BRANCHING RULES FOR MODULAR FUNDAMENTAL REPRESENTATIONS OF SYMPLECTIC GROUPS

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To our teacher A. E. Zalesskii on the occasion of his sixtieth birthday

Abstract. In this paper branching rules for the fundamental representations of the symplectic groups in positive characteristic are found. The submodule structure of the restrictions of the fundamental modules for the group $Sp_{2n}(K)$ to the naturally embedded subgroup $Sp_{2n-2}(K)$ is determined. As a corollary, inductive systems of fundamental representations for $Sp_\infty(K)$ are classified. The submodule structure of the fundamental Weyl modules is refined.

1. Introduction

The article is devoted to finding branching rules for the fundamental representations of the sympplectic groups in positive characteristic. The classical branching rules are concerned with the restrictions of representations of the classical algebraic and symmetric groups in characteristic 0 to naturally embedded subgroups of smaller ranks. For a group of rank n and its fixed irreducible representation φ they yield the composition factors of the restriction of φ to a naturally embedded subgroup of rank $n-1$ and hence to similar subgroups of smaller ranks, at least algorithmically. These rules provide a basis for induction on rank and have found numerous applications. In positive characteristic one cannot expect to obtain complete branching rules in an explicit form in a near future since this problem is closely connected with that of finding the dimensions of arbitrary irreducible representations and the composition factors of the Weyl modules. So it is worth to investigate important particular cases where such rules can be found and to seek for asymptotic analogs of these rules. The notion of an inductive system of representations (see the definition below) introduced by Zalesskii in [11] yields an asymptotic version of the branching rules. It proved to be useful for the study of ideals in group algebras of locally finite groups as well, see, for instance, Zalesskii's survey [12]. We classify the inductive systems of the fundamental representations for the infinite-dimensional symplectic group $Sp_\infty(K)$. This class of representations yields an example of representations of a simple form for which the branching rules in positive characteristic differ from the characteristic 0 case.

Let $K \subset F$ be fields of characteristic $p > 0$, $\bar{F}$ be the algebraic closure of F, and $\mathbb{Z}^+$ be the set of nonnegative integers. Let $G_n = Sp_{2n}(K)$. Denote by ω_i^n, $0 \le i \le n$, the ith fundamental module and representation of G_n over F where ω_0^n is the trivial one. Let $W_0^n = \omega_0^n, W_1^n, \dots, W_n^n$ be the corresponding Weyl modules. Set $W_i^n = \omega_i^n = 0$ for $i < 0$ and for $i > n$. The labeling of the fundamental modules is standard, the fundamental and the Weyl modules for G_n are the F-modules affording the restrictions to G_n of the relevant representations of the group $Sp_{2n}(\bar{F})$ (it is well known that these restrictions can be realized over F). For an integer $z > 0$ we denote by $\mathrm{lp}(z)$ the maximal i such that

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$p^i \mid z$. We have $\mathrm{lp}(z) = 0$ if $p \not| z$. Let M be a G_n-module. The restriction of M to G_{n-1} is denoted by $M{\downarrow}G_{n-1}$. We shall write

$$M \sim N_1 + \cdots + N_q$$

if there is a series $0 = M_0 \subset M_1 \subset \cdots \subset M_q = M$ of submodules of M and a permutation σ such that $N_{\sigma(i)} \cong M_i/M_{i-1}$ for all $i = 1, \ldots, q$. Moreover, if in addition M_i/M_{i-1} coincides with the socle of M/M_{i-1} for $i = 1, \ldots, q$, then the sequence

$$N_{\sigma(1)} \prec_s N_{\sigma(2)} \prec_s \ldots \prec_s N_{\sigma(q)}$$

is called the *socle series* of M.

Theorem 1.1 below describes the branching rules for the fundamental G_n-modules and the submodule structure of the restrictions of these modules to G_{n-1}.

Theorem 1.1. *Let $n \geq 2$ and $0 \leq i \leq n$. Set $d = \mathrm{lp}(n - i + 1)$; $\varepsilon = 0$ if $n - i + 1 \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise. Then*

(i) $\omega_i^n{\downarrow}G_{n-1} \sim \omega_i^{n-1} + 2\omega_{i-1}^{n-1} + \left(\sum_{t=0}^{d-1} 2\omega_{i-2p^t}^{n-1}\right) + \varepsilon\omega_{i-2p^d}^{n-1}$ *(the sum in the brackets is zero whenever $d = 0$);*

(ii) $\omega_i^n{\downarrow}G_{n-1} = \omega_{i-1}^{n-1} \oplus \omega_{i-1}^{n-1} \oplus D$ *and the series*

$$\omega_{i-2}^{n-1} \prec_s \omega_{i-2p}^{n-1} \prec_s \cdots \prec_s \omega_{i-2p^{d-1}}^{n-1} \prec_s \omega \prec_s \omega_{i-2p^{d-1}}^{n-1} \prec_s \cdots \prec_s \omega_{i-2p}^{n-1} \prec_s \omega_{i-2}^{n-1}$$

with $\omega = \omega_i^{n-1} \oplus \varepsilon\omega_{i-2p^d}^{n-1}$ and ω_j^{n-1} omitted for $j < 0$ is the socle series of D. In particular, $D = \omega$ if $i = 0, 1$ or $p \not| n + 1 - i$.

Corollary 1.2. *For $n \geq 2$ the restriction $\omega_i^n{\downarrow}G_{n-1}$ is completely reducible if and only if $i = 0, 1$ or $p \not| n + 1 - i$.*

The proof of Theorem 1.1 is based on the description of the composition factors of the fundamental Weyl modules (Premet and Suprunenko [6, Theorem 2] for $p > 2$ and independently Adamovich [1, Theorem 2 and its Corollary 1] for arbitrary p) and Adamovich's results [2] on the submodule structure of these Weyl modules. In Section 2 these results are refined (Theorem 2.13). In particular, a new irreducibility criterion for the fundamental Weyl modules is obtained (Corollary 2.14) and it is proved that their socles are always simple (Corollary 2.15).

For $n - p + 2 \leq i \leq n$ Gow [5] has given an explicit construction of the modules ω_i^n and has described the submodule structure of the restrictions $\omega_i^n{\downarrow}G_1 \times G_{n-1}$ (the natural embedding) ([5, Theorem 2.2]). This implies our Theorem 1.1 for these modules. In [5] a certain explicitly determined operator δ on the exterior algebra $\wedge V$ of the natural G_n-module V is considered and it is proved that for $n - p + 2 \leq i \leq n$ the module ω_i^n can be realized as the quotient $\ker\delta \cap \wedge^i V/\delta^{p-1}(\wedge^{i+2p-2}V)$ ([5, Corollary 2.4]). This nice construction gives a realization for an important class of modules without complicated representation-theoretic machinery. However, it cannot be extended to other fundamental modules since according to [5, Theorem 4.2], the quotient above is zero for $i < n - p + 2$.

