ASYMPTOTIC RESULTS ON MODULAR REPRESENTATIONS
OF SYMMETRIC GROUPS AND ALMOST
SIMPLE MODULAR GROUP ALGEBRAS

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1. Main results

Let $G$ be a group and $F$ be a field. When is the lattice of the (two-sided) ideals of
the group algebra $FG$ as simple as possible? More precisely, every group algebra has the
augmentation ideal. It is natural to ask when $FG$ has no other proper nonzero ideals. In
this case the group algebra $FG$ is called \textit{almost simple}. The following problem goes back
to Kaplansky (1965).

\textbf{Problem.} For a field $F$, find a group-theoretic characterization of the groups $G$ for which
$FG$ is almost simple.

In fact, this is a question about infinite simple groups because it is easy for $G$ finite, and
because a non-trivial normal subgroup gives rise to a non-trivial ideal different from the
augmentation ideal. Also note that the problem reduces easily to the question on when
the augmentation ideal is simple as a ring.

The first interesting class of groups with almost simple $FG$ was discovered in [3]. This
class is rather exotic and contains groups like the universal Hall group and algebraically
closed groups. For locally finite groups $G$ a substantial progress was achieved recently
by using representation theory of finite groups, see [18, 19, 16], and others. The rep-
resentation theory approach transforms the problems on ideals to certain problems on
asymptotic behavior of representations of finite groups, which are often of independent
interest. The method is more effective over fields of characteristic zero as the theory of
ordinary representations is much better elaborated than the modular theory.

This paper is devoted to the modular aspect of the theory. One of our main results
deals with the case where $G$ is a direct limit of alternating groups and $F$ is any field of
characteristic $p > 2$. We note that the case where $\text{char } F = 0$ was considered in [18].

Let $\mathbb{N}$ be the set of natural numbers. Denote by $\text{Alt}(\Omega)$ and $\text{Sym}(\Omega)$ (or, simply, $A_n$
and $\Sigma_n$) the alternating and symmetric groups, respectively, on a set $\Omega$ with $|\Omega| = n$. Let

$$\text{Alt}(\Omega_1) \subset \text{Alt}(\Omega_2) \subset \cdots \subset \text{Alt}(\Omega_i) \subset \cdots$$

be a chain of (strict) embeddings of finite alternating groups. Then we can consider the
union (more precisely, the direct limit)

$$G = \bigcup_{i=1}^{\infty} \text{Alt}(\Omega_i),$$
which is a locally finite group. We call such $G$ a limit alternating group. We emphasize that the embeddings in (1) do not have to be natural embeddings. If the embeddings are natural then $G$ is the finitary alternating group $\text{Alt}_{\infty}$, which consists of all permutations of $\mathbb{N}$ fixing all but finitely many numbers. In general $G$ has no natural permutation representation, like in the case of non-diagonal limit permutation groups defined below, see [18] for more details. We note that for diagonal $G$ the natural permutation set may be identified with $\mathbb{N}$ but the action of each $\text{Alt}(\Omega_i)$ has infinitely many non-trivial orbits unless $G$ is the finitary alternating group, see [16].

**Definition 1.1.** (i) An embedding $\text{Alt}(\Omega_1) \to \text{Alt}(\Omega_2)$ is called diagonal if the orbits of $\text{Alt}(\Omega_1)$ on $\Omega_2$ have lengths 1 or $|\Omega_1|$.  
(ii) A limit alternating group $G$ is called diagonal if all but finitely many embeddings $\text{Alt}(\Omega_i) \to \text{Alt}(\Omega_{i+1})$ are diagonal. Otherwise $G$ is called non-diagonal.

The next theorem explains the ring theoretic importance of non-diagonal groups.

**Theorem A.** Let $\text{char } F \neq 2$, and $G$ be a limit alternating group. Then $FG$ is almost simple if and only if $G$ is non-diagonal.

The proof of Theorem A (see also a more general Theorem 3.13) relies on four powerful tools. The first one is a general result of Passman and Zalesskii [15, 14] which claims that the Jacobson radical of the group algebra of a locally finite simple group is trivial. In particular, the Jacobson radical of $FG$ is trivial. The second one is the inductive systems techniques introduced by Zalesskii (see e.g. the exposition [20] and references there). These tools allow us to reduce the problem to some questions on the asymptotic behavior of representations of finite alternating groups. Our main results on inductive systems are Theorems 2.30 and 3.12. To answer these questions we need one of the main results of [16]. This result shows that if almost all embeddings in (1) are non-diagonal, then for any $n \in \mathbb{N}$ there exists $N > n$ such that, for any $i > N$, the group $\text{Alt}(\Omega_n)$ has a regular orbit on $\Omega_i$, i.e. an orbit of length $|\text{Alt}(\Omega_n)|$. Finally, we need to prove some new asymptotic results on the branching rules for symmetric groups in characteristic $p$. We believe these results might be of independent interest so we present them in this section.

Recall that the representations $1_{\Sigma_n} : g \mapsto 1$ and $\text{sgn}_n : g \mapsto \text{sign}(g)$ are the only 1-dimensional representations of $\Sigma_n$. All the other irreducible representations of $\Sigma_n$ are faithful. Suppose first that $\text{char } F = 0$. Then the natural permutation $F\Sigma_n$-module $M$ (of dimension $n$) splits as a direct sum $M \cong 1_{\Sigma_n} \oplus V$ where $V$ is the natural irreducible $\Sigma_n$-module. Denote by $V^\sigma$ the module $V \otimes \text{sgn}_n$. Using the classical branching rule one can easily deduce the following simple but useful fact.

**Asymptotic Theorem.** Let $\text{char } F = 0$.

(i) For any $n \in \mathbb{N}$ there exists $N > n$ such that, for every $i \geq N$, the restriction of any $\Sigma_i$-module to $\Sigma_n$ has a 1-dimensional composition factor.

(ii) For any $n \in \mathbb{N}$ there exists $N > n$ such that, for every $i \geq N$, the restriction of any faithful $\Sigma_i$-module to $\Sigma_n$ has either the natural irreducible module $V$ or the module $V^\sigma$ as a composition factor.

It is the analogue of this theorem which we need to find in characteristic $p$ in order to prove Theorem A. Note that in positive characteristic the result is wrong as stated.

From now on let $\text{char } F = p > 0$.

If $p$ does not divide $n$ then the natural permutation module $M$ splits, as in characteristic 0, into a direct sum of $1_{\Sigma_n}$ and an irreducible module $V$ of dimension $n - 1$. Otherwise
\( M \) is uniserial with composition factors \( 1_{\Sigma_{n}} V, 1_{\Sigma_{n}} \) where \( V \) is an irreducible module of dimension \( n - 2 \). In any case we still call \( V \) the natural irreducible \((\Sigma_{n})\)-module.

We recall from [9] that the irreducible \( F\Sigma_{n} \)-modules are labelled by the \( p \)-regular partitions of \( n \), i.e., the partitions of \( n \) in which each part appears with multiplicity \( < p \). If \( \lambda \) is such a partition, we denote the corresponding irreducible module by \( D^{\lambda} \). We make use of an important class of irreducible \( \Sigma_{n} \)-modules introduced in [11].

**Definition 1.2.** Let \( \lambda = (\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{s} > 0) \) be a \( p \)-regular partition. The number of rows \( s \) is called the height of \( \lambda \) and is denoted \( h(\lambda) \). We say \( \lambda \) is completely splittable if \( \lambda_{1} - \lambda_{s} + s \leq p \). In this case the corresponding irreducible module \( D^{\lambda} \) is also called completely splittable. (Clearly, \( 1 \leq s \leq p - 1 \).

Observe that the natural irreducible module \( V = D^{(n-1,1)} \) and the module \( V^{\sigma} \) are not completely splittable (for \( n > p \)). Also note that the 1-dimensional modules \( 1_{\Sigma_{n}} = D^{(n)} \) and \( \text{sgn}_{n} \) are the only completely splittable modules of heights 1 and \( p - 1 \), respectively (see Section 2). On the other hand, all completely splittable modules of heights 2, 3, \ldots, \( p - 2 \) are faithful. These facts and the following result proved in [11] show that the straightforward generalization of the Asymptotic Theorem above to characteristic \( p \) is wrong.

**Proposition 1.3.** [11, 2.8] Let \( N > n > (p - 1)^{2} \), and let \( \lambda \) be a completely splittable partition of \( N \) of height \( s \). Then the restriction \( D^{\lambda} \downarrow_{\Sigma_{n}}^{\Sigma_{N}} \) is a direct sum of completely splittable modules of height \( s \).

Proposition 1.3 shows that essentially the best result one can hope for as a modular analogue of the Asymptotic Theorem above is

**Theorem B (Modular Asymptotic Theorem).** Let \( n > (p - 1)^{2} \).

(i) Assume \( p > 2 \). There exists \( N = N(n) > n \) such that, for any \( i \geq N \), the restriction of any \( \Sigma_{i} \)-module to \( \Sigma_{n} \) contains a completely splittable \( \Sigma_{n} \)-module as a composition factor.

