BRANCHING RULES FOR MODULAR FUNDAMENTAL REPRESENTATIONS OF SYMPLECTIC GROUPS

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To our teacher A. E. Zalesskii on the occasion of his sixtieth birthday

Abstract. In this paper branching rules for the fundamental representations of the symplectic groups in positive characteristic are found. The submodule structure of the restrictions of the fundamental modules for the group $Sp_{2n}(K)$ to the naturally embedded subgroup $Sp_{2n-2}(K)$ is determined. As a corollary, inductive systems of fundamental representations for $Sp_{\infty}(K)$ are classified. The submodule structure of the fundamental Weyl modules is refined.

1. Introduction

The article is devoted to finding branching rules for the fundamental representations of the symplectic groups in positive characteristic. The classical branching rules are concerned with the restrictions of representations of the classical algebraic and symmetric groups in characteristic 0 to naturally embedded subgroups of smaller ranks. For a group of rank $n$ and its fixed irreducible representation $\varphi$ they yield the composition factors of the restriction of $\varphi$ to a naturally embedded subgroup of rank $n-1$ and hence to similar subgroups of smaller ranks, at least algorithmically. These rules provide a basis for induction on rank and have found numerous applications. In positive characteristic one cannot expect to obtain complete branching rules in an explicit form in a near future since this problem is closely connected with that of finding the dimensions of arbitrary irreducible representations and the composition factors of the Weyl modules. So it is worth to investigate important particular cases where such rules can be found and to seek for asymptotic analogs of these rules. The notion of an inductive system of representations (see the definition below) introduced by Zalesskii in [11] yields an asymptotic version of the branching rules. It proved to be useful for the study of ideals in group algebras of locally finite groups as well, see, for instance, Zalesskii’s survey [12]. We classify the inductive systems of the fundamental representations for the infinite-dimensional symplectic group $Sp_{\infty}(K)$. This class of representations yields an example of representations of a simple form for which the branching rules in positive characteristic differ from the characteristic 0 case.

Let $K \subset F$ be fields of characteristic $p > 0$, $\bar{F}$ be the algebraic closure of $F$, and $\mathbb{Z}^+$ be the set of nonnegative integers. Let $G_n = Sp_{2n}(K)$. Denote by $\omega^n_i$, $0 \leq i \leq n$, the $i$th fundamental module and representation of $G_n$ over $F$ where $\omega^n_0$ is the trivial one. Let $W^n_0 = \omega^n_0, W^n_1, \ldots, W^n_n$ be the corresponding Weyl modules. Set $W^n_i = \omega^n_i = 0$ for $i < 0$ and for $i > n$. The labeling of the fundamental modules is standard, the fundamental and the Weyl modules for $G_n$ are the $F$-modules affording the restrictions to $G_n$ of the relevant representations of the group $Sp_{2n}(F)$ (it is well known that these restrictions can be realized over $F$). For an integer $z > 0$ we denote by $l(p(z))$ the maximal $i$ such that

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Let $p^d \mid z$. We have $lp(z) = 0$ if $p \not| z$. Let $M$ be a $G_n$-module. The restriction of $M$ to $G_{n-1}$ is denoted by $M \downarrow G_{n-1}$. We shall write

$$M \sim N_1 + \cdots + N_q$$

if there is a series $0 = M_0 \subset M_1 \subset \cdots \subset M_q = M$ of submodules of $M$ and a permutation $\sigma$ such that $N_{\sigma(i)} \cong M_i / M_{i-1}$ for all $i = 1, \ldots, q$. Moreover, if in addition $M_i / M_{i-1}$ coincides with the socle of $M / M_{i-1}$ for $i = 1, \ldots, q$, then the sequence

$$N_{\sigma(1)} \prec_s N_{\sigma(2)} \prec_s \cdots \prec_s N_{\sigma(q)}$$

is called the socle series of $M$.

Theorem 1.1 below describes the branching rules for the fundamental $G_n$-modules and the submodule structure of the restrictions of these modules to $G_{n-1}$.

**Theorem 1.1.** Let $n \geq 2$ and $0 \leq i \leq n$. Set $d = lp(n - i + 1)$; $\varepsilon = 0$ if $n - i + 1 \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise. Then

(i) $\omega_i^n \downarrow G_{n-1} \sim \omega_i^{n-1} + 2\omega_{i-1}^{n-1} + \left(\sum_{t=0}^{d-1} 2\omega_{i-2p^t}^{n-1}\right) + \varepsilon\omega_{i-2p^d}^{n-1}$ (the sum in the brackets is zero whenever $d = 0$);

(ii) $\omega_i^n \downarrow G_{n-1} = \omega_i^{n-1} \oplus \omega_{i-1}^{n-1} \oplus D$ and the series

$$\omega_{i-2}^{n-1} \prec_s \omega_{i-2p}^{n-1} \prec_s \cdots \prec_s \omega_{i-2p^{d-1}} \prec_s \omega \prec_s \omega_{i-2p^{d-1}} \prec_s \cdots \prec_s \omega \prec_s \omega_{i-2p} \prec_s \omega_{i-2} \prec_s \omega_i^{n-1}$$

with $\omega = \omega_i^{n-1} \oplus \varepsilon\omega_{i-2p}^{n-1}$ and $\omega_{i}^{n-1}$ omitted for $j < 0$ is the socle series of $D$. In particular, $D = \omega$ if $i = 0, 1$ or $p \not| n + 1 - i$.

**Corollary 1.2.** For $n \geq 2$ the restriction $\omega_i^n \downarrow G_{n-1}$ is completely reducible if and only if $i = 0, 1$ or $p \not| n + 1 - i$.

The proof of Theorem 1.1 is based on the description of the composition factors of the fundamental Weyl modules (Premet and Suprunenko [6, Theorem 2] for $p > 2$ and independently Adamovich [1, Theorem 2 and its Corollary 1] for arbitrary $p$) and Adamovich’s results [2] on the submodule structure of these Weyl modules. In Section 2 these results are refined (Theorem 2.13). In particular, a new irreducibility criterion for the fundamental Weyl modules is obtained (Corollary 2.14) and it is proved that their socles are always simple (Corollary 2.15).

For $n - p + 2 \leq i \leq n$ Gow [5] has given an explicit construction of the modules $\omega_i^n$ and has described the submodule structure of the restrictions $\omega_i^n \downarrow G_1 \times G_{n-1}$ (the natural embedding) ([5, Theorem 2.2]). This implies our Theorem 1.1 for these modules. In [5] a certain explicitly determined operator $\delta$ on the exterior algebra $\wedge V$ of the natural $G_n$-module $V$ is considered and it is proved that for $n - p + 2 \leq i \leq n$ the module $\omega_i^n$ can be realized as the quotient $\ker \delta \cap \wedge^i V / \delta^{p-1}(\wedge^{i+2p^2-2} V)$ ([5, Corollary 2.4]). This nice construction gives a realization for an important class of modules without complicated representation-theoretic machinary. However, it cannot be extended to other fundamental modules since according to [5, Theorem 4.2], the quotient above is zero for $i < n - p + 2$.