In [7] Sheth has found the branching rules for modular representations of symmetric groups corresponding to two part partitions. The composition factors occurring in the relevant restrictions are similar to those of the module D in Theorem 1.1(ii). We conjecture that the submodule structure of these restrictions is also similar to that of D. The authors plan to consider this question as well as the similar one for representations of special linear groups with highest weights $\omega_i + \omega_j$ in a subsequent paper.

In Section 4 Theorem 1.1 is applied to classify the inductive systems of fundamental F-representations for $Sp_\infty(K)$. Let

$$H_1 \subset H_2 \subset \ldots \subset H_n \subset \ldots \tag{1}$$

be a sequence of groups, and Ψ_n, $n = 1, 2, \dots$, be a nonempty finite set of (inequivalent) irreducible representations of H_n over a fixed field. The system $\Psi = \{\Psi_n \mid n = 1, 2, \dots\}$ is called an *inductive system* (of representations) for the group $H = \bigcup_{n=1}^{\infty} H_n$ if each Ψ_n coincides with the union of the sets of composition factors (up to equivalence) of the restrictions $\pi{\downarrow}H_n$ where π runs over Ψ_{n+1}. In this article (1) is the sequence of the naturally embedded groups $G_n = Sp_{2n}(K)$, so $\bigcup_{n=1}^{\infty} G_n = Sp_\infty(K)$. Set

$$\begin{array}{rclrcl} \mathcal{F}_n & = & \{\omega_i^n \mid 0 \le i \le n\}, & \mathcal{F} & = & \{\mathcal{F}_n \mid n = 1, 2, \dots\}; \\ \mathcal{L}_n^s & = & \{\omega_i^n \mid 0 \le i \le s\}, & \mathcal{L}^s & = & \{\mathcal{L}_n^s \mid n = 1, 2, \dots\}; \\ \mathcal{R}_n^u & = & \{\omega_i^n \mid n+1-u \le i \le n\}, & \mathcal{R}^u & = & \{\mathcal{R}_n^u \mid n = 1, 2, \dots\}. \end{array}$$

Theorem 1.3. *The inductive systems of fundamental representations over F for $Sp_\infty(K)$ are exhausted by the systems $\mathcal{F}$, $\mathcal{L}^s$, $\mathcal{R}^{p^t-1}$, and $\mathcal{L}^s \cup \mathcal{R}^{p^t-1}$ ($s \ge 0$, $t \ge 1$).*

It is clear that $\mathcal{L}^0$ (which consists of the trivial representations) and $\mathcal{R}^{p-1}$ are minimal inductive systems. However, the question on the minimal inductive systems for $Sp_\infty(K)$ is far from solution. For $p > 2$ Zalesskii and Suprunenko [10] have described the inductive system $\Phi = \{\Phi_n \mid n = 1, 2, \dots\}$ where for each n the set Φ_n consists of two irreducible representations with highest weights $\omega_{n-1} + \frac{1}{2}(p-3)\omega_n$ and $\frac{1}{2}(p-1)\omega_n$. The system Φ coincides with $\mathcal{R}^2$ for $p = 3$ and yields another example of a minimal inductive system for $p > 3$.

For other classical groups the questions investigated in this paper present no problems since the situation is the same as in characteristic 0.

The authors [3] have found the minimal and the minimal nontrivial inductive systems for the group $SL_\infty(\bar{K})$. For this group the system consisting of the trivial representation is the only minimal inductive system, and the minimal nontrivial ones are exhausted by the systems $\mathbf{L}^j = \{L_n^j \mid n = 1, 2, \dots\}$ and $\mathbf{R}^j = \{R_n^j \mid n = 1, 2, \dots\}$ where L_n^j consists of two irreducible representations of $SL_{n+1}(\bar{K})$ with highest weights 0 and $p^j\omega_1$ and R_n^j of those with highest weights 0 and $p^j\omega_n$. The picture is similar for the groups SL_∞ and SU_∞ over locally finite fields.

Until Proposition 4.2 we assume that $K = F = \bar{F}$. At the end Proposition 4.2 transfers the results to arbitrary fields.

2. The structure of the fundamental Weyl modules

In this section we refine the results of [6], [1], and [2] on the structure of the fundamental Weyl modules for G_n.

Throughout the paper we set $\pi_i^n = \omega_{n+1-i}^n$ and $V_i^n = W_{n+1-i}^n$. We denote by $[a, b]$ the set of all $j \in \mathbb{Z}^+$ with $a \le j \le b$. For an integer $k \in \mathbb{Z}^+$ write its p-adic expansion $k = k_0 + k_1 p + \dots + k_s p^s$ with $0 \le k_i < p$ and set $k_i = 0$ for all such $i \in \mathbb{Z}^+$ that $p^i > k$. We shall write $k = (k_0, k_1, \dots, k_s)$. We say that an integer m *contains k to base p* and write $k \subset_p m$ if and only if for each i either $k_i = 0$, or $k_i = m_i$. Set $d_k^m = 1$ if $k \subset_p m$, and $d_k^m = 0$ otherwise.

Theorem 2.1. [6, Theorem 2] *Let $p > 2$. Then $W_i^n \sim \sum_{k=0}^{\infty} d_k^{n+1-i+2k} \omega_{i-2k}^n$.*

We need some more notation to state Adamovich's results. For $\lambda \in \mathbb{Z}^+$ define maps $s'_\lambda : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $s_\lambda : \mathbb{Z}^+ \to \mathbb{Z}^+$ setting

$s'_\lambda(l) = l + 2k'$ where $l + 1 = a'p^\lambda - k'$, $a' \in \mathbb{Z}^+$, $0 \le k' < p^\lambda$;

$s_\lambda(l) = l + 2k$ where $l = ap^\lambda - k$, $a \in \mathbb{Z}^+$, $0 \le k < p^\lambda$.

We say that the *reflection* s'_λ or s_λ is *l-admissible* if $k' \ne 0$ and $p \nmid a'$ or $k \ne 0$ and $p \nmid a$, respectively. We denote by $S(l)$ the set of all $m > l$ that can be written in the form $m = s_{\lambda_u} \dots s_{\lambda_1}(l)$ where $\lambda_u < \dots < \lambda_1$ and for each $i = 0, 1, \dots, u-1$ the reflection $s_{\lambda_{i+1}}$ is $s_{\lambda_i} \dots s_{\lambda_1}(l)$-admissible. Similarly we define $S'(l)$ (writing s'_{λ_i} instead of s_{λ_i}).

Theorem 2.2. [1] *Let* $0 \le l \le n$. *Then* $V_{l+1}^n \sim \pi_{l+1}^n + \sum_{m \in S'(l)} \pi_{m+1}^n$.

As $s'_\lambda(x-1) = s_\lambda(x) - 1$, the following theorem yields an equivalent statement.

Theorem 2.3. *Let* $1 \le l \le n+1$. *Then* $V_l^n \sim \pi_l^n + \sum_{m \in S(l)} \pi_m^n$.

