of $D$. As in Section 2, let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of $D$. Since $D$ is semisimple, $\Omega$ can be expressed as a disjoint union $\Omega_0 \sqcup \Omega'$ such that $D' = \bigoplus_{\zeta \in \Omega'} D_\zeta$. Observe that $\Omega_0$ is nonempty if and only if $\mathbf{1}_n \notin A$. In that case $\Omega_0$ consists of a single element, say 0. As $DAD = A$, we have a decomposition as in Proposition 2.5. If $\mathbf{1}_n \in A$, then $\mathrm{Null}(A) = 0$. Assume that $\mathbf{1}_n \notin A$. Then, obviously, $\mathrm{Null}(A) \subseteq \Lambda_A(0,0)$. Set

$$\Lambda_A^0 = \sum_{\zeta \in \Omega'} \Lambda_A(0,\zeta)\Lambda_A(\zeta,0) \subseteq \Lambda_A(0,0).$$

**Proposition 3.7** *Let $A \subseteq M_n(\mathbb{F})$ be as above and $\mathbf{1}_n \notin A$. Then $\Lambda_A^0 = \Lambda_A(0,0)$. In particular,* $\mathrm{Null}(A) \subseteq W_{00} \otimes \Lambda_A^0$.

*Proof.* It suffices to note that

$$A' = \left( \bigoplus_{(\zeta,\xi) \neq (0,0)} W_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi) \right) \oplus (W_{00} \otimes \Lambda_A^0)$$

is an ideal of $A$ containing the Levi subalgebra $D'$, so $A/A'$ is nilpotent. $\square$

Let $\zeta, \xi \in \Omega$. We write $\zeta \sim \xi$ if there exist $\eta_1, \ldots, \eta_m \in \Omega'$ such that $\eta_1 = \zeta$, $\eta_m = \xi$, and for each $1 \leq i \leq m-1$ the $D \otimes D^{op}$-module $A$ contains a nonzero $D_{\eta_i \eta_{i+1}}$- or $D_{\eta_{i+1}\eta_i}$-module, equivalently, $\Lambda_A(\eta_i, \eta_{i+1})$ or $\Lambda_A(\eta_{i+1}, \eta_i)$ is nonzero. Obviously, $\sim$ is an equivalence relation. Let $\Omega' = \Omega_1 \sqcup \cdots \sqcup \Omega_m$ be the partition of $\Omega'$ into equivalence classes and let $A_i$ be the subalgebra of $A$ generated by all $W_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi)$ with $\zeta, \xi \in \Omega_i \cup \{0\}$ and $(\zeta,\xi) \neq (0,0)$.

**Proposition 3.8** *Set* $\Gamma_i = (\Omega_i \times \Omega_i) \cup (\{0\} \times \Omega_i) \cup (\Omega_i \times \{0\})$. *Then*

$$A_i = \left( \bigoplus_{(\zeta,\xi)\in\Gamma_i} W_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi) \right) \oplus W_{00} \otimes \sum_{\zeta\in\Omega_i} \Lambda_A(0,\zeta)\Lambda_A(\zeta,0) \quad (1 \le i \le m). \tag{5}$$

*Moreover,* $A = A_1 \uplus \cdots \uplus A_n$ *is the complete weak decomposition of* $A$.

*Proof.* First observe that $A_iA_j = 0$ for $i \neq j$ (since this is true for the generating sets). Now as in the proof of Proposition 3.7, we get (5) and conclude that all $A_i$ are perfect. It follows from Proposition 3.7 and (5) that $A = A_1 + \cdots + A_n$. Therefore $A = A_1 \uplus \cdots \uplus A_n$. It remains to show that each $A_i$ is weakly indecomposable. Assume that $A_i = B_1 \uplus B_2$ is a nontrivial weak decomposition. Note that each $B_k$ is an ideal of $A$. Therefore by Lemma 2.7, there is a partition $\Omega_i = \Omega_i^1 \sqcup \Omega_i^2$ such that for each $\zeta \in \Omega_i^k$ we have $D_\zeta \subseteq B_k$. By assumption, there exist $\zeta \in \Omega_i^1$ and $\xi \in \Omega_i^2$ such that $\Lambda_A(\zeta,\xi) \neq 0$ or $\Lambda_A(\xi,\zeta) \neq 0$. Therefore $0 \neq B_1A_iB_2 \subseteq B_1B_2$ or $0 \neq B_2A_iB_1 \subseteq B_2B_1$, which yields a contradiction. □

One can introduce a notion of weakly indecomposable Lie algebras as well. The role of the null-radical for a Lie algebra is played by its center. The following is well known.

**Lemma 3.9** *Let* $L$ *be a perfect Lie algebra and let* $L_1, \dots, L_n$ *be perfect ideals of* $L$ *such that* $L = L_1 + \cdots + L_n$. *Then the following conditions are equivalent.*
(L1) $L_i \cap L_j \subseteq Z(L)$ *for all* $i \neq j$.
(L2) $[L_i, L_j] = 0$ *for all* $i \neq j$.

**Definition 3.10** A perfect Lie algebra $L$ is called *weakly decomposable* if there is a nontrivial (weak) decomposition as in Lemma 3.9 (which will be denoted by $L = L_1 \uplus \cdots \uplus L_n$) such that (L1) or (L2) holds. Otherwise, $L$ is called *weakly indecomposable*. If each algebra $L_i$ in the decomposition is weakly indecomposable, then the decomposition is called *complete*.

**Proposition 3.11** *Each finite dimensional perfect Lie algebra has a unique weak decomposition* $L = L_1 \uplus \cdots \uplus L_n$ *into weakly indecomposable ideals* $L_i$.

# 4 Normal submodules of $[M_n(\mathbb{F})]$

Let $S$ be a semisimple Lie subalgebra of $[M_n(\mathbb{F})] = \mathfrak{gl}_n(\mathbb{F})$. In this section we describe irreducible $S$-submodules in $[M_n(\mathbb{F})]$ under the adjoint action (under some additional assumptions).

Below $S_1, \dots, S_k$ are the simple components of $S$, so $S = S_1 \oplus \cdots \oplus S_k$. Let $V$ be the natural module for $M_n(\mathbb{F})$. As $V$ is completely reducible, $V = V_1 \oplus \cdots \oplus V_t$ where $V_1, \dots, V_t$ are irreducible $S$-modules. Let $\{W_\zeta \mid \zeta \in \Omega\}$ be a set of representatives of the isomorphism classes of irreducible $S$-submodules of $V$. Set $\Omega' = \{\zeta \in \Omega \mid W_\zeta \text{ is nontrivial}\}$. If $\Omega \neq \Omega'$, then the set $\Omega \backslash \Omega'$ consists of a single element, which will be denoted by 0. In other words, we use the notation $W_0$ for the trivial one-dimensional $S$-module. We denote by $\mathcal{E}(S)$ the enveloping algebra of $S$ in $M_n(\mathbb{F})$, i.e. the associative subalgebra generated by $S$. Set $D = \mathcal{E}(S) + \mathbb{F}\mathbf{1}_n$. Clearly, $D$ is semisimple, and the simple components of $D$ are in a bijective correspondence with isomorphism classes of irreducible submodules of $V$. So

$$D = \bigoplus_{\zeta\in\Omega} D_\zeta$$

with $D_\zeta \cong \operatorname{End} W_\zeta$. The enveloping algebra $\mathcal{E}(S)$ is a subalgebra of $D$ and

$$\mathcal{E}(S) = \bigoplus_{\zeta \in \Omega'} D_\zeta.$$

The Lie algebra $[M_n(\mathbb{F})]$ is an $S$-module under the adjoint action. Let $\varphi : M_n(\mathbb{F}) \to \mathcal{M}_D$ be the canonical isomorphism (see Proposition 2.2). Observe that the adjoint action of $S$ on $\mathcal{M}_D$ induced by $\varphi$ agrees with the natural action of $S$ on the direct summands $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$ of $\mathcal{M}_D$ (see (3)): we just view $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$ as the tensor product of the $S$-module $W_{\zeta\xi} = W_\zeta \otimes W_\xi^*$ and the trivial $S$-module $\Lambda(\zeta, \xi)$.

**Definition 4.1** A nonzero $S$-submodule $M$ of $M_n(\mathbb{F})$ is called *normal* if there exist $\zeta, \xi \in \Omega$, a matrix $\Lambda_M \in \Lambda(\zeta, \xi)$, and an $S$-submodule $W_M$ of $W_{\zeta\xi}$ such that $M = W_M \otimes \Lambda_M$. Note that $M \cong W_M$ as $S$-modules. If $M$ is normal then we denote by $\bar{M}$ the module $W_{\zeta\xi} \otimes \Lambda_M$.

Note that normal $S$-module can be trivial. If $M$ is normal, then the corresponding pair $(\zeta, \xi)$ is determined uniquely and is called the *type* of $M$. The matrix $\Lambda_M$ is unique up to a scalar multiple.

Recall that $D = \mathcal{E}(S) + \mathbb{F}\mathbf{1}_n$. Therefore Lemma 2.3 implies the following.