In [7] Sheth has found the branching rules for modular representations of symmetric groups corresponding to two part partitions. The composition factors occurring in the relevant restrictions are similar to those of the module $D$ in Theorem 1.1(ii). We conjecture that the submodule structure of these restrictions is also similar to that of $D$. The authors plan to consider this question as well as the similar one for representations of special linear groups with highest weights $\omega_j + \omega_j$ in a subsequent paper.

In Section 4 Theorem 1.1 is applied to classify the inductive systems of fundamental $F$-representations for $Sp_{n\omega}(K)$. Let

$$H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots \quad (1)$$
be a sequence of groups, and $\Psi_n, n = 1, 2, \ldots$, be a nonempty finite set of (inequivalent) irreducible representations of $H_n$ over a fixed field. The system $\Psi = \{\Psi_n \mid n = 1, 2, \ldots\}$ is called an inductive system (of representations) for the group $H = \bigcup_{n=1}^{\infty} H_n$ if each $\Psi_n$ coincides with the union of the sets of composition factors (up to equivalence) of the restrictions $\pi_n H_n$ where $\pi$ runs over $\Psi_{n+1}$. In this article (1) is the sequence of the naturally embedded groups $G_n = Sp_{2n}(K)$, so $\bigcup_{n=1}^{\infty} G_n = Sp_{\infty}(K)$. Set
\[
\mathcal{F}_n = \{\omega_i^n \mid 0 \leq i \leq n\}, \quad \mathcal{F} = \{\mathcal{F}_n \mid n = 1, 2, \ldots\};
\[
\mathcal{L}_n^s = \{\omega_i^n \mid 0 \leq i \leq s\}, \quad \mathcal{L}^s = \{\mathcal{L}_n^s \mid n = 1, 2, \ldots\};
\[
\mathcal{R}_n^u = \{\omega_i^n \mid n + 1 - u \leq i \leq n\}, \quad \mathcal{R}^u = \{\mathcal{R}_n^u \mid n = 1, 2, \ldots\}.
\]

**Theorem 1.3.** The inductive systems of fundamental representations over $F$ for $Sp_{\infty}(K)$ are exhausted by the systems $\mathcal{F}, \mathcal{L}^s, \mathcal{R}^{p-1}$, and $\mathcal{L}^s \cup \mathcal{R}^{p-1}$ ($s \geq 0, t \geq 1$).

It is clear that $\mathcal{L}^0$ (which consists of the trivial representations) and $\mathcal{R}^{p-1}$ are minimal inductive systems. However, the question on the minimal inductive systems for $Sp_{\infty}(K)$ is far from solution. For $p > 2$ Zalesskii and Suprunenko [10] have described the inductive systems $\Phi = \{\Phi_n \mid n = 1, 2, \ldots\}$ where for each $n$ the set $\Phi_n$ consists of two irreducible representations with highest weights $\omega_{n+1} - \frac{1}{2}(p-3)\omega_n$ and $\frac{1}{2}(p-1)\omega_n$. The system $\Phi$ coincides with $\mathcal{R}^2$ for $p = 3$ and yields another example of a minimal inductive system for $p > 3$.

For other classical groups the questions investigated in this paper present no problems since the situation is the same as in characteristic 0.

The authors [3] have found the minimal and the minimal nontrivial inductive systems for the group $SL_{\infty}(K)$. For this group the system consisting of the trivial representation is the only minimal inductive system, and the minimal nontrivial ones are exhausted by the systems $L^n = \{L^n_n \mid n = 1, 2, \ldots\}$ and $R^n = \{R^n_n \mid n = 1, 2, \ldots\}$ where $L^n_n$ consists of two irreducible representations of $SL_{n+1}(K)$ with highest weights 0 and $p^i\omega_1$ and $R^n_n$ of those with highest weights 0 and $p^i\omega_n$. The picture is similar for the groups $SL_{\infty}$ and $SU_{\infty}$ over locally finite fields.

Until Proposition 4.2 we assume that $K = F = \bar{F}$. At the end Proposition 4.2 transfers the results to arbitrary fields.

2. THE STRUCTURE OF THE FUNDAMENTAL WEYL MODULES

In this section we refine the results of [6], [1], and [2] on the structure of the fundamental Weyl modules for $G_n$.

Throughout the paper we set $\pi_n^n = \omega_{n+1-n}^n$ and $V^n_j = W^n_{n+1-n}$. We denote by $[a, b]$ the set of all $j \in \mathbb{Z}^+$ with $a \leq j \leq b$. For an integer $k \in \mathbb{Z}^+$ write its $p$-adic expansion $k = k_0 + k_1 p + \cdots + k_s p^s$ with $0 \leq k_i < p$ and set $k_i = 0$ for all such $i \in \mathbb{Z}^+$ that $p^i > k$. We shall write $k = (k_0, k_1, \ldots, k_s)$. We say that an integer $m$ contains $k$ to base $p$ and write $k \subset_p m$ if and only if for each $i$ either $k_i = 0$, or $k_i = m_i$. Set $d_k^n = 1$ if $k \subset_p m$, and $d_k^n = 0$ otherwise.

**Theorem 2.1.** [6, Theorem 2] Let $p > 2$. Then $W^n_i \sim \sum_{k=0}^{\infty} d_k^{n+1-i+2k} \omega_i^{n-2k}$.

We need some more notation to state Adamovich’s results. For $\lambda \in \mathbb{Z}^+$ define maps $s^n_{\lambda}: \mathbb{Z}^+ \to \mathbb{Z}^+$ and $s^\lambda: \mathbb{Z}^+ \to \mathbb{Z}^+$ setting
\[
s^n_{\lambda}(l) = l + 2k' \quad \text{where} \quad l + 1 = d' p^\lambda - k', \quad d' \in \mathbb{Z}^+, \quad 0 \leq k' < p^\lambda;
\[
s^\lambda(l) = l + 2k \quad \text{where} \quad l = a p^\lambda - k, \quad a \in \mathbb{Z}^+, \quad 0 \leq k < p^\lambda.
\]

We say that the reflection $s^\lambda_{\lambda}$ or $s^n_{\lambda}$ is $l$-admissible if $k' \neq 0$ and $p \nmid a'$ or $k \neq 0$ and $p \nmid a$, respectively. We denote by $S(l)$ the set of all $m > l$ that can be written in the form $m = s_{\lambda_n} \cdots s_{\lambda_1}(l)$ where $\lambda_n < \cdots < \lambda_1$ and for each $i = 0, 1, \ldots, u - 1$ the reflection $s_{\lambda_{i+1}}$ is $s_{\lambda_i}, \ldots, s_{\lambda_1}(l)$-admissible. Similarly we define $S'(l)$ (writing $s'_{\lambda_i}$ instead of $s_{\lambda_i}$).
Theorem 2.2. [1] Let $0 \leq l \leq n$. Then $V_{l+1}^n \sim \pi_{l+1}^n + \sum_{m \in S(l)} \pi_{m+1}^n$.