Let us rewrite Theorem 2.1 in terms of π_m^n and V_l^n (without restrictions on p).

Theorem 2.4. *Let* $1 \le l \le n+1$. *Then* $V_l^n \sim \sum_{k=0}^{\infty} d_k^{l+2k} \pi_{l+2k}^n$.

Now our goal is to show that Theorems 2.3 and 2.4 are equivalent, so Theorem 2.4 (and 2.1) holds for $p = 2$. For this purpose we prove some technical facts on the triples k, l, m with $k \subset_p m = l + 2k$ and admissible reflections.

Until the end of the section $l \ge 1$. For each $m \in S(l)$ the tuple $(\lambda_1; \ldots; \lambda_u)$ is uniquely determined (see the comments before the Theorem in [2]). If u is odd for some m, set $\lambda_{u+1} = \mathrm{lp}(m)$. Then $s_{\lambda_{u+1}}(m) = m$ and $\lambda_{u+1} < \lambda_u$. Now for every $m \in S(l)$ we have a uniquely determined sequence of reflections $s_{\lambda_1}, \ldots, s_{\lambda_{2t}}$. Such sequences will be called *l-admissible*. For an integer $0 \le a \le p-1$ set $\bar{a} = p - 1 - a$. The following lemma is straightforward.

Lemma 2.5. *Set* $q = \mathrm{lp}(l)$. *The reflection* s_λ *is* l*-admissible if and only if* $\lambda > q$ *and* $l_\lambda \ne p-1$. *In that case* $s_\lambda(l) = p^q(\bar{l}_q + 1, \bar{l}_{q+1}, \ldots \bar{l}_{\lambda-1}, l_\lambda + 1, l_{\lambda+1}, \ldots)$.

Two consequent applications of Lemma 2.5 yield

Proposition 2.6. *Let* $\mathrm{lp}(l) \le \mu < \lambda$, $m = s_\mu s_\lambda(l)$, *and* $k = (m-l)/2$. *The pair* s_λ, s_μ *is* l*-admissible if and only if* $l_\lambda \ne p-1$ *and* $l_\mu \ne 0$. *In that case*

$$\begin{array}{rcl} m & = & (l_0, \ \ldots, \ l_{\mu-1}, \ \bar{l}_\mu + 1, \ \bar{l}_{\mu+1}, \ \ldots, \ \bar{l}_{\lambda-1}, \ l_\lambda + 1, \ l_{\lambda+1}, \ \ldots), \\ k & = & (0, \ \ldots, \ 0, \ \bar{l}_\mu + 1, \ \bar{l}_{\mu+1}, \ \ldots, \ \bar{l}_{\lambda-1}). \end{array}$$

In particular, $k \subset_p m = l + 2k$.

We call a tuple $\sigma = (\lambda_1; \ldots; \lambda_{2t})$ *l-admissible* if $\lambda_i \in \mathbb{Z}^+$, $\lambda_1 > \cdots > \lambda_{2t}$, $l_{\lambda_{2j-1}} \ne p-1$ and $l_{\lambda_{2j}} \ne 0$ for $j = 1, \ldots, t$. For an l-admissible σ set

$$\mathfrak{Q}_l(\sigma) = \bigcup_{j=1}^{t} [\lambda_{2j}, \lambda_{2j-1} - 1],$$

$\delta_{i\sigma} = 1$ if $i = \lambda_j$ for some j and $\delta_{i\sigma} = 0$ otherwise. Define $l^\sigma \in \mathbb{Z}^+$ putting

$$l_i^\sigma = \begin{cases} \bar{l}_i + \delta_{i\sigma}, & i \in \mathfrak{Q}_l(\sigma) \\ l_i + \delta_{i\sigma}, & i \notin \mathfrak{Q}_l(\sigma). \end{cases}$$

Proposition 2.6 yields the following corollary.

Corollary 2.7. *A sequence* $s_{\lambda_1}, \ldots, s_{\lambda_{2t}}$ *is* l*-admissible if and only if the tuple* $\sigma = (\lambda_1; \ldots; \lambda_{2t})$ *is* l*-admissible. In that case* $s_{\lambda_{2t}} \ldots s_{\lambda_1}(l) = l^\sigma$.

Proposition 2.8. *An integer* $m \in S(l)$ *if and only if* $m - l = 2k > 0$ *and* $k \subset_p m$.

Proof. Let $\sigma = (\lambda_1; \ldots; \lambda_{2t})$ be an l-admissible tuple and $m = s_{\lambda_{2t}} \ldots s_{\lambda_1}(l)$. Set $m^0 = l$, $m^j = s_{\lambda_{2j}} \ldots s_{\lambda_1}(l)$, and $k^j = (m^j - m^{j-1})/2$, $1 \le j \le t$. Using Proposition 2.6, one deduces that $k^j \subset_p m^j$; $m^j = l^{\sigma^j}$ with $\sigma^j = (\lambda_1; \ldots; \lambda_{2j})$; $k_i^j = 0$ and $m_i^j = m_i^{j-1}$ for $i \notin [\lambda_{2j}, \lambda_{2j-1}]$. Therefore $k = k^1 + \ldots + k^t \subset_p m$.

Assume now that $m - l = 2k > 0$ and $k \subset_p m$. Choose integers $\tau_1, \tau_2, \ldots, \tau_{2t}$ as follows. Set $\tau_0 = -1$. Assume that τ_{2j} is chosen. If there is no $i > \tau_{2j}$ such that $k_i = m_i \ne 0$, we set $t = j$ and stop the process. Otherwise we choose for τ_{2j+1} minimal $i > \tau_{2j}$ with $k_i = m_i \ne 0$. As $m > 2k$, there exists $f > \tau_{2j+1}$ with $k_f \ne m_f$ (observe that in this case

$k_f = 0$). We choose minimal such f for τ_{2j+2}. Set $\lambda_q = \tau_{2t+1-q}$. Since $k \subset_p m$, using Corollary 2.7 and analyzing the p-adic expansions of k and m, one can conclude that the tuple $\sigma = (\lambda_1; \dots; \lambda_{2t})$ is l-admissible and $m = l^\sigma$. □

Corollary 2.9. *Theorems 2.3 and 2.4 are equivalent, so Theorems 2.1 and 2.4 are valid in characteristic 2 as well.*

Now we rewrite Adamovich's results [2] on the submodule structure of the Weyl modules in our terms. We fix n and write V_l and π_m instead of V_l^n and π_m^n. For $m \in S(l)$ or $m = l$ we denote by $P_l(m)$ the smallest submodule of V_l that has a composition factor π_m. Since V_l is multiplicity-free, $P_l(m)$ is correctly defined and each submodule of V_l is a sum of $P_l(m)$ for some m. Hence the submodule structure of V_l is determined by the inclusion relations between the submodules $P_l(m)$ (see also comments at the beginning of [2]). We shall write $\pi_m \prec \pi_q$ if $P_l(m) \subset P_l(q)$. Let $\sigma = (\lambda_1; \dots; \lambda_{2t})$ be an l-admissible tuple. For $m = l^\sigma$ set

$$\mathfrak{Q}_l(m) = \mathfrak{Q}_l(\sigma) = \bigcup_{j=1}^{t} [\lambda_{2j}, \lambda_{2j-1} - 1].$$