**Lemma 4.2** *Let $M$ be a normal $S$-submodule of $M_n(\mathbb{F})$. Then $(\mathcal{E}(S) + \mathbb{F}\mathbf{1}_n) M (\mathcal{E}(S) + \mathbb{F}\mathbf{1}_n) = \bar{M}$.*

If $M$ and $N$ are $S$-submodules in $M_n(\mathbb{F})$, we denote by $MN$ and $[M, N]$ the vector spaces spanned by all products $xy$ and $[x, y] = xy - yx$, respectively ($x \in M$, $y \in N$). In fact, $MN$ and $[M, N]$ are $S$-modules, since for $s \in S$ we have

$$\begin{aligned} [s, xy] &= [s, x]y + x[s, y], \\ [s, [x, y]] &= [[s, x], y] + [x, [s, y]]. \end{aligned}$$

**Lemma 4.3** *Let $M = W_M \otimes \Lambda_M$ and $N = W_N \otimes \Lambda_N$ be normal submodules of $M_n(\mathbb{F})$ of types $(\zeta_1, \xi_1)$ and $(\zeta_2, \xi_2)$, respectively. If $MN \neq 0$, then $\xi_1 = \zeta_2$ and $MN = W_M W_N \otimes \Lambda_M \Lambda_N$ is normal of type $(\zeta_1, \xi_2)$.*

*Proof.* This follows from the properties of multiplication in $\mathcal{M}_D$. □

Note that in view of Lemma 4.3 the normal submodules form a semigroup under the multiplication (if we add the zero module).

**Proposition 4.4** *Assume that there is a bijection between the simple components of $S$ and isomorphism classes of nontrivial irreducible submodules of $V$ such that for each $\zeta \in \Omega' = \{1, \dots, k\}$ the following conditions hold:*

(I) $\operatorname{Ann}_S W_\zeta = \bigoplus_{m \neq \zeta} S_m$*; in particular, $W_\zeta$ can be considered as an $S_\zeta$-module;*

(II) $W_{\zeta\zeta} = W_\zeta \otimes W_\zeta^*$ *contains no submodule isomorphic to $W_\zeta$ or $W_\zeta^*$;*

(III) $W_\zeta \not\cong W_\zeta^*$ *as $S$-modules (and $S_\zeta$-modules);*

(IV) *each nontrivial composition factor of $W_{\zeta\zeta}$ appears with multiplicity 1.*

*Then each nontrivial irreducible $S$-submodule $M$ of $M_n(\mathbb{F})$ is normal, i.e. $M = W_M \otimes \Lambda_M$ where $W_M \subseteq W_{\zeta\xi}$ and $\Lambda_M \in \Lambda(\zeta, \xi)$ for some $\zeta, \xi \in \Omega$. Moreover, $W_M = W_{\zeta\xi}$ if $\zeta \neq \xi$.*

*Proof.* Observe that (3) is a direct sum decomposition of $M_n(\mathbb{F})$ into $S$-submodules $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$. It follows from assumptions (II) and (III) that any two nontrivial irreducible submodules of different summands are not isomorphic. Therefore there exist unique $\zeta, \xi \in \Omega$ such that $M \subseteq W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$. If $\zeta \neq \xi$, then $W_{\zeta\xi}$ is irreducible, so $M = W_{\zeta\xi} \otimes \Lambda_M$ for some $\Lambda_M$. If $\zeta = \xi$, then the proposition follows from (IV). $\square$

## 5 Enveloping algebras of plain Lie algebras

Let $L \subset M_n(\mathbb{F})$ be a perfect Lie algebra. Then $L = S \oplus R$ where $S$ is a maximal semisimple subalgebra and $R$ is the radical of $L$. As above, we denote by $S_1, \dots, S_k$ the simple components of $S$ and by $\mathcal{E}(L)$ the enveloping algebra of $L$ in $M_n(\mathbb{F})$. Let $V$ be the natural $M_n(\mathbb{F})$-module, $\{W_\zeta \mid \zeta \in \Omega\}$ be a set of representatives of the isomorphism classes of all irreducible $L$-submodules of $V$, and let $\Omega' = \{\zeta \in \Omega \mid W_\zeta \text{ is nontrivial}\}$. In this section we assume that $L$ is perfect $\mathfrak{sl}_2$-free and $V$ is a plain $L$-module. In other words, we assume that the set $\Omega'$ is identified with the set $\{1, \dots, k\}$ (which labels the simple components of $S$) such that

- $\operatorname{Ann}_S W_\zeta = \bigoplus_{m \neq \zeta} S_m$ for $\zeta = 1, \dots, k$;
- $S_\zeta \cong \mathfrak{sl}_{n_\zeta}(\mathbb{F})$ with $n_\zeta = \dim W_\zeta \geq 3$ for $\zeta = 1, \dots, k$.

In particular, $W_\zeta$ is the natural $S_\zeta$-module or dual to it. Then one can easily verify that the conditions (I)–(IV) of Proposition 4.4 hold. Therefore each nontrivial irreducible $S$-submodule of $M_n(\mathbb{F})$ is normal. Note that we exclude the components isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ in order to satisfy (III) of Proposition 4.4 (the natural $\mathfrak{sl}_2(\mathbb{F})$-module is isomorphic to its dual). The aim of this section is to prove that $[\mathcal{E}(L), \mathcal{E}(L)] = L$.

We denote by ${}_LL$ the adjoint module for $L$ and by ${}_SL$ its restriction to $S$. Let $R_0$ be the sum of all trivial submodules of ${}_SL$ and let $L_1$ be the sum of all nontrivial irreducible submodules. Since ${}_SL$ is completely reducible, ${}_SL = L_1 \oplus R_0$. Observe that $R_0 \subset R$.

Let $M$ be a nontrivial irreducible submodule of ${}_SL$. By Proposition 4.4, $M$ is normal, i.e. there exist $\zeta, \xi \in \Omega$, $\Lambda_M \in \Lambda(\zeta, \xi)$, and an $S$-submodule $W_M$ of $W_{\zeta\xi}$ isomorphic to $M$ such that $M = W_M \otimes \Lambda_M$. If $\zeta \neq \xi$, then $W_{\zeta\xi}$ is irreducible, so $M = \bar{M} = W_{\zeta\xi} \otimes \Lambda_M$ (see Definition 4.1). Denote by $W'_{\zeta\zeta}$ the subspace of matrices in $W_{\zeta\zeta}$ with zero traces and by $I$ the identity matrix. Then $W'_{\zeta\zeta}$ and $\mathbb{F}I$ are irreducible $S$-submodules of $W_{\zeta\zeta}$ for $\zeta \neq 0$ and $W_{\zeta\zeta} = W'_{\zeta\zeta} \oplus \mathbb{F}I$. If $\zeta = \xi$, then we have $M = W'_{\zeta\zeta} \otimes \Lambda_M$, since $M$ is nontrivial. It is convenient to set $W'_{\zeta\xi} = W_{\zeta\xi}$ for $\zeta \neq \xi$, so $M = W'_{\zeta\xi} \otimes \Lambda_M$ for any $\zeta$ and $\xi$. We have proved the following lemma.

**Lemma 5.1** *Let $M$ be a nontrivial irreducible submodule of ${}_SL$. Then $M = W'_{\zeta\xi} \otimes \Lambda_M$ for some $\zeta, \xi \in \Omega$ and $\Lambda_M \in \Lambda(\zeta, \xi)$.*

Let $\mathfrak{M}$ be the set of all nontrivial irreducible submodules of ${}_SL$. Put

$$\bar{L}_1 = \sum_{M \in \mathfrak{M}} \bar{M}.$$

**Lemma 5.2** *Let $\zeta, \xi \in \Omega$ and $x \in L$. Let $x_{\zeta\xi}$ be the projection of $x$ into $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$. Assume $x_{\zeta\xi} \neq 0$. Express $x_{\zeta\xi} = X_1 \otimes \Lambda_1 + \dots + X_m \otimes \Lambda_m$ where $X_1, \dots, X_m \in W_{\zeta\xi}$ are linearly independent. If $\zeta = \xi$, assume additionally that the matrices $I, X_1, \dots, X_m$ are linearly independent. Then $W_{\zeta\xi} \otimes \Lambda_i \subseteq \bar{L}_1$ for each $i = 1, \dots, m$.*

