Minimal inductive systems of modular representations for naturally embedded algebraic and finite groups of type A

A. A. Baranov and I. D. Suprunenko
Institute of Mathematics
National Academy of Sciences of Belarus
Surganova 11, Minsk, 220072, Belarus
baranov@im.bas-net.by
suprunenko@im.bas-net.by

Abstract

The article is devoted to the classification of the minimal and minimal nontrivial inductive systems of modular representations for naturally embedded algebraic and finite groups of type A and related locally finite groups. It occurs that the minimal systems consist of the trivial representations for the relevant groups and the minimal nontrivial ones are connected with Frobenius twists of the standard representations and their duals. These results are applied to the description of the maximal ideals in group algebras of the locally finite groups SL_∞ and SU_∞ in describing characteristic.

It is also proved that for an arbitrary classical algebraic group, the restriction of an irreducible module with highest weight large enough to a naturally embedded finite Chevalley group of the same type, but a smaller rank contains the regular module.

1 Introduction

Let $\mathbb{F}$ be any field of characteristic $p > 0$, $\mathbb{K}$ be an algebraically closed field of characteristic p, and $\mathbb{N}$ be the set of natural numbers. Let

$$\Gamma_1 \to \Gamma_2 \to \cdots \to \Gamma_n \to \dots \tag{1}$$

be a sequence of embeddings where either all Γ_i are finite or all of them are algebraic groups over $\mathbb{K}$ and for algebraic Γ_i all embeddings in (1) are rational. Throughout the paper $\operatorname{Irr}\mathbb{F}\Gamma_n$ (or $\operatorname{Irr}\Gamma_n$ if $\mathbb{F}$ is fixed) is the set of equivalence classes of irreducible (rational for Γ_n algebraic) Γ_n-modules over $\mathbb{F}$ ($\mathbb{F} = \mathbb{K}$ for Γ_n algebraic); $\operatorname{Irr} M$ is the set of composition factors of a module M; M^* is the dual module of M; τ is the trivial one-dimensional module for any group. For groups $H \subset G$ and a G-module M denote by $M{\downarrow}H$ the restriction of M to H. Let Φ_n be a nonempty finite subset of $\operatorname{Irr}\Gamma_n$. We say that the collection $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is an *inductive system* for the sequence (1) if

$$\bigcup_{M\in\Phi_{n+1}} \operatorname{Irr}(M{\downarrow}\Gamma_n) = \Phi_n$$

for all $n \in \mathbb{N}$ (see also the definition in Section 5). The system Φ is called *trivial* if $\Phi_n = \{\tau\}$ for all n; otherwise Φ is nontrivial.

Set $G_n = SL_{n+1}(\mathbb{K})$ (the algebraic group of type A_n). Let Fr be the Frobenius morphism of G_n associated with raising elements of $\mathbb{K}$ to the pth power. For a morphism ρ of G_n and $M \in \operatorname{Irr} G_n$ denote by $\rho(M)$ the G_n-module affording the representation $G_n \to \rho(G_n) \to GL(M)$. Let V_n be the standard G_n-module and V_n^* be its dual. Set

$$\mathcal{L}_n^j = \{\mathrm{Fr}^j(V_n), \tau\} \quad \text{and} \quad \mathcal{R}_n^j = \{\mathrm{Fr}^j(V_n^*), \tau\} \quad (j = 0, 1, 2, \dots).$$

Let $\mathfrak{A}$ be the sequence of the natural embeddings

$$G_1 \to G_2 \to \cdots \to G_n \to \dots.$$

It is clear that $\mathcal{L}^j = \{\mathcal{L}_n^j\}_{n\in\mathbb{N}}$ and $\mathcal{R}^j = \{\mathcal{R}_n^j\}_{n\in\mathbb{N}}$ are inductive systems for $\mathfrak{A}$. The following theorem shows that they exhaust the minimal nontrivial inductive systems.

Theorem 1.1 *Let Φ be a nontrivial inductive system for $\mathfrak{A}$. Then Φ contains one of the systems $\mathcal{L}^j$ or $\mathcal{R}^j$ ($j \geq 0$). In particular, any inductive system for $\mathfrak{A}$ contains the trivial one.*

The first result of the article on finite groups concerns not only groups of type A. We prove that the restriction of an irreducible representation φ of a classical algebraic group over $\mathbb{K}$ to a Chevalley group of the same type and a smaller rank over a finite subfield of $\mathbb{K}$ contains the regular representation provided the highest weight of φ is large enough. Let $\Gamma_1 \subset \ldots \subset \Gamma_n \subset \ldots$ be the sequence of the naturally embedded simply connected classical groups $\Gamma_i = \Gamma_i(\mathbb{K})$ of rank i with the root systems of the same type $\Gamma = A, B, C$, or D. Transfer the notation Fr to the groups Γ_i. Denote by ω_j^i, $1 \leq j \leq i$, the fundamental weights of the group Γ_i labeled in the standard way. Let $M \in \operatorname{Irr}\Gamma_n$ and let λ be the highest weight of M. Represent λ in the form $\lambda = \sum_{j=0}^{u} p^j\lambda_j$ where the weights λ_j are p-restricted. Let $\lambda_j = \sum_{i=1}^{n} a_{ij}\omega_i^n$. Set $S(M) = \max\{\sum_{i=1}^{n} a_{ij} \mid 0 \leq j \leq u\}$.

Theorem 1.2 *Let $H_m \subset \Gamma_m$ be a finite ordinary or twisted Chevalley group of rank m which is the fixed subgroup of some Frobenius morphism of Γ_m and let $l = |H_m|-1$. Then for each $n > m+1$ and each $M \in \operatorname{Irr}\Gamma_n$ with $S(M) \geq p + p^2 + \ldots + p^l + (p-1)(m+1)$ the restriction $M{\downarrow}H_m$ contains the regular $\mathbb{K}H_m$-module. In particular, $\operatorname{Irr}(M{\downarrow}H_m) = \operatorname{Irr}\mathbb{K}H_m$.*

Remark 1.3 It is clear that Theorem 1.2 can be transferred to irreducible $\mathbb{K}$-representations of finite Chevalley groups associated with Γ_n which contain H_m.

Now let $q = p^s$ and let

$$H_1 \to H_2 \to \dots \to H_n \to \dots \qquad (2)$$

be the sequence of finite groups $H_n = SL_{n+1}(q)$ or $SU_{n+1}(q^2)$ (all the groups are of the same type) naturally embedded each into another. Set $H = \varinjlim H_n$. Then H is isomorphic to one of the groups $SL_\infty(q)$ or $SU_\infty(q^2)$. One can consider these groups as the groups of infinite matrices of the form

$$A = \mathrm{diag}(A_n, 1, 1, \dots)$$

where $A_n \in SL_n(q)$ or $SU_n(q^2)$. So we can define an automorphism Fr of H as above. We say that Ψ is an *inductive system* for H if Ψ is an inductive system for the sequence (2). Let M be an $\mathbb{F}H$-module. One can define the corresponding inductive system $\Phi = \Phi(M)$ as follows. Set $\Phi_n = \mathrm{Irr}(M{\downarrow}H_n)$. One easily checks that $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is an inductive system for H. Let $\mathbb{F}$ contain the field of order q for $H = SL_\infty(q)$ and the field of order q^2 for $H = SU_\infty(q^2)$. We say that $\mathbb{F}$ is a *splitting field* for H. Denote by V the natural module for H over $\mathbb{F}$ and by V^* the dual to V. Set $\mathcal{L}^j = \Phi(\mathrm{Fr}^j(V))$, and $\mathcal{R}^j = \Phi(\mathrm{Fr}^j(V^*))$, $0 \le j < s$. We keep the notation $\mathcal{L}^j$ and $\mathcal{R}^j$ used for algebraic groups since the relevant inductive systems can be obtained by restriction from algebraic groups.

Theorem 1.4 *Let $\mathbb{F}$ be a splitting field for $H = SL_\infty(q)$ or $SU_\infty(q^2)$. Then each nontrivial inductive system for H over $\mathbb{F}$ contains either $\mathcal{L}^j$ or $\mathcal{R}^j$ $(0 \le j < s)$. In particular, any inductive system for H contains the trivial one.*

The notion of an inductive system has been introduced by Zalesskii in [11] and has been developed in [12]. Inductive systems yield an asymptotic version of the branching rules for relevant embeddings. Classical branching rules for algebraic groups in characteristic 0 have found numerous applications. It is quite easy to deduce an analog of Theorem 1.1 from these rules. Since one cannot expect to find their explicit modular analogs in the general case, it is worth to seek for an asymptotic version. Moreover, inductive systems can be applied to the study of ideals in group algebras of locally finite groups. It is proved in [12] that there exists a bijective correspondence between the inductive systems for a locally finite group and the semiprimitive ideals of the corresponding group algebra. This enabled the authors to obtain some results on the ideals of $\mathbb{F}H$.