Note that $\mathfrak{Q}_l(m) = \bigcup_{j=1}^{t} [\tau_{2j-1}, \tau_{2j} - 1]$ where τ_i are as in the proof of Proposition 2.8. For instance, for $p = 3$,

$$\begin{aligned} m &= (0,\ 1,\ \underline{2,\ 2,\ 0},\ 1,\ 0,\ \underline{2,\ 1,\ 0,\ 0},\ 2,\ 1), \\ k &= (0,\ 0,\ \underline{2,\ 2,\ 0},\ 0,\ 0,\ \underline{2,\ 1,\ 0,\ 0},\ 0,\ 0), \end{aligned}$$

we have $\mathfrak{Q}_{m-2k}(m) = [2, 4] \cup [7, 10]$. Put also $\mathfrak{Q}_l(l) = \emptyset$.

Theorem 2.10. [2] *For $1 \le m, q \le n + 1$ and $m, q \in S(l) \cup \{l\}$ the module $\pi_m \prec \pi_q$ (as composition factors of V_l) if and only if $\mathfrak{Q}_l(q) \subset \mathfrak{Q}_l(m)$.*

Remark 2.11. Actually the sets $\mathfrak{P}_l(m)$ which are considered in [2] differ slightly from $\mathfrak{Q}_l(m)$. For $m \in S(l)$ one has $\mathfrak{P}_l(m) = \cup_{j=1}^{t} [\mu_{2j} + 1, \mu_{2j-1}]$ where $\mu_s = \lambda_s$ for $s < 2t$, $\mu_{2t} = \lambda_{2t}$ if $m \ne s_{\lambda_{2t-1}} \dots s_{\lambda_1}(l)$, and $\mu_{2t} = 0$ otherwise. However, Lemma 2.5 enables one to deduce that $\mathfrak{P}_l(m) \subset \mathfrak{P}_l(q)$ if and only if $\mathfrak{Q}_l(m) \subset \mathfrak{Q}_l(q)$. The crucial point is that $\mathrm{lp}(l) = \mathrm{lp}(m)$ for $m \in S(l)$.

For l-admissible tuples $\sigma = (\lambda_1; \dots; \lambda_{2t})$ and $\sigma' = (\lambda'_1; \dots; \lambda'_{2s})$ we say that $\sigma \le \sigma'$ if there exists $f \le 2t, 2s$ such that $\lambda_i = \lambda'_i$ for $1 \le i \le f$ and either $f = 2t$, or $f < 2t, 2s$ and $\lambda_{f+1} < \lambda'_{f+1}$. It is convenient to assume that the empty tuple $\emptyset$ is l-admissible, $\emptyset \le \sigma$ for all σ, $l^\emptyset = l$, and $\mathfrak{Q}_l(\emptyset) = \emptyset$. The following is obvious.

Lemma 2.12. *Let σ and σ' be l-admissible tuples. Then $l^\sigma \le l^{\sigma'}$ if and only if $\sigma \le \sigma'$.*

Set $n' = n + 1$. Construct an l-admissible tuple $\sigma^{\max} = (\mu_1; \dots; \mu_{2t})$ as follows. Put $\mu_0 = +\infty$. Assume that μ_{2j} is chosen. Set $\mu = \mu_{2j} - 1$. If there is no l-admissible tuple $(\alpha; \beta)$ such that $\mu \ge \alpha > \beta$ and

$$(l_0, \dots, l_\mu)^{(\alpha;\beta)} = (l_0, \dots, \bar{l}_\beta + 1, \bar{l}_{\beta+1}, \dots, \bar{l}_{\alpha-1}, l_\alpha + 1, \dots, l_\mu) \le (n'_0, \dots, n'_\mu),$$

we stop the process and set $t = j$ ($\sigma^{\max} = \emptyset$ if $t = 0$). Otherwise we choose maximal such pair $(\alpha; \beta)$ (with respect to $\le$); set $\mu_{2j+1} = \alpha$ and $\mu_{2j+2} = \beta$; and if

$$(\bar{l}_\beta + 1, \bar{l}_{\beta+1}, \dots, \bar{l}_{\alpha-1}, l_\alpha + 1) < (n'_\beta, \dots, n'_\alpha),$$

we stop the process and determine $(\mu_{2j+3}; \dots; \mu_{2t})$ as the maximal l-admissible tuple with $\mu_{2j+3} < \beta$. Obviously, $l^{\sigma^{\max}}$ is the maximal integer m such that π_m^n is a composition factor of V_l^n.

For l-admissible tuples σ and σ' we write $\sigma \prec \sigma'$ if and only if $\mathfrak{Q}_l(\sigma) \supset \mathfrak{Q}_l(\sigma')$. Using Corollary 2.7, Theorem 2.10, and Lemma 2.12, we get our main result on the structure of fundamental Weyl modules.

Theorem 2.13. *The map $\sigma \mapsto \pi^n_{l^\sigma}$ is a poset isomorphism between the l-admissible tuples $\sigma \leq \sigma^{\max}$ and the composition factors of V^n_l with the partial orders $\prec$.*

If $l < n'$, we denote by v the maximal integer such that $l_v \neq n'_v$. If $l_v + 1 = n'_v$, we denote by u the maximal integer $< v$ such that $\bar{l}_u \neq n'_u$ setting $u = -1$ if $\bar{l}_i = n'_i$ for all $i < v$. Put $s = \mathrm{lp}(l)$.

Corollary 2.14. *Let $n' = n + 1$ and $1 \leq l \leq n'$. Then V^n_l is irreducible (i.e. $\sigma^{\max} = \emptyset$) if and only if one of the following holds.*
(1) $l = n'$;
(2) $l < n'$ and $s \geq v$;
(3) $l < n'$, $s < v$, $l_v + 1 = n'_v$, $\bar{l}_s \geq n'_s$; $l_i = p - 1$ and $n'_i = 0$ for $s < i < v$.

Proof. This follows from Proposition 2.6 and Corollary 2.7. □

Corollary 2.15. *Let $n' = n + 1$ and $1 \leq l \leq n'$. The socle of V^n_l is always simple. For reducible V^n_l it has the form π_{l^γ} with $\gamma = (t; s)$ and t as follows.*
(1) $t = v$ if $s < v$ and either $l_v + 1 < n'_v$, or $\bar{l}_u < n'_u$;
(2) $t = w$ if $l_v + 1 = n'_v$; $u = -1$ or $\bar{l}_u > n'_u$; $s < w < v$; $l_w \neq p - 1$; and $l_j = p - 1$ for $w < j < v$.