(ii) Assume \( p > 3 \). There exists \( N = N(n) > n \) such that, for any \( i \geq N \), the restriction of any faithful \( \Sigma_{i} \)-module to \( \Sigma_{n} \) contains either the natural irreducible module \( V \), or the module \( V^{\sigma} \), or a faithful completely splittable \( \Sigma_{n} \)-module as a composition factor.

(iii) Assume \( p = 3 \). There exists \( N = N(n) > n \) such that, for any \( i \geq N \), the restriction of any faithful \( \Sigma_{i} \)-module to \( \Sigma_{n} \) contains a composition factor either the natural irreducible module \( V \), or the module \( V^{\sigma} \), or an irreducible \( \Sigma_{n} \)-module \( D^{\lambda} \) or \( D^{\lambda} \otimes \text{sgn}_{n} \) with \( \lambda = (\lambda_{1}, \lambda_{2}) \), \( 2 \leq \lambda_{1} - \lambda_{2} \leq 7 \).

**Remark.** The proof of the Modular Asymptotic Theorem relies on the modular branching rules for symmetric groups [10] and the Mullineux Conjecture proved in [7] (see also [2]). The Mullineux bijection becomes trivial in characteristic 2. This is the reason why our methods fail in this case. In characteristic 3 we use the results of Sheth [17].

2. Inductive systems and asymptotic branching

In this section we prove the Modular Asymptotic Theorem.

**Notation.** Throughout the paper \( F \) is an arbitrary field of characteristic \( p > 0 \).

If \( G \) is a group and \( M \) is an \( FG \)-module, we denote by \( \text{Irr}(M) \) the set of the isomorphism classes of the composition factors of \( M \). If \( G \) is a subgroup of a group \( H \) and \( \Phi = \{ M_{1}, M_{2}, \ldots, M_{k} \} \) is a set of \( H \)-modules, we denote by \( \Phi \downarrow_{G}^{H} \) the set \( \bigcup_{i=1}^{k} \text{Irr}(M_{i} \downarrow_{G}^{H}) \).
If \( \lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0) \) is a partition we write \( h(\lambda) \) for \( s \) and call it the height of \( \lambda \). We do not distinguish between \( \lambda \) and its Young diagram

\[ \lambda = \{(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid j \leq \lambda_i \}. \]

Elements \((i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}\) are called nodes.

Let \( m, f \in \mathbb{N} \). Express \( m \) in the form

\[ m = (p - 1)d + r, \quad d \in \mathbb{Z}, \quad 0 < r \leq p - 1. \]

We need to consider certain classes of partitions. Set

\[ \beta(m, f) = ((f + d)^{p-1}, (f + d - 1)^{p-1}, \ldots, (f + 1)^{p-1}, f^r), \]

\[ \gamma(m, f) = (f + (m - 1)(p - 1), f + (m - 2)(p - 1), \ldots, f + (p - 1), f), \]

\[ \delta(m, f) = (f + (m - 2)(p - 1), f + (m - 2)(p - 1), f + (m - 3)(p - 1), \ldots, f + (p - 1), f). \]

Obviously all these partitions are of height \( m \). If \( \lambda_i = (\lambda_i^1, \ldots, \lambda_i^k), \ i = 1, \ldots, k, \) are partitions and \( \lambda_i^1 \geq \lambda_i^{i+1} \) for all \( i < k \), put

\[ (\lambda^1; \ldots; \lambda^k) = (\lambda_1^1, \ldots, \lambda_1^k; \ldots; \lambda_k^1, \ldots, \lambda_k^k) \]

(glue \( \lambda_2 \) to the bottom of \( \lambda_1 \), \( \lambda_3 \) to the bottom of \( \lambda_2 \), and so on). We write \( \beta(m, f; \mu) \) and \( \gamma(m, f; \mu) \) instead of \((\beta(m, f); \mu)\) and \((\gamma(m, f); \mu)\), respectively.

Denote by \( \varepsilon_n \) the partition corresponding to the sign representation of \( \Sigma_n \). It follows from [8, Theorem A] that

\[ \varepsilon_n = (l^{a_1}, (l - 1)^{a_2}) \]

where \( l, a_1 \) and \( a_2 \) are determined from \( n = (p - 1)(l - 1) + a_1, 0 < a_1 \leq p - 1, a_1 + a_2 = p - 1. \) So

\[ \beta(m, f) = (\varepsilon_{k_1}; \ldots; \varepsilon_{k_d}; (f^r)); \]

\[ \gamma(m, f) = ((n_1); (n_2); \ldots; (n_m)); \]

\[ \delta(m, f) = ((n_2); (n_2); (n_3); \ldots; (n_m)). \]

where \( k_i = (f + d + 1 - i)(p - 1) \) and \( n_j = f + (p - 1)(m - j) \).

**Some properties of the Mullineux bijection.** To prove the Modular Asymptotic Theorem we will need some facts concerning the Mullineux bijection. We briefly recall main definitions referring the reader to [7, 2, 13] for details. For a \( p \)-regular partition \( \lambda \) we denote by \( M(\lambda) \) its Mullineux image, which is the \( p \)-regular partition defined by

\[ D^\lambda \otimes \sgn_n \cong D^{M(\lambda)}. \]

The map \( \lambda \mapsto M(\lambda) \) is called the Mullineux bijection. We will use an explicit description of the bijection \( M \), conjectured in [13] and proved in [7] (see also [2]). To explain the construction of \( M \), we need the notion of a \( p \)-edge of the Young diagram of \( \lambda \). So fix an arbitrary partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) of \( n \). The rim of a Young diagram \( \lambda \) is its southeast border — in other words, a node \((i, j)\) of \( \lambda \) belongs to its rim if and only if the node \((i + 1, j + 1)\) does not belong to \( \lambda \). Let us number the nodes of the rim moving from the “top-right” to the “left-bottom”. Define the first \( p \)-segment of the rim as the set consisting of the nodes whose numbers do not exceed \( p \). If the last node \( B \) of the first \( p \)-segment is in the last row of \( \lambda \) then \( \lambda \) has only one \( p \)-segment. If not, let \( r \) be the row containing \( B \). The first node of the second \( p \)-segment is the node which has the smallest number, say \( j \), among the nodes of the rim lying in row \( r + 1 \). The second \( p \)-segment is now defined as the set consisting of the nodes whose numbers \( i \) satisfy \( j \leq i \leq j + p - 1 \). Repeating this procedure sufficiently many times we reach the bottom row of the diagram. It is clear that
all $p$-segments except possibly the last one contain $p$ nodes. The $p$-edge is defined as the union of the $p$-segments. Set $\lambda^{(1)} = \lambda$, and define $\lambda^{(i)}$ to be $\lambda^{(i-1)} \setminus \{\text{the } p\text{-edge of } \lambda^{(i-1)}\}$. Let $j$ be the largest number such that $\lambda^{(j)} \neq \emptyset$. Then the Mullineux symbol of $\lambda$ is the array

$$G(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_j \\ r_1 & r_2 & \cdots & r_j \end{pmatrix}$$

where $a_i$ is the number of the nodes of the $p$-edge of $\lambda^{(i)}$ and $r_i = h(\lambda^{(i)})$ is the height of $\lambda^{(i)}$. The partition can be uniquely reconstructed from its Mullineux symbol, see [13]. Define $\varepsilon_i = 0$ if $p$ divides $a_i$ and $\varepsilon_i = 1$ otherwise.

**Theorem 2.1.** [7, 2] If $G(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_j \\ r_1 & r_2 & \cdots & r_j \end{pmatrix}$ then $G(M(\lambda)) = \begin{pmatrix} a_1 & a_2 & \cdots & a_j \\ s_1 & s_2 & \cdots & s_j \end{pmatrix}$ where $s_i = a_i + \varepsilon_i - r_i$.

For a partition $\mu$ we denote by $\mu_1$ the first part of $\mu$.

**Lemma 2.2.** Let $m \geq p - 1$, $\varepsilon_m = (\varepsilon_1, (l - 1)^{a_2})$ and $\mu$ be a partition such that $\mu_1 \leq l - 1$. If the partition $\alpha = (\varepsilon_m; \mu)$ is $p$-regular then $M(\alpha) = ((m); M(\mu))$.

**Proof.** Recall that $M(\varepsilon_m) = (m)$. We shall prove by induction on $m$ that if $\mu$ satisfies the assumptions of the lemma, then $M(\alpha) = (M(\varepsilon_m); M(\mu))$ and $M(\mu_1) \leq m - (p - 1)$ where $M(\mu_1)$ is the length of the first row in $M(\mu)$. If $m = p - 1$, then $\mu = \emptyset$ and everything is clear.

Assume that $p \leq m < 2p - 1$. Then $l = 2$ and $\mu_1 \leq 1$, so $\mu = (1^b)$, and $b \leq m - (p - 1)$ as $\alpha$ must be $p$-regular. We may assume that $b > 0$. Then the Mullineux symbol of $\alpha$ is

$$G(\alpha) = \begin{pmatrix} p + b \\ p - 1 + b \\ m - p \end{pmatrix}$$

or

$$G(\alpha) = \begin{pmatrix} p + b \\ m - p \end{pmatrix}.$$ (the second case arises if $m = p$). Therefore

$$G(M(\alpha)) = \begin{pmatrix} p + b \\ 2 \\ m - p \end{pmatrix}.$$ As $b \leq a_1 = m - (p - 1)$, $G(M(\alpha))$ is the Mullineux symbol of the partition $(m, b)$ in both cases. Thus $M(\alpha) = (m, b)$.