Theorem 1.5 *Let $H = SL_\infty(q)$ or $SU_\infty(q^2)$ and $\mathbb{F}$ be a splitting field for H. Then any proper ideal of $\mathbb{F}H$ is contained in the augmentation ideal* $\mathrm{Aug}(\mathbb{F}H)$*. Moreover, the annihilators* $\mathrm{Ann}_{\mathbb{F}H}\,\mathrm{Fr}^j(V)$ *and* $\mathrm{Ann}_{\mathbb{F}H}\,\mathrm{Fr}^j(V^*)$ *$(0 \le j < s)$, are exactly all distinct maximal ideals of* $\mathrm{Aug}(\mathbb{F}H)$.

Theorems 1.4 and 1.5 are particular cases of Theorems 5.3 and 5.5, respectively, which for an arbitrary locally finite field $\mathbb{L}$ and a field $\mathbb{F}$ with $\mathrm{char}\,\mathbb{F} = \mathrm{char}\,\mathbb{L}$ describe the minimal nontrivial inductive systems and the maximal ideals in $\mathrm{Aug}(\mathbb{F}H)$ for $H = SL_\infty(\mathbb{L})$ or $SU_\infty(\mathbb{L})$. To note this, see Remark 3.3 as well.

For other classical groups the question on the minimal inductive systems seems substantially more difficult. Analogs of Theorems 1.1 and 1.4 are not valid there. Indeed, for the groups of types B and D there are inductive systems consisting of the spinor and semispinor representations, respectively. Denote by $\varphi(\omega)$ the irreducible representation of a fixed algebraic group with highest weight ω. For the natural embeddings of the groups of type C_n and $p > 2$ Zalesskii and Suprunenko [10] have described an inductive system Ψ with $\Psi_n = \{\varphi(\omega_{n-1}^n + \frac{1}{2}(p-3)\omega_n^n), \varphi(\frac{1}{2}(p-1)\omega_n^n)\}$, $n = 1, 2, \dots$. The authors [1] have obtained the branching rules for modular fundamental representations of C_n and classified the inductive systems consisting of such and trivial representations. In particular, there exists a minimal inductive system $\mathcal{R}^{p-1} = \{\mathcal{R}_n^{p-1}\}$, $n = 1, 2, \dots$, with $\mathcal{R}_n^{p-1} = \{\varphi(\omega_j^n) \mid n + 1 - j \leq p - 1\}$.

2 Inductive systems for algebraic groups

In this section the minimal nontrivial inductive systems for the sequence $\mathfrak{A}$ of the natural embeddings $G_1 \to G_2 \to \dots \to G_n \to \dots$ are determined and Theorem 1.2 is proved. Let $\alpha_1, \dots, \alpha_n$ be the simple roots of G_n. We identify these roots with the relevant simple roots of G_{n+1}. Denote by X_n^+ the set of dominant weights of G_n and let $\omega_1^n, \dots, \omega_n^n$ (or $\omega_1, \dots, \omega_n$ if n is fixed) be the fundamental weights of G_n. Recall that $X_n^+ = \{a_1\omega_1 + \dots + a_n\omega_n \mid a_i \in \mathbb{N}_0\}$ where $\mathbb{N}_0$ is the set of nonnegative integers. Set $X_n^q = \{a_1\omega_1 + \dots + a_n\omega_n \mid 0 \leq a_i < q\}$. The weights in X_n^q are called q-restricted. Denote by $\langle \cdot, \cdot \rangle$ the canonical pairing on the set of weights of G_n. In what follows $M(\lambda)$ is the irreducible G_n-module with highest weight λ. Recall that $V_n = M(\omega_1^n)$ and $V_n^* = M(\omega_n^n)$. For $\lambda = a_1\omega_1 + \dots + a_n\omega_n \in X_n^+$ set $\delta(\lambda) = a_1 + \dots + a_n$. The following lemmas are used in the proof of Theorem 1.1.

Lemma 2.1 *For each $\lambda \in X_n^+$ and each root α of G_n we have*

$$\delta(\lambda) = \max\langle \mu, \alpha \rangle$$

where μ runs over the weights of $M(\lambda)$.

Proof. Let β be the maximal root of G_n. It is well known that $\beta = \alpha_1 + \dots + \alpha_n$. Since the Weyl group acts transitively on the set of roots, $\max\langle \mu, \alpha \rangle = \max\langle \mu, \beta \rangle$ for every root α. As β is a dominant weight, $\langle \alpha_i, \beta \rangle \geq 0$. This forces $\delta(\lambda) = \langle \lambda, \beta \rangle = \max\langle \mu, \beta \rangle$, as required. □

Lemma 2.2 *Let $k, m, n \in \mathbb{N}$ be such that $n \geq m(k+1)$. Assume that $\lambda \in X_n^+ \setminus \{0\}$ and $\delta(\lambda) \leq k$. Then $\tau \in \mathrm{Irr}(M(\lambda)\downarrow G_m)$. Moreover, if λ is p-restricted, then* $\mathrm{Irr}(M(\lambda)\downarrow G_m)$ *contains V_m or V_m^*.*

Proof. Denote by $H(i_1, \dots, i_k)$ the subgroup generated by the root subgroups associated with the roots $\pm\alpha_{i_1}, \dots, \pm\alpha_{i_k}$. Let $\lambda = a_1\omega_1^1 + \dots + a_n\omega_n^n$. Obviously, at most k of the coefficients a_i are nonzero. Now one can observe that there exists i such that $a_i \neq 0$ and either $a_{i+1} = \dots = a_{i+m} = 0$, or $a_{i-m} =$

$\cdots = a_{i-1} = 0$. Put $a = a_i$, $H = H(i, i+1, \dots, i+m)$ in the first case and $H = H(i-m, \dots, i-1, i)$ in the second one. It is clear that H is conjugate to G_{m+1} in G_n. Hence $\mathrm{Irr}(M(\lambda){\downarrow}G_{m+1}) = \mathrm{Irr}(M(\lambda){\downarrow}H)$. Now Smith's theorem [7] yields that $\mathrm{Irr}(M(\lambda){\downarrow}G_{m+1})$ contains $M(a\omega_1^{m+1})$ or $M(a\omega_{m+1}^{m+1})$. Similar arguments show that $\tau \in \mathrm{Irr}(M(\lambda){\downarrow}G_m)$.

Now assume that λ is p-restricted. Then $a < p$ and $M(a\omega_1^{m+1})$ is isomorphic to the ath symmetric power $S^a(V_{m+1})$ of V_{m+1} (see, for instance, [6, (1.14)]). Let e be a highest weight vector of V_{m+1} and $f \in V_{m+1}$ be a nonzero vector fixed by G_m. It is not difficult to see that the vector $ef^{a-1} \in S^a(V_{m+1})$ generates the standard G_m-module. Therefore $\mathrm{Irr}(M(a\omega_1^{m+1}){\downarrow}G_m)$ contains V_m. Considering the dual modules, we get that $\mathrm{Irr}(M(a\omega_{m+1}^{m+1}){\downarrow}G_m)$ contains V_m^*. This yields our claim for $\mathrm{Irr}(M(\lambda){\downarrow}G_m)$. □

Lemma 2.3 *Let $k, m, n,$ and $\lambda = a_1\omega_1^1 + \cdots + a_n\omega_n^n$ be such as in Lemma 2.2. Let i be the maximal power of p such that p^i divides all the coefficients a_j. Then* $\mathrm{Irr}(M(\lambda){\downarrow}G_m)$ *contains $M(p^i\omega_1^m)$ or $M(p^i\omega_m^m)$.*

Proof. One can write $\lambda = p^i\lambda_1 + p^{i+1}\lambda_2$ where $\lambda_1 \in X_n^p \backslash \{0\}$ and $\lambda_2 \in X_n^+$. By Steinberg's tensor product theorem [8],

$$M(\lambda) = \mathrm{Fr}^i(M(\lambda_1)) \otimes \mathrm{Fr}^{i+1}(M(\lambda_2)).$$

Obviously, $\delta(\lambda_1), \delta(\lambda_2) \leq k$. By Lemma 2.2, $\mathrm{Irr}(M(\lambda_1){\downarrow}G_m)$ contains either V_m or V_m^*, and $\mathrm{Irr}(M(\lambda_2){\downarrow}G_m)$ contains τ. It remains to observe that $\mathrm{Fr}^i(V_m) = M(p^i\omega_1^m)$ and $\mathrm{Fr}^i(V_m^*) = M(p^i\omega_m^m)$. □