*Proof.* Let $M$ be the $S$-submodule of $L$ generated by $x$. Then $M = M^0 \oplus M'$ where $M^0$ is a trivial $S$-module and $M' = M_1 \oplus \cdots \oplus M_l$ is a sum of nontrivial irreducible $S$-modules. Set $\bar{M}' = \bar{M}_1 + \cdots + \bar{M}_l \subseteq \bar{L}_1$. Let $M^0_{\zeta\xi}$ and $\bar{M}'_{\zeta\xi}$ be the projections of $M^0$ and $\bar{M}'$, respectively, to $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)$. It follows from the definition that $\bar{M}'_{\zeta\xi} = W_{\zeta\xi} \otimes \Lambda_M(\zeta, \xi) \subseteq \bar{L}_1$ where $\Lambda_M(\zeta, \xi)$ is a subspace of $\Lambda(\zeta, \xi)$. Clearly, it suffices to show that $\Lambda_i \in \Lambda_M(\zeta, \xi)$ for each $i = 1, \dots, m$. Assume that this is not the case. Reordering the indices one can assume that $\{1, \dots, m'\}$ is a maximal subset of $\{1, \dots, m\}$ such that $\Lambda_1, \dots, \Lambda_{m'}$ are linearly independent modulo $\Lambda_M(\zeta, \xi)$. Then the image of $x_{\zeta\xi}$ in the quotient $W_{\zeta\xi} \otimes \Lambda(\zeta, \xi)/W_{\zeta\xi} \otimes \Lambda_M(\zeta, \xi)$ is $X'_1 \otimes \Lambda'_1 + \cdots + X'_{m'} \otimes \Lambda'_{m'}$ where $X'_i = X_i + \sum_{j=m'+1}^{m} \lambda_i^j X_j$ and $\Lambda'_i$ is the image of $\Lambda_i$ in $\Lambda(\zeta, \xi)/\Lambda_M(\zeta, \xi)$. Observe that $M^0_{\zeta\xi}$ is either 0 (for $\zeta \neq \xi$) or lies in $\mathbb{F}I \otimes \Lambda(\zeta, \xi)$ (for $\zeta = \xi$). As $X'_1, \dots, X'_m$ (resp. $I, X'_1, \dots, X'_m$) are linearly independent for $\zeta \neq \xi$ (resp. $\zeta = \xi$), we get that $x_{\zeta\xi} \notin M^0_{\zeta\xi} + \bar{M}'_{\zeta\xi}$, which contradicts the assumption. The lemma follows. □

**Lemma 5.3** *Let $M_1$ and $M_2$ be nontrivial irreducible submodules of ${}_SL$. Then $\bar{M}_1\bar{M}_2 \subset \bar{L}_1$, except possibly in the case where $M_1$ and $M_2$ are of types $(0, \xi)$ and $(\xi, 0)$, respectively. In the exceptional case $\bar{M}_1\bar{M}_2 \subset \bar{L}_1 + R_0$.*

*Proof.* By Lemma 5.1, $M_i = W'_{\zeta_i\xi_i} \otimes \Lambda_i$ where $i = 1, 2$ and $(\zeta_i, \xi_i)$ is the type of $M_i$. One can assume that $\bar{M}_1\bar{M}_2 \neq 0$. Since $\bar{M}_i = W_{\zeta_i\xi_i} \otimes \Lambda_i$ for $i = 1, 2$ is normal, by Lemma 4.3, $\xi_1 = \zeta_2 = \xi$, $\Lambda = \Lambda_1\Lambda_2 \neq 0$, and $\bar{M}_1\bar{M}_2 \subseteq W_{\zeta_1\xi_2} \otimes \Lambda$.

Assume that $(\zeta_1, \xi_2) \neq (0, 0)$. We shall use Lemma 5.2, so it suffices to find $x \in L$ with nonzero projection $X \otimes \Lambda$ into $W_{\zeta_1\xi_2} \otimes \Lambda(\zeta_1, \xi_2)$ such that if $\zeta_1 = \xi_2$, then $X$ is not scalar.

For $i = 1, 2$ let $x_i = X_i \otimes \Lambda_i \in M_i$ with $X_i \in W'_{\zeta_i\xi_i}$. Then

$$[x_1, x_2] = X_1X_2 \otimes \Lambda - X_2X_1 \otimes \Lambda' \in L,$$

where $\Lambda' = \Lambda_2\Lambda_1$. Recall that we identify all $W_{\zeta\xi}$ with the corresponding subspaces of $M_{n'}(\mathbb{F})$ and $\{e_{ij}\}_{i,j=1}^{n'}$ is the basis of $M_{n'}(\mathbb{F})$ consisting of matrix units (see Section 2). Assume that $\zeta_1 \neq \xi_2$. Then $\Lambda' = 0$. Set

$$X_i = e_{11} - \delta_{\zeta_i\xi_i} e_{22}. \tag{6}$$

Then $X_i \in W'_{\zeta_i\xi_i}$ for $i = 1, 2$, and $X_1X_2 = e_{11} \neq 0$ (as $\delta_{\zeta_1\xi_1}\delta_{\zeta_2\xi_2} = 0$). Hence $x = [x_1, x_2] = e_{11} \otimes \Lambda \in L$, as required. Therefore, one can suppose that $\zeta_1 = \xi_2 = \zeta$ (recall also that $\xi_1 = \zeta_2 = \xi$). Assume that $\zeta = \xi$. Since $M$ and $N$ are nontrivial, $\xi \neq 0$ and $W_{\xi\xi}$ is the set of all $n_\xi \times n_\xi$ matrices with $n_\xi \geq 3$. Set $X_1 = e_{12}$ and $X_2 = e_{23}$. Then $X_1X_2 = e_{13}$ and $X_2X_1 = 0$, so $x = [x_1, x_2] = e_{13} \otimes \Lambda \in L$, as required. Therefore one can suppose that $\zeta \neq \xi$. Recall also that by assumption, $\zeta \neq 0$. Set $X_1 = X_2 = e_{11}$. We have $x_i = X_i \otimes \Lambda_i \in M_i$ and $X_1X_2 = e_{11} \neq 0$. Since $\Lambda \in \Lambda(\zeta, \zeta)$, $\Lambda' \in \Lambda(\xi, \xi)$, and $\zeta \neq \xi$, the projection of $x = [x_1, x_2]$ into $W_{\zeta\zeta} \otimes \Lambda(\zeta, \zeta)$ is $e_{11} \otimes \Lambda \neq 0$, as required.

Now consider the exceptional case: $\zeta_1 = \xi_2 = 0$ and $\xi_1 = \zeta_2 = \xi \neq 0$. Arguing as above one can assume that $\Lambda' \neq 0$. Set $X_1 = X_2 = e_{11}$. Then $X_1X_2 = X_2X_1 = e_{11} \neq 0$. As $\xi \neq 0$, by Lemma 5.2, $W_{\xi\xi} \otimes \Lambda' \subseteq \bar{L}_1$. Since $[x_1, x_2]$ has a nonzero projection to $W_{00} \otimes \Lambda$, we have

$$\bar{M}_1\bar{M}_2 = W_{00} \otimes \Lambda \subseteq \mathbb{F}[x_1, x_2] + W_{\xi\xi} \otimes \Lambda' \subseteq R_0 + \bar{L}_1,$$

as required. □

Set

$$\mathcal{E}_0 = \sum PQ$$

where $P$ and $Q$ run over all pairs of irreducible submodules in $L_1$ of types $(0, \xi)$ and $(\xi, 0)$, respectively, with $\xi$ running over $\Omega'$.

**Lemma 5.4** *Let $M_1, \dots, M_d$ be nontrivial irreducible submodules of $L$. Then $\bar{M}_1 \dots \bar{M}_d \subseteq \bar{L}_1 + \mathcal{E}_0$.*

*Proof.* Proceed by induction on $d$, the case $d = 1$ being trivial. The case $d = 2$ immediately follows from Lemma 5.3. Suppose that $d \geq 3$. Let $(\zeta, \xi)$ be the type of $M_2$. Using Lemma 5.3, we obtain that either $\xi \neq 0$ and $\bar{M}_1 \bar{M}_2 \subseteq \bar{L}_1$ or $\zeta \neq 0$ and $\bar{M}_2 \bar{M}_3 \subseteq \bar{L}_1$, so we can apply the induction assumption. □

**Lemma 5.5** *Let $L$ be a finite dimensional perfect Lie algebra, $S$ a Levi subalgebra of $L$. Then there exist nontrivial irreducible $S$-submodules of $L$ that generate $L$ as an algebra.*

*Proof.* Express $L$ in the form $L = S \oplus R$ where $R = \operatorname{Rad} L$. Pick irreducible $S$-submodules $M_1, \dots, M_k$ of $R$ such that $R = M_1 \oplus \dots \oplus M_k \oplus [R, R]$. Since $L$ is perfect, all $M_i$ are nontrivial and $R$ is nilpotent. As $R$ is nilpotent, the vector space $M_1 + \dots + M_k$ generates $R$ as an algebra (see [3, ch. I, §4, exercise 4]). Therefore, the simple components of $S$ and $M_1, \dots, M_k$ generate $L$. □

**Proposition 5.6** *We have*

(i) $\mathcal{E}(L) = \bar{L}_1 + \mathcal{E}_0$;

(ii) $\mathcal{E}(L) = \bar{L}_1 + R_0$;

(iii) $[\mathcal{E}(L), \mathcal{E}(L)] = L$.

*Proof.* (i). Set $A = \bar{L}_1 + \mathcal{E}_0$. By Lemma 4.2, for each nontrivial irreducible submodule $M$ of $L$ we have $\bar{M} \subset \mathcal{E}(L)$. Therefore, $A \subseteq \mathcal{E}(L)$. By Lemma 5.5, $L$ is generated by nontrivial irreducible $S$-submodules, hence $\mathcal{E}(L)$ does. Thus by Lemma 5.4, $\mathcal{E}(L) \subseteq A$, so $\mathcal{E}(L) = A$.