As $s'_{\lambda}(x-1) = s_{\lambda}(x) - 1$, the following theorem yields an equivalent statement.

Theorem 2.3. Let $1 \leq l \leq n + 1$. Then $V_l^n \sim \pi_l^n + \sum_{m \in S(l)} \pi_m^n$.

Let us rewrite Theorem 2.1 in terms of $\pi_m^n$ and $V_l^n$ (without restrictions on $p$).

Theorem 2.4. Let $1 \leq l \leq n + 1$. Then $V_l^n \sim \sum_{k=0}^{\infty} d_k^{l+2k} \pi_{l+2k}^n$.

Now our goal is to show that Theorems 2.3 and 2.4 are equivalent, so Theorem 2.4 (and 2.1) holds for $p = 2$. For this purpose we prove some technical facts on the triples $k, l, m$ with $k \subset_p m = l + 2k$ and admissible reflections.

Until the end of the section $l \geq 1$. For each $m \in S(l)$ the tuple $(\lambda_1; \ldots; \lambda_u)$ is uniquely determined (see the comments before the Theorem in [2]). If $u$ is odd for some $m$, set $\lambda_{u+1} = l \rho(m)$. Then $s_{\lambda_{u+1}}(m) = m$ and $\lambda_{u+1} < \lambda_u$. Now for every $m \in S(l)$ we have a uniquely determined sequence of reflections $s_{\lambda_1}, \ldots, s_{\lambda_{2t}}$. Such sequences will be called $l$-admissible. For an integer $0 \leq a \leq p - 1$ set $\bar{a} = p - 1 - a$. The following lemma is straightforward.

Lemma 2.5. Set $q = l \rho(l)$. The reflection $s_{\lambda}$ is $l$-admissible if and only if $\lambda > q$ and $l_\lambda \neq p - 1$. In that case $s_{\lambda}(l) = \rho \left( l_q + 1, l_{q+1}, \ldots, l_{\lambda-1}, l_\lambda + 1, l_{\lambda+1}, \ldots, l_t \right)$.

Two consequent applications of Lemma 2.5 yield

Proposition 2.6. Let $l \rho(l) \leq \mu \leq l_\mu$, $m = s_{l_\mu} s_{\lambda}(l)$, and $k = (m - l)/2$. The pair $s_{\lambda}, s_{\mu}$ is $l$-admissible if and only if $l_{\lambda} \neq p - 1$ and $l_{\mu} \neq 0$. In that case

$$
\begin{align*}
  m &= (l_0, \ldots, l_{\mu-1}, l_{\mu} + 1, l_{\mu+1}, \ldots, l_{\lambda-1}, l_{\lambda}, l_{\lambda+1}, \ldots), \\
  k &= (0, \ldots, 0, l_{\mu} + 1, l_{\mu+1}, \ldots, l_{\lambda-1}).
\end{align*}
$$

In particular, if $l_0 \subset_p m = l + 2k$. We call a tuple $\sigma = (\lambda_1; \ldots; \lambda_{2t})$ l-admissible if $\lambda_i \in \mathbb{Z}^+$, $\lambda_1 > \cdots > \lambda_{2t}, l_{\lambda_{j-1}} \neq p - 1$ and $l_{\lambda_{2j}} \neq 0$ for $j = 1, \ldots, t$. For an l-admissible $\sigma$ set

$$
\Omega_l(\sigma) = \bigcup_{j=1}^{t} [\lambda_{2j}, \lambda_{2j-1} - 1],
$$

$\delta_{i,\sigma} = 1$ if $i = \lambda_j$ for some $j$ and $\delta_{i,\sigma} = 0$ otherwise. Define $l^\sigma \subset \mathbb{Z}^+$ putting

$$
l^\sigma_i = \begin{cases} 
  l_i + \delta_{i,\sigma}, & i \in \Omega_l(\sigma) \\
  l_i + \delta_{i,\sigma}, & i \notin \Omega_l(\sigma).
\end{cases}
$$

Proposition 2.6 yields the following corollary.

Corollary 2.7. A sequence $s_{\lambda_1}, \ldots, s_{\lambda_{2t}}$ is l-admissible if and only if the tuple $\sigma = (\lambda_1; \ldots; \lambda_{2t})$ is l-admissible. In that case $s_{\lambda_1} \cdots s_{\lambda_t}(l) = l^\sigma$.

Proposition 2.8. An integer $m \in S(l)$ if and only if $m - l = 2k > 0$ and $k \subset_p m$.

Proof. Let $\sigma = (\lambda_1; \ldots; \lambda_{2t})$ be an l-admissible tuple and $m = s_{\lambda_2} \cdots s_{\lambda_t}(l)$. Set $m^0 = l$, $m^j = s_{\lambda_{2j}} \cdots s_{\lambda_{t}}(l)$, and $k^j = (m^j - m^{j-1})/2, 1 \leq j \leq t$. Using Proposition 2.6, one deduces that $k^j \subset_p m^j; m^j = l^{\sigma^j}$ with $\sigma^j = (\lambda_1; \ldots; \lambda_{2j}); k_{i,\sigma}^j = 0$ and $m_{i,\sigma}^j = m_{i,\sigma}^j - 1$ for $i \notin [\lambda_{2j}, \lambda_{2j-1}]$. Therefore $k = k^1 + \cdots + k^t \subset_p m$.

Assume now that $m - l = 2k > 0$ and $k \subset_p m$. Choose integers $\tau_1, \tau_2, \ldots, \tau_{2t}$ as follows. Set $\tau_0 = -1$. Assume that $\tau_{2j}$ is chosen. If there is no $i > \tau_{2j}$ such that $k_i = m_i \neq 0$, we set $\tau_{2j+1} = \tau_{2j} + 1$ and stop the process. Otherwise we choose for $\tau_{2j+1}$ minimal $i > \tau_{2j}$ with $k_i = m_i \neq 0$. As $m > 2k$, there exists $f > \tau_{2j+1}$ with $k_f \neq m_f$ (observe that in this case
We choose minimal such \( f \) for \( \tau_{2j+2} \). Set \( \lambda_q = \tau_{2q+1} - q \). Since \( k \subset p \), using Corollary 2.7 and analyzing the \( p \)-adic expansions of \( k \) and \( m \), one can conclude that the tuple \( \sigma = (\lambda_1; \ldots; \lambda_{2t}) \) is \( l \)-admissible and \( m = l^q \).