Let Ψ_n be a finite subset of $\mathrm{Irr}\, G_n$ and $\Psi = \{\Psi_n\}_{n\in\mathbb{N}}$. Set $\delta(\Psi_n) = \max\{\delta(\lambda) \mid M(\lambda) \in \Psi_n\}$ and

$$\delta(\Psi) = \min\{\delta(\Psi_n) \mid n \in \mathbb{N}\}.$$

Lemma 2.4 *Let $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ be an inductive system for $\mathfrak{A}$. Then $\delta(\Phi_n) = \delta(\Phi)$ for all n. In particular, $\delta(\lambda) \leq \delta(\Phi)$ if $M(\lambda) \in \Phi_n$.*

Proof. It suffices to show that $\delta(\Phi_n) = \delta(\Phi_{n+1})$ for all n. Let λ and ρ be such that $M(\lambda) \in \Phi_{n+1}$, $M(\rho) \in \Phi_n$, and $\delta(\lambda) = \delta(\Phi_{n+1})$ and $\delta(\rho) = \delta(\Phi_n)$. By Lemma 2.1, there exists a weight μ of $M(\lambda)$ such that

$$\delta(\lambda) = \langle \mu, \alpha_1 + \cdots + \alpha_n \rangle = \langle \nu, \alpha_1 + \cdots + \alpha_n \rangle$$

where ν is the restriction of the weight μ to G_n. Hence

$$\delta(\Phi_{n+1}) = \delta(\lambda) \leq \delta(\mathrm{Irr}(M(\lambda){\downarrow}G_n)) \leq \delta(\Phi_n).$$

Since $\Phi_n = \mathrm{Irr}(\Phi_{n+1}{\downarrow}G_n)$, there exist weights λ_1 and λ_2 of G_{n+1} such that ρ is the restriction of λ_1 to G_n and λ_1 is a weight of $M(\lambda_2) \in \Phi_{n+1}$. Hence

$$\delta(\Phi_n) = \delta(\rho) = \langle \rho, \alpha_1 + \cdots + \alpha_n \rangle = \langle \lambda_1, \alpha_1 + \cdots + \alpha_n \rangle \leq \delta(\lambda_2) \leq \delta(\Phi_{n+1}),$$

so $\delta(\Phi_n) = \delta(\Phi_{n+1})$. □

Proof of Theorem 1.1. Clearly, $\mathrm{Fr}^j(V_n) = M(p^j\omega_1^n)$ and $\mathrm{Fr}^j(V_n^*) = M(p^j\omega_n^n)$ for all n. Fix any $m \in \mathbb{N}$. Set $k = \delta(\Phi)$. Take any n such that $n \geq m(k+1)$ and any nonzero λ such that $M(\lambda) \in \Phi_n$. By Lemma 2.4, $\delta(\lambda) \leq k$. Therefore by Lemma 2.3, Φ_m contains $M(p^i\omega_1^m)$ or $M(p^i\omega_m^m)$ for some $i = i(m) \geq 0$. Observe that $p^i \leq k$. Now it is clear that there exists $j \geq 0$ and an infinite sequence $m_1 < m_2 < \cdots < m_r < \ldots$ such that either for each r the set Φ_{m_r} contains $M(p^j\omega_1^{m_r})$, or each of them contains $M(p^j\omega_{m_r}^{m_r})$. Applying Smith's theorem [7], we obtain the required assertion. □

Now we prove a lemma that holds for arbitrary classical groups. Transfer to other classical groups the notation for roots and weight systems introduced at the beginning of the section for the groups G_n. Until the end of this section $\Gamma_1 \subset \ldots \Gamma_n \subset \ldots$ is the sequence of the simply connected classical groups $\Gamma_i = \Gamma_i(\mathbb{K})$ with the root systems of the same type where the group Γ_i of rank i is identified with the subgroup $H(2, 3, \ldots, i+1) \subset \Gamma_{i+1}$. So the roots $\alpha_2, \ldots, \alpha_{i+1}$ of Γ_{i+1} yield a basis of the root system for Γ_i. Similarly the roots $\alpha_{n-m+1}, \ldots, \alpha_n$ yield a basis of the root system for the group Γ_m embedded into Γ_n. Below we consider the weights of restrictions of Γ_n-modules to Γ_m with respect to this basis.

Lemma 2.5 *Let $\lambda = a_1\omega_1^n + \ldots + a_n\omega_n^n \in X_n^p$ and $m < n$. Then for each integer b with $0 \leq b \leq a_1 + a_2 + \ldots + a_{n-m}$ the restriction $M(\lambda)\downarrow\Gamma_m$ has a composition factor with highest weight $\lambda_b = (a_{n-m+1} + b)\omega_1^m + a_{n-m+2}\omega_2^m + \ldots + a_n\omega_m^m$.*

Proof. For a root α of Γ_n and $t \in \mathbb{K}$ let $x_\alpha(t) \in \Gamma_n$ and $\mathcal{X}_\alpha \subset \Gamma_n$ be the root element and the root subgroup of Γ_n associated with α and t. Set $M = M(\lambda)$. Let $M_\nu \subset M$ be the weight subspace of weight μ. By [2, Proposition 5.13], there exist $X_{\alpha,d} \in \mathrm{End}\, M$ $(d = 0, 1, 2, \ldots)$ with the following properties: $X_{\alpha,d} = 0$ for d large enough, $X_{\alpha,0} = 1$, $x_\alpha(t) = \sum_{d=0}^{\infty} t^d X_{\alpha,d}$ (in $\mathrm{End}\, M$), $X_{\alpha,d}M_\nu \subset M_{\nu+d\alpha}$ for all weights ν of M. Let b be such as in the assertion of the lemma. Set $s = n - m + 1$. If $b > a_{s-1}$, represent b in the form $b = c + a_{s-k+1} + \ldots + a_{s-1}$ where $0 \leq c \leq a_{s-k}$; if $b \leq a_{s-1}$, put $c = b$ and $k = 1$. Let $v \in M$ be a nonzero highest weight vector. Now we shall construct a vector $u_b \in M$ that generates an indecomposable Γ_m-module with highest weight λ_b. First we define the integers $d_1, \ldots, d_k$ as follows: $d_k = c$, $d_j = a_{s-j} + d_{j+1}$ for $1 \leq j < k$. Set $u_b = X_{-\alpha_{s-1},d_1} \ldots X_{-\alpha_{s-k},d_k} v$. Observe that $d_1 = b$. By [9, Lemma 2.9], the vector u_b is nonzero and is fixed by the root subgroups $\mathcal{X}_{\alpha_i}$ for $i \neq n - m$.

The properties of the operators $X_{\alpha,d}$ imply that u_b has the weight λ_b with respect to Γ_m. Now it is clear that u_b generates an indecomposable Γ_m-module with highest weight λ_b and so $M\downarrow\Gamma_m$ has the required composition factor. □

Proof of Theorem 1.2. Set $p + p^2 + \ldots + p^l = y$. Let $\lambda = \sum_{j=0}^{u} p^j\lambda_j$ with $\lambda_j = \sum_{i=1}^{n} a_{ij}\omega_i^n \in X_p^n$ be the highest weight of M. Our assumptions yield that $\sum_{i=1}^{n-m-1} a_{it} \geq y$ for some t, $0 \leq t \leq u$. Hence, by Lemma 2.5 and Steinberg's tensor product theorem [8], $M(\lambda_t)\downarrow\Gamma_{m+1}$ has a composition factor of the form $M(y\omega_1^{m+1}) \otimes M(\mu)$ with p-restricted μ. The same theorem implies that

$$M(y\omega_1^{m+1})\downarrow\Gamma_m = (\tau \oplus M(p\omega_1^m)) \otimes (\tau \oplus M(p^2\omega_1^m)) \otimes \cdots \otimes (\tau \oplus M(p^l\omega_1^m)).$$

Now it is clear that $M(y\omega_1^{m+1})\downarrow H_m = M'$ where $M' = (\tau \oplus M_1) \otimes (\tau \oplus M_2) \otimes \ldots \otimes (\tau \oplus M_l)$ for some irreducible $\mathbb{K}H_m$-modules $M_1, \ldots, M_l$. Hence $(\mathrm{Fr}^t(M') \otimes N)$ is a quotient of a submodule of $M\downarrow H_m$ for some H_m-module N. By [4, ch.3, Corollary 2.17], the regular H_m-module $\mathbb{K}H_m$ is a direct summand of M' as each nonidentity element of H_m acts nontrivially on the modules M_j, $1 \le j \le l$. So $\mathbb{K}H_m$ is a direct summand of $\mathrm{Fr}^t(M')$ as well since $\mathrm{Fr}^t(\mathbb{K}H_m) \cong \mathbb{K}H_m$. Let Q be an irreducible submodule of N. By [3, Corollary 10.20], $\mathbb{K}H_m \otimes Q$ is isomorphic to the direct sum of $\dim Q$ copies of $\mathbb{K}H_m$. Now the projectivity of $\mathbb{K}H_m$ implies that $\mathbb{K}H_m$ is a submodule of $M\downarrow H_m$. □

3 Inductive systems over splitting fields

In this section $q = p^s$, $s \in \mathbb{N}$, and $H_1 \to H_2 \to \cdots \to H_n \to \ldots$ is the sequence of finite groups $H_n = SL_{n+1}(q)$ or $SU_{n+1}(q^2)$ ($\cong {}^2A_n(q)$) naturally embedded each into another; $H = \varinjlim H_n$; $\mathbb{F}$ is a splitting field for H. Here the minimal nontrivial inductive systems for H over $\mathbb{F}$ are determined.