(ii). By Lemma 5.3, $\mathcal{E}_0 \subseteq \bar{L}_1 + R_0$. Thus

$$\mathcal{E}(L) = \bar{L}_1 + \mathcal{E}_0 \subseteq \bar{L}_1 + R_0 \subseteq \mathcal{E}(L).$$

Therefore, $\mathcal{E}(L) = \bar{L}_1 + R_0$.

(iii). Since $L$ is perfect, it suffices to show that $[\bar{L}_1, \bar{L}_1] \subseteq L$ and $[\bar{L}_1, R_0] \subseteq L$. We first show that the latter inclusion follows from the former one. So assume that $[\bar{L}_1, \bar{L}_1] \subseteq L$. We are going to show that $[\bar{L}_1, R_0] \subseteq L$, i.e. that $[\bar{M}, R_0] \subseteq L$ for each nontrivial irreducible submodule $M$ of $L$. Let $(\zeta_1, \xi_1)$ be the type of $M$. If $\zeta_1 \neq \xi_1$, then $\bar{M} = M$ and $[M, R_0] \subseteq L$, as required. Suppose that $\zeta_1 = \xi_1 \neq 0$. Note that each element $x \in \mathcal{E}_0$ is represented in the form $x = X \otimes \Lambda$ where $X \in \mathbb{F}$ and $\Lambda \in \Lambda(0, 0)$. Hence $x\bar{M} = \bar{M}x = 0$, so $[\bar{M}, \mathcal{E}_0] = 0$. Therefore, we have

$$[\bar{M}, R_0] \subseteq [\bar{M}, \bar{L}_1 + \mathcal{E}_0] \subseteq [\bar{L}_1, \bar{L}_1] \subseteq L.$$

It remains to prove that $[\bar{L}_1, \bar{L}_1] \subseteq L$, i.e. $[\bar{M}, \bar{N}] \subseteq L$ for all nontrivial irreducible $M$ and $N$. Let $(\zeta_1, \xi_1)$ and $(\zeta_2, \xi_2)$ be the types of $M$ and $N$, respectively. If $\zeta_1 \neq \xi_1$ and $\zeta_2 \neq \xi_2$, then $\bar{M} = M$ and $\bar{N} = N$, and we are done. Next, suppose that $\zeta_1 = \xi_1 = \zeta \neq 0$. If $\zeta_2 \neq \zeta$ (resp. $\xi_2 \neq \zeta$), then by Lemma 5.3,

$$[\bar{M}, \bar{N}] \subseteq \bar{M}\bar{N} + \bar{N}\bar{M} \subseteq \bar{Q}_1 + \dots + \bar{Q}_d$$

where $Q_i$ are nontrivial irreducible submodules of type $(\zeta_2, \zeta)$ (resp. $(\zeta, \xi_2)$). Since $\bar{Q}_i = Q_i \subseteq L$, we have $[\bar{M}, \bar{N}] \subseteq L$. Hence we assume that $\zeta_1 = \xi_1 = \zeta_2 = \xi_2 = \zeta$. We have

$\bar{M} = W_{\zeta\zeta} \otimes \Lambda_M = M \oplus \mathbb{F}I_M$ and $\bar{N} = W_{\zeta\zeta} \otimes \Lambda_N = N \oplus \mathbb{F}I_N$ where $I_M = I \otimes \Lambda_M$ and $I_N = I \otimes \Lambda_N$ with $I$ being the identity matrix. So

$$[\bar{M}, \bar{N}] = [M, N] + [M, I_N] + [N, I_M] + \mathbb{F}[I_M, I_N].$$

Clearly, $[M, N] \subseteq L$. We now show that $M' = [M, I_N] \subseteq L$. Assume that $M' \neq 0$. Since $S$ acts trivially on $I_N$, the $S$-module $M'$ is a homomorphic image of $M$ (the homomorphism is given by the map $x \mapsto [x, I_N]$ for $x \in M$). As $M$ is irreducible and $M' \neq 0$, we have $M \cong M'$. By Lemma 5.3, $M'$ is a submodule of $\bar{M}\bar{N} + \bar{N}\bar{M} \subseteq \bar{L}_1$. Since $M'$ is nontrivial and irreducible, $M' \subseteq L$, as required. Similarly, $[N, I_M] \subseteq L$.

It remains to check that $[I_M, I_N] \in L$. We have

$$[I_M, I_N] = I \otimes \Lambda_M \Lambda_N - I \otimes \Lambda_N \Lambda_M.$$

Recall that $n_\zeta = \dim W_\zeta > 2$. Fix any nonzero elements $\alpha_1, \dots, \alpha_{n_\zeta} \in \mathbb{F}$ such that $\sum \alpha_i = 0$ and $\sum \alpha_i^{-1} = 0$ (as $\mathbb{F}$ is algebraically closed, one can take all $n_\zeta$th roots of unity). Set $X_1 = \operatorname{diag}(\alpha_1, \dots, \alpha_{n_\zeta})$ and $X_2 = \operatorname{diag}(\alpha_1^{-1}, \dots, \alpha_{n_\zeta}^{-1})$. Then $X_1 X_2 = X_2 X_1 = I$. Since $X_1$ and $X_2$ have zero traces, the elements $x_1 = X_1 \otimes \Lambda_M$ and $x_2 = X_2 \otimes \Lambda_N$ lie in $M$ and $N$, respectively. It remains to observe that $[I_M, I_N] = [x_1, x_2] \in L$. □

The following obvious lemma (cf. Proposition 3.7) is useful for describing the null-radical of $\mathcal{E}(L)$.

**Lemma 5.7** (1) *The null-radical of $\mathcal{E}(L)$ is contained in $\mathcal{E}_0$;*

(2) $\dim \mathcal{E}_0 \leq \sum_{\zeta \in \Omega'} d_{0\zeta} d_{\zeta 0}$ *where $d_{\zeta\xi}$ is the composition length of the sum of all irreducible $L$-submodules of $L$ of type $(\zeta, \xi)$.*

# 6 Main results

Let $A$ be an associative algebra, $L$ a Lie algebra, and let

$$\iota : L \to [A]$$

be a Lie algebra monomorphism. A pair $(\iota, A)$ is called an *enveloping algebra* for $L$ if $\iota(L)$ generates $A$ as an algebra. In view of the universal property, the map $\iota$ uniquely extends to an associative algebra homomorphism

$$\bar{\iota} : \mathcal{A}(L) \to A$$

where $\mathcal{A}(L)$ is the augmentation ideal of the universal enveloping algebra $\mathcal{U}(L)$, i.e. the ideal of codimension 1 generated by $L$. Note that we do not require $A$ to have the identity, so it is more convenient for us to deal with $\mathcal{A}(L)$ instead of $\mathcal{U}(L)$. Two enveloping algebras $(\iota_1, A_1)$ and $(\iota_2, A_2)$ are isomorphic if there is an algebra isomorphism $\delta : A_1 \to A_2$ such that $\delta\iota_1 = \iota_2$. Observe that each enveloping algebra $(\iota, A)$ is uniquely determined (up to isomorphism) by the corresponding kernel $H_A = \operatorname{Ker}(\bar{\iota})$ in $\mathcal{A}(L)$. In particular, there exists a 1–1 correspondence between the enveloping algebras for $L$ and the ideals $H$ in $\mathcal{A}(L)$ such that $H \cap L = 0$. This gives a partial ordering on the set of enveloping algebras of $L$: we say that $(\iota_1, A_1) \leq (\iota_2, A_2)$ if and only if $H_{A_2} \subseteq H_{A_1}$. Given an enveloping algebra $(\iota, A)$ for a Lie algebra $L$, we often identify $L$ with its image $\iota(L)$ in $A$, and call $A$ the enveloping algebra, keeping in mind that there is a relevant monomorphism $\iota$.

Let $A$ be a finite dimensional associative algebra. We say that $A$ is *perfect* if $A^2 = A$, and *strongly perfect* if $A$ has no proper ideals of codimension $\leq 4$. Set $\bar{A} = A$ if $\mathbf{1}_A \in A$ and $\bar{A} = A \oplus \mathbb{F}\mathbf{1}_{\bar{A}}$ (the algebra that is obtained from $A$ by external adjointing the identity) otherwise. Assume that $A$ is an enveloping algebra of a perfect Lie algebra $L$. One can easily see that $A$ is *perfect*. Clearly, $\bar{A}$ is a quotient of $\mathcal{U}(L)$. The algebra $\bar{A}$ can be viewed as an $L$-module ${}_L\bar{A}$ (under the regular action). Note that the enveloping algebra of the image of $L$ in the algebra of transformations of the space $\bar{A}$ is isomorphic to $A$.