Assume now that $m \geq 2p - 1$. Observe that $\alpha^{(1)} = (\varepsilon_m; \mu^{(1)})$. Note that $m - p \geq p - 1$.

By inductive hypothesis,

$$M(\alpha^{(1)}) = (M(\varepsilon_m); M(\mu^{(1)})) = ((m - p); M(\mu^{(1)}))$$

and

$$M(\mu^{(1)})_1 \leq (m - p) - (p - 1).$$

(2)

We have

$$G(\alpha) = \begin{pmatrix} p + a \\ p - 1 + b \\ G(\alpha^{(1)}) \end{pmatrix}$$

where

$$\begin{pmatrix} a \\ b \\ G(\mu^{(1)}) \end{pmatrix} = G(\mu).$$

Therefore

$$G(M(\alpha)) = \begin{pmatrix} p + a \\ 1 + c \\ c \\ G(M(\alpha^{(1)})) \end{pmatrix}$$

where

$$\begin{pmatrix} a \\ c \\ G(M(\mu^{(1)})) \end{pmatrix} = G(M(\mu)).$$
Now, in view of (2), in constructing $\mathbf{M}(\alpha)$ one glues $a$ nodes to the rows of $\mathbf{M}(\mu(1))$ (obtaining $\mathbf{M}(\mu)$) and $p$ nodes to the first row (obtaining $(m)$). Therefore $\mathbf{M}(\alpha) = (\mathbf{M}(\varepsilon_m); \mathbf{M}(\mu))$, and $\mathbf{M}(\mu)_1 \leq m - (p - 1)$, as required.

**Lemma 2.3.** Let $\alpha = (\varepsilon_{m_1}; \ldots; \varepsilon_{m_k}; \mu)$ be a $p$-regular partition with $m_i \leq m_{i+1} - (p - 1)$ for all $i$. Then $\mathbf{M}(\alpha) = ((m_1, \ldots, m_k); \mathbf{M}(\mu))$.

*Proof.* Apply Lemma 2.2 (k times).

**Lemma 2.4.** Let $m = d(p - 1) + r$ where $0 \leq r < p - 1$. Then $\mathbf{M}(\beta(m, f)) = \gamma(d, e; \nu)$ where $e = (f + 1)(p - 1)$, $\nu = \mathbf{M}((f^r))$.

*Proof.* This follows from Lemma 2.3 with $\mu = (f^r)$.

**Lemma 2.5.** Let $\mu = (\mu_1 \geq \cdots \geq \mu_k)$ be a $p$-regular partition, and $m$ be an integer satisfying $0 \leq m \leq \mu_k - (p - 1)$. Set $\alpha = (\mu; (m))$. Then $\mathbf{M}(\alpha) = (\mathbf{M}(\mu); \varepsilon_m)$.

*Proof.* The proof is similar to that of Lemma 2.2. Proceed by induction on $m$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition with $\lambda_i \geq \mu_i$ for all $i$. Observe that $\alpha^{(1)} = (\mu^{(1)}; (m - \mu_1))$ (or $\mu^{(1)}$ if $m \leq \mu$). Moreover, $\mu_k^{(1)} - (p - 1) \geq \mu_k - (p - 1) \geq m - p$ if $m > p$. Therefore by inductive hypothesis, $\mathbf{M}(\alpha^{(1)}) = (\mathbf{M}(\mu^{(1)}); \varepsilon_{m-p})$ (or $\mu^{(1)}$ if $m \leq \mu$). We consider the generic case $m > p$ (the case $m \leq p$ being similar). We have

$$G(\alpha) = \begin{pmatrix} a + p & G(\alpha^{(1)}) \\ b + 1 & G(\mu^{(1)}) \end{pmatrix}$$

where $\begin{pmatrix} a & G(\mu^{(1)}) \\ b & G(\alpha^{(1)}) \end{pmatrix} = G(\mu)$.

Note that the assumption $\mu_k - (p - 1) \geq m > 0$ implies $\mu_k \geq p$, whence $a$ is divisible by $p$. Therefore

$$G(\mathbf{M}(\alpha)) = \begin{pmatrix} a + p & G(\mathbf{M}(\alpha^{(1)})) \\ a - b + p - 1 & G(\mathbf{M}(\mu^{(1)})) \end{pmatrix}$$

Now in constructing $\mathbf{M}(\alpha)$, one glues $p$ nodes to the $p - 1$ rows of $\varepsilon_{m-p}$ (obtaining $\varepsilon_m$) and $a$ nodes to the rows of $\mathbf{M}(\mu^{(1)})$ (obtaining $\mathbf{M}(\mu)$). Therefore $\mathbf{M}(\alpha) = (\mathbf{M}(\mu); \varepsilon_m)$, as required.

**Lemma 2.6.** Let $\mu = (\mu_1 \geq \cdots \geq \mu_1)$ be a $p$-regular partition, $\alpha = (\mu; (m_1, \ldots, m_k))$ where $\mu_i - (p - 1) \geq m_1$, and $m_i - (p - 1) \geq m_{i+1}$ for $1 \leq i < k$. Then $\mathbf{M}(\alpha) = (\mathbf{M}(\mu); \varepsilon_{m_1}; \ldots; \varepsilon_{m_k})$.

*Proof.* Apply Lemma 2.5 (k times).

**Lemma 2.7.** Let $f, d \in \mathbb{N}$, $e = f(p - 1)$, and $\mu = (\mu_1, \ldots, \mu_k)$ be a $p$-regular partition with $\mu_k \geq e + d(p - 1)$. Then

$$\mathbf{M}((\mu; \gamma(d, e))) = (\mathbf{M}(\mu); \beta((d - 2)(p - 1), f))$$

In particular,

$$\mathbf{M}(\delta(d, e)) = (\nu; \beta((d - 2)(p - 1), f))$$

where $\nu = \mathbf{M}((l^2))$ with $l = e + (d - 2)(p - 1)$.

*Proof.* This follows from Lemma 2.6.

**Lemma 2.8.** Let $k \in \mathbb{N}$, and $\lambda = (\lambda_1, \ldots, \lambda_{p-1})$ be a partition with $\lambda_{p-1} \geq kp$ and $\lambda_1 - \lambda_{p-1} > 1$. If $\mathbf{M}(\lambda) = (\mu_1, \mu_2, \ldots)$, then $\mu_{p+1} \geq k$.
Proposition 2.13. Let \( \lambda_1 - \lambda_{p-1} > 1 \), the number of nodes of the \( p \)-edge of \( \lambda^{(1)} \) is \( n_1 p \) with \( n_1 \geq 2 \). This implies \( \lambda_1^{(2)} - \lambda_{p-1}^{(2)} > 1 \). Also \( \lambda_{p-1}^{(2)} \geq \lambda_{p-1}^{(1)} - p = (k-1)p \). Therefore we have
\[
G(\lambda) = \begin{pmatrix}
n_1 p & n_2 p & \cdots & n_k p \\
p-1 & p-1 & \cdots & p-1
\end{pmatrix}
\]
where \( n_i \geq 2 \) (\( 1 \leq i \leq k \)). Hence
\[
G(M(\lambda)) = \begin{pmatrix}
n_1 p & n_2 p & \cdots & n_k p \\
(n_1-1)p + 1 & (n_2-1)p + 1 & \cdots & (n_k-1)p + 1
\end{pmatrix}.
\]
Since \( (n_i - 1)p + 1 \geq p + 1 \) for \( 1 \leq i \leq k \), we have \( \mu_{p+1} \geq k \). \( \square \)

Lemmas on branching. The results of this section rely on the modular branching rules proved in [10, 12], especially Theorem 0.4(ii) from [10] and Theorem 1.4 from [12]. To describe a version of the branching rule needed here we recall the notion of a normal node. The \((p-)residue\), \( \text{res} \), of a node \( A = (i, j) \) is defined to be the \((p-)residue\) class \((j - i) \pmod{p}\). A node \((i, \lambda_i) \in \lambda\) is called removable if \( \lambda_i > \lambda_{i+1} \). A node \((i, \lambda_i + 1)\) is called addable if \( i = 1 \) or \( i > 1 \) and \( \lambda_i < \lambda_{i-1} \). If \( A = (i, \lambda_i) \) is a removable node then
\[
\lambda_A := \lambda \setminus \{ A \} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)
\]
is a partition of \( n - 1 \) obtained from \( \lambda \) by removing \( A \). A removable node \( A \) of \( \lambda \) is called normal if for every addable node \( B \) above \( A \) with \( \text{res} B = \text{res} A \) there exists a removable node \( C(B) \) strictly between \( A \) and \( B \) with \( \text{res} C(B) = \text{res} A \), and such that \( B \neq B' \) implies \( C(B) \neq C(B') \).

Theorem 2.9. [12, Theorem 1.4] Let \( A \) be a removable node of a \( p \)-regular partition \( \lambda \) such that \( \lambda_A \) is also \( p \)-regular. Then \( D^{\lambda_A} \) is a composition factor of \( D^{\lambda} \downarrow_{\Sigma_n} \) if and only if \( A \) is normal.