Assume that $\mathbb{K}$ is the algebraic closure of $\mathbb{F}$, so $H_n \subset G_n$ for all n. By Steinberg's theorem [8], the restrictions to H_n of the q-restricted irreducible G_n-modules yield the complete set of inequivalent irreducible $\mathbb{K}H_n$-modules. Since $\mathbb{F}$ is a splitting field for H_n, the extension of the ground field yields a natural bijection from $\mathrm{Irr}\,\mathbb{F}H_n$ to $\mathrm{Irr}\,\mathbb{K}H_n$. Therefore the irreducible $\mathbb{F}H_n$-modules can be parametrized by q-restricted dominant weights of G_n. For such a weight λ we denote by $N(\lambda)$ the corresponding irreducible $\mathbb{F}H_n$-module, so

$$M(\lambda)\downarrow \mathbb{K}H_n \cong N(\lambda) \otimes_{\mathbb{F}} \mathbb{K}. \qquad (3)$$

Therefore it will cause no confusion if we use the same symbol $\mathrm{Irr}\,H_n$ both for $\mathrm{Irr}\,\mathbb{F}H_n$ and $\mathrm{Irr}\,\mathbb{K}H_n$.

Remark 3.1 Sometimes we shall consider modules $N(\lambda)$ with $\lambda = q^l\lambda_0$ and q-restricted λ_0. In this case $M(\lambda)\downarrow \mathbb{K}H_n \cong N(\mu) \otimes_{\mathbb{F}} \mathbb{K}$ with q-restricted μ and we set $N(\lambda) = N(\mu)$. Here $\mu = \lambda_0$ except the case where $H = SU_{n+1}(q^2)$ and l is odd. In the exceptional case $\mu = -w_0(\lambda_0)$ where w_0 is the longest element of the Weyl group.

Lemma 3.2 *Let $m < n$, $\lambda \in X_n^q$, $\mu \in X_m^q$, and $M(\mu) \in \mathrm{Irr}(M(\lambda)\downarrow G_m)$. Then $N(\mu) \in \mathrm{Irr}(N(\lambda)\downarrow H_m)$.*

Proof. In view of (3),

$$\mathrm{Irr}(N(\lambda) \otimes_{\mathbb{F}} \mathbb{K}\downarrow \mathbb{K}H_m) = \mathrm{Irr}((M(\lambda)\downarrow \mathbb{K}G_m)\downarrow \mathbb{K}H_m) \ni M(\mu)\downarrow \mathbb{K}H_m = N(\mu) \otimes_{\mathbb{F}} \mathbb{K}.$$

Since $\mathbb{F}$ is a splitting field for H_m, the set $\mathrm{Irr}(N(\lambda)\downarrow H_m)$ contains $N(\mu)$. □

Proof of Theorem 1.4. Let $\mathcal{L}^j = \{\mathcal{L}_n^j\}_{n\in\mathbb{N}}$ and $\mathcal{R}^j = \{\mathcal{R}_n^j\}_{n\in\mathbb{N}}$ $(0 \le j < s)$ be the inductive systems for H defined earlier. One easily observes that $\mathcal{L}_n^j = \{N(p^j\omega_1^n), \tau\}$ and $\mathcal{R}_n^j = \{N(p^j\omega_n^n), \tau\}$.

Let $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ where $\Phi_n = \{N(\lambda_1^n), \dots, N(\lambda_{k_n}^n)\}$ and λ_i^n, $1 \le i \le k_n$, are q-restricted. Set $\Psi_n = \{M(\lambda_1^n), \dots, M(\lambda_{k_n}^n)\}$ and $\delta_n = \max\{\delta(\lambda_i^n) \mid i = 1, \dots, k_n\}$. The proof will be divided into two subcases.

Case 1: $\sup\{\delta_n \mid n \in \mathbb{N}\} = k < \infty$. Fix $m, n \in \mathbb{N}$ such that $n \ge m(k+1)$. By Lemma 2.3, $\mathrm{Irr}(\Psi_n{\downarrow}G_m)$ contains $M(p^i\omega_1^m)$ or $M(p^i\omega_m^m)$ for some $i = i(m) < s$. Now Lemma 3.2 implies that Ψ_m contains $N(p^i\omega_1^m)$ or $N(p^i\omega_m^m)$. So there exists $j < s$ and an infinite sequence $m_1 < m_2 < \dots < m_r < \dots$ such that either for each r the set Φ_{m_r} contains $N(p^j\omega_1^{m_r})$, or each of them contains $N(p^j\omega_{m_r}^{m_r})$. This completes Case 1 since $\mathcal{L}^j$ and $\mathcal{R}^j$ are inductive systems.

Case 2: $\sup\{\delta_n \mid n \in \mathbb{N}\} = \infty$. We shall prove that $\Phi_n = \mathrm{Irr}\, H_n$, so Φ contains all the systems $\mathcal{L}^j$ and $\mathcal{R}^j$. Fix $m \in \mathbb{N}$. Set $l = |H_m| - 1$ and $c = p + p^2 + \ldots + p^l + (p-1)(m+1)$. Since $\sup\{\delta_n \mid n \in \mathbb{N}\} = \infty$, there exist $n > m+1$ and $M = M(\lambda) \in \Psi_n$ such that $\delta(\lambda) \ge (1+p+\ldots+p^{s-1})c$. Recall that for each $M(\lambda) \in \Psi_n$ the weight λ is q-restricted and so can be represented in the form $\lambda = \lambda_0 + p\lambda_1 + \dots + p^{s-1}\lambda_{s-1}$ where $\lambda_j \in X_n^p$. Observe that $\delta(\lambda) = \delta(\lambda_0) + p\delta(\lambda_1) + \dots + p^{s-1}\delta(\lambda_{s-1})$. Therefore $s(M) \ge c$ in the notation of Theorem 1.2. Now Theorem 1.2 yields that $\mathrm{Irr}(M{\downarrow}H_m) = \mathrm{Irr}\, H_m$ and completes the proof. □

Remark 3.3 By Remark 3.1, one can consider the inductive systems $\mathcal{L}^j$ and $\mathcal{R}^j$ for arbitrary $j = 0, 1, 2, \ldots$ Note that $\mathcal{R}^j \cong \mathcal{L}^{j+s}$ for $H = SU_\infty(q^2)$. Hence any nontrivial inductive system for $H = SU_\infty(q^2)$ contains one of the systems $\mathcal{L}^j$ ($0 \le j < 2s$).

4 Nonsplitting fields

We shall heavily use the following fact.