Let $L$ be a perfect Lie algebra and $V$ be an $L$-module. We denote by $\operatorname{Irr} L$ the set of nontrivial irreducible $L$-modules (up to equivalence), and by $\operatorname{Irr}(V)$ the set of inequivalent nontrivial composition factors of $V$. We say that $L$ is *strongly plain* if $L$ is plain and $\mathfrak{sl}_2$-free. For each plain Lie algebra $L$ we fix a faithful plain module $\mathcal{V}(L)$ and set $\mathfrak{S}(L) = \operatorname{Irr}(\mathcal{V}(L))$. Although $\mathfrak{S}(L)$ depends on $\mathcal{V}$, the choice of $\mathcal{V}$ is not essential. By the *type* of an enveloping algebra $A$ we mean the set $\operatorname{Irr}({}_L\bar{A})$. Recall that an enveloping algebra $A$ of $L$ is called $\mathfrak{P}$-*enveloping* if $[A, A] = L$. We denote by $\mathfrak{P}(L)$ the poset of $\mathfrak{P}$-enveloping algebras of a Lie algebra $L$.

We shall use some results of [1], however, the definitions of $\operatorname{Irr} L$ and $\operatorname{Irr}(V)$ slightly differ from those in [1], since we ignore trivial modules. (This is because we prefer to deal with $\mathcal{A}(L)$ instead of $\mathcal{U}(L)$.) In the theorem below we reformulate results from [1] to the case of ideals in $\mathcal{A}(L)$.

**Theorem 6.1 ([1, 3.4])** *Let $L$ be a perfect finite dimensional Lie algebra, $\Phi$ a finite subset of $\operatorname{Irr} L$, $\mathfrak{F}(\Phi)$ the set of all ideals $X$ of $\mathcal{A}(L)$ such that $\mathcal{A}(L)/X$ is finite dimensional and $\operatorname{Irr}(\mathcal{A}(L)/X) = \Phi$. Then $\mathfrak{F}(\Phi)$ is nonempty and has a smallest element $N(\Phi)$ and a largest element $M(\Phi)$ such that $N(\Phi) \subseteq X \subseteq M(\Phi)$ for all $X \in \mathfrak{F}(\Phi)$. The algebra $\mathcal{A}(L)/M(\Phi)$ is semisimple, the algebra $M(\Phi)/N(\Phi)$ is nilpotent.*

**Corollary 6.2** *Let $L$ be a perfect Lie algebra, $\Phi$ be a finite subset of $\operatorname{Irr} L$, and let $\mathfrak{E}_\Phi(L)$ be the poset of enveloping algebras for $L$ of type $\Phi$. Assume that $\mathfrak{E}_\Phi(L) \neq \emptyset$. Then $\mathfrak{E}_\Phi(L)$ has the largest (universal) element.*

**Theorem 6.3** *Let $A$ be a strongly perfect associative algebra. Then*

(1) *$L = [A, A]$ is a strongly plain Lie algebra, ${}_L\bar{A}$ is a plain $L$-module, and $L$ generates $A$.*

(2) *If $A = A_1 \uplus \cdots \uplus A_n$ is the complete weak decomposition of $A$ and $L_i = [A_i, A_i]$, then $L = L_1 \uplus \cdots \uplus L_n$ is the complete weak decomposition of $L$.*

*Proof.* (1) Recall that $\bar{A} = A$ if $\mathbf{1}_A \in A$ and $\bar{A} = A \oplus \mathbb{F}\mathbf{1}_{\bar{A}}$ otherwise. We consider $\bar{A}$ as an algebra of endomorphisms of the vector space $\bar{A}$ induced by the left multiplications, so $\bar{A}$ is represented as a subalgebra of $M_n(\mathbb{F})$ with the same identity. Let $D$ be a Levi subalgebra of $\bar{A}$. Let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of $D$. Since $A$ is strongly perfect, $D$ contains at most one component $D_0$ of dimension 1. Clearly, $D' = \bigoplus_{\zeta \neq 0} D_\zeta$ is a Levi subalgebra of $A$. Since $D_\zeta \cong M_{n_\zeta}(\mathbb{F})$, we have $[D_\zeta, D_\zeta] \cong \mathfrak{sl}_{n_\zeta}(\mathbb{F})$ and

$$S = [D, D] = \bigoplus_{\zeta \in \bar{\Omega}} [D_\zeta, D_\zeta]$$

Let $N = \operatorname{Rad} A$ be the radical of $A$. As $N$ is nilpotent, $[N]$ is a nilpotent ideal of $[A]$. Clearly, $[A]/[N] \cong [D']$. As $S$ is a semisimple Lie algebra, the radical $R$ of $L$ coincides with $L \cap N$ and $L/R \cong S$. Therefore, $S$ is a plain Levi subalgebra of $L'$ and ${}_S\bar{A}$ is a plain $S$-module. Moreover, since $A$ is strongly perfect, $D'$ has no component of dimension 4, so $S$ is strongly plain. Using Proposition 2.5, we get the decomposition

$$A = \bigoplus_{\zeta,\xi\in\Omega} W_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi)$$

Let $L'$ be a Lie subalgebra of $A$ generated by all $W'_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi)$. Note that $L'$ is perfect and $S$ is a Levi subalgebra of $L'$, so $L'$ is plain and ${}_{L'}\bar{A}$ is a plain $L'$-module. It remains to show that $L' = L$ and $L$ generates $A$. Let $A'$ be the enveloping algebra of $L'$ in $M_n(\mathbb{F})$. Clearly, $A'$ is generated by all $W_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi)$ with $(\zeta,\xi) \neq (0,0)$. Therefore $A'$ is an ideal of $A$. Since $A'$ contains the Levi subalgebra $D'$, $A/A'$ is nilpotent. As $A$ is perfect, $A' = A$. Therefore by Proposition 5.6(iii), $L = [A,A] = L'$ and $L$ generates $A$, as required.
(2) By (1), each $L_i$ is perfect. Obviously, the decomposition $L = L_1 \uplus \cdots \uplus L_n$ is weak. Let $L = M_1 \uplus \cdots \uplus M_k$ ($k \geq n$) be the complete weak decomposition of $L$. We have to prove that $k = n$. Since each $M_i$ is an $L$-submodule of $L$, there is a partition $\Omega' = \Omega_1 \sqcup \cdots \sqcup \Omega_k$ such that $M_i$ is generated by all $W'_{\zeta\xi} \otimes \Lambda_A(\zeta,\xi)$ with $\zeta,\xi \in \Omega_i \cup \{0\}$. Since $[M_i, M_j] = 0$ for all $i \neq j$, we have $\zeta \not\sim \xi$ (see Proposition 3.8) for each pair $(\zeta,\xi) \in \Omega_i \times \Omega_j$. This implies that the complete weak decomposition of $A$ has at least $k$ components. Therefore $k = n$. □

**Corollary 6.4** *Let $A$ be a finite dimensional associative algebra having no proper ideals of codimension $\leq 4$. Then $[A,A] = [[A,A],[A,A]]$.*

**Theorem 6.5** *Let $L$ be a perfect finite dimensional $\mathfrak{sl}_2$-free Lie algebra.*

(1) *Assume that $\mathfrak{P}(L) \neq \emptyset$. Then for each $A \in \mathfrak{P}(L)$, ${}_L\bar{A}$ is a faithful plain $L$-module. In particular, $L$ is plain.*

(2) *Assume that $L$ is plain. Then for each faithful plain $L$-module $V$ the enveloping algebra of $L$ in $\operatorname{End} V$ is $\mathfrak{P}$-enveloping. In particular, $\mathfrak{P}(L) \neq \emptyset$.*

*Proof.* (1) It suffices to observe that each $A \in \mathfrak{P}(L)$ is strongly perfect, so by Theorem 6.3(1), $L = [A,A]$ is plain and ${}_L\bar{A}$ is a plain $L$-module. (In fact this can be easily proved directly because we know that $L = [A,A]$ is perfect, see the proof of Theorem 6.3(1)).
(2) This is proved in Proposition 5.6(iii) □

*Proof of Theorem 1.4.* It suffices to observe that $L$ is quasispecial if and only if $\mathfrak{P}(L) \neq \emptyset$. □

*Proof of Theorem 1.5.* This is proved in Theorem 6.5(2). □

**Proposition 6.6** *Let $L$ be a strongly plain Lie algebra and $A \in \mathfrak{P}(L)$. Let $L = L_1 \uplus \cdots \uplus L_n$ be the complete weak decomposition and let $A_i$ be the subalgebra of $A$ generated by $L_i$. Then $A_i \in \mathfrak{P}(L_i)$ and $A = A_1 \uplus \cdots \uplus A_n$ is the complete weak decomposition of $A$.*

*Proof.* Since $A \in \mathfrak{P}(L)$, we can identify $L$ with the Lie algebra $[A,A]$. Let $A = B_1 \uplus \cdots \uplus B_n$ be the complete weak decomposition of $A$. Since $L$ is strongly plain, $A$ and all $B_i$ are strongly perfect. Therefore, by Theorem 6.3(2), $L = [B_1,B_1] \uplus \cdots \uplus [B_n,B_n]$ is the complete weak decomposition of $L$. In view of uniqueness, one can assume that $[B_i,B_i] = L_i$ for all $i$. It remains to note that $L_i$ generates $B_i$, i.e. $B_i = A_i$. □

**Lemma 6.7** *Let $L$ be a strongly plain Lie algebra, let $V$ be a faithful plain $L$-module, and $\operatorname{Irr}(V) = \{W_\zeta \mid \zeta \in \Omega'\}$. Then $L$ is weakly indecomposable if and only if for each nontrivial partition $\Omega' = \Omega_1 \sqcup \Omega_2$ there exist $\zeta_1 \in \Omega_1$ and $\zeta_2 \in \Omega_2$ such that the adjoint $L$-module has a composition factor isomorphic to $W_\zeta \otimes W_\xi^*$ or $W_\xi \otimes W_\zeta^*$.*

*Proof.* Let $A$ be the enveloping algebra of $L$ in $\operatorname{End} V$. By Proposition 6.6 and Theorem 6.3(2), $L$ is weakly indecomposable if and only if $A$ is. We identify $L$ with $[A, A]$. Clearly, $L \supset W_{\zeta\xi} \otimes \Lambda_A(\zeta, \xi)$ for all $\zeta \neq \xi$. It remains to apply Proposition 3.8. □

Recall that $\mathfrak{S}(L)$ is the set $\operatorname{Irr}(\mathcal{V}(L))$ where $\mathcal{V}(L)$ is a fixed faithful plain $L$-module. For a subset $\Phi$ of $\operatorname{Irr} L$ we denote by $\Phi^*$ the set $\{\varphi^* \mid \varphi \in \Phi\}$ of dual modules.