We start with two easy consequences of Theorem 2.9.

Lemma 2.10. \( D^{\gamma(d, f)} \in \text{Irr}(D^{\gamma(d, f+1)} \downarrow_{\Sigma_n}) \) where \( n = |\gamma(d, f)| \).

Proof. This follows from Theorem 2.9 and the fact that all removable nodes of \( \gamma(d, f) \) have the same residue. \( \square \)

Lemma 2.11. \( D^{\delta(d, f)} \in \text{Irr}(D^{\gamma(d, f)} \downarrow_{\Sigma_n}) \) where \( n = |\delta(d, f)| \).

Proof. This follows from Theorem 2.9 and the fact that the top removable node of any partition is always normal. \( \square \)

Lemma 2.12 ([1, 4.10]). Let \( \mu = (\mu_1, \mu_2, \ldots) \) be a \( p \)-regular partition \((\mu \text{ may be } \emptyset)\), \( f > \mu_1 + 1 \) and \( m \in \mathbb{N} \). Set \( \beta = \beta(m, f; \mu), \alpha = \beta(m, f-1; \mu), n = |\alpha| \). Then
\[
D^\alpha \in \text{Irr}(D^{\beta} \downarrow_{\Sigma_n}).
\]

Proposition 2.13. Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a \( p \)-regular partition, \( m \in \{1, 2, \ldots, k\} \), and
\[
\lambda_m > 2 + \sum_{i=m+1}^{k} \lambda_i.
\]
Set \( f = \lambda_m - 1 - \sum_{i=m+1}^{k} \lambda_i, \beta = \beta(m, f), n = |\beta|, \beta' = \beta(m, f + 1; (1)), n' = |\beta'| \). Then
\[
D^{\beta} \in \text{Irr}(D^{\lambda} \downarrow_{\Sigma_n}).
\]
Moreover, if $\lambda_{m+1} \neq 0$, then
\[ D^n \in \text{Irr}(D^n) \).

Proof. The first part is [1, 4.14]. The second part is contained in the proof of [1, 4.14].

Lemma 2.14. Let $\mu = (\mu_1, \mu_2, \ldots)$ be a non-empty $p$-regular partition, $f > |\mu| + 1$, $m \in \mathbb{N}$, and let $A$ be the top removable node of (the Young diagram of) $\mu$ such that $\mu_A$ is $p$-regular. Set $\gamma = \gamma(m; f; \mu)$, $\alpha = \gamma(m; f - 1; \mu_A)$, $n = |\alpha|$. Then
\[ D^n \in \text{Irr}(D^n) \).

Proof. The proof is identical to [1, 4.11].

Lemma 2.15. Let $\mu = (\mu_1, \mu_2, \ldots)$ be a non-empty $p$-regular partition, $f > |\mu| + 1$, $e = f - |\mu|$, $n = |\gamma(m, e)|$, $n' = |\gamma(m, e + 1; (1))|$, and $m \in \mathbb{N}$. Then
\[ D^n \in \text{Irr}(D^n) \).

Proof. Apply Lemma 2.14 sufficiently many times.

Lemma 2.16. Let $\mu = (\mu_1, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots, \lambda_h)$ be $p$-regular partitions. Set $c = \lambda_{m+1} + \cdots + \lambda_h$. Assume that $D^n \in \text{Irr}(D^n) \). Then $\rho_m \leq \lambda_m + c$.

Proof. If $D^n \in \text{Irr}(D^n) \), then $\nu$ dominates $\rho_A$ for some removable node $A$ of $\rho$. Therefore
\[ \nu_m + \nu_{m+1} + \cdots \leq (\rho_A)_m + (\rho_A)_{m+1} + \cdots \leq \rho_m + \rho_{m+1} + \cdots .
\]

The lemma follows by induction.

Inductive systems. Let $G$ be a countable locally finite group. Then $G$ can be considered as a union
\[ G = \bigcup_{n=1}^{\infty} G_n \]
where $G_1 \subset G_2 \subset \ldots$ are finite subgroups of $G$.

Definition 2.17. [20, 1.1] Let $\Phi_n$ be a non-empty set of the isomorphism classes of irreducible $FG_n$-modules ($n = 1, 2, \ldots$). We say that the collection $\Phi = \{\Phi_n\}_{n \in \mathbb{N}}$ is an inductive system (for $G$) if for any $n \in \mathbb{N}$ we have
\[ \Phi_n = \text{Irr}(\Phi_{n+1} \downarrow_{G_{n+1}}) .
\]

The following result explains our interest in inductive systems. Note that $G$ always has at least two inductive systems: the inductive system $\Gamma = \{\Gamma_n\}_{n \in \mathbb{N}}$ with $\Gamma_n = \text{Irr} G_n$ for all $n$ and the unitary inductive system with $\Phi_n = \{1_{G_n}\}$ for all $n$. Any inductive system distinct from $\Gamma$ is called proper. The following result was proved by the third author, see [20].

Theorem 2.18. Let $G = \bigcup_{n=1}^{\infty} G_n$ be a locally finite group as above. Assume additionally that $G$ is simple. Then the group algebra $FG$ is almost simple if and only there are no proper inductive systems for $G$, except for the unitary one.

Proof. Assume there are no proper inductive systems, except for the unitary one. Then by [19, Theorem 1], (see also [20, 1.25]) every proper subideal of the augmentation ideal is contained in the Jacobson radical of $FG$. But this radical is trivial by a theorem of Passman and Zalesskii [15, 14] claiming that the group algebra of a simple locally finite group is semiprimitive. Now since the augmentation ideal has codimension 1, we conclude that $FG$ is almost simple. The converse follows immediately from [20, 1.25].
We denote by \(\Sigma_\infty\) the group of all finitary permutations of \(\mathbb{N}\). Clearly, \(\Sigma_\infty\) is a locally finite group and

\[
\Sigma_\infty = \cup_{n>N} \Sigma_n
\]

where \(\Sigma_n\) is the group of all permutations of the set \(\{1,2,\ldots,n\}\), and \(N\) is any natural number. So we may consider the inductive systems for \(\Sigma_\infty\).

**Definition 2.19.** Let \(\Phi = \{\Phi_n\}_{n>N}\) be an inductive system for \(\Sigma_\infty\). Define the inductive system \(\Phi^\sigma = \{\Phi^\sigma_n\}_{n>N}\) by

\[
\Phi^\sigma_n = \{D^\lambda \otimes \text{sgn}_n \mid D^\lambda \in \Phi_n\}.
\]

One can easily check that \(\Phi^\sigma\) so defined is indeed an inductive system.

We shall use some special inductive systems introduced in [11,1]. For \(n > (p-1)^2\) let \(\Phi(s)_n\) be the set of all completely splittable \(\Sigma_n\)-modules of height \(s\) (see Definition 1.2), and let \(\Phi(s) = \{\Phi(s)_n\}_{n>(p-1)^2}\). It has been proved in [1] that \(\Phi(1), \Phi(2), \ldots, \Phi(p-1)\) are the minimal inductive systems for \(\Sigma_\infty = \cup_{n>(p-1)^2} \Sigma_n\).

**Lemma 2.20.** [1,5,6] Let \(s \in \{1,2,\ldots,p-1\}\). Then \(\Phi(s)^\sigma = \Phi(p-s)\).

Note that \(\Phi(1)_n = \{1_{\Sigma_n}\}\). It follows from Lemma 2.20 that \(\Phi(p-1)_n = \{\text{sgn}_n\}\).

It is convenient to call the inductive systems \(\Phi(1), \Phi(p-1),\) and \(\Phi(1) \cup \Phi(p-1)\) (for \(\Sigma_\infty\)) trivial. One of our goals in the next subsection is to describe the minimal non-trivial inductive systems for \(\Sigma_\infty\).

Put

\[
\Pi_n = \{D^{(n)}, D^{(n-1,1)}\}, \quad n > (p-1)^2.
\]

The following result is straightforward.

**Lemma 2.21.** \(\Pi = \{\Pi_n\}_{n>(p-1)^2}\) is an inductive system for \(\Sigma_\infty\).

The inductive systems for \(\Sigma_\infty\) consisting of modules \(D^\lambda\) with \(\lambda\) having at most two parts have been classified by Sheth [17]. We need a very special corollary of his result:

**Lemma 2.22.** [17, Theorem 7.7] Let \(p > 2\). Set

\[
\Theta_n = \{D^{(l_1,l_2)} \mid l_1 + l_2 = n, \quad \text{\(0 \leq l_1 - l_2 \leq p^2 - 2\)}\}. \quad n\in\mathbb{N}
\]

Then \(\Theta = \{\Theta_n\}_{n\in\mathbb{N}}\) is an inductive system for \(\Sigma_\infty\), \(\Theta \supset \Phi(2)\), and there are no inductive systems between \(\Theta\) and \(\Phi(2)\).