**Corollary 2.9.** Theorems 2.3 and 2.4 are equivalent, so Theorems 2.1 and 2.4 are valid in characteristic 2 as well.

Now we rewrite Adamovich’s results [2] on the submodule structure of the Weyl modules in our terms. We fix \( n \) and write \( V_l \) and \( \pi_m \) instead of \( V_l^n \) and \( \pi_m^n \). For \( m \in S(l) \) or \( m = l \) we denote by \( P_l(m) \) the smallest submodule of \( V_l \) that has a composition factor \( \pi_m \). Since \( V_l \) is multiplicity-free, \( P_l(m) \) is correctly defined and each submodule of \( V_l \) is a sum of \( P_l(m) \) for some \( m \). Hence the submodule structure of \( V_l \) is determined by the inclusion relations between the submodules \( P_l(m) \) (see also comments at the beginning of [2]). We shall write \( \pi_m \prec \pi_q \) if \( P_l(m) \subset P_l(q) \). Let \( \sigma = (\lambda_1; \ldots; \lambda_{2t}) \) be an \( l \)-admissible tuple. For \( m = l^q \) set

\[
\Omega_l(m) = \Omega_l(\sigma) = \bigcup_{j=1}^{l} [\lambda_{2j}, \lambda_{2j-1} - 1].
\]

Note that \( \Omega_l(m) = \bigcup_{j=1}^{f} [\tau_{2j-1}, \tau_{2j} - 1] \) where \( \tau_i \) are as in the proof of Proposition 2.8. For instance, for \( p = 3 \),

\[
m = (0, 1, 2, 2, 0, 1, 0, 2, 1, 0, 0, 2, 1),
m = (0, 0, 0, 2, 0, 0, 0, 2, 1, 0, 0, 0),
\]

we have \( \Omega_{m-2k}(m) = [2, 4] \cup [7, 10] \). Put also \( \Omega_l(l) = \emptyset \).

**Theorem 2.10.** [2] For \( 1 \leq m, q \leq n + 1 \) and \( m, q \in S(l) \cup \{l\} \) the module \( \pi_m \prec \pi_q \) (as composition factors of \( V_l \)) if and only if \( \Omega_l(q) \subset \Omega_l(m) \).

**Remark 2.11.** Actually the sets \( \mathfrak{P}_l(m) \) which are considered in [2] differ slightly from \( \Omega_l(m) \). For \( m \in S(l) \) one has \( \mathfrak{P}_l(m) = \bigcup_{j=1}^{f} [\mu_{2j} + 1, \mu_{2j-1}] \) where \( \mu_s = \lambda_s \) for \( s < 2t \), \( \mu_{2t} = \lambda_{2t} \) if \( m \neq s\lambda_{2t-1} \ldots s\lambda_1(l) \), and \( \mu_{2t} = 0 \) otherwise. However, Lemma 2.5 enables one to deduce that \( \mathfrak{P}_l(m) \subset \mathfrak{P}_l(q) \) if and only if \( \Omega_l(m) \subset \Omega_l(q) \). The crucial point is that \( \operatorname{lp}(l) = \operatorname{lp}(m) \) for \( m \in S(l) \).

For \( l \)-admissible tuples \( \sigma = (\lambda_1; \ldots; \lambda_{2t}) \) and \( \sigma' = (\lambda'_1; \ldots; \lambda'_{2s}) \) we say that \( \sigma \leq \sigma' \) if there exists \( f \leq 2t, 2s \) such that \( \lambda_i = \lambda'_i \) for \( 1 \leq i \leq f \) and either \( f = 2t \), or \( f < 2t, 2s \) and \( \lambda_{f+1} < \lambda'_{f+1} \). It is convenient to assume that the empty tuple \( \emptyset \) is \( l \)-admissible, \( \emptyset \leq \sigma \) for all \( \sigma \), \( l^0 = l \), and \( \Omega_l(\emptyset) = \emptyset \). The following is obvious.

**Lemma 2.12.** Let \( \sigma \) and \( \sigma' \) be \( l \)-admissible tuples. Then \( l^\sigma \leq l^{\sigma'} \) if and only if \( \sigma \leq \sigma' \).

Set \( n' = n + 1 \). Construct an \( l \)-admissible tuple \( \sigma^\text{max} = (\mu_1; \ldots; \mu_{2t}) \) as follows. Put \( \mu_0 = +\infty \). Assume that \( \mu_{2j} \) is chosen. Set \( \mu = \mu_{2j} - 1 \). If there is no \( l \)-admissible tuple \( (\alpha; \beta) \) such that \( \mu \geq \alpha > \beta \) and

\[
(\bar{l}_0, \ldots, \bar{l}_{\mu})(\alpha; \beta) = (\bar{l}_0, \ldots, \bar{l}_{\beta} + 1, \bar{l}_{\beta+1}, \ldots, \bar{l}_{\alpha-1}, \bar{l}_{\alpha} + 1, \ldots, \bar{l}_{\mu}) \leq (n'_0, \ldots, n'_\mu),
\]

we stop the process and set \( t = j \) \( (\sigma^\text{max} = \emptyset \) if \( t = 0 \)). Otherwise we choose maximal such pair \( (\alpha; \beta) \) (with respect to \( \leq \)) set \( \mu_{2j+1} = \alpha \) and \( \mu_{2j+2} = \beta \); and if

\[
(\bar{l}_\beta + 1, \bar{l}_{\beta+1}, \ldots, \bar{l}_{\alpha-1}, \bar{l}_{\alpha} + 1) < (n'_0, \ldots, n'_\alpha),
\]

we stop the process and determine \( (\mu_{2j+3}; \ldots; \mu_{2t}) \) as the maximal \( l \)-admissible tuple with \( \mu_{2j+3} < \beta \). Obviously, \( l^\sigma^\text{max} \) is the maximal integer \( m \) such that \( \pi_m^n \) is a composition factor of \( V_l^n \).
For $l$-admissible tuples $\sigma$ and $\sigma'$ we write $\sigma \prec \sigma'$ if and only if $\Omega_l(\sigma) \supset \Omega_l(\sigma')$. Using Corollary 2.7, Theorem 2.10, and Lemma 2.12, we get our main result on the structure of fundamental Weyl modules.

**Theorem 2.13.** The map $\sigma \mapsto \pi^n_\tau$ is a poset isomorphism between the $l$-admissible tuples $\sigma \leq \sigma^{\max}$ and the composition factors of $V^n_\tau$ with the partial orders $\prec$.

If $l < n'$, we denote by $v$ the maximal integer such that $l_v \neq n'_v$. If $l_v + 1 = n'_v$, we denote by $u$ the maximal integer $< v$ such that $l_u \neq n'_u$ setting $u = -1$ if $l_i = n'_i$ for all $i < v$. Put $s = \text{lp}(l)$.