Proof. Applying Results 2.6, 2.7, and 2.13, we conclude that π_{l^γ} is a composition factor of V^n_l and for each l-admissible tuple $\tau \leq \sigma^{\max}$ the set $\mathfrak{Q}_l(\tau) \subset [s, t-1]$, so $\pi_{l^\gamma} \prec \pi_{l^\tau}$. □

3. Branching rules and the submodule structure of the restrictions

In this section the main results of the article are proved. We shall need the following simple lemma.

Lemma 3.1. *Assume that $d^{l+2k}_k = 1$ (i.e. $k \subset_p l + 2k$).*
(i) If $p^s \mid l + 2k$, then $p^s \mid k$ and $p^s \mid l$.
(ii) If $p^s \mid l$, then $p^s \mid k$ and $p^s \mid l + 2k$.

Proof. One can assume that $s \geq 1$.
(i) Let $p^s \mid l + 2k$. Since $k \subset_p l + 2k$, we have $p^s \mid k$. This implies that $p^s \mid l$.
(ii) Let $p^s \mid l$. Then $l_0 = \cdots = l_{s-1} = 0$. Let $r = \mathrm{lp}(k)$. Assume that $r < s$. Since $k \subset_p l + 2k$, we have $k_r = (2k)_r \neq 0$, which is impossible. Therefore $r \geq s$, so $p^s \mid k$ and $p^s \mid l + 2k$. □

As in Section 2 , we shall omit the superscript n in our notation for modules when it is known which group is considered. Replacing ω_i by π_{n+1-i} and W_i by V_{n+1-i}, one immediately concludes that Theorem 1.1(i) is equivalent to the following

Theorem 3.2. *Let $1 \leq i \leq n + 1$ and $d = \mathrm{lp}(i)$. Then*

$$\pi^n_i \downarrow G_{n-1} \sim \pi_{i-1} + 2\pi_i + \left(\sum_{t=0}^{d-1} 2\pi_{i-1+2p^t} \right) + \varepsilon\pi_{i-1+2p^d}$$

where $\varepsilon = 0$ if $i \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise.

Proof. One can rewrite the formula in Theorem 3.2 as follows.

$$\pi^n_i \downarrow G_{n-1} \sim \pi_{i-1} + 2\pi_i + \sum_{t=0}^{\infty} b^i_t \pi_{i-1+2p^t} \tag{2}$$

where

$$b_t^i = \begin{cases} 2, & i \equiv 0 \pmod{p^{t+1}}, \\ 1, & i \equiv ap^t \pmod{p^{t+1}} \text{ and } a \not\equiv 0, -1 \pmod{p}, \\ 0, & i \not\equiv 0 \pmod{p^t} \text{ or } i \equiv -p^t \pmod{p^{t+1}}. \end{cases} \tag{3}$$

Recall that by convention $\pi_i^n = 0$ for all $i > n+1$, and π_{n+1}^n is the trivial one-dimensional G_n-module. So (2) holds for $i \geq n+1$. Assume now that $1 \leq l < n+1$ and (2) is valid for all $i > l$. We shall prove it for $i = l$. Then the theorem will follow by induction.

It follows from [4, Proposition 3.3.2 and Theorem 4.3.1] that $V_l{\downarrow}G_{n-1}$ has a filtration by Weyl modules for G_{n-1}. Then the classical branching rules for characteristic 0 [13] and Theorem 2.4 imply

$$V_l^n{\downarrow}G_{n-1} \sim V_{l-1} + 2V_l + V_{l+1} \sim 2\mathcal{V} + \sum_{t=0}^{\infty} f_t^{l+2t-1}\pi_{l+2t-1} \tag{4}$$

where $\mathcal{V} = \sum_{k=0}^{\infty} d_k^{l+2k}\pi_{l+2k}$, $\quad f_0^{l-1} = d_0^{l-1}$, and

$$f_t^{l+2t-1} = d_t^{l+2t-1} + d_{t-1}^{l+2t-1} \quad \text{for } t \geq 1.$$

On the other hand, by Theorem 2.4, $V_l{\downarrow}G_{n-1} \sim \sum_{k=0}^{\infty} d_k^{l+2k}(\pi_{l+2k}{\downarrow}G_{n-1})$. Since $d_0^l = 1$ and the branching rules for π_i with $i > l$ are assumed to satisfy (2), one can determine the branching of π_l. Therefore it suffices to check that the right part of (4) is equal to $\sum_{k=0}^{\infty} d_k^{l+2k}U_{l+2k}$ where U_i is the right part of (2). The latter sum can be rewritten as follows:

$$2\mathcal{V} + \sum_{k=0}^{\infty} d_k^{l+2k}(\pi_{l+2k-1} + \sum_{s=0}^{\infty} b_s^{l+2k}\pi_{l+2k+2p^s-1}) = 2\mathcal{V} + \sum_{t=0}^{\infty} e_t^{l+2t-1}\pi_{l+2t-1}$$

where

$$e_t^{l+2t-1} = d_t^{l+2t} + \sum_{k,s \geq 0,\ k+p^s=t} d_k^{l+2k}b_s^{l+2k}. \tag{5}$$

We have to show that $e_t^{l+2t-1} = f_t^{l+2t-1}$ for all $t \geq 0$. Note that $f_t^{l+2t-1} \leq 2$. We proceed by steps.

Step 1. At most one summand in (5) is nonzero. In particular, $e_t^{l+2t-1} \leq 2$.

Assume that $d_k^{l+2k}b_s^{l+2k} \neq 0$ and $d_{k'}^{l+2k'}b_{s'}^{l+2k'} \neq 0$ with $t = k+p^s = k'+p^{s'}$ and $s > s'$. Since $b_s^{l+2k} \neq 0$, we have $p^s \mid l+2k$. As $d_k^{l+2k}, d_{k'}^{l+2k'} \neq 0$, by Lemma 3.1, p^s divides k, l, and k'. Hence $p^s \mid k+p^s-k' = p^{s'}$, which yields a contradiction.

Now assume that $d_t^{l+2t} \neq 0$ and $d_k^{l+2k}b_s^{l+2k} \neq 0$ with $k+p^s = t$. As above, we get that p^s divides k, l, and t. Let $r = \mathrm{lp}(l)$. Then by Lemma 3.1 (ii), $p^r \mid k$ and $p^r \mid t$. Since $k+p^s = t$, we have $r = s$, so $l_s \neq 0$. Consider the following cases.

Case 1. $k \not\equiv 0$ and $t \not\equiv 0 \pmod{p^{s+1}}$. Then $l+2t \equiv t$ and $l+2k \equiv k \pmod{p^{s+1}}$, so $p^s = t-k \equiv 2(t-k) \pmod{p^{s+1}}$, which is impossible.

Case 2. $k \equiv 0 \pmod{p^{s+1}}$. Then $t_s = 1$. Since $l+2t \equiv t \pmod{p^{s+1}}$, we have $l_s = p-1$, so $(l+2k)_s = p-1$. This implies that $b_s^{l+2k} = 0$ and yields a contradiction.