**Minimal non-trivial inductive systems.** Set

\[
\Phi(\beta, m)_n = \cup_{f > n} \text{Irr}(\beta(m,f)\downarrow_{\Sigma_n}),
\]

\[
\Phi(\beta, m; (1))_n = \cup_{f > n} \text{Irr}(\beta(m,f; (1))\downarrow_{\Sigma_n});
\]

\[
\Phi(\beta, m) = \{\Phi(\beta, m)_n\}_{n>(p-1)^2};
\]

\[
\Phi(\beta, m; (1)) = \{\Phi(\beta, m; (1))_n\}_{n>(p-1)^2}.
\]

It follows from Lemma 2.12 that \(\Phi(\beta, m)\) and \(\Phi(\beta, m; (1))\) are inductive systems for the group \(\Sigma_\infty = \cup_{n>(p-1)^2} \Sigma_n\) (for any \(m \in \mathbb{N}\)). Recall the systems \(\Phi(1), \Phi(2), \ldots, \Phi(p-1),\) and \(\Pi\), introduced before.

**Lemma 2.23.** Let \(1 \leq m \leq p - 1\). Then \(\Phi(\beta, m) = \Phi(m)\).

**Proof.** Clearly, \(D^{\beta(m,f)} \in \Phi(m)\) for any \(f\). So the definition of \(\Phi(\beta, m)\) ensures that \(\Phi(\beta, m) \subseteq \Phi(m)\). Now the result follows from the minimality of \(\Phi(m)\).

**Lemma 2.24.** \(\Phi(\beta, p - 1; (1)) = \Pi^\sigma\).
Proof. Let $n = |\beta(p - 1, f; (1))|$. By Lemma 2.2 we have $M(\beta(p - 1, f; (1))) = (n - 1, 1)$. The lemma follows. 

**Definition 2.25.** Let $\Phi = \{\Phi_n\}_{n > N}$ be an inductive system for $\Sigma_\infty = \cup_{n > N} \Sigma_n$. We define its height $h(\Phi)$ as

$$h(\Phi) = \sup\{h(\lambda) \mid D^\lambda \in \Phi_n \text{ for some } n > N\}.$$ 

**Proposition 2.26** ([1, 5.2]). Let $\Phi = \{\Phi_n\}_{n > N}$ be an inductive system for the group $\Sigma_\infty = \cup_{n > N} \Sigma_n$. Assume that $h(\Phi) = +\infty$. Then $\Phi_n = \text{Irr} \Sigma_n$ for all $n > N$.

The proposition above shows that all proper inductive systems for $\Sigma_\infty$ have finite heights. So assume from now on that $\Phi$ is an inductive system with $h(\Phi) < +\infty$. It follows that for any $M \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $D^\lambda \in \Phi_n$ such that $\lambda_1 > M$ (in other words, the first part of the partitions involved gets arbitrarily large). This shows that the set

$$V(\Phi) = \{k \in \mathbb{N} \mid \text{for any } M \in \mathbb{N} \text{ there exists } n \in \mathbb{N} \text{ and } D^\lambda \in \Phi_n \text{ with } \lambda = (\lambda_1, \lambda_2, \ldots) \text{ such that } \lambda_k > M\}$$

is not empty (it contains $k = 1$). Since also $V(\Phi) \subseteq \{1, 2, \ldots, h(\Phi)\}$ we can define

$$m(\Phi) = \max V(\Phi).$$

It follows from the definition of $m = m(\Phi)$ and $h = h(\Phi)$ that there exists $c = c(\Phi) \in \mathbb{N}$ such that

$$\sum_{j=m+1}^h \lambda_j \leq c \quad (3)$$

for any $D^\lambda \in \Phi$. Let $N_0 \geq c + 2$. By definition of $m$, there exists $n \in \mathbb{N}$ and $D^\lambda \in \Phi_n$ such that $\lambda_m > N_0$. Set

$$f = \lambda_m - \sum_{j=m+1}^h \lambda_j - 1, \quad \beta = \beta(m, f), \quad n = |\beta|, \quad \beta' = \beta(m, f + 1; (1)), \quad n' = |\beta'|.$$

By Proposition 2.13,

$$D^\beta \in \text{Irr}(D^\lambda \downarrow \Sigma_n) \subseteq \Phi_n.$$

Moreover, if $\lambda_{m+1} \neq 0$, then

$$D^{\beta'} \in \text{Irr}(D^\lambda \downarrow \Sigma_{n'}).$$

So we have

**Lemma 2.27.** Let $\Phi$ be an inductive system for $\Sigma_\infty$, and let $m = m(\Phi)$. Then $\Phi \supseteq \Phi(\beta, m)$. Moreover, if for each $M \in \mathbb{N}$ there exists a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $D^\lambda \in \Phi_{|\lambda|}$, $\lambda_m > M$, and $\lambda_{m+1} \neq 0$, then $\Phi \supseteq \Phi(\beta, m; (1))$.

**Lemma 2.28.** Let $p > 2$, $m = m(\Phi)$. Assume that $m = d(p - 1) + r$ for $d \geq 1$ and $0 < r < p - 1$. Then $\Phi \supseteq \Phi(\beta, d(p - 1))$. 

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Proof. By Lemma 2.27, \( \Phi \supseteq \Phi(\beta, m) \). It follows from Lemma 2.4 that the inductive system \( \Phi^\sigma \) contains \( D^{\gamma(d,(f+1)(p-1); \mu(f))} \) for all \( f \in \mathbb{N} \), where \( \mu(f) = M((f^r)^+) \). Note that \( |\mu(f)| = fr \), whence \( (f+1)(p-1) > |\mu(f)| + 1 \). So, by Lemma 2.15, \( \Phi^\sigma \) contains \( D^{\gamma(d,(f+1)(p-1)-fr)} \) for all \( f \in \mathbb{N} \). Since \( (f+1)(p-1)-fr \to +\infty \) as \( f \to +\infty \), Lemma 2.10 implies that \( \Phi^\sigma \) contains all \( D^{\beta(d,p,f)} \), \( f \in \mathbb{N} \). Therefore, by Lemma 2.7, the inductive system \( \Phi = (\Phi^\sigma)^\sigma \) contains all \( D^{\beta(d,p-1,f)} \), \( f \in \mathbb{N} \), so \( \Phi \supseteq \Phi(\beta,d,(p-1)) \). \( \square \)

**Lemma 2.29.** Let \( p > 2 \), \( m = m(\Phi) = d(p-1) \) for \( d \geq 2 \). Then \( \Phi \supseteq \Phi(\beta, m-p) \).

Proof. As in the proof of Lemma 2.28 we see that \( \Phi^\sigma \) contains \( D^{\gamma(d,f(p-1))} \) for all \( f \in \mathbb{N} \). By Lemma 2.11, \( \Phi^\sigma \) contains all \( D^{\beta(d,f(p-1))} \), \( f \in \mathbb{N} \). Set

\[
\mu(f) = M(\delta(d,f(p-1))).
\]

By Lemma 2.7,

\[
\mu(f) = (\nu(f); \beta((d-2)(p-1), f))
\]

where

\[
\nu(f) = M((l^2)) \in \Phi(2)^\sigma, \quad l = f(p-1) + (d-2)(p-1).
\]

By Lemma 2.20, \( h(\nu(f)) = p - 2 \), so

\[
h(\mu(f)) = p - 2 + (d-2)(p-1) = m - p.
\]

Observe that the length of the last row of \( \mu(f) \) is \( f \). By Proposition 2.13 (applied with \( m = k \)), \( D^{\beta(m-p,f-1)} \in \text{Irr}(D^{\mu(f)} \Sigma_{(m-p,f-1)}^{\sigma}) \). Therefore \( \Phi \supseteq \Phi(\beta, m-p) \). \( \square \)

**Theorem 2.30.** Let \( F \) be a field of characteristic \( p > 2 \).

(i) Let \( p > 3 \). Then any non-trivial inductive system contains one of the following systems:

\[
\Pi, \quad \Pi^\sigma, \quad \Phi(s) \ (1 < s < p-1).
\]

(ii) Let \( p = 3 \). Then any non-trivial inductive system contains one of the following systems:

\[
\Pi, \quad \Pi^\sigma, \quad \Theta, \quad \Theta^\sigma.
\]

Proof. Let \( \Phi \) be a proper non-trivial inductive system. In view of Lemma 2.20, it suffices to prove that either \( \Phi \) or \( \Phi^\sigma \) contains one of the inductive systems (4) or (5) if \( p = 3 \). So we may assume without loss of generality that \( m(\Phi) \geq m(\Phi^\sigma) \). Set \( m = m(\Phi) \). Consider the following cases.

**Case 1:** \( m = 1 \). Since \( \Phi \neq \Phi(1) \), we have \( h(\Phi) > 1 \). Therefore for each \( i \) there exists a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( D^\lambda \in \Phi_i \) and \( \lambda_2 \neq 0 \). Hence by Lemma 2.27, \( \Phi \supseteq \Phi(\beta; 1; (1)) = \Pi \).

**Case 2:** \( 1 < m < p-1 \). By Lemmas 2.27 and 2.23, we have \( \Phi \supseteq \Phi(\beta, m) = \Phi(m) \).

**Case 3:** \( m = p - 1 \). We may assume that \( \Phi \) does not contain \( \Phi(\beta, p-1; (1)) = \Pi^\sigma \) (see Lemma 2.24 for the last equality). So, in view of Lemma 2.27, there exists \( N_1 \in \mathbb{N} \) such that for any partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) with \( D^\lambda \in \Phi_{|\lambda|} \) and \( \lambda_p \neq 0 \) we have \( \lambda_{p-1} < N_1 \).