**Corollary 2.14.** Let $n' = n + 1$ and $1 \leq l \leq n'$. Then $V^n_l$ is irreducible (i.e. $\sigma^{\max} = \emptyset$) if and only if one of the following holds.
1. $l = n'$;
2. $l < n'$ and $s \geq v$;
3. $l < n'$, $s < v$, $l_v + 1 = n'_v$, $\bar{l}_s \geq n'_s$; $l_i = p - 1$ and $n'_i = 0$ for $s < i < v$.

**Proof.** This follows from Proposition 2.6 and Corollary 2.7.

**Corollary 2.15.** Let $n' = n + 1$ and $1 \leq l \leq n'$. The socle of $V^n_l$ is always simple. For reducible $V^n_l$ it has the form $\pi^n_\tau$ with $\gamma = (t; s)$ and $t$ as follows.
1. $t = v$ if $s < v$ and either $l_v + 1 < n'_v$, or $\bar{l}_u < n'_u$;
2. $t = w$ if $l_w + 1 = n'_w$; $u = -1$ or $\bar{l}_u > n'_u$; $s < w < v$; $l_w \neq p - 1$; and $l_j = p - 1$ for $w < j < v$.

**Proof.** Applying Results 2.6, 2.7, and 2.13, we conclude that $\pi^n_\tau$ is a composition factor of $V^n_l$ and for each $l$-admissible tuple $\tau \leq \sigma^{\max}$ the set $Q_l(\tau) \subset [s, t - 1]$, so $\pi^n_\tau \prec \pi^{n'}_\tau$.  

3. BRANCHING RULES AND THE SUBMODULE STRUCTURE OF THE RESTRICTIONS

In this section the main results of the article are proved. We shall need the following simple lemma.

**Lemma 3.1.** Assume that $d_k^{i+2k} = 1$ (i.e. $k \subset_p l + 2k$).
1. If $p^s \mid l + 2k$, then $p^s \mid k$ and $p^s \mid l$.
2. If $p^s \mid l$, then $p^s \mid k$ and $p^s \mid l + 2k$.

**Proof.** One can assume that $s \geq 1$.
1. Let $p^s \mid l + 2k$. Since $k \subset_p l + 2k$, we have $p^s \mid k$. This implies that $p^s \mid l$.
2. Let $p^s \mid l$. Then $l_0 = \cdots = l_{s-1} = 0$. Let $r = \text{lp}(k)$. Assume that $r < s$. Since $k \subset_p l + 2k$, we have $k_r = (2k)_r \neq 0$, which is impossible. Therefore $r \geq s$, so $p^s \mid k$ and $p^s \mid l + 2k$.

As in Section 2, we shall omit the superscript $n$ in our notation for modules when it is known which group is considered. Replacing $\omega_i$ by $\pi_{n+1-i}$ and $W_i$ by $V_{n+1-i}$, one immediately concludes that Theorem 1.1(i) is equivalent to the following

**Theorem 3.2.** Let $1 \leq i \leq n + 1$ and $d = \text{lp}(i)$. Then
\[ \pi^n_i \downarrow G_{n-1} \sim \pi_{i-1} + 2\pi_i + \left( \sum_{t=0}^{d-1} 2\pi_{i-1+2p^t} \right) + \varepsilon \pi_{i-1+2p^t} \]
where $\varepsilon = 0$ if $i \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise.

**Proof.** One can rewrite the formula in Theorem 3.2 as follows.
\[ \pi^n_i \downarrow G_{n-1} \sim \pi_{i-1} + 2\pi_i + \sum_{t=0}^{\infty} b_t^i \pi_{i-1+2p^t} \]
(2)}
where
\[
b_i^t = \begin{cases} 
2, & i \equiv 0 \pmod{p^{t+1}}, \\
1, & i \equiv ap^t \pmod{p^{t+1}} \text{ and } a \neq 0, -1 \pmod{p}, \\
0, & i \not\equiv 0 \pmod{p^t} \text{ or } i \equiv -p^t \pmod{p^{t+1}}. 
\end{cases}
\] (3)

Recall that by convention \(\pi_i^n = 0\) for all \(i > n + 1\), and \(\pi_i^{n+1}\) is the trivial one-dimensional \(G_n\)-module. So (2) holds for \(i \geq n + 1\). Assume now that \(1 \leq l < n + 1\) and (2) is valid for all \(i > l\). We shall prove it for \(i = l\). Then the theorem will follow by induction.

It follows from [4, Proposition 3.3.2 and Theorem 4.3.1] that \(V_{l+1}G_{n-1}\) has a filtration by Weyl modules for \(G_{n-1}\). Then the classical branching rules for characteristic 0 [13] and Theorem 2.4 imply
\[
V_l^n \downarrow G_{n-1} \sim V_{l-1} + 2V_l + V_{l+1} \sim 2V + \sum_{t=0}^{\infty} f_{l+2t-1}^{l+2t-1} \pi_{l+2t-1} 
\] (4)

where \(V = \sum_{k=0}^{\infty} d_k^{l+2k} \pi_{l+2k-1} + \sum_{s=0}^{\infty} b_s^{l+2k} \pi_{l+2k+2p^s-1} \) and
\[
f_{l+2t-1} = d_{l+2t}^{l+2t-1} + \sum_{k,s \geq 0, k+p^s = t} d_k^{l+2k} b_s^{l+2k}.
\] (5)

We have to show that \(e_{l+2t-1} = f_{l+2t-1}^{l+2t-1}\) for all \(t \geq 0\). Note that \(f_{l+2t-1}^{l+2t-1} \leq 2\). We proceed by steps.

**Step 1. At most one summand in (5) is nonzero. In particular, \(e_{l+2t-1}^{l+2t-1} \leq 2\).**

Assume that \(d_k^{l+2k} b_s^{l+2k} \neq 0\) and \(d^{l+2k} b_s^{l+2k} \neq 0\) with \(t = k + p^s = k' + p^{s'}\) and \(s > s'\). Since \(b_s^{l+2k} \neq 0\), we have \(p^s | l + 2k\). As \(d_k^{l+2k}, d_k^{l+2k} \neq 0\), by Lemma 3.1, \(p^s\) divides \(k\), \(l\), and \(k'\). Hence \(p^s | k + p^s - k' = p^{s'}\), which yields a contradiction.

Now assume that \(d_k^{l+2t} \neq 0\) and \(d_k^{l+2k} b_s^{l+2k} \neq 0\) with \(k + p^s = t\). As above, we get that \(p^s\) divides \(k\), \(l\), and \(t\). Let \(r = \text{lp}(l)\). Then by Lemma 3.1 (ii), \(p^r | k\) and \(p^r | t\). Since \(k + p^s = t\), we have \(r = s\), and \(s \neq 0\). Consider the following cases.