**Lemma 6.8** *Let $L$ be a weakly indecomposable plain Lie algebra and let $V$ be a faithful plain $L$-module. Then* $\operatorname{Irr}(V) = \mathfrak{S}(L)$ *or* $\mathfrak{S}(L)^*$.

*Proof.* Indeed, assume that $\operatorname{Irr}(V) \neq \mathfrak{S}(L), \mathfrak{S}(L)^*$. Set $\Phi_1 = \operatorname{Irr}(V) \cap \mathfrak{S}(L)$ and $\Phi_2 = \operatorname{Irr}(V) \backslash \Phi_1$. Since $V$ and $\mathcal{V}(L)$ are plain and faithful, $\mathfrak{S}(L) = \Phi_1 \sqcup \Phi_2^*$. By Lemma 6.7, there exists $\varphi_1 \in \Phi_1$ and $\varphi_2 \in \Phi_2$ such that the adjoint $L$-module contains a composition factor isomorphic to $\varphi_1 \otimes \varphi_2^*$ or $\varphi_2 \otimes \varphi_1^*$. It remains to observe that the adjoint $L$-module is a submodule of $\mathcal{V}(L) \otimes \mathcal{V}(L)^*$ and the latter $L$-module contains none of these composition factors. The contradiction obtained proves the lemma. □

**Proposition 6.9** *Let $L$ be a perfect Lie algebra. Let $H$ be an ideal of a $\mathfrak{P}$-enveloping algebra $A$ of $L$. Assume that $H \cap L \subseteq Z(L)$. Then $HA = AH = 0$, i.e. $H \subseteq \operatorname{Null}(A)$.*

*Proof.* Since $[A, A] = L$, we have $[L, H] \subseteq L \cap H \subseteq Z(L)$. As $[L, L] = L$, we have

$$[L, H] = [[L, L], H] \subseteq [L, [L, H] \subseteq [L, Z(L)] = 0,$$

i.e. $L$ commutes with $H$. Therefore,

$$HL = LH = [L, L]H = [L, LH] \subseteq [L, H] = 0.$$

It remains to note that $L$ generates $A$. □

Let $L$ be a strongly plain finite dimensional Lie algebra and let $L = L_1 \uplus \cdots \uplus L_n$ be the complete weak decomposition of $L$. Clearly, each $L_i$ is strongly plain and the restriction of $\mathcal{V}(L)$ to $L_i$ is plain. Without loss of generality one may assume that

$$\mathfrak{S}(L) = \mathfrak{S}(L_1) \sqcup \cdots \sqcup \mathfrak{S}(L_n)$$

(viewing irreducible $L_i$-modules as $L$-modules). Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$. For $i = 1, \ldots, n$ set $\mathfrak{S}(L_i)^{\varepsilon_i} = \mathfrak{S}(L_i)$ if $\varepsilon_i = 1$ and $\mathfrak{S}(L_i)^{\varepsilon_i} = \mathfrak{S}(L_i)^*$ if $\varepsilon_i = -1$. Put

$$\begin{aligned} \mathfrak{S}(L)^\varepsilon &= \mathfrak{S}(L_1)^{\varepsilon_1} \sqcup \cdots \sqcup \mathfrak{S}(L_n)^{\varepsilon_n} \\ \mathfrak{P}_\varepsilon(L) &= \{A \in \mathfrak{P}(L) \mid \operatorname{Irr}({}_L\bar{A}) = \mathfrak{S}(L)^\varepsilon\}. \end{aligned}$$

Note that $\mathfrak{P}_\varepsilon(L)$ may be empty for some $\varepsilon$. If $L$ is weakly indecomposable, we shall write $\mathfrak{P}_\pm(L)$ instead of $\mathfrak{P}_{(\pm 1)}(L)$. The following theorem describes the poset of $\mathfrak{P}$-enveloping algebras for strongly plain Lie algebras.

**Theorem 6.10** *Let $L$ be a strongly plain finite dimensional Lie algebra and let $L = L_1 \uplus \cdots \uplus L_n$ be the complete weak decomposition of $L$. Then the following holds.*

(1) *Each $\mathfrak{P}$-enveloping algebra of $L$ is strongly perfect.*

(2) *Let $A \in \mathfrak{P}(L)$ and let $A_i$ be the subalgebra of $A$ generated by $L_i$. Then $A_i \in \mathfrak{P}(L_i)$ and $A = A_1 \uplus \cdots \uplus A_n$ is the complete weak decomposition of $A$.*

(3) *For each faithful plain $L$-module $V$ one has* $\mathrm{Irr}(V) = \mathfrak{S}(L)^{\varepsilon}$ *for some* $\varepsilon$.

(4) $\mathfrak{P}(L) = \bigsqcup_{\varepsilon=(\pm1,\dots,\pm1)} \mathfrak{P}_{\varepsilon}(L)$.

(5) *The map* $\nu : (\iota, A) \mapsto (-\iota, A^{op})$ *is an isomorphism between the posets* $\mathfrak{P}_{\varepsilon}(L)$ *and* $\mathfrak{P}_{-\varepsilon}(L)$.

(6) *If* $\mathfrak{P}_{\varepsilon}(L) \neq \emptyset$, *then the class* $\mathfrak{P}_{\varepsilon}(L)$ *contains a universal object* $\mathcal{N}_{\varepsilon}(L)$. *More exactly, there is an algebra* $\mathcal{N}_{\varepsilon}(L) \in \mathfrak{P}_{\varepsilon}(L)$ *such that for each* $A \in \mathfrak{P}_{\varepsilon}(L)$ *one has* $A \leq \mathcal{N}_{\varepsilon}(L)$, *the kernel* $H_A$ *of the corresponding homomorphism* $\mathcal{N}_{\varepsilon}(L) \to A$ *is a subspace of* $\mathrm{Null}(\mathcal{N}_{\varepsilon}(L))$, *and* $H_A \cap L = 0$. *In particular, the poset* $\mathfrak{P}_{\varepsilon}(L)$ *is antiisomorphic to the poset of all subspaces* $H \subseteq \mathrm{Null}(\mathcal{N}_{\varepsilon}(L))$ *with* $H \cap L = 0$.

(7) *Set* $\mathcal{N}_i = \mathcal{N}_+(L_i)$. *Then for each* $\varepsilon$ *and each* $A \in \mathfrak{P}_{\varepsilon}(L)$ *there exists an ideal* $\hat{H}_A$ *of the algebra* $\mathcal{N}^{\varepsilon} = \mathcal{N}_1^{\varepsilon_1} \oplus \cdots \oplus \mathcal{N}_n^{\varepsilon_n}$ *such that* $\hat{H}_A \subseteq \mathrm{Null}(\mathcal{N}^{\varepsilon})$ *and* $A \cong \mathcal{N}^{\varepsilon}/\hat{H}_A$. *In particular,*

$$A/\mathrm{Null}(A) \cong \mathcal{N}^{\varepsilon}/\mathrm{Null}(\mathcal{N}^{\varepsilon}) \cong (\mathcal{N}_1/\mathrm{Null}(\mathcal{N}_1))^{\varepsilon_1} \oplus \cdots \oplus (\mathcal{N}_n/\mathrm{Null}(\mathcal{N}_n))^{\varepsilon_n}.$$