Lemma 4.1 *Let $\mathbb{F}$ be a field of characteristic p and $\mathbb{E}$ be any extension field of $\mathbb{F}$. Let G be a finite group and M be an irreducible $\mathbb{F}G$-module. Then $M \otimes_{\mathbb{F}} \mathbb{E}$ is a direct sum of irreducible $\mathbb{F}G$-modules no two of which are isomorphic. If $\mathbb{E}$ is a finite Galois extension of $\mathbb{F}$ and a splitting field for G, then there is a bijection between* $\mathrm{Irr}\,\mathbb{F}G$ *and* $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$*-orbits in* $\mathrm{Irr}\,\mathbb{E}G$*; $M \otimes_{\mathbb{F}} \mathbb{E}$ is a direct sum of all modules in the corresponding orbit $O(M)$; and $O(M)$ is the set of all $N \in \mathrm{Irr}\,\mathbb{E}G$ such that the $\mathbb{F}G$-module N is a sum of copies of M.*

Proof. This follows from [3, Corollaries 7.11 and 7.19] and the fact that the $\mathbb{F}G$-module $M \otimes_{\mathbb{F}} \mathbb{E}$ is a sum of copies of M. □

Set $t = s$ for $H = SL_\infty(p^s)$ and $t = 2s$ for $H = SU_\infty(p^{2s})$; $\mathbb{F}_H = \mathbb{F}_{p^t}$ (the field of order p^t). As above, the irreducible $\mathbb{F}_H H_n$-modules can be parametrized by q-restricted weights λ and will be denoted by $N(\lambda)$. In this section we assume that $\mathbb{F}$ is not a splitting field for H, i.e. $\mathbb{F}_H \not\subset \mathbb{F}$. Then $\mathbb{F}_H \cap \mathbb{F}$ is a finite field $\mathbb{F}_0 = \mathbb{F}_{p^u}$. Note that $u < t$ and $u|t$. We identify $\mathbb{F}_H$ with the isomorphic subfield of the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. Let $\mathbb{F}_1 = \mathbb{F}_H\mathbb{F}$ be the composite of $\mathbb{F}_H$ and $\mathbb{F}$. It is well known that $\mathbb{F}_1$ is a Galois extension of $\mathbb{F}$ and the Galois group $\mathrm{Gal}(\mathbb{F}_1/\mathbb{F})$ is isomorphic to $\Sigma = \mathrm{Gal}(\mathbb{F}_H/\mathbb{F}_0) \cong \mathbb{Z}_{t/u}$ (the cyclic group of order t/u); moreover, we can obtain this isomorphism restricting the automorphisms of $\mathbb{F}_1$ to $\mathbb{F}_H$.

The following proposition allows us to reduce the study of inductive systems over $\mathbb{F}$ to the similar problem for $\mathbb{F}_0$.

Proposition 4.2 *The map $f : M \mapsto \bar{M} = M \otimes_{\mathbb{F}_0} \mathbb{F}$ yields a bijection between $\mathrm{Irr}\,\mathbb{F}_0 H_n$ and $\mathrm{Irr}\,\mathbb{F} H_n$. Moreover, for $n > 1$*

$$\mathrm{Irr}(\bar{M}{\downarrow}H_{n-1}) = \{\bar{N} \mid N \in \mathrm{Irr}(M{\downarrow}H_{n-1})\}.$$

Proof. First we claim that $\bar{M}$ is irreducible. Indeed, by Lemma 4.1, $\bar{M}$ is a direct sum of k irreducible inequivalent H_n-modules. Since $\mathbb{F}_1$ is a Galois extension of $\mathbb{F}$ and is a splitting field for H_n, the same lemma implies that $\bar{M} \otimes_{\mathbb{F}} \mathbb{F}_1$ is the direct sum of all modules involved in some different k $\mathrm{Gal}(\mathbb{F}_1/\mathbb{F})$-orbits in $\mathrm{Irr}\,\mathbb{F}_1 H_n$. On the other hand, $\bar{M} \otimes_{\mathbb{F}} \mathbb{F}_1 \cong (M \otimes_{\mathbb{F}_0} \mathbb{F}_H) \otimes_{\mathbb{F}_H} \mathbb{F}_1$. By Lemma 4.1, $M \otimes_{\mathbb{F}_0} \mathbb{F}_H = M_1 \oplus \cdots \oplus M_l$ where $M_1, \dots, M_l$ constitute a Σ-orbit in $\mathrm{Irr}\,\mathbb{F}_H H_n$. Since $\mathbb{F}_H$ is a splitting field for H_n, every M_i remains irreducible after a field extension. As each element of $\mathrm{Gal}(\mathbb{F}_1/\mathbb{F})$ is completely determined by its action on $\mathbb{F}_H$ and irreducible $\mathbb{F}_1 H_n$-representations can be realized over $\mathbb{F}_H$, one can identify the action of $\mathrm{Gal}(\mathbb{F}_1/\mathbb{F})$ on $\mathrm{Irr}\,\mathbb{F}_1 H_n$ with that of Σ on $\mathrm{Irr}\,\mathbb{F}_H H_n$. Hence $M_1 \otimes_{\mathbb{F}_H} \mathbb{F}_1, \dots, M_l \otimes_{\mathbb{F}_H} \mathbb{F}_1$ constitute a $\mathrm{Gal}(\mathbb{F}_1/\mathbb{F})$-orbit in $\mathrm{Irr}\,\mathbb{F}_1 H_n$. This implies $k = 1$, so $\bar{M}$ is irreducible.

The same arguments with field extensions show that the map f is injective. By Lemma 4.1, $|\mathrm{Irr}\,\mathbb{F}_0 H_n| = |\mathrm{Irr}\,\mathbb{F} H_n|$ since the relevant Galois groups have equal numbers of orbits on $\mathrm{Irr}\,\mathbb{F}_H H_n$ and $\mathrm{Irr}\,\mathbb{F}_1 H_n$, respectively. Hence f is a bijection. Since taking field extensions commutes with restricting to subgroups, the irreducibility of $\bar{M}$ yields the assertion of the lemma for restrictions. □

Let $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ be an inductive system for H over $\mathbb{F}_0$. Set $\bar{\Phi}_n = \{\bar{M} \mid M \in \Phi_n\}$; $\bar{\Phi} = \{\bar{\Phi}_n\}_{n\in\mathbb{N}}$.

Corollary 4.3 *The map $\Phi \to \bar{\Phi}$ yields a bijection between the sets of inductive systems for H over $\mathbb{F}_0 = \mathbb{F} \cap \mathbb{F}_H$ and $\mathbb{F}$, respectively.*

In view of Corollary 4.3, we restrict our attention to the modules and the inductive systems over $\mathbb{F}_0 = \mathbb{F}_{p^u}$. One easily observes that the orbit of $N(p^j\omega_1^n)$, $0 \le j < u$, under Σ is the set $\{N(p^{j+ku}\omega_1^n) \mid k = 0, 1, \dots, t/u - 1\}$. The orbit of $N(p^j\omega_n^n)$ has a similar form.

Denote by $\mathcal{N}_n^j$ and $\mathcal{N}_n^{*j}$ the H_n-modules $N(p^j\omega_1^n)$ and $N(p^j\omega_n^n)$, respectively, considered over the field $\mathbb{F}_0$. Since we are interested in inductive systems for $H = \varinjlim H_n$, we may and shall assume hereinafter that $n > 1$.

Lemma 4.4 (i) *The modules $\mathcal{N}_n^j$ and $\mathcal{N}_n^{*j}$ $(0 \le j < s)$ are irreducible.*

(ii) *For $H_n = SL_{n+1}(q)$ the modules $\mathcal{N}_n^i \cong \mathcal{N}_n^j$ $(\mathcal{N}_n^{*i} \cong \mathcal{N}_n^{*j})$ if and only if $i \equiv j \pmod u$; $\mathcal{N}_n^i \not\cong \mathcal{N}_n^{*j}$ for all i and j.*

(iii) *For $H_n = SU_{n+1}(q^2)$ the modules $\mathcal{N}_n^i \cong \mathcal{N}_n^j$ $(\mathcal{N}_n^{*i} \cong \mathcal{N}_n^{*j})$ if and only if $i \equiv j \pmod u$; $\mathcal{N}_n^i \cong \mathcal{N}_n^{*j}$ if and only if $i \equiv s + j \pmod u$.*

Proof. (i) By [5, Lemma 4.3.2], $\mathcal{N}_n^0$ is irreducible. This implies the irreducibility of $\mathcal{N}_n^j$ and $\mathcal{N}_n^{*j}$ $(0 \le j < s)$ since the relevant linear groups coincide.

(ii) It follows from Lemma 4.1 and (i) that $\mathcal{N}_n^i \otimes_{\mathbb{F}_0} \mathbb{F}_H$ $(0 \le i < s)$ is a direct sum of modules in $\operatorname{Irr} \mathbb{F}_H H_n$ that constitute a Σ-orbit and this orbit consists of all $N(p^k \omega_1^n)$ with $0 \le k < s$ and $k \equiv i \pmod u$. Since $\operatorname{Irr} \mathbb{F}_0 H_n$ is parametrized by Σ-orbits on $\operatorname{Irr} \mathbb{F}_H H_n$, we obtain the required assertion.