**Case 1.** \(k \neq 0\) and \(t \equiv 0 \pmod{p^{s+1}}\). Then \(l + 2t \equiv t\) and \(l + 2k \equiv k \pmod{p^{s+1}}\), so \(p^s = t - k \equiv 2(t - k) \pmod{p^{s+1}}\), which is impossible.

**Case 2.** \(k \equiv 0 \pmod{p^{s+1}}\). Then \(t = 1\). Since \(l + 2t \equiv t\) (mod \(p^{s+1}\)), we have \(s = p - 1\), so \((l + 2k)_s = p - 1\). This implies that \(b_s^{l+2k} = 0\) and yields a contradiction.

**Case 3.** \(t \equiv 0 \pmod{p^{s+1}}\). Then \(k_s = p - 1\). Therefore \((l + 2k)_s = k_s = p - 1\), and so above, \(b_s^{l+2k} = 0\).

**Step 2.** \(e_{l+2t-1}^{l+2t-1} = 2\) if and only if \(f_{l+2t-1}^{l+2t-1} = 2\) (equivalently, \(d_{l+2t-1}^{l+2t-1} = d_{l-1+2t-1}^{l+2t-1} = 1\)).

Assume that \(e_{l+2t-1}^{l+2t-1} = 2\). By Step 1, this is equivalent to the following: there exist \(k, s \geq 0\) with \(k + p^s = t\) such that \(b_s^{l+2k} = 2 \neq d_k^{l+2k} = 1\). Hence \(p^{s+1} | l + 2k\) and \(k \equiv 0 \pmod{l+2k}\). By Lemma 3.1 (i), \(p^{s+1}\) divides \(l + 2k + p^s(p^s - 1)\).
Therefore $t = k + p^s \subset_p l + 2t - 1$ and $t - 1 = k + (p^s - 1) \subset_p l + 2t - 1$, so $d^{t+2t-1}_t = d^{t+2t-1}_{t-1} = 1$, as required.

Assume now that $d^{t+2t-1}_t = d^{t+2t-1}_{t-1} = 1$. Let $s = \text{lp}(t)$. Then $t_s \neq 0$ and $(t - 1)_s = t_{s-1}$. Since both $t$ and $t - 1$ are contained in $l + 2t - 1$, we have $t_s = 1$. Moreover, we have $(l - 1)_s = \cdots = (l - 1)_{s-1} = (t - 1)_s = \cdots = (t - 1)_{s-1} = p - 1$. Since $(l + 2t - 1)_s = t_s = 1$, we get $(l - 1)_s = p - 1$. Hence $p^{s+1} \mid l$. Set $k = t - p^s$. Then $p^{s+1} \mid l + 2k$ and $b^{k+2k}_s = 2$. It remains to observe that $k = t - p^s \subset_p (l + 2t - 1) - p^s - (p^s - 1)$, so $d^{k+2k}_k = 1$ and $e^{t+2t-1}_t = d^{k+2k}_{k+2k}$.

**Step 3.** If $e^{t+2t-1}_t \neq 0$, then $f^{t+2t-1}_t \neq 0$, i.e. $t \subset_p l + 2t - 1$ or $t - 1 \subset_p l + 2t - 1$.

Let $r = \text{lp}(l)$. First assume that $d^{t+2t}_t = 1$ (see (5)), i.e. $t \subset_p l + 2t$. Then by Lemma 3.1 (ii), $p^s \mid t$. One easily checks that if $p^{s+1} \nmid t$, then $t - 1 \subset_p l + 2t - 1$, and if $p^{s+1} \mid t$, then $t \subset_p l + 2t - 1$, as required.

Assume now that there exist $k, s \geq 0$ with $k + p^s = t$ such that $d^{k+2k}_k = 1$ and $b^{k+2k}_s \neq 0$. By Lemma 3.1, $p^s \mid k$ and $\text{lp}(l + 2k) = r$. If $b^{k+2k}_s = 2$, then by Step 2, $f^{t+2t-1}_t = 2 \neq 0$. Hence we can assume that $b^{k+2k}_s = 1$. By (3), $p^s \mid l + 2k$ and $(l + 2k)_s \neq 0$, $p - 1$, so $r = s$. Assume that $k_s = 0$. Then $t - 1)_s = 0$ and $t - 1 = k + p^s - 1 \subset_p l + 2k + p^s + (p^s - 1) = l + 2t - 1$. If $k_s \neq 0$, we have $k_s = (l + 2k)_s \neq -p - 1$, so $t = k + p^s \subset_p l + 2k + p^s + (p^s - 1) = l + 2t - 1$, as required.

**Step 4.** If $f^{t+2t-1}_t \neq 0$, then $e^{t+2t-1}_t \neq 0$.

If $t = 0$, then $e^{t-1}_t = 1$, so assume that $t \geq 1$. We have either $t \subset_p l + 2t - 1$, or $t - 1 \subset_p l + 2t - 1$. One needs to show that either $t \subset_p l + 2t$ (i.e. $d^{t+2t}_t = 1$), or there exist $k, s \geq 0$ with $k + p^s = t$ such that $k \subset_p l + 2k$, $p^s \mid l + 2k$, and $(l + 2k)_s \neq p - 1$ (i.e. $d^{k+2k}_k = 1$ and $b^{k+2k}_s \neq 0$). First assume that $t \subset_p l + 2t - 1$. Let $s = \text{lp}(t)$. We have $(l + 2t - 1)_s = t_s \neq 0$. If $p^s \mid l + 2t$, then $k = t - p^s \subset_p l + 2t - 1 - p^s - (p^s - 1) = l + 2k$, $p^s \mid l + 2t - 2p^s = l + 2k$, and $(l + 2k)_s = (l + 2t - 1)_s \neq p - 1$, as required. If $p^s \nmid l + 2t$, one gets $t \subset_p (l + 2t - 1) + 1 = l + 2t$, as desired.

Assume now that $t - 1 \subset_p l + 2t - 1$. Consider the following cases.

**Case 1.** $(t - 1)_s = 0$. If $(l + 2t - 1)_s = 0$, then $t \subset_p l + 2t$. Assume that $(l + 2t - 1)_s = 0$. Set $s = 0$, $k = t - 1$. Then $k = t - 1 \subset_p (l + 2t - 1) - 1 = l + 2k$ and $(l + 2k)_s = (l + 2t - 1)_s \neq p - 1$, as required.