Furthermore, take \( D^\lambda \in \Phi_{|\lambda|} \) with \( \lambda_{p-1} \) very large. By Lemma 2.8, \( D^\lambda \otimes \text{sgn} = D^\mu \) with \( \mu_{p+1} \) very large, unless \( \lambda_1 - \lambda_{p-1} \leq 1 \). But \( \Phi^\sigma \) may not contain \( D^\mu \) with arbitrarily large \( \mu_{p+1} \) since then \( m(\Phi^\sigma) \geq p + 1 > m(\Phi) \) which contradicts our assumptions. Thus, there exists \( N_2 > N_1 \) such that for each \( \lambda \) with \( D^\lambda \in \Phi_{|\lambda|} \) and \( \lambda_{p-1} \geq N_2 \) we have \( \lambda_1 - \lambda_{p-1} \leq 1 \), and so \( \lambda = \varepsilon_{|\lambda|} \) as in addition \( \lambda_p = 0 \).
By Lemma 2.27, \( \Phi \supseteq \Phi(\beta, p-1) \). But \( \Phi(\beta, p-1)_n = \{ \text{sgn}_n \} \). Set
\[
\Psi_n = \Phi_n \setminus \{ \text{sgn}_n \}.
\]
We are going to show that \( \Psi = \{ \Psi_n \}_{n>N} \) is an inductive system for \( \Sigma_\infty = \cup_{n>N} \Sigma_n \), provided \( N \) is large enough. It suffices to check that \( \text{sgn}_j \notin \text{Irr}(\Psi_i \cup \Sigma_j) \) for any pair \( i > j > N \). Assume not, i.e. there exists \( \lambda = (\lambda_1, \lambda_2, \ldots) \) with \( \text{sgn}_j \in \text{Irr}(D^\lambda \cup \Sigma_j) \). Let \( c = c(\Phi) \), see (3). Since \( N \) is large enough, we may assume that the length of the \( (p-1) \)-st row of \( \varepsilon_j \) is at least \( N_2 + c \). So, by Lemma 2.16, we have \( \lambda_{p-1} \geq N_2 \). Hence by the choice of \( N_2 \), \( D^\lambda = \text{sgn}_j \notin \Psi_i \). This yields a contradiction. So \( \Psi \) is an inductive system. Observe that \( \Phi \cup \Phi(p-1) = \Phi \). Therefore \( \Psi \) is non-trivial. Since \( \lambda_{p-1} < N_2 \) for each partition \( \lambda \) with \( D^\lambda \in \Psi_{|\lambda|} \), we have \( m(\Psi) < p-1 \). So \( \Psi \) (and \( \Phi \)) contains one of the systems (4) by what has already been proved.

Case 4: \( p-1 < m < 2(p-1) \). Set \( r = m-(p-1) \). By Lemma 2.27, \( \Phi \supseteq \Phi(\beta, m) \). It follows from Lemma 2.4 that \( \Phi^{\sigma} \) contains all \( D^\gamma(1, n(f);(1)) \), \( f \in \mathbb{N} \), where \( n(f) = (f+1)(p-1) - fr + 1 \). Note that
\[
D^\gamma(1, n(f);(1)) = D^{(n(f),1)} \in \Pi_{n(f)+1}.
\]
Since \( n(f) \rightarrow +\infty \) as \( f \rightarrow +\infty \), \( \Phi^{\sigma} \supseteq \Pi \), so \( \Phi \supseteq \Pi^{\sigma} \), as desired.

Case 5: \( p > 3 \) and \( m = 2(p-1) \). Then by Lemmas 2.29 and 2.23, \( \Phi \supseteq \Phi(\beta, p-2) = \Phi(p-2) \), as desired.

Case 6: \( p = 3 \) and \( m = 2(p-1) = 4 \). By Lemma 2.27, \( \Phi \supseteq \Phi(\beta, 4) \). Recall that \( \Phi(\beta, 4) \) is generated by all \( \beta(4, f) = (\varepsilon_2(f+1), \varepsilon_2 f) \), \( f \in \mathbb{N} \). Therefore by Lemma 2.3, \( \Phi(\beta, 4)^\sigma \) is generated by all partitions \( \eta(f) = (2f+2, 2f) \), \( f \in \mathbb{N} \). We shall use Lemma 2.22. Note that
\[
D^n(f) \in (\Theta_{4f+2}, \Phi(2)_{4f+2})
\]
for all \( f \). Since there are no inductive systems between \( \Theta \) and \( \Phi(2) \), we have \( \Phi(\beta, 4)^\sigma = \Theta \). It follows that \( \Phi \supseteq \Theta^{\sigma} \), as required.

Case 7: \( m > 2(p-1) \). Represent \( m \) in the form \( m = d(p-1) + r \) where \( 0 \leq r < p-1 \). Assume that \( r = 0 \). Then \( d \geq 3 \). By Lemma 2.29, \( \Phi(\beta, m-p) \subset \Phi \). Since \( m(\Phi(\beta, m-p)) = m-p > p-1 \), \( \Phi(\beta, m-p) \) is non-trivial, and we can apply induction. If \( r \neq 0 \), then the theorem follows from Lemma 2.28.

Set
\[
\mathcal{T}_n = \left\{ \bigcup_{s=2}^{p-2} \Phi(s)_n \cup \{ D^{(n-1,1)}, D^{(n-1,1)} \otimes \text{sgn}_n \} \right\} \cup \Theta_n \cup \Theta_{n} \cup \{ D^{(n-1,1)}, D^{(n-1,1)} \otimes \text{sgn}_n \}, \quad \text{if} \ p > 3,
\]
\[
\mathcal{T}_n = \{ \Theta_n \}, \quad \text{if} \ p = 3.
\]

Corollary 2.31. Let \( p > 2 \) and \( n > (p-1)^2 \). Then there exists \( M = M(n) > n \) such that, for any \( i \geq M \), the restriction of any faithful \( \Sigma_i \)-module to \( \Sigma_n \) contains a composition factor \( D \in \mathcal{T}_n \).

Proof. It is enough to prove that the restriction of any faithful irreducible \( \Sigma_i \)-module to \( \Sigma_n \) contains a composition factor \( D \in \mathcal{T}_n \).

For each \( i \geq n \) denote by \( \Psi_i \) the set of all \( D^\lambda \in \text{Irr} \Sigma_i \) such that
\[
\text{Irr}(D^\lambda \mid_{\Sigma_n}) \cap \mathcal{T}_n = \emptyset
\]
Observe that for all $i \geq n$ we have $\text{Irr}(\Psi_{i+1} \downarrow_{\Sigma_i}) \subseteq \Psi_i$. For $i \geq n$ we set

$$\Phi_i = \cap_{j \geq i} \text{Irr}(\Psi_j \downarrow_{\Sigma_i})$$

We show that $\Phi = \{ \Phi_i \}_{i \geq n}$ is an inductive system. Let $i \geq n$. Note that

$$\Psi_i \supseteq \text{Irr}(\Psi_{i+1} \downarrow_{\Sigma_i}) \supseteq \text{Irr}(\Psi_{i+2} \downarrow_{\Sigma_i}) \supseteq \ldots$$

Since the set $\Psi_i$ is finite, there exists $k = k(i)$ such that for any $j \geq k$ we have

$$\Phi_i = \text{Irr}(\Psi_j \downarrow_{\Sigma_i}).$$  \hspace{1cm} (6)

Choose $j \geq \max(k(i), k(i+1))$. Then

$$\text{Irr}(\Psi_{i+1} \downarrow_{\Sigma_i}) = \text{Irr}(\text{Irr}(\Psi_j \downarrow_{\Sigma_i+1}) \downarrow_{\Sigma_i}) = \text{Irr}(\Psi_j \downarrow_{\Sigma_i}) = \Phi_i,$$

so $\Phi$ is an inductive system.

If the corollary is false, there exists $j \geq k(n)$ such that $\Psi_j$ contains a faithful module $D$. In view of (6), $\Phi_n$ contains a faithful module. Therefore $\Phi$ is non-trivial. By Theorem 2.30, $\Phi$ contains one of the systems (4). Therefore $\Phi_n \cap \mathcal{T}_n \neq \emptyset$. On the other hand, since $\Phi_n \subseteq \Psi_n$,

$$\Phi_n \cap \mathcal{T}_n \subseteq \Psi_n \cap \mathcal{T}_n = \emptyset.$$

The contradiction obtained proves the corollary. \hfill \Box

Now parts (ii) and (iii) of the Modular Asymptotic Theorem stated in Section 1 follow from Corollary 2.31. Part (i) follows from the following proposition.

**Proposition 2.32.** Let $p > 2$ and $n > (p-1)^2$. There exists $N = N(n) > n$ such that, for any $i \geq N$, the restriction of any $\Sigma_i$-module to $\Sigma_n$ contains a completely splittable $D^\lambda$ as a composition factor (see Section 1).