Case 3. $t \equiv 0 \pmod{p^{s+1}}$. Then $k_s = p-1$. Therefore $(l+2k)_s = k_s = p-1$, so as above, $b_s^{l+2k} = 0$.

Step 2. $e_t^{l+2t-1} = 2$ *if and only if* $f_t^{l+2t-1} = 2$ *(equivalently,* $d_t^{l+2t-1} = d_{t-1}^{l+2t-1} = 1$*).*

Assume that $e_t^{l+2t-1} = 2$. By Step 1, this is equivalent to the following: there exist $k, s \geq 0$ with $k+p^s = t$ such that $b_s^{l+2k} = 2$ and $d_k^{l+2k} = 1$. Hence $p^{s+1} \mid l+2k$ and $k \subset_p l+2k$. By Lemma 3.1 (i), p^{s+1} divides l and k. Note that $l+2t-1 = l+2k+p^s+(p^s-1)$.

Therefore $t = k+p^s \subset_p l+2t-1$ and $t-1 = k+(p^s-1) \subset_p l+2t-1$, so $d_t^{l+2t-1} = d_{t-1}^{l+2t-1} = 1$, as required.

Assume now that $d_t^{l+2t-1} = d_{t-1}^{l+2t-1} = 1$. Let $s = \mathrm{lp}(t)$. Then $t_s \neq 0$ and $(t-1)_s = t_s-1$. Since both t and $t-1$ are contained in $l+2t-1$, we have $t_s = 1$. Moreover, we have $(l-1)_0 = \cdots = (l-1)_{s-1} = (t-1)_0 = \cdots = (t-1)_{s-1} = p-1$. Since $(l+2t-1)_s = t_s = 1$, we get $(l-1)_s = p-1$. Hence $p^{s+1} \mid l$. Set $k = t-p^s$. Then $p^{s+1} \mid k$, so $p^{s+1} \mid l+2k$ and $b_s^{l+2k} = 2$. It remains to observe that $k = t-p^s \subset_p (l+2t-1)-p^s-(p^s-1)$, so $d_k^{l+2k} = 1$ and $e_t^{l+2t-1} = d_k^{l+2k} b_s^{l+2k} = 2$.

Step 3. If $e_t^{l+2t-1} \neq 0$, then $f_t^{l+2t-1} \neq 0$, i.e. $t \subset_p l+2t-1$ or $t-1 \subset_p l+2t-1$.

Let $r = \mathrm{lp}(l)$. First assume that $d_t^{l+2t} = 1$ (see (5)), i.e. $t \subset_p l+2t$. Then by Lemma 3.1 (ii), $p^r \mid t$. One easily checks that if $p^{r+1} \nmid t$, then $t-1 \subset_p l+2t-1$, and if $p^{r+1} \mid t$, then $t \subset_p l+2t-1$, as required.

Assume now that there exist $k, s \geq 0$ with $k+p^s = t$ such that $d_k^{l+2k} = 1$ and $b_s^{l+2k} \neq 0$. By Lemma 3.1, $p^r \mid k$ and $\mathrm{lp}(l+2k) = r$. If $b_s^{l+2k} = 2$, then by Step 2, $f_t^{l+2t-1} = 2 \neq 0$. Hence we can assume that $b_s^{l+2k} = 1$. By (3), $p^s \mid l+2k$ and $(l+2k)_s \neq 0, p-1$, so $r = s$. Assume that $k_s = 0$. Then $(t-1)_s = 0$ and $t-1 = k+p^s-1 \subset_p l+2k+p^s+(p^s-1) = l+2t-1$. If $k_s \neq 0$, we have $k_s = (l+2k)_s \neq p-1$, so $t = k+p^s \subset_p l+2k+p^s+(p^s-1) = l+2t-1$, as required.

Step 4. If $f_t^{l+2t-1} \neq 0$, then $e_t^{l+2t-1} \neq 0$.

If $t = 0$, then $e_0^{l-1} = 1$, so assume that $t \geq 1$. We have either $t \subset_p l+2t-1$, or $t-1 \subset_p l+2t-1$. One needs to show that either $t \subset_p l+2t$ (i.e. $d_t^{l+2t} = 1$), or there exist $k, s \geq 0$ with $k+p^s = t$ such that $k \subset_p l+2k$, $p^s \mid l+2k$, and $(l+2k)_s \neq p-1$ (i.e. $d_k^{l+2k} = 1$ and $b_s^{l+2k} \neq 0$). First assume that $t \subset_p l+2t-1$. Let $s = \mathrm{lp}(t)$. We have $(l+2t-1)_s = t_s \neq 0$. If $p^s \mid l+2t$, then $k = t-p^s \subset_p l+2t-1-p^s-(p^s-1) = l+2k$, $p^s \mid l+2t-2p^s = l+2k$, and $(l+2k)_s = (l+2t-1)_s - 1 \neq p-1$, as required. If $p^s \nmid l+2t$, one gets $t \subset_p (l+2t-1)+1 = l+2t$, as desired.

Assume now that $t-1 \subset_p l+2t-1$. Consider the following cases.

Case 1. $(t-1)_0 = 0$. If $(l+2t-1)_0 = 0$, then $t \subset_p l+2t$. Assume that $(l+2t-1)_0 \neq 0$. Set $s = 0$, $k = t-1$. Then $k = t-1 \subset_p (l+2t-1)-1 = l+2k$ and $(l+2k)_0 = (l+2t-1)_0 - 1 \neq p-1$, as required.

Case 2. $(t-1)_0 \neq 0, p-1$. Then $(l+2t-1)_0 = (t-1)_0 \neq p-1$, so $t \subset_p l+2t$.

Case 3. $p \mid t$. Let $s = \mathrm{lp}(t)$. Since $t-1 \subset_p l+2t-1$, we have $p^s \mid l+2t$, so $p^s \mid l$. As $p^{s+1} \nmid t$, the integer $(t-1)_s \neq p-1$. If $(t-1)_s \neq 0$, then $(l+2t-1)_s = (t-1)_s \neq p-1$, so $t \subset_p l+2t$. Assume now that $(t-1)_s = 0$. This implies $t_s = 1$. If $(l+2t-1)_s = 0$, then $t \subset_p l+2t$. Therefore one can suppose that $(l+2t-1)_s \geq 1$. Then $k = t-p^s \subset_p (l+2t-1)-p^s-(p^s-1) = l+2k$, $p^s \mid l+2t-2p^s = l+2k$, and $(l+2k)_s = (l+2t-1)_s - 1 \neq p-1$, as required. $\square$