*Proof.* (1) This is obvious.
(2) This is proved in Proposition 6.6.
(3) Since for each $i$ the restriction $V{\downarrow}L_i$ is plain, this follows from Lemma 6.8.
(4) This follows from Theorem 6.5(1) and Theorem 6.10(3).
(5) Let $(\iota, A) \in \mathfrak{P}_{\varepsilon}(L)$. Recall that $(\iota, A)$ is the enveloping algebra of $L$ in $\mathrm{End}\,\bar{A}$. Therefore $(-\iota, A^{op})$ is the enveloping algebra of $L$ in $\mathrm{End}\,\bar{A}^*$. Hence

$$\mathrm{Irr}(V_{A^{op}}) = \mathrm{Irr}({}_L\bar{A}^*) = \mathrm{Irr}({}_L\bar{A})^* = (\mathfrak{S}(L)^{\varepsilon})^* = \mathfrak{S}(L)^{-\varepsilon},$$

so $(-\iota, A^{op}) \in \mathfrak{P}_{-\varepsilon}(L)$. Clearly, $\nu$ is a poset homomorphism. Since $\nu^2 = 1$, the map $\nu$ is an isomorphism.
(6) It follows from Theorem 6.5(2) that the set $\mathfrak{P}_{\varepsilon}(L)$ coincides with the set $\mathfrak{E}_{\Phi}(L)$ for $\Phi = \mathfrak{S}(L)^{\varepsilon}$ (see Corollary 6.2). Therefore the existence of the universal $\mathfrak{P}$-enveloping algebra $\mathcal{N}_{\varepsilon}(L)$ follows from Corollary 6.2. Let $A \in \mathfrak{P}_{\varepsilon}(L)$ and $H_A$ be the kernel of the corresponding homomorphism $\mathcal{N}_{\varepsilon}(L) \to A$. Then $H_A \cap L = 0$. Therefore by Proposition 6.9, $H_A \subseteq \mathrm{Null}(\mathcal{N}_{\varepsilon}(L))$.
(7) Let $A_i$ be the subalgebra of $A$ generated by $L_i$. By Theorem 6.10(2), $A_i \in \mathfrak{P}(L_i)$ and $A = A_1 \uplus \cdots \uplus A_n$ is the complete weak decomposition of $A$. By Theorem 6.10(5) and (6), $A_i \cong \mathcal{N}_i^{\varepsilon_i}/H_i$ where $H_i \subseteq \mathrm{Null}(\mathcal{N}_i^{\varepsilon_i})$. Therefore $\bar{A} = A_1 \oplus \cdots \oplus A_n \cong \mathcal{N}^{\varepsilon}/H'$ where $H' = H_1 \oplus \cdots \oplus H_n \subseteq \mathrm{Null}(\mathcal{N}^{\varepsilon})$. It remains to observe that the kernel of the natural homomorphism $\bar{A} \to A$ lies in $\mathrm{Null}(\bar{A})$ and to apply Proposition 3.1. □

*Proof of Theorem 1.7.* By Theorem 6.3(1), $L$ is strongly plain. Since $A_1$ and $A_2$ are $\mathfrak{P}$-enveloping algebras for $L$, the result follows from Theorem 6.10. □

**Remark 6.11** Assume that $L$ is weakly indecomposable.

(1) It follows from Theorem 6.10(2) and (5) that the algebras $\mathcal{N}_+(L)$ and $\mathcal{N}_-(L)$ are weakly indecomposable and antiisomorphic.

(2) The example below shows that the set $\mathfrak{P}_+(L)$ may have more than one element. This implies that there are non-universal $\mathfrak{P}$-enveloping algebras.

(3) The poset $\mathfrak{P}_+(L)$ may not contain the smallest element.

Below is an example of a perfect weakly indecomposable Lie algebra $L$ such that $\mathfrak{P}_+(L)$ has more than one element.

**Example 6.12** Let $n \geq 3$ and $L_n$ be a Lie algebra of $(2n+1) \times (2n+1)$ matrices of the form

$$x = \begin{pmatrix} x_0 & x_1 & x_3 \\ & 0 & x_2 \\ & & x_0 \end{pmatrix} \tag{7}$$

where $x_0$ runs over all $n \times n$ matrices with zero traces; and $x_1, x_2, x_3$ run over all matrices of sizes $n \times 1$, $1 \times n$, $n \times n$, respectively (all empty spaces are zero matrices). We are going to describe all $\mathfrak{P}$-enveloping algebras for $L_n$. Let us denote by $V_1$ the natural module for $M_{2n+1}(\mathbb{F})$ and by $\varphi_1 : L_n \to \operatorname{End} V_1$ the representation defined by (7). For $L_n$ we construct another faithful matrix representation $\varphi_2$ as follows.

$$\varphi_2(x) = \begin{pmatrix} 0 & x_2 & x_4 & & & \\ & x_0 & x_1 & & & \\ & & 0 & & & \\ & & & x_0 & x_1 & x_3 \\ & & & & 0 & x_2 \\ & & & & & x_0 \end{pmatrix} \tag{8}$$

where $x_4 = -\operatorname{tr} x_3$. Let $V_2$ be the $(3n+3)$-dimensional $L_n$-module corresponding to $\varphi_2$. Let $A_1$ and $A_2$ be the enveloping algebras of $L$ in $\operatorname{End} L_1$ and $\operatorname{End} L_2$, respectively. Clearly, $A_1$ and $A_2$ consist of all matrices of the forms (7) and (8), respectively, where the $x_i$ run over all matrices of the corresponding sizes ($x_4$ does not depend on $x_3$). Let $R$ be the subspace of $A_2$ consisting of matrices (8) with $x_0 = x_1 = x_2 = x_3 = 0$. Then $R$ is an ideal of $A_2$ of dimension 1 and $A_2/R \cong A_1$. Obviously, $A_1, A_2 \in \mathfrak{P}_+(L_n)$. Let $\mathcal{N}_+(L_n)$ be a universal $\mathfrak{P}$-enveloping. By Lemma 5.7(2), the dimension of $\mathcal{E}_0$ for $\mathcal{N}_+(L_n)$ does not exceed 1. Therefore by Lemma 5.7(1), $\mathcal{N}_+(L_n)$ has at most one nonzero ideal $H$ lying in $\operatorname{Null}(\mathcal{N}_+(L_n))$. It follows that $\mathcal{N}_+(L_n) = A_2$ and $H = R$, so $\mathfrak{P}_+(L_n) = \{A_1, A_2\}$. By Theorem 6.10(5), $\mathfrak{P}_-(L_n) = \{A_1^{op}, A_2^{op}\}$.

*Proof of Theorem 1.2.* Set

$$\mathfrak{I}(L) = \{\operatorname{Irr}(V) \mid V \text{ is a plain } L\text{-module } \}.$$

Let $S_1, \dots, S_k$ be the simple components of $L/\operatorname{Rad} L$. Clearly, for each non-trivial composition factor $W$ of a plain $L$-module $V$ there exists $i$ such that $S_i = L/\operatorname{Ann}_L(W) \cong \mathfrak{sl}(W)$ and $S_j W = 0$ for $j \neq i$. This implies that $|\mathfrak{I}(L)| \leq 3^k$. Let $\mathfrak{M}(L)$ be the set of maximal (with respect to $\subseteq$) elements of $\mathfrak{I}(L)$. Let $\Phi_1, \Phi_2 \in \mathfrak{M}(L)$ and let $V_1, V_2$ be plain $L$-modules such that $\Phi_i = \operatorname{Irr}(V_i)$. Set $\Phi_i^* = \operatorname{Irr}(V_i^*)$. Assume that $\Phi_1 \neq \Phi_2$. Then

$$\Phi_1 \cap \Phi_2^* \neq \emptyset. \tag{9}$$

Indeed, otherwise $V_1 \oplus V_2$ is a plain $L$-module and $\Phi_1 \cup \Phi_2 = \operatorname{Irr}(V_1 \oplus V_2) \in \mathfrak{I}(L)$, so $\Phi_1$ and $\Phi_2$ are not maximal. Equation (9) implies that $|\mathfrak{M}(L)| \leq 2^k$.

Fix any $\Phi \in \mathfrak{M}(L)$. Since $\operatorname{Ann}_L(V_1 \oplus V_2) = \operatorname{Ann}_L(V_1) \cap \operatorname{Ann}_L(V_2)$, there exists a plain $L$-module $V_\Phi$ with $\operatorname{Irr}(V_\Phi) = \Phi$ such that $\operatorname{Ann}_L(V_\Phi)$ annihilates each plain $L$-module $V$ with $\operatorname{Irr}(V) = \Phi$. Set $L_\Phi = L/\operatorname{Ann}_L(V_\Phi)$. Let $A_\Phi = \mathcal{N}_\Phi(L_\Phi)$ be the universal $\mathfrak{P}$-enveloping algebra of $L_\Phi$ corresponding to $\Phi$ (see Theorem 6.10(6)). We assert that the associative algebras $A_\Phi$, $\Phi \in \mathfrak{M}(L)$, with the representations $\eta_\Phi : L \to L_\Phi \to A_\Phi$ satisfy the theorem. Indeed, the restrictions of all finite dimensional $A_\Phi$-modules to $L_\Phi$ are plain. Let now $V$ be a plain $L$-module. Take arbitrary $\Phi \in \mathfrak{M}(L)$ such that $\operatorname{Irr}(V) \subseteq \Phi$. Consider the $L$-module

$V' = V \oplus V_\Phi$. Since $\mathrm{Irr}(V') = \Phi$, $\mathrm{Ann}_L(V_\Phi)$ annihilates $V'$, so $V'$ is a plain $L_\Phi$-module. Moreover, by Theorem 6.10, the enveloping algebra of $L_\Phi$ in $\mathrm{End}\, V'$ is a homomorphic image of $A_\Phi$. Since $V'$ contains $V$, the enveloping algebra ist $A_\Phi$. It remains to note that $V$ is an $A_\Phi$-submodule of $V'$. $\square$

Let $A$ be an associative algebra. Set $\bar{A} = A$ if $\mathbf{1}_A \in A$ and $\bar{A} = A \oplus \mathbb{F}\mathbf{1}_{\bar{A}}$ (the algebra that is obtained from $A$ by external adjointing the identity) otherwise. If $\mathbf{1}_A \in A$, we denote by $U(A)$ the group of invertible elements of $A$. Observe that the external adjointing the identity to $A$ does not change the commutator subgroup of $U(A)$, i.e. $U(A)' = U(\bar{A})'$. This allows us to define $U(A)'$ even if $A$ has no the identity. We say that $A$ is *reduced* if $A$ has no a proper ideal $C$ such that $A = \bar{C}$. Obviously, for each finite dimensional algebra $A$ there exists a reduced algebra $A_0$ (not necessary containing the identity) such that $U(A)' = U(A_0)'$.