**Proof.** The proof is similar to that of Corollary 2.31, but uses [1, 5.7] instead of Theorem 2.30. \hfill \Box

### 3. Diagonal and Non-diagonal groups

Throughout this section we denote by $M_n$ and $V_n$ the natural permutation module (of dimension $n$) and the natural irreducible module, respectively, for both $FA_n$ and $F\Sigma_n$. If $G$ and $H$ are two groups, $L$ is an $FG$-module and $M$ is an $FH$-module we write $L \otimes M$ for the outer tensor product of $L$ and $M$ (which is a module over $G \times H$). If $N$ is another $FG$-module we write $L \otimes N$ for the inner tensor product of $L$ and $N$ (which is a $G$-module).

**Diagonal embeddings.** In this subsection we prove the easier direction of Theorem A.

Any diagonal embedding $A_n \to A_m$ is obtained by composing a diagonal embedding $A_n \to A_n \times \cdots \times A_n$, the embedding $A_n \times \cdots \times A_n \to A_{kn}$ as a ‘Young subgroup’, and the natural embedding $A_{kn} \to A_m$ ($m \geq kn$). So

**Lemma 3.1.** Let $\varphi : A_n \to A_m$ be a diagonal embedding. Then the only possible composition factors of the restriction $V_n \downarrow_{A_n}$ are $V_n$ and $1_{A_n}$. Moreover, $V_n$ always occurs.

**Proof.** As our embedding is diagonal, the restriction of the natural permutation module is a direct sum of natural permutation modules and trivial modules. The subgroup is non-trivial, so there is at least one non-trivial orbit. So $V_n$ occurs. \hfill \Box

The next lemma is only needed in the proof of Proposition 3.10.

**Lemma 3.2.** $M_k \otimes F \otimes F$ is a subfactor of the restriction $V_{3k} \downarrow_{\Sigma_k \times \Sigma_k \times \Sigma_k}$. \hfill 13
Proof. Let \( \{e_1, e_2, \ldots, e_{3k}\} \) be the natural basis of \( M_{3k} \). Set \( L \) to be the 1-dimensional submodule \( F(e_1 + e_2 + \cdots + e_{3k}) \) if the characteristic \( p \) divides \( 3k \), and set \( L = (0) \) otherwise. Denote \( \bar{e}_j = e_j + L \in M_{3k}/L \). Then \( V_{3k} \) is a submodule of \( M_{3k}/L \) (of codimension 1) generated by the elements \( \bar{e}_i - \bar{e}_{3k}, 1 \leq i \leq 3k - 1 \). Set
\[
N_1 = \text{span}\{\bar{e}_i - \bar{e}_{3k} \mid 2k + 1 \leq i \leq 3k - 1\}, \quad N_2 = \text{span}\{N_1, \bar{e}_i - \bar{e}_{3k} \mid 1 \leq i \leq k\}.
\]
Then \( N_1 \subseteq N_2 \) are submodules of \( V_{3k} \subseteq \Sigma_k \times \Sigma_k \times \Sigma_k \), and \( N_2/N_1 \cong M_k \boxtimes F \boxtimes F \). \( \square \)

**Proposition 3.3.** Assume that \( G \) is a diagonal limit alternating group. Then \( FG \) is not almost simple.

**Proof.** By Theorem 2.18, it is enough to construct a proper inductive system for \( G \), different from the unitary one. We may assume that all embeddings in (1) are diagonal (just take the union starting from \( i \) large enough, which does not change the group \( G \)). Set \( \Phi_n = \{1_{\text{Alt}(\Omega_n)}, V_{\{e_n\}}\} \) for all \( n \). Then \( \Phi = \{\Phi_n\}_{n \in \mathbb{N}} \) is a (proper non-unitary) inductive system, thanks to Lemma 3.1. \( \square \)

**Non-diagonal embeddings.** The main result of this subsection is Theorem 3.12, which claims that a non-diagonal limit alternating group \( G \) has no proper inductive systems different from the unitary one. Since \( G \) is simple, Theorem A follows from Proposition 3.3, Theorem 3.12 and Theorem 2.18.

**Lemma 3.4.** Let \( X \) be a finite group. Then each irreducible \( \Sigma_n \times X \)-module is of the form \( D \boxtimes L \) with \( D \in \text{Irr} \Sigma_n \) and \( L \in \text{Irr} X \).

**Proof.** Since any field is a splitting field for \( \Sigma_n \), the lemma follows from [5, 7.10 and 10.38]. \( \square \)

We cite an important result of [16]

**Lemma 3.5.** [16, Theorem 1.7] Let \( \text{Alt}(\Omega_1) \subset \text{Alt}(\Omega_2) \subset \cdots \subset \text{Alt}(\Omega_i) \subset \cdots \) be non-diagonal embeddings. Then for every \( i \in \mathbb{N} \) there exists \( N = N(i) > i \) such that, for any \( n > N \), the group \( \text{Alt}(\Omega_i) \) has a regular orbit on \( \Omega_n \).

We shall use the following result of Bryant and Kovacs [4] (the proof can also be found in [6, III.2.16]).

**Lemma 3.6.** [4] Let \( X \) be a finite group and let \( V_g \) be an \( FX \)-module on which \( g \) does not act as a scalar, for all \( g \in X \setminus \{1\} \). Then the regular module \( FX \) is a direct summand of \( \otimes_{g \in X \setminus \{1\}} V_g \).

**Lemma 3.7.** Let \( k, n \in \mathbb{N} \), \( n > (p-1)^2 \), and \( \lambda \) be a completely splittable partition of \( kn \) of height \( s \). Denote by \( H \) the Young subgroup \( \Sigma_n \times \cdots \times \Sigma_n < \Sigma_{kn} \). Then any composition factor of the restriction \( D^{\lambda}|_H \) is of the form
\[
D^{\lambda(1)} \boxtimes \cdots \boxtimes D^{\lambda(k)}
\]
where \( \lambda(1), \ldots, \lambda(k) \) are completely splittable partitions of \( n \) of height \( s \).

**Proof.** It is proved in [11] that the composition factors of the restriction \( D^{\lambda}|_{\Sigma_n} \) are of the form \( D^\mu \) where \( \mu \) is a completely splittable partition of \( n \) of height \( s \). Hence the composition factors of the restriction \( D^{\lambda}|_{\Sigma_n \times \cdots \times \Sigma_n} \) are as claimed. \( \square \)

The case \( p = 3 \) in the Asymptotic Theorem is somewhat exceptional. We need the following lemma for this case only. We say a partition \( \lambda \) is nice if \( \lambda = (\lambda_1, \lambda_2) \) and \( 1 < \lambda_1 - \lambda_2 \leq 7 \).
Proof. Without loss of generality we may assume that $m$ with multiplicities $l$. Then for any faithful $D$ result from [17]: if $m_1D_1 + \cdots + m_lD_l$ mean that $D_1, \ldots, D_l$ are the composition factors of the module $N$ with multiplicities $m_1, \ldots, m_l$, respectively. We use the following special case of the main result from [17]: if $r > 9$ then

$$D_{\lambda_1}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_1}^{(r-1)};$$
$$D_{\lambda_2}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_2}^{(r-1)};$$
$$D_{\lambda_3}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_3}^{(r-1)} + 2D_{\lambda_3}^{(r-1)};$$
$$D_{\lambda_4}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_4}^{(r-1)} + D_{\lambda_4}^{(r-1)};$$
$$D_{\lambda_5}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_5}^{(r-1)} + 2D_{\lambda_5}^{(r-1)};$$
$$D_{\lambda_6}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_6}^{(r-1)} + D_{\lambda_6}^{(r-1)};$$
$$D_{\lambda_7}^{\downarrow_{\Sigma_r-1}} \sim D_{\lambda_7}^{(r-1)}.$$  

It follows that the composition factors of $D_{\lambda_1}^{\downarrow_{\Sigma_r \times \Sigma_n}}$ are all of the form $D_{\lambda_1}^{(n)} \otimes D_{\lambda_2}^{(n)}$ where $s_1, s_2 \leq 7$. Moreover, for any $1 < s \leq 7$ and $9 \leq m < 2n$, we get by induction on $2n - m$ that the restriction $D_{\lambda_1}^{(2n)} \downarrow_{\Sigma_m}$ contains more composition factors $D_{\lambda_1}^{(m)}$ with $1 < t \leq 7$ than composition factors $D_{\lambda_1}^{(m)}$ with $0 \leq t < 1$. Since $D_{\lambda_1}^{(m)}$ and $D_{\lambda_1}^{(m)}$ are the sign modules $sgn_m$ (for $m$ odd and even, respectively), they are 1-dimensional. Now we can conclude that some $D_{\lambda_1}^{(n)} \otimes D_{\lambda_2}^{(n)}$ with $1 < s_1, s_2 \leq 7$ is a composition factor of $D_{\lambda_1}^{\downarrow_{\Sigma_r \times \Sigma_n}}$. 

Let $\varphi_1, \ldots, \varphi_{d-1} : X \to \Sigma_k$ be embeddings of a group $X$ of order $d$ into symmetric groups $\varphi : X \to \Sigma_r$ be any homomorphism. Consider the embedding

$$X \to \underbrace{\Sigma_k \times \cdots \times \Sigma_k}_{d-1 \text{ times}}, \ g \mapsto (\varphi_1(g), \ldots, \varphi_{d-1}(g), \varphi(g)).$$

Then the following lemma holds.