Now we investigate the submodule structure of the restriction $\pi_i^n{\downarrow}G_{n-1}$. Let $n > 1$ and $1 \leq i \leq n$. As π_i^n is the top composition factor of V_i^n, it follows from (4) that $\pi_i^n{\downarrow}G_{n-1}$ is a quotient of the G_{n-1}-module $V_i^n{\downarrow}G_{n-1} \sim V_{i-1} + 2V_i + V_{i+1}$. Applying Smith's theorem [8] both to V_i and π_i, we conclude that $V_i^n{\downarrow}G_{n-1} = V_i \oplus V_i \oplus V$ where $V \sim V_{i-1} + V_{i+1}$, and $\pi_i^n{\downarrow}G_{n-1} = \pi_i \oplus \pi_i \oplus D$ where D is a quotient of V. Now Theorem 1.1(ii) and Corollary 1.2 follow immediately from

Theorem 3.3. *Let $d = \mathrm{lp}\, i$. Set $\varepsilon = 0$ if $i \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise; $j_q = i-1+2p^q$. Choose minimal $t \in \mathbb{Z}^+$ such that $j_t > n$. Put $d' = \min\{d, t\}$. Then*

$$\pi_{j_0} \prec_s \pi_{j_1} \prec_s \cdots \prec_s \pi_{j_{d'-1}} \prec_s \pi_{i-1} \oplus \varepsilon\pi_{j_d} \prec_s \pi_{j_{d'-1}} \prec_s \cdots \prec_s \pi_{j_1} \prec_s \pi_{j_0}$$

is the socle series of D. *In particular,* $D = \pi_{i-1} \oplus \varepsilon\pi_{j_d}$ *if* $d' = 0$.

Proof. By Theorem 3.2, $D \sim \pi_{i-1} + 2\pi_{j_0} + \dots + 2\pi_{j_{d'-1}} + \varepsilon\pi_{j_d}$. It follows from Theorem 2.4 that the factors $\pi_{i-1}, \pi_{j_0}, \dots, \pi_{j_{d'-1}}$ come from V_{i-1} and the factors $\pi_{j_0}, \dots, \pi_{j_{d'-1}}$, and $\varepsilon\pi_{j_d}$ if nonzero come from V_{i+1}. Note that

$$j_k = i - 1 + 2p^k = i + p^k + (p^k - 1) = i + (\underbrace{p-1, \dots, p-1}_{k}, 1),$$

so $\mathfrak{Q}^{i-1}(j_k) = [k, d-1]$ for all $0 \le k \le d-1$. Therefore by Theorem 2.10,

$$\pi_{j_0} \prec \pi_{j_1} \prec \cdots \prec \pi_{j_{d'-1}} \prec \pi_{i-1} \quad \text{in } V_{i-1}. \tag{6}$$

Similarly, we get $\mathfrak{Q}^{i+1}(j_k) = [0, k-1]$ for $1 \le k < d$ (and for $k = d$ if $\varepsilon \ne 0$ and $d > 0$). Hence

$$\varepsilon\pi_{j_d} \prec \pi_{j_{d'-1}} \prec \cdots \prec \pi_{j_1} \prec \pi_{j_0} \quad \text{in } V_{i+1}. \tag{7}$$

(Here the symbol $\prec$ is extended to the zero module in the natural way.) Since π_i^n is selfdual, D is selfdual also. Let $D_1 \prec_s \cdots \prec_s D_m$ be the socle series of D. Recall that D has a filtration by quotients of V_{i-1} and V_{i+1}. As the factor π_{i-1} has multiplicity 1 and D is selfdual, (6) implies that π_{i-1} is a factor of D_q with $d'+1 \le q \le m - d'$, so $m \ge 2d'+1$. If $\varepsilon\pi_{j_d} = 0$, then $m = 2d'+1$ is the composition length of D and the theorem follows from (6) and (7). Assume that $\varepsilon\pi_{j_d} \ne 0$. As above, by the selfduality of D and (7), π_{j_d} is a factor of $D_{q'}$ with $d'+1 \le q' \le m - d'$. Assume that $q' \ne q$. Then $m = 2d'+2$, so D is uniserial, which contradicts the selfduality of D. Hence $q' = q$ and the theorem follows from (6) and (7). □

Remark 3.4. Obviously, if $\varepsilon\pi_{j_d} = 0$ (i.e. $d' < d$ or $i \equiv -p^d \pmod{p^{d+1}}$), then the module D is uniserial, so has exactly $2d'+2$ different submodules. Since D is selfdual, one can easily observe that D has exactly $2d+4$ different submodules in the case where $\varepsilon\pi_{j_d} \ne 0$ (i.e. $d = d'$ and $i \not\equiv -p^d \pmod{p^{d+1}}$).

4. Inductive systems and the transfer to an arbitrary field

In this section the inductive systems of fundamental representations for $Sp_\infty(K)$ are classified and the results of the paper are transferred to an arbitrary field.

Proposition 4.1. (i) *Let* Ψ *be an inductive system and* $\mathcal{R}^{p^{t-1}} \subset \Psi$. *Then* $\mathcal{R}^{p^t-1} \subset \Psi$.
(ii) $\mathcal{R}^k$ *is an inductive system if and only if* $k = p^t - 1$, $t \ge 1$.

Proof. (i) Assume that $\mathcal{R}^{p^t-1} \not\subset \Psi$. Choose maximal l such that $\mathcal{R}^l \subset \Psi$. Then $p^{t-1} \le l < p^t - 1$. Take minimal s such that $l_s \ne p-1$ and set $i = p^s(l_s, l_{s+1}, \dots)$. Then $i > 0$ and $\mathcal{R}^i \subset \Psi$. One has $\mathrm{lp}(i) \ge s$. Moreover, if $\mathrm{lp}(i) = s$, then $i \not\equiv -p^s \pmod{p^{s+1}}$. Therefore Theorem 3.2 implies that $\pi^{n-1}_{i-1+2p^s}$ is a composition factor of $\pi_i^n{\downarrow}G_{n-1}$ for $n \ge i-1+2p^s$, and $\mathcal{R}^{i-1+2p^s} \subset \Psi$. As $i - 1 + 2p^s > l$, we get a contradiction.

(ii) In view of (i), it suffices to verify that $\mathcal{R}^{p^t-1}$ is an inductive system. By Theorem 3.2, we need only to check that if $i < p^t - 1$ and $s = \mathrm{lp}(i)$, then $i - 1 + 2p^s \le p^t - 1$ if $i_s \ne p-1$, and $i - 1 + 2p^{s-1} \le p^t - 1$ if $i_s = p-1$ and $s > 0$. But this is clear since $i \le (p-2)p^s + (p-1)p^{s+1} + \ldots + (p-1)p^{t-1}$ in the first case and $i \le (p-1)p^s + (p-1)p^{s+1} + \ldots + (p-1)p^{t-1}$ in the second one. □

Proof of Theorem 1.3. Theorem 1.1(i) and Proposition 4.1 yield that $\mathcal{F}$, $\mathcal{L}^s$, $\mathcal{R}^{p^t-1}$, and $\mathcal{L}^s \cup \mathcal{R}^{p^t-1}$, $s \ge 0$, $t \ge 1$, are inductive systems for $Sp_\infty(K)$.