**Case 2.** $(t - 1)_s = 0, p - 1$. Then $(l + 2t - 1)_s = 0$, so $t \subset_p l + 2t$.

**Case 3.** $p \mid t$. Let $s = \text{lp}(t)$. Since $t - 1 \subset_p l + 2t - 1$, we have $p^s \mid l + 2t$, so $p^s \mid l$. As $p^{s+1} \nmid t$, the integer $(t - 1)_s \neq p - 1$. If $(t - 1)_s = 0$, then $(l + 2t - 1)_s = (t - 1)_s \neq p - 1$, so $t \subset_p l + 2t$. Assume now that $(t - 1)_s = 0$. This implies $t_s = 1$. If $(l + 2t - 1)_s = 0$, then $t \subset_p l + 2t$. Therefore one can suppose that $(l + 2t - 1)_s \geq 1$. Then $k = t - p^s \subset_p (l + 2t - 1) - p^s - (p^s - 1) = l + 2k$, $p^s \mid l + 2t - 2p^s = l + 2k$, and $(l + 2k)_s = (l + 2t - 1)_s \neq p - 1$, as required.

Now we investigate the submodule structure of the restriction $\pi^n_i \downarrow G_{n-1}$. Let $n > 1$ and $1 \leq i \leq n$. As $\pi^n_i$ is the top composition factor of $V_i^n$, it follows from (4) that $\pi^n_i \downarrow G_{n-1}$ is a quotient of the $G_{n-1}$-module $V_i^n \downarrow G_{n-1} \sim V_{i-1} + 2V_i + V_{i+1}$. Applying Smith’s theorem [8] both to $V_i$ and $\pi_i$, we conclude that $V_i^n \downarrow G_{n-1} = V_i \oplus V_i \oplus V$ where $V \sim V_{i-1} + V_{i+1}$, and $\pi_i^n \downarrow G_{n-1} = \pi_i \oplus \pi_i \oplus D$ where $D$ is a quotient of $V$. Now Theorem 1.1(ii) and Corollary 1.2 follow immediately from

**Theorem 3.3.** Let $d = \text{lp} i$. Set $\varepsilon = 0$ if $i = - p^d (\mod p^{d+1})$ and $\varepsilon = 1$ otherwise; $j_q = i - 1 + 2p^q$. Choose minimal $t \in \mathbb{Z}^+$ such that $j_t > n$. Put $d' = \min \{d, t\}$. Then

$$\pi_{j_0} \prec_\pi \pi_{j_1} \prec_\pi \cdots \prec_\pi \pi_{j_{d-1}} \prec_\pi \pi_{i-1} \oplus \varepsilon \pi_{j_d} \prec_\pi \cdots \prec_\pi \pi_{j_{d'-1}} \prec_\pi \cdots \prec_\pi \pi_{j_1} \prec_\pi \pi_{j_0}$$
is the socle series of $D$. In particular, $D = \pi_{i-1} \oplus \varepsilon \pi_{jd}$ if $d' = 0$.

**Proof.** By Theorem 3.2, $D \sim \pi_{i-1} + 2\pi_{j_0} + \cdots + 2\pi_{j_{d'-1}} + \varepsilon \pi_{jd'}$. It follows from Theorem 2.4 that the factors $\pi_{i-1}, \pi_{j_0}, \ldots, \pi_{j_{d'-1}}$ come from $V_{i-1}$ and the factors $\pi_{j_0}, \ldots, \pi_{j_{d'-1}}$, and $\varepsilon \pi_{jd}$ if nonzero come from $V_{i+1}$. Note that

$$j_k = i - 1 + 2p^k = i + p^k + (p^k - 1) = i + \underbrace{p - 1, \ldots, p - 1, 1}_k,$$

so $\Omega^{i-1}(j_k) = [k, d - 1]$ for all $0 \leq k \leq d - 1$. Therefore by Theorem 2.10,

$$\pi_{j_0} \prec \pi_{j_1} \prec \cdots \prec \pi_{j_{d'-1}} \prec \pi_{i-1}$$

in $V_{i-1}$. Similarly, we get $\Omega^{i+1}(j_k) = [0, k - 1]$ for $1 \leq k < d$ (and for $k = d$ if $\varepsilon \neq 0$ and $d > 0$). Hence

$$\varepsilon \pi_{jd} \prec \pi_{jd'} \prec \cdots \prec \pi_{j_1} \prec \pi_{j_0}$$

in $V_{i+1}$.

(Here the symbol $\prec$ is extended to the zero module in the natural way.) Since $\pi^n_0$ is selfdual, $D$ is selfdual also. Let $D_1 \prec \cdots \prec D_m$ be the socle series of $D$. Recall that $D$ has a filtration by quotients of $V_{i-1}$ and $V_{i+1}$. As the factor $\pi_{i-1}$ has multiplicity 1 and $D$ is selfdual, (6) implies that $\pi_{i-1}$ is a factor of $D_q$ with $d' + 1 \leq q \leq m - d'$, so $m \geq 2d' + 1$. If $\varepsilon \pi_{jd} = 0$, then $m = 2d' + 1$ is the composition length of $D$ and the theorem follows from (6) and (7). Assume that $\varepsilon \pi_{jd} \neq 0$. As above, by the selfduality of $D$ and (7), $\pi_{jd}$ is a factor of $D_q'$ with $d' + 1 \leq q' \leq m - d'$. Assume that $q' \neq q$. Then $m = 2d' + 2$, so $D$ is uniserial, which contradicts the selfduality of $D$. Hence $q' = q$ and the theorem follows from (6) and (7). □

**Remark 3.4.** Obviously, if $\varepsilon \pi_{jd} = 0$ (i.e. $d' < d$ or $i \equiv -p^d \pmod{p^{d+1}}$), then the module $D$ is uniserial, so has exactly $2d + 2$ different submodules. Since $D$ is selfdual, one can easily observe that $D$ has exactly $2d + 4$ different submodules in the case where $\varepsilon \pi_{jd} \neq 0$ (i.e. $d = d'$ and $i \equiv -p^d \pmod{p^{d+1}}$).

4. **Inductive systems and the transfer to an arbitrary field**

In this section the inductive systems of fundamental representations for $Sp_\infty(K)$ are classified and the results of the paper are transferred to an arbitrary field.

**Proposition 4.1.** (i) Let $\Psi$ be an inductive system and $R^{j'-1} \subset \Psi$. Then $R^{j'-1} \subset \Psi$.

(ii) $R^\Sigma$ is an inductive system if and only if only if $k = p^t - 1$, $t \geq 1$.

**Proof.** (i) Assume that $R^{j'-1} \not\subset \Psi$. Choose maximal $l$ such that $R^l \subset \Psi$. Then $p^{l-1} \leq l < p^l - 1$. Take minimal $s$ such that $l_s \neq p - 1$ and set $i = p^s(l_s, l_{s+1}, \ldots)$. Then $i > 0$ and $R^i \subset \Psi$. One has $lp(i) \geq s$. Moreover, if $lp(i) = s$, then $i \equiv -p^s \pmod{p^{s+1}}$. Therefore Theorem 3.2 implies that $\pi^{n-1}_{i-1+2p^s}$ is a composition factor of $\pi^n_{i-1-2p^s}$ for $n \geq i - 1 + 2p^s$, and $R^{i-1+2p^s} \subset \Psi$. As $i - 1 + 2p^s > l$, we get a contradiction.