(iii) The arguments are quite similar to those of item (ii). We only have to take into account that the $\mathbb{F}_H H_n$-modules $N(p^{k+s}\omega_1^n)$ and $N(p^k \omega_n^n)$ are isomorphic. $\square$

Set $\tilde{\mathcal{L}}_n^j = \{\mathcal{N}_n^j, \tau\}$, $\tilde{\mathcal{R}}_n^j = \{\mathcal{N}_n^{*j}, \tau\}$, $\tilde{\mathcal{L}}^j = \{\tilde{\mathcal{L}}_n^j\}_{n \in \mathbb{N}}$, and $\tilde{\mathcal{R}}^j = \{\tilde{\mathcal{R}}_n^j\}_{n \in \mathbb{N}}$. Note that $\tilde{\mathcal{L}}^j$ and $\tilde{\mathcal{R}}^j$ are in fact the systems $\mathcal{L}^j$ and $\mathcal{R}^j$ for H considered as the systems of modules over $\mathbb{F}_0$. It is not difficult to see that $\tilde{\mathcal{L}}^j$ and $\tilde{\mathcal{R}}^j$ are inductive systems.

Lemma 4.5 *Let $\pi(j)$ be the residue of j modulo u.*

(i) *For $H = SL_\infty(q)$ the inductive systems $\tilde{\mathcal{L}}^0, \dots, \tilde{\mathcal{L}}^{u-1}, \tilde{\mathcal{R}}^0, \dots, \tilde{\mathcal{R}}^{u-1}$ are pairwise nonequivalent; $\tilde{\mathcal{L}}^j \cong \tilde{\mathcal{L}}^{\pi(j)}$ and $\tilde{\mathcal{R}}^j \cong \tilde{\mathcal{R}}^{\pi(j)}$.*

(ii) *For $H = SU_\infty(q^2)$ the inductive systems $\tilde{\mathcal{L}}^0, \dots, \tilde{\mathcal{L}}^{u-1}$ are pairwise nonequivalent; $\tilde{\mathcal{L}}^j \cong \tilde{\mathcal{L}}^{\pi(j)}$ and $\tilde{\mathcal{R}}^j \cong \tilde{\mathcal{L}}^{\pi(j+s)}$.*

Proof. This follows from Lemma 4.4. $\square$

Theorem 4.6 *Let Ψ be a nontrivial inductive system for $H = SL_\infty(q)$ or $SU_\infty(q^2)$ over the field $\mathbb{F}_0 = \mathbb{F}_{p^u} \subset \mathbb{F}_H$ ($\mathbb{F}_0 \neq \mathbb{F}_H$). Then Ψ contains one of the following systems:*

$\tilde{\mathcal{L}}^0, \dots, \tilde{\mathcal{L}}^{u-1}, \tilde{\mathcal{R}}^0, \dots, \tilde{\mathcal{R}}^{u-1}$ for $H = SL_\infty(q)$;

$\tilde{\mathcal{L}}^0, \dots, \tilde{\mathcal{L}}^{u-1}$ for $H = SU_\infty(q^2)$.

Proof. Set
$$\bar{\Psi}_n = \bigcup_{M \in \Psi_n} \operatorname{Irr}(M \otimes_{\mathbb{F}_0} \mathbb{F}_H).$$
Obviously, $\bar{\Psi} = \{\bar{\Psi}_n\}_{n \in \mathbb{N}}$ is a nontrivial inductive system for H over $\mathbb{F}_H$. Therefore by Theorem 1.4, $\bar{\Psi}$ contains either $\mathcal{L}^j$ or $\mathcal{R}^j$ for some $j < s$. Now Lemma 4.1 implies that for each $n \in \mathbb{N}$ the set $\bar{\Psi}_n$ contains the Σ-orbit of the module $N(p^j \omega_1^n)$ or $N(p^j \omega_n^n)$. Therefore it follows from the proof of Lemma 4.4 that Ψ_n contains either $\mathcal{N}_n^j$, or $\mathcal{N}_n^{*j}$, so Ψ contains either $\tilde{\mathcal{L}}^j$, or $\tilde{\mathcal{R}}^j$. Now Lemma 4.5 yields the theorem. $\square$

Let σ be an automorphism of the field $\mathbb{F}_0$. For an irreducible $\mathbb{F}_0 H_n$-module N denote by N^σ the module obtained from N by applying σ to all matrix entries of the relevant matrix representation. Define an action of σ on $\operatorname{Irr} \mathbb{F} H_n = \{N \otimes_{\mathbb{F}_0} \mathbb{F} \mid N \in \operatorname{Irr} \mathbb{F}_0 H_n\}$ (see Proposition 4.2) via $(N \otimes_{\mathbb{F}_0} \mathbb{F})^\sigma = N^\sigma \otimes_{\mathbb{F}_0} \mathbb{F}$. Similarly, one can define Ψ^σ for an inductive system Ψ for $\mathbb{F}H$. Set $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^0$ and $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}^0$. Let $\theta \in \operatorname{Aut} \mathbb{F}_0$ raise elements to the power p. Observe that $\tilde{\mathcal{L}}^i = \tilde{\mathcal{L}}^{\theta^i}$ and $\tilde{\mathcal{R}}^i = \tilde{\mathcal{R}}^{\theta^i}$ for $i = 0, 1, \dots, u-1$.

Corollary 4.7 *Let Ψ be a nontrivial inductive system for $H = SL_\infty(q)$ or $SU_\infty(q^2)$ over a field $\mathbb{E} \subset \mathbb{F}_H$. Then Ψ contains one of the following systems:*

$\tilde{\mathcal{L}}^\sigma$ or $\tilde{\mathcal{R}}^\sigma$ ($\sigma \in \operatorname{Aut} \mathbb{E}$) for $H = SL_\infty(q)$;

$\tilde{\mathcal{L}}^\sigma$ ($\sigma \in \operatorname{Aut} \mathbb{E}$) for $H = SU_\infty(q^2)$.

Proof. The case $\mathbb{E} \neq \mathbb{F}_H$ follows from Theorem 4.6; for $\mathbb{E} = \mathbb{F}_H$ the corollary follows from Theorem 1.4 and Remark 3.3. □

5 Finitary groups over locally finite fields

Throughout this section $\mathbb{F}$ and $\mathbb{L}$ are fields of characteristic p and $\mathbb{L}$ is locally finite. One can represent $\mathbb{L}$ as the union of finite subfields

$$\mathbb{L}_1 \subset \mathbb{L}_2 \subset \cdots \subset \mathbb{L}_i \subset \ldots \tag{4}$$

Denote by $SL_\infty(\mathbb{L})$ the union of the groups

$$SL_\infty(\mathbb{L}_1) \subset SL_\infty(\mathbb{L}_2) \subset \cdots \subset SL_\infty(\mathbb{L}_i) \subset \ldots$$

Let $\mathbb{L}$ have an automorphism α of order 2. Then α fixes all $\mathbb{L}_i$. Removing a finite number of subfields in sequence (4) if necessary, one can assume that α acts nontrivially on $\mathbb{L}_1$. Denote by $SU_\infty(\mathbb{L})$ the union of the groups

$$SU_\infty(\mathbb{L}_1) \subset SU_\infty(\mathbb{L}_2) \subset \cdots \subset SU_\infty(\mathbb{L}_i) \subset \ldots$$

One can observe that this construction does not depend upon the choice of sequence (4).

Remark 5.1 It is not difficult to see that $\mathbb{L}$ has an automorphism of order 2 if and only if $\mathbb{L}$ has a subfield of order p^2 and has no subfields of order p^{2^l} for l large enough.

The groups $SL_\infty(\mathbb{L})$ and $SU_\infty(\mathbb{L})$ are locally finite. Recall that a set $\{G_\alpha\}_{\alpha\in A}$ of finite subgroups of a locally finite group G is called a *local system* if $G = \bigcup_{\alpha\in A} G_\alpha$ and for each pair $\alpha, \beta \in A$ there is $\gamma \in A$ such that $G_\alpha, G_\beta \subset G_\gamma$. Set $\alpha \leq \beta$ if $G_\alpha \subset G_\beta$. Then A is a directed set and $G = \varinjlim G_\alpha$. Let Φ_α be a finite subset of $\operatorname{Irr} \mathbb{F}G_\alpha$. The collection $\Phi = \{\Phi_\alpha\}_{\alpha\in A}$ is called an *inductive system* for G if

$$\bigcup_{M\in\Phi_\beta} \operatorname{Irr}(M{\downarrow}G_\alpha) = \Phi_\alpha$$

for all $\alpha < \beta$ in A.