Let $\Psi = \{\Psi_i,\ i = 1, 2, \dots\}$ be an inductive system of fundamental representations. It is clear that either for every $s, u \in \mathbb{Z}^+$ there exist n and l such that $\omega_l^n \in \Psi_n$, $l > s$, and $n + 1 - l > u$, or $\Psi \subset \mathcal{L}^s \cup \mathcal{R}^u$ for some s and u. In the first case we claim that

$\Psi = \mathcal{F}$. Indeed, fix m and l, $0 \le l \le m$. Then one can choose k and n such that $\omega_k^n \in \Psi_n$, $k \ge l$, and $n-k \ge m-l$. Since Ψ is an inductive system, Theorem 1.1(i) implies that $\omega_l^{n+l-k} \in \Psi_{n+l-k}$ and $\omega_l^m \in \Psi_m$. Hence $\Psi = \mathcal{F}$.

Next, suppose that $\Psi \subset \mathcal{L}^s \cup \mathcal{R}^u$. Choose minimal s and u with this property assuming that $s=-1$ if $\Psi \subset \mathcal{R}^u$ and $u=0$ if $\Psi \subset \mathcal{L}^s$. (Observe that for all s and u, $(\mathcal{L}^s \cap \mathcal{R}^u)_n = \emptyset$ for n large enough.) We shall prove that $\Psi = \mathcal{L}^s \cup \mathcal{R}^u$ and $u = p^t - 1$ with $t \in \mathbb{Z}^+$ (in particular, $\Psi = \mathcal{R}^u$ for $s=-1$ and $\Psi = \mathcal{L}^s$ for $u=0$).

First let $u > 0$. We claim that $\mathcal{R}^u \subset \Psi$ and $u = p^t - 1$. As $\Psi \not\subset \mathcal{L}^s$ and Ψ and $\mathcal{L}^s$ are inductive systems, $\Psi_n \cap \mathcal{R}_n^u \neq \emptyset$ for infinitely many integers n. So there exists $v \le u$ such that $\pi_v^n \in \Psi_n$ for infinitely many n. Choose maximal such v. Theorem 3.2 yields that $\pi_v^n \in \Psi_n$ for all $n \ge v-1$ and $\mathcal{R}^v \subset \Psi$. Now Proposition 4.1 and the choice of v imply that $v = p^t - 1$ and $\mathcal{R}^v$ is an inductive system. It remains to show that $v = u$. Suppose this is not the case. As $\Psi \not\subset \mathcal{L}^s \cup \mathcal{R}^v$, there exist l and t such that $v < l \le u$, $t > s+l-1$, and $\pi_l^t \in \Psi_t$. Since Ψ, $\mathcal{L}^s$, and $\mathcal{R}^v$ are inductive systems, this implies that for every $k > t$ there exists $\pi_{m_k}^k \in \Psi_k$ with $v < m_k \le u$ which contradicts the choice of v. Hence $v = u = p^t - 1$ and $\mathcal{R}^u \subset \Psi$.

Now we show that $\mathcal{L}^s \subset \Psi$ if $s \ge 0$. As $\Psi \not\subset \mathcal{L}^{s-1} \cup \mathcal{R}^u$, for some $n > s+u-1$ we have $\omega_s^n \in \Psi_n$. Since Ψ, $\mathcal{R}^u$, and $\mathcal{L}^{s-1}$ for $s \ge 1$ are inductive systems, this forces $\omega_s^n \in \Psi_n$ for all $n \ge s$. Now Theorem 1.1 yields that $\mathcal{L}^s \subset \Psi$, as desired. □

Proposition 4.2. *All theorems of the paper hold for arbitrary $F \supset K$.*

Proof. Since the restrictions of the fundamental representations of a semisimple algebraic group over an algebraically closed field to relevant Chevalley groups over arbitrary subfields remain irreducible and can be realized over these subfields, only Theorems 1.1(ii) (or 3.3), 2.10, and 2.13 require some analysis. Let M be the G_n-module $\omega_i^{n+1}{\downarrow}G_n$ or W_i^n.

First assume that $F = \bar{F}$ is algebraically closed. Set $H = Sp_{2n}(F)$. Let L be the Lie algebra of H. For a root α of H and $t \in F$ denote by $x_\alpha(t) \in H$ and $X_\alpha \in L$ the root elements in H and L associated with α. It is well known that $x_\alpha(t)(m) = (1 + tX_\alpha)m$ for $m \in M$ and long α (see, for instance, [6, Lemma 1]). For $g \in G_n$ set $x_\alpha^g(t) = g x_\alpha(t) g^{-1}$ and $X_\alpha^g = g X_\alpha g^{-1}$. It is clear that $X_\alpha^g \in L$. It suffices to show that each G_n-submodule $N \subset M$ is an H-submodule. Obviously, $x_\alpha^g(t)N = N$ and $X_\alpha^g N \subset N$ for all long roots α, $g \in G_n$, and $t \in K$. But this forces $x_\alpha^g(t)N = N$ for all $t \in F$. However, using the commutator relations for the Chevalley groups of type C (see, for instance, [9, Lemma 15]), one can deduce that the subgroup generated by all $x_\alpha^g(t)$ with $g \in G_n$, $t \in F$, and long α coincides with H. (Here, in fact, it suffices to make computations within subgroups of type C_2 and show that our subgroup contains all short root subgroups). Hence N is an H-module, as desired.

Now let $F \supset K$ be arbitrary. For a finite dimensional FG_n-module S set $\bar{S} = S \otimes_F \bar{F}$ and denote the socle of S by $\mathrm{soc}(S)$. Since $\dim \mathrm{Hom}_{FG_n}(E, S) = \dim \mathrm{Hom}_{\bar{F}G_n}(\bar{E}, \bar{S})$ for any FG_n-module E, we have $\mathrm{soc}(\bar{S}) = \overline{\mathrm{soc}(S)}$ if all composition factors of S are absolutely irreducible. The same holds for other members of the socle series of S. If $M = W_i^n$, then M and $\bar{M}$ are multiplicity-free and their submodules are completely determined by the sets of composition factors. Therefore the arguments on socles allow us to conclude that each submodule of $\bar{M}$ has the form $\bar{S}$ for some submodule $S \subset M$.

Let $M = \omega_i^{n+1}{\downarrow}G_n$ with $i, n \ge 1$. Then the socle of M contains a submodule $V \cong \omega_{i-1}^n \oplus \omega_{i-1}^n$. Since M and V are selfdual and M has only two composition factors isomorphic to ω_{i-1}^n, there exists a submodule D of M such that $M = V \oplus D$ and $\bar{D}$ is the unique submodule in $\bar{M}$ with $\bar{M} = \bar{V} \oplus \bar{D}$. Now one can see that the socle series of D is determined by that of $\bar{D}$ and is described by Theorem 1.1(ii). □

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