(ii) In view of (i), it suffices to verify that $R^{j'-1}$ is an inductive system. By Theorem 3.2, we need only to check that if $i < p^l - 1$ and $s = lp(i)$, then $i - 1 + 2p^s \leq p^l - 1$ if $i_s \neq p - 1$, and $i - 1 + 2p^{s-1} \leq p^l - 1$ if $i_s = p - 1$ and $s > 0$. But this is clear since $i \leq (p - 2)p^s + (p - 1)p^{s-1} + \cdots + (p - 1)p^l - 1$ in the first case and $i \leq (p - 1)p^s + (p - 1)p^{s+1} + \cdots + (p - 1)p^l - 1$ in the second one. □

**Proof of Theorem 1.3.** Theorem 1.1(i) and Proposition 4.1 yield that $F$, $L^\Sigma$, $R^{j'-1}$, and $L^\Sigma \cup R^{j'-1}$, $s \geq 0$, $t \geq 1$, are inductive systems for $Sp_\infty(K)$.

Let $\Psi = \{\Psi_i, i = 1, 2, \ldots\}$ be an inductive system of fundamental representations. It is clear that either for each $s, u \in \mathbb{Z}^+$ there exist $n$ and $l$ such that $\omega^n_l \in \Psi_n$, $l > s$, and $n + 1 - l > u$, or $\Psi \subset L^\Sigma \cup R^u$ for some $s$ and $u$. In the first case we claim that
Ψ = F. Indeed, fix m and l, 0 ≤ l ≤ m. Then one can choose k and n such that ω^n_k ∈ Ψ_n, k ≥ l, and n − k ≥ m − l. Since Ψ is an inductive system, Theorem 1.1(i) implies that ω^{n+l-k} ∈ Ψ_{n+l-k} and ω^n_l ∈ Ψ_m. Hence Ψ = F.

Next, suppose that Ψ ⊂ L^s ∪ R^u. Choose minimal s and u with this property assuming that s = −1 if Ψ ⊂ R^u and u = 0 if Ψ ⊂ L^s. (Observe that for all s and u, (L^s ∩ R^u)_m = 0 for n large enough.) We shall prove that Ψ = L^s ∪ R^u and u = p^t − 1 with t ∈ Z^+(in particular, Ψ = R^u for s = −1 and Ψ = L^s for u = 0).

First let u > 0. We claim that R^u ⊂ Ψ and u = p^t − 1. As Ψ ⊄ L^s and Ψ and L^s are inductive systems, Ψ_n ∩ R^u_n = 0 for infinitely many integers n. So there exists v ≤ u such that π^n_v ∈ Ψ_n for infinitely many n. Choose maximal such v. Theorem 3.2 yields that π^n_v ∈ Ψ_n for all n ≥ v − 1 and R^v_n ⊂ Ψ. Now Proposition 4.1 and the choice of v imply that v = p^t − 1 and R^v is an inductive system. It remains to show that v = u. Suppose this is not the case. As Ψ ⊄ L^s ∪ R^v, there exist l and t such that v < l ≤ u, t > s + l − 1, and π^t_l ∈ Ψ_l. Since Ψ, L^s, and R^u are inductive systems, this implies that for every k > t there exists π^t_{mk} ∈ Ψ_k with v < mk ≤ u which contradicts the choice of v. Hence v = u = p^t − 1 and R^u ⊂ Ψ.

Now we show that L^s ⊂ Ψ if s ≥ 0. As Ψ ⊄ L^{s−1} ∪ R^u, for some n > s + u − 1 we have ω^n_s ∈ Ψ_n. Since Ψ, R^u, and L^{s−1} for s ≥ 1 are inductive systems, this forces ω^n_s ∈ Ψ_n for all n ≥ s. Now Theorem 1.1 yields that L^s ⊂ Ψ, as desired. □

Proposition 4.2. All theorems of the paper hold for arbitrary F ⊃ K.

Proof. Since the restrictions of the fundamental representations of a semisimple algebraic group over an algebraically closed field to relevant Chevalley groups over arbitrary subfields remain irreducible and can be realized over these subfields, only Theorems 1.1(ii) (or 3.3), 2.10, and 2.13 require some analysis. Let M be the G_n-module ω_{i}^{n+1}↓G_n or W_{i}^n.

First assume that F = F̄ is algebraically closed. Set H = Sp_{2n}(F). Let L be the Lie algebra of H. For a root α of H and t ∈ F denote by x_α(t) ∈ H and X_α ∈ L the root elements in H and L associated with α. It is well known that x_α(t)(m) = (1 + tX_α)m for m ∈ M and long α (see, for instance, [6, Lemma 1]). For g ∈ G_n set x_α^g(t) = gx_α(t)g^{-1} and X_α^g = gx_αg^{-1}. It is clear that X_α^g ∈ L. It suffices to show that each G_n-submodule N ⊂ M is an H-submodule. Obviously, x_α^g(t)N = N and X_α^gN ⊂ N for all long roots α, g ∈ G_n, and t ∈ K. But this forces x_α^g(t)N = N for all t ∈ F. However, using the commutator relations for the Chevalley groups of type C (see, for instance, [9, Lemma 15]), one can deduce that the subgroup generated by all x_α^g(t) with g ∈ G_n, t ∈ F, and long α coincides with H. (Here, in fact, it suffices to make computations within subgroups of type C_2 and show that our subgroup contains all short root subgroups). Hence N is an H-module, as desired.

Now let F ⊃ K be arbitrary. For a finite dimensional FG_n-module M set S = S ⊗_F F̄ and denote the socle of S by soc(S). Since dim Hom_{FG_n}(E, S) = dim Hom_{FG_n}(Ē, S) for any FG_n-module E, we have soc(S) = soc(S̄) if all composition factors of S are absolutely irreducible. The same holds for other members of the socle series of S. If M = W^n_i, then M and M̄ are multiplicity-free and their submodules are completely determined by the sets of composition factors. Therefore the arguments on socles allow us to conclude that each submodule of M has the form S̄ for some submodule S ⊂ M.

Let M = ω_{i}^{n+1}↓G_n with i, n ≥ 1. Then the socle of M contains a submodule V = ω_{i−1}^{n+1} ⊕ ω_{i−1}^{n}. Since M and V are selfdual and M has only two composition factors isomorphic to ω_{i−1}^{n}, there exists a submodule D of M such that M = V ⊕ D and D̄ is the unique submodule in M̄ with M̄ = V ⊕ D̄. Now one can see that the socle series of D is determined by that of D̄ and is described by Theorem 1.1(ii). □
REFERENCES


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