In this section we shall denote by H one of the groups $SL_\infty(\mathbb{L})$ or $SU_\infty(\mathbb{L})$, by H_i the subgroup $SL_\infty(\mathbb{L}_i)$ or $SU_\infty(\mathbb{L}_i)$, and by $H_{n,i}$ the subgroup $SL_n(\mathbb{L}_i)$ or $SU_n(\mathbb{L}_i)$, respectively. Observe that the set $\{H_{n,i}\}_{n,i\in\mathbb{N}}$ is a *local system* for H, i.e. a directed set of subgroups of H with $\varinjlim H_{n,i} = H$. Set $\mathbb{E}_i = \mathbb{F} \cap \mathbb{L}_i$. Denote by $\mathcal{N}^j_{n,i}$ and $\mathcal{N}^{*j}_{n,i}$ the $H_{n,i}$-modules $N(p^j\omega_1^n)$ and $N(p^j\omega_n^n)$, respectively, considered over the field $\mathbb{E}_i$ (cf. the notation before Lemma 4.4). In the lemma below $j = 0, 1 \ldots,$ but one takes into account that the relevant modules can coincide for different j. Observe that by Proposition 4.2, each irreducible $\mathbb{E}_{k+1}H_{n,k}$-module can be represented in the form $S \otimes_{\mathbb{E}_k} \mathbb{E}_{k+1}$ where $S \in \operatorname{Irr} \mathbb{E}_k H_{n,k}$ as $\mathbb{L}_k$ is a splitting field for $H_{n,k}$ and $\mathbb{L}_k \cap \mathbb{E}_{k+1} = \mathbb{E}_k$.

Lemma 5.2 *Let M be an irreducible $\mathbb{E}_{k+1}H_{n,k+1}$-module and*

$$\operatorname{Irr}(M{\downarrow}H_{n,k}) = \{S_1 \otimes_{\mathbb{E}_k} \mathbb{E}_{k+1}, \dots , S_l \otimes_{\mathbb{E}_k} \mathbb{E}_{k+1}\}$$

where $S_t \in \operatorname{Irr}\mathbb{E}_k H_{n,k}$, $1 \le t \le l$. Then

$$\operatorname{Irr}(M \otimes_{\mathbb{E}_{k+1}} \mathbb{F}{\downarrow}H_{nk}) = \{S_1 \otimes_{\mathbb{E}_k} \mathbb{F}, \dots , S_l \otimes_{\mathbb{E}_k} \mathbb{F}\}.$$

In particular,

$$\begin{aligned}\operatorname{Irr}(\mathcal{N}^j_{n,k+1} \otimes_{\mathbb{E}_{k+1}} \mathbb{F}{\downarrow}H_{nk}) &= \{\mathcal{N}^j_{nk} \otimes_{\mathbb{E}_k} \mathbb{F}\};\\ \operatorname{Irr}(\mathcal{N}^{*j}_{n,k+1} \otimes_{\mathbb{E}_{k+1}} \mathbb{F}{\downarrow}H_{nk}) &= \{\mathcal{N}^{*j}_{nk} \otimes_{\mathbb{E}_k} \mathbb{F}\}.\end{aligned}$$

Proof. Let $\mathbb{F}_1$ be the composite of the fields $\mathbb{F}$ and $\mathbb{L}_{k+1}$. Set $\Sigma_1 = \operatorname{Gal}(\mathbb{F}_1/\mathbb{F})$. Recall that $\operatorname{Gal}(\mathbb{F}_1/\mathbb{F}) \cong \operatorname{Gal}(\mathbb{L}_{k+1}/\mathbb{E}_{k+1})$. Put $H^k = H_{n,k}$, $H^{k+1} = H_{n,k+1}$, $\bar{M} = (M \otimes_{\mathbb{E}_{k+1}} \mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}_1$, $\bar{S}_t = (S_t \otimes_{\mathbb{E}_k} \mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}_1$. By Lemma 4.1, the set $\operatorname{Irr}(M \otimes_{\mathbb{E}_{k+1}} \mathbb{F}{\downarrow}H^k)$ is completely determined by the Σ_1-orbits of the elements of $\operatorname{Irr}(\bar{M}{\downarrow}H^k) = \cup_{t=1}^{l} \operatorname{Irr}(\bar{S}_t)$. Proposition 4.2 implies that $S_1 \otimes_{\mathbb{E}_k} \mathbb{F}$ are nonequivalent irreducible $\mathbb{F}H^k$-modules. Hence by Lemma 4.1, $\operatorname{Irr}(\bar{S}_t)$ is an Σ_1-orbit on $\operatorname{Irr}\mathbb{F}_1 H^k$. This yields the first assertion of our lemma. It remains to observe that $\mathcal{N}^j_{n,k+1}$ and $\mathcal{N}^{*j}_{n,k+1}$ are irreducible $\mathbb{E}_{k+1}H^{k+1}$-modules by Lemma 4.4(i) and consider their restrictions to H^k. □

Set $\mathcal{L}_{n,i} = \{\mathcal{N}^0_{n,i} \otimes_{\mathbb{E}_i} \mathbb{F}, \tau\}$, $\mathcal{R}_{n,i} = \{\mathcal{N}^{*0}_{n,i} \otimes_{\mathbb{E}_i} \mathbb{F}, \tau\}$, $\mathcal{L} = \{\mathcal{L}_{n,i}\}_{n,i\in\mathbb{N}}$, $\mathcal{R} = \{\mathcal{R}_{n,i}\}_{n,i\in\mathbb{N}}$, and $\mathbb{E} = \mathbb{F} \cap \mathbb{L}$. Using Lemma 5.2, one easily observes that $\mathcal{L}$ and $\mathcal{R}$ are inductive systems for H. Furthermore, if $\sigma \in \operatorname{Aut}\mathbb{E}$ and $\sigma_i = \sigma|_{\mathbb{E}_i}$, then $\mathcal{L}^\sigma = \{\{\mathcal{N}^{\sigma_i}_{n,i} \otimes_{\mathbb{E}_i} \mathbb{F}, \tau\} \mid n, i \in \mathbb{N}\}$ and $\mathcal{R}^\sigma = \{\{\mathcal{N}^{*\sigma_i}_{n,i} \otimes_{\mathbb{E}_i} \mathbb{F}, \tau\} \mid n, i \in \mathbb{N}\}$ are inductive systems. Here $\mathcal{N}^{\sigma_i}_{n,i} = \mathcal{N}^j_{n,i}$ and $\mathcal{N}^{*\sigma_i}_{n,i} = \mathcal{N}^{*j}_{n,i}$ if σ_i raises elements to the p^jth power.

Theorem 5.3 *Let $\mathbb{F}$ and $\mathbb{L}$ be fields of characteristic p, $\mathbb{L}$ be locally finite, $H = SL_\infty(\mathbb{L})$ or $SU_\infty(\mathbb{L})$, and $\mathbb{E} = \mathbb{F} \cap \mathbb{L}$. Let $\Psi = \{\Psi_{n,i}\}_{n,i\in\mathbb{N}}$ be a nontrivial inductive system for $H = SL_\infty(\mathbb{L})$ or $SU_\infty(\mathbb{L})$ over $\mathbb{F}$. Then Ψ contains one of the following systems:*

$\mathcal{L}^\sigma$ or $\mathcal{R}^\sigma$ ($\sigma \in \operatorname{Aut}\mathbb{E}$) for $H = SL_\infty(\mathbb{L})$;

$\mathcal{L}^\sigma$ ($\sigma \in \operatorname{Aut}\mathbb{E}$) for $H = SU_\infty(\mathbb{L})$.

In particular, Ψ contains the trivial inductive system.

Proof. Denote by Ψ_i the set $\{\Psi_{n,i}\}_{n\in\mathbb{N}}$. Clearly, Ψ_i is a nontrivial inductive system for $H_i = \bigcup_{n\in\mathbb{N}} H_{n,i}$. Using Corollary 4.3, we shall identify inductive systems for H_i over $\mathbb{E}_i$ with those over $\mathbb{F}$. By Corollary 4.7, Ψ_i contains one of the following systems: $\tilde{\mathcal{L}}^{\sigma_i}$ or $\tilde{\mathcal{R}}^{\sigma_i}$ ($\sigma_i \in \operatorname{Aut}\mathbb{E}_i$) for $H_i = SL_\infty(\mathbb{L}_i)$; $\tilde{\mathcal{L}}^{\sigma_i}$ ($\sigma_i \in \operatorname{Aut}\mathbb{E}_i$) for $H_i = SU_\infty(\mathbb{L}_i)$. Assume, for definiteness, that $\tilde{\mathcal{L}}^{\sigma_i} \subset \Psi_i$ for $H_i = SL_\infty(\mathbb{L}_i)$ and infinitely many i. Then by Lemma 5.2, $\tilde{\mathcal{L}}^{\rho} \subset \Psi_{i-1}$ where $\rho = \sigma_i|_{\mathbb{E}_{i-1}}$. Therefore we shall assume that for each i there exists $\sigma_i \in \operatorname{Aut}\mathbb{E}_i$ such that $\tilde{\mathcal{L}}^{\sigma_i} \subset \Psi_i$. The case where $\tilde{\mathcal{R}}^{\sigma_i} \subset \Psi_i$ for all i can be considered quite similarly. Let Δ_i be the set of all $\theta \in \operatorname{Aut}\mathbb{E}_i$ such that $\tilde{\mathcal{L}}^\theta \subset \Psi_i$. For $i < j$ set

$$\Delta_{ij} = \{\theta|_{\mathbb{E}_i} \mid \theta \in \Delta_j\} \subset \operatorname{Aut}\mathbb{E}_i.$$