Let $G$ be a perfect group, $A$ be a reduced algebra, and

$$\iota : G \to U(A)'$$

be a monomorphism. The pair $(\iota, A)$ is called an *enveloping algebra* for $G$ if $\iota(G)$ generates $\bar{A}$ as an algebra. In view of the universal property, the map $\iota$ uniquely extends to an associative algebra homomorphism

$$\bar{\iota} : \mathbb{F}G \to \bar{A}.$$

Let $\varphi : G \to \mathrm{End}\, V$ be a faithful representation of a perfect group $G$ and let $B$ be a subalgebra of $\mathrm{End}\, V$ generated by $\varphi(G)$. Then one can easily check that there exists a unique reduced subalgebra $A$ of $B$ such $B = A + \mathbb{F}I$ where $I$ is the identity transformation of $V$. This subalgebra $A$ is called the enveloping algebra of $G$ in $\mathrm{End}\, V$. Conversely, let $A$ be an enveloping algebra for a perfect group $G$. The algebra $\bar{A}$ can be viewed as an $G$-module ${}_G\bar{A}$ (under the regular action). One can check that the enveloping algebra of the image of $G$ in $\mathrm{End}\, \bar{A}$ is isomorphic to $A$.

An enveloping algebra $A$ of a perfect group $G$ is called $\mathfrak{P}$-*enveloping* if

$$G = U(A)'.$$

We denote by $\mathfrak{P}(G)$ the set of $\mathfrak{P}$-enveloping algebras for a group $G$. Let $V$ be a vector space over $\mathbb{F}$ and let $G$ be a perfect algebraic subgroup of $GL(V)$. Let $\{W_1, \ldots, W_k\}$ be the set of nontrivial composition factors of the $G$-module $V$ and let $\varphi_i : G \to GL(W_i)$ be the corresponding representations of $G$. The $G$-module $V$ is called *plain* if

$$G/\mathrm{Ker}\, \varphi_i \cong SL(W_i) \quad (1 \leq i \leq k)$$

and $W_i \not\cong W_j^*$ if $i \neq j$ and $\dim W_i > 2$. Note that $\dim W_i \geq 2$ for each $i$ since $G$ is perfect. A perfect group $G$ is called *plain* if it has faithful plain module $V$. If, in addition, $\dim W_i > 2$ for all $i$, then $G$ is *strongly plain* (or *$SL_2$-free plain*).

Theorem 1.6 is an immediate consequence of the following analog of Theorem 6.5.

**Theorem 6.13** *Let $G$ be a perfect $SL_2$-free algebraic group.*

(1) *Assume that $\mathfrak{P}(G) \neq \emptyset$. If $A \in \mathfrak{P}(G)$, then ${}_G\bar{A}$ is a faithful plain $G$-module. In particular, $G$ is plain.*

(2) *Assume that $G$ is plain. Then for each faithful plain $G$-module $V$ the enveloping algebra of $G$ in* $\mathrm{End}\, V$ *is $\mathfrak{P}$-enveloping. In particular, $\mathfrak{P}(G) \neq \emptyset$.*

*Proof.* (1) We view $\bar{A}$ as an algebra of endomorphisms of the vector space $\bar{A}$, so $\bar{A}$ is represented as a subalgebra of $M_n(\mathbb{F})$ with the same identity (here $n = \dim \bar{A}$). Let $D$ be a Levi subalgebra of $\bar{A}$. Let $\{D_\zeta \mid \zeta \in \Omega\}$ be the set of simple components of $D$. Since $D_\zeta \cong M_{n_\zeta}(\mathbb{F})$, we have $U(D_\zeta) \cong GL_{n_\zeta}(\mathbb{F})$ and $U(D_\zeta)' \cong SL_{n_\zeta}(\mathbb{F})$. Clearly, $U(D)'$ is the direct product of the groups $U(D_\zeta)'$. Let $N = \operatorname{Rad} \bar{A}$ be the radical of $\bar{A}$. As $N$ is nilpotent, $U(N)$ is a unipotent normal subgroup of $U(\bar{A})$. Clearly, $U(\bar{A}) = U(D)U(N)$. Therefore $G = U(A)' = U(\bar{A})' = U(D)'R$ where $R$ is the unipotent radical of $U(A)'$. It remains to observe that $U(D)'$ is a plain subgroup of $GL(V)$ and $\bar{A}$ is a plain $U(D)'$-module.
(2) Let $\varphi : G \to GL(V)$ be a faithful plain representation of $G$ and let $W_1, \dots, W_n$ list the nontrivial composition factors of $V$. Let $\varphi_i : G \to GL(W_i)$ denote the restriction of $\varphi$ to $W_i$ and let $\tau_i : SL(W_i) \to PSL(W_i)$ be the natural projection. Then $G_i = G/\operatorname{Ker} \varphi_i$ is isomorphic to $SL(W_i)$, and $H_i = G_i/\operatorname{Ker} \tau_i \circ \varphi_i$ is isomorphic to $PSL(W_i)$. Set $H = \cap_{i=1}^n \operatorname{Ker} \varphi_i$ and $M = \cap_{i=1}^n \operatorname{Ker} \tau \circ \varphi_i$. Then $H$ acts trivially on each composition factor of $V$ so $H$ is unipotent, hence $H$ is nilpotent. As $\operatorname{Ker} \tau_i$ is abelian, $M/N$ is abelian so $M$ is solvable. As $G/M$ is a finite subdirect product of simple groups, it is in fact a direct product. Clearly, $G/M$ have no non-trivial finite quotient. Therefore, if $X$ is a finite index subgroup of $G$ then $G = XM$. If $X \neq G$, then $G/X$ is not solvable as $G$ is perfect. Besides, $G/X \cong MX/X \cong M/(M \cap X)$. This is a contradiction as $M$ is solvable. This means that $G$ contains no proper subgroup of finite index. Therefore, $G$ is connected.

Let $A$ be the enveloping algebra of $G$ in $\operatorname{End} V$ and let $L$ be the Lie algebra of $G$. We identify $L$ with the corresponding subalgebra of $\mathfrak{gl}(V)$. Since $G$ is perfect, $L$ is perfect as well. Observe that $L$ is strongly plain and $V$ is a plain $L$-module. By Theorem 6.5(2), $L = [B, B]$ where $B$ is the enveloping algebra of $L$ in $\operatorname{End} V$. Since $G$ is connected, by [4, Corollary 2 of Theorem II.12.8], $\mathbb{F}I + B = \mathbb{F}I + A$ where $I$ is the identity of $\operatorname{End} V$. Hence $L = [A, A]$. Set $H = U(A + \mathbb{F}I)$. One can easily show that $H$ is a connected algebraic group (see, for instance, [2, Example I.1.6(9)]) and the Lie algebra of $H$ is $[A + \mathbb{F}I]$. By [2, I.2.3], the commutator subgroup $H'$ is a connected algebraic subgroup of $H$, and by [2, II.7.8], $L = [A, A]$ is the Lie algebra of $H'$. Since $H' \supseteq G$ are connected and have the same Lie algebras, by [2, II.7.1], $G = H' = U(A)'$, as required. □

One can easily describe the properties of $\mathfrak{P}$-enveloping algebras for $G$ in the spirit of Theorem 6.10. Observe that in Theorem 1.7 only reduced associative algebras are considered. We have the following analog of Theorem 1.7 for algebraic groups.

**Theorem 6.14** *Let $A_1$ and $A_2$ be finite dimensional associative algebras. Assume that $U(A_1)' \cong U(A_2)'$ and each $A_1, A_2$ has no nontrivial homomorphism into $M_2(\mathbb{F})$. Then $A_1 \sim A_2$. Moreover, if $A_1$ or $A_2$ is weakly indecomposable, then there is an algebra $A$ and ideals $H_1, H_2 \subseteq$ $\operatorname{Null}(A)$ such that for $i = 1, 2$ we have $U(A)' \cong U(A_i)'$ and $A_i \cong A/H_i$ or $A_i \cong (A/H_i)^{op}$.*


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