Lemma 3.9. Let $D_1, \ldots, D_{d-1}$ be faithful irreducible $\Sigma_k$-modules ($k > 4$), and $D$ be a $\Sigma_r$-module. Then the restriction $(D_1 \otimes \cdots \otimes D_{d-1} \otimes D)^{\downarrow_X}$ contains the regular $FX$-module as a direct summand.

Proof. We have

$$D_1 \otimes \cdots \otimes D_{d-1} \otimes D^{\downarrow_X} \cong (D_1^{\downarrow_{\varphi_1(X)}}) \otimes \cdots \otimes (D_{d-1}^{\downarrow_{\varphi_{d-1}(X)}}) \otimes (D^{\downarrow_{\varphi(X)}}).$$

Since $D_i$ is faithful, $\Sigma_k$ is a subgroup of $\text{End} D_i$. As $\Sigma_k$ has a trivial center, each $1 \neq g \in X$ does not act as a scalar on $D_i$. Therefore by Lemma 3.6, $(D_1^{\downarrow_{\varphi_1(X)}}) \otimes \cdots \otimes (D_{d-1}^{\downarrow_{\varphi_{d-1}(X)}})$ contains the regular $X$-module as a direct summand. 

Proposition 3.10. Let $p > 2$ and let $X$ be a subgroup of $\text{Sym}(\Omega)$, $d = |X|$ and $d > (p-1)^2$. Assume that $X$ has at least $d-1$ regular orbits on the set $\Omega$, and $|\Omega| \gg d(d-1)$. Then for any faithful $\text{Sym}(\Omega)$-module $D$ we have $\text{Irr}(D^{\downarrow_X}) = \text{Irr}(X)$. 

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Proof. Set \( n = |\Omega| \). We may assume that \( \Omega = \{1,2,\ldots, n\} \), and
\[
\{1,2,\ldots, d\}, \{d+1,\ldots, 2d\}, \ldots, \{(d-2)d+1,\ldots, (d-1)d\}
\]
are regular orbits of \( X \) on \( \Omega \). Moreover, we may assume that \( D \) is irreducible.

Assume first that \( p > 3 \). By Theorem B, the restriction \( D|_{\Sigma_{d(d-1)}} \) contains (as a composition factor) either the natural module \( V = V_{d(d-1)} \) or \( V^\sigma \) or a completely splittable module \( C \) of height \( s \) with \( 1 < s < p-1 \). So, if we put \( r = n - d(d-1) \) then the restriction of \( D \) to the Young subgroup \( \Sigma_{d(d-1)} \times \Sigma_r \) must contain a module of the form \( V \boxtimes B \) or \( V^\sigma \boxtimes B \) or \( C \boxtimes B \) where \( B \) is some \( \Sigma_r \)-module.

In the first two cases it follows from Lemma 3.2 that the restriction of \( D \) to the Young subgroup \( H = \Sigma_d \times \cdots \times \Sigma_d \times \Sigma_r \) contains a subfactor \( M_d \boxtimes F \boxtimes \cdots \boxtimes F \boxtimes B \) or \( M^n_\iota \boxtimes F \boxtimes \cdots \boxtimes F \boxtimes B \), \((d-1)\) times. Note that the image of \( X \) in \( \text{Sym}(\Omega) \) is contained in \( H \). Since \( \{1,2,\ldots, d\} \) is a regular orbit of \( X \), the restriction of \( D \) to \( X \) contains a tensor product of the regular module with some other module, which is a direct sum of the regular modules. In particular, we get the claim of the lemma.

In the second case, Proposition 1.3 shows that the restriction of \( D \) to \( H \) contains a composition factor of the form \( C_1 \boxtimes \cdots \boxtimes C_{d-1} \boxtimes B \) where all modules \( C_i \) are faithful completely splittable. So the result follows from Lemma 3.9.

If \( p = 3 \) then, by Theorem B, we have two possibilities: either the restriction \( D|_{\Sigma_{d(d-1)}} \) contains the natural module \( V = V_{d(d-1)} \) or \( V^\sigma \) or one of the modules module \( D^\lambda, D^\lambda \boxtimes \text{sgn}_n \) where \( \lambda = (\lambda_1, \lambda_2) \) with \( 1 < \lambda_1 - \lambda_2 \leq 7 \). The first case is considered as for \( p > 3 \). In the second case Lemma 3.8 ensures that the restriction of \( D \) to \( H \) contains a composition factor of the form \( C_1 \boxtimes \cdots \boxtimes C_{d-1} \boxtimes B \) with faithful modules \( C_1, \ldots, C_{d-1} \). So we again can argue as in the case \( p > 3 \).

Corollary 3.11. Let \( p > 2 \), and let \( X \) be a subgroup of \( \text{Alt}(\Omega) \), \( d = |X| \) and \( d > (p-1)^2 \). Assume that \( X \) has at least \( 2(d-1) \) regular orbits on the set \( \Omega \), and \( |\Omega| \gg 2d(d-1) \). Then, for any faithful \( \text{Alt}(\Omega) \)-module \( E \) we have \( \text{Irr}(E_{\downarrow X}) = \text{Irr}(X) \).

Proof. Obviously, there are \( X \)-invariant subsets \( \Omega_1, \Omega_2, \) and \( \Omega_3 \) of \( \Omega \) such that \( |\Omega_1| = |\Omega_2| = n \gg d(d-1), \Omega_i (i = 1,2) \) involves at least \( d-1 \) regular \( X \)-orbits, and the permutation representations \( X \to \text{Sym}(\Omega_1) \) and \( X \to \text{Sym}(\Omega_2) \) are equivalent. Therefore \( X < H = \{ (\sigma, \tau) \mid \sigma \in \Sigma_n, \tau \in \text{Alt}(\Omega_3) \} < \text{Alt}(\Omega) \).

Denote \( \Sigma = \{ (\sigma, 1) \mid \sigma \in \Sigma_n \} \). Then \( \Sigma \cong \Sigma_n \) and \( H = \Sigma \times \text{Alt}(\Omega_3) \). Clearly, the restriction \( E_{\downarrow X} \) has a faithful composition factor. Therefore the restriction of \( E \) to \( H \) contains a subfactor \( D \boxtimes B \) with faithful \( D \in \Sigma \). Now by Proposition 3.10, \( \text{Irr}(D_{\downarrow X}) = \text{Irr}(X) \). Therefore \( \text{Irr}(E_{\downarrow X}) = \text{Irr}(X) \), as required.

Theorem 3.12. Let \( p > 2 \) and \( G \) be a non-diagonal limit alternating group. Then \( G \) has no proper inductive systems, different from the unitary one.

Proof. Let \( G_i = \text{Alt}(\Omega_i), i=1,2, \ldots, \) and \( \Phi = \{ \Phi_i \}_{i \in \mathbb{N}} \) be an inductive system for \( G = \bigcup_{i \in \mathbb{N}} G_i \). Assume that \( \Phi_i \) contains a faithful \( G_i \)-module (i.e. a module different from \( 1_{G_i} \)). Then \( \Phi_j \) contains a faithful \( G_j \)-module for any \( j \geq i \). We have to prove that in this case \( \Phi_i = \text{Irr}(G_i) \) for every \( i \in \mathbb{N} \). We may assume that \( i \) is large enough since for any \( k < i \) we have \( \text{Irr}(\{(\text{Irr}(G_i))\}_{i=k}^{i-1}) = \text{Irr}(G_k) \). By Lemma 3.5, the group \( G_i \) has arbitrarily many regular orbits on \( \Omega_j \), provided \( j \) is big enough. Now Corollary 3.11 shows that the restriction of any faithful irreducible \( G_j \)-module to \( G_i \) contains all irreducible \( G_i \)-modules as composition factors, provided \( j \) is big enough.
It is well known that any countably infinite, locally finite, simple group $G$ is a union of finite subgroups $\{G_i | i \in \mathbb{N}\}$ such that for each $i$ we have $G_i \subset G_{i+1}$ and $G_i \cap N_{i+1} = \{1\}$ for some maximal normal subgroup $N_{i+1}$ of $G_{i+1}$. The set of pairs $\mathcal{K} = \{(G_i, N_i) | i \in \mathbb{N}\}$ is called a Kegel sequence for $G$. Assume that $G_i/N_i \cong \text{Alt}(\Omega_i)$ for all $i$. We say that $G$ is diagonal if there exists $n \in \mathbb{N}$ such that for all $i > n$ the embedding $G_i \subset G_{i+1}$ is diagonal, i.e. for every point $\alpha \in \Omega_{i+1}$ the index $|G_i : (G_i)_\alpha N_i|$ is 1 or $|\Omega_i|$, and non-diagonal otherwise. Here $(G_i)_\alpha$ denotes the stabilizer of $\alpha$ in the natural action of $G_i$ on $\Omega_{i+1}$. Assume that char $F = p > 0$ and all $N_i$ are $p$-groups. Then $N_i$ acts trivially on any irreducible $G_i$-module, in particular $\text{Irr} G_i$ can be identified with $\text{Irr Alt}(\Omega_i)$. Clearly, if $G$ is diagonal then it has a proper inductive system different from the unitary one (cf. Proposition 