It follows from Lemma 5.2 that $\Delta_{ij} \subset \Delta_i$. Note that

$$\Delta_i \supset \Delta_{i,i+1} \supset \Delta_{i,i+2} \supset \dots$$

Therefore one can define $\bar{\Delta}_i = \bigcap_{j=i+1}^{\infty} \Delta_{ij}$. Observe that $\bar{\Delta}_i$ is nonempty and there exists $c = c(i) \in \mathbb{N}$ such that $\bar{\Delta}_i = \Delta_{ik}$ for all $k \geq c$. Take any $k > c(i), c(i+1)$. We have $\bar{\Delta}_{i+1}|_{\mathbb{E}_i} = \Delta_{i+1,k}|_{\mathbb{E}_i} = \Delta_{ik} = \bar{\Delta}_i$. Therefore there exists a sequence of automorphisms $\theta_i \in \Delta_i$ such that $\theta_{i+1}|_{\mathbb{E}_i} = \theta_i$ for all i. This sequence determines a unique automorphism σ of $\mathbb{E}$ such that $\sigma|_{\mathbb{E}_i} = \theta_i$. Now it is clear that $\mathcal{L}^\sigma$ is contained in Ψ. □

Let G be a locally finite group and $\{G_\alpha\}_{\alpha \in A}$ be a local system of G. Let I be a (two-sided) proper ideal of the group algebra $\mathbb{F}G$. Then the quotient $\mathbb{F}G/I$ can be considered as an $\mathbb{F}G$-module. Set

$$\Phi(I)_\alpha = \operatorname{Irr}(\mathbb{F}G/I{\downarrow}G_\alpha).$$

One can easily check that the collection $\Phi(I) = \{\Phi(I)_\alpha\}_{\alpha \in A}$ is an inductive system for G. Recall that an ideal I of an algebra R is called *semiprimitive* if the Jacobson radical $\operatorname{Rad}(R/I) = 0$. The following Zalesskii's result reduces the problem of describing lattices of ideals in $\mathbb{F}G$ to inductive systems.

Theorem 5.4 ([12, 1.25]) *The map $I \mapsto \Phi(I)$ is a bijection between the poset of proper semiprimitive ideals of $\mathbb{F}G$ and the poset of inductive systems for G over $\mathbb{F}$. Let M_Φ be the semiprimitive ideal corresponding to an inductive system Φ. Then for each proper ideal I of $\mathbb{F}G$ we have $I \subset M_{\Phi(I)}$ and the quotient $M_{\Phi(I)}/I$ is locally nilpotent.*

Theorem 1.5 (see Introduction) is a particular case of the following theorem (see also Remark 3.3).

Theorem 5.5 *Let $\mathbb{F}$ and $\mathbb{L}$ be fields of characteristic p, $\mathbb{L}$ be locally finite, $H = SL_\infty(\mathbb{L})$ or $SU_\infty(\mathbb{L})$, and $\mathbb{E} = \mathbb{F} \cap \mathbb{L}$. Let V be the natural H-module and V^σ be the module V twisted by $\sigma \in \operatorname{Aut}\mathbb{E}$. Then any proper ideal of $\mathbb{F}H$ is contained in the augmentation ideal $\operatorname{Aug}(\mathbb{F}H)$. Moreover, the annihilators*

$\operatorname{Ann}_{\mathbb{F}H}V^\sigma$ *and* $\operatorname{Ann}_{\mathbb{F}H}(V^*)^\sigma$ *($\sigma \in \operatorname{Aut}\mathbb{E}$) for $H = SL_\infty(\mathbb{L})$;*

$\operatorname{Ann}_{\mathbb{F}H}V^\sigma$ *($\sigma \in \operatorname{Aut}\mathbb{E}$) for $H = SU_\infty(\mathbb{L})$*

are exactly all distinct maximal ideals of the algebra $\operatorname{Aug}(\mathbb{F}H)$.

Proof. Let M be an ideal of $\mathbb{F}H$ such that $\Phi(M)$ is the trivial inductive system for H. Since $\operatorname{Irr}(\mathbb{F}H/M{\downarrow}H_{n,i}) = \{\tau\}$ and $H_{n,i}$ is perfect at least for $n > 2$, $M \cup \mathbb{F}H_{n,i} = \operatorname{Aug}\mathbb{F}H_{n,i}$ for all n and i. This implies that $M = \operatorname{Aug}\mathbb{F}H$. Therefore in view of Theorem 5.4, the maximal ideals of $\operatorname{Aug}(\mathbb{F}H)$ (= the ideals of $\mathbb{F}H$ that are maximal among those properly lying in Aug($\mathbb{F}H$)) are exactly the ideals M_Φ where Φ runs over the minimal nontrivial inductive systems for H.

Let W be the natural H-module over the field $\mathbb{L}$ and W_0 be the module W considered over the field $\mathbb{E}$. Recall that $V = W_0 \otimes_{\mathbb{E}} \mathbb{F}$ and $V^\sigma = W_0^\sigma \otimes_{\mathbb{E}} \mathbb{F}$ for every $\sigma \in \operatorname{Aut}\mathbb{E}$. Let $I = \operatorname{Ann}_{\mathbb{F}H} V^\sigma$ and $J = \operatorname{Ann}_{\mathbb{F}H}(V^*)^\sigma$. Obviously, $\Phi(I) = \mathcal{L}^\sigma$ and $\Phi(J) = \mathcal{R}^\sigma$ (see Theorem 5.3) since the annihilators of V^σ and $\mathbb{F}H/I$ (respectively, $(V^*)^\sigma$ and $\mathbb{F}H/J$) coincide. It remains to observe that the ideals I and J are semiprimitive (as V^σ and $(V^*)^\sigma$ are completely reducible) and to apply Theorems 5.3 and 5.4. □

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References

[1] Baranov, A.A.; Suprunenko, I.D. Branching rules for modular fundamental representations of symplectic groups. Bull. London Math. Soc. **2000**, *32* (4), 409–420.

[2] Borel, A. Properties and linear representations of Chevalley groups. In *Seminar on Algebraic Groups and Related Finite Groups*; Lecture Notes in Math. vol. 131; Springer-Verlag, 1970; 1-55.

[3] Curtis, Ch.; Reiner, I. *Methods of representation theory—with applications to finite groups and orders, vol. I*; Wiley: New York, 1990.

[4] Feit, W. *The representation theory of finite groups*; North-Holland Publ. Comp.: Amsterdam, New York, Oxford; 1982.

[5] Kleidman, P.; Liebeck, M. *The subgroup structure of the finite classical groups*; London Math. Soc. Lecture Notes, no 129; Cambridge Univ. Press, 1990.

[6] Seitz, G.M. *The maximal subgroups of classical algebraic groups*; Mem. Amer. Math. Soc. no 365; 1987.

[7] Smith, S. Irreducible modules and parabolic subgroups. J. Algebra **1982**, *75*, 286–289.

[8] Steinberg, R. Representations of algebraic groups. Nagoya Math. J. **1963**, *22*, 33-56.

[9] Suprunenko, I.D. On Jordan blocks of elements of order p in irreducible representations of classical groups with p-large highest weights. J. Algebra **1997**, *191*, 589-627.

[10] Suprunenko, I.D.; Zalesskii, A.E. Representations of dimensions $(p^n \mp 1)/2$ of a symplectic group of degree $2n$ over a finite field (in Russian). Vestsi AN BSSR, ser. fiz.-mat. n. **1987**, (6), 9–15.

[11] Zalesskii, A. E. Group rings of locally finite groups and representation theory. In *Proceedings of the International Conference on Algebra*, Novosibirsk, 1989; Contemporary Math. vol. 131, part 1; Amer. Math. Soc., 1992; 453–472.

[12] Zalesskii, A.E. Group rings of simple locally finite groups. In *Finite and locally finite groups*; Hartley, B., Seitz, G.M., Borovik, A.V., Bryant, R.M., Eds.; NATO ASI Series, C, vol. 471; Kluwer, 1995; 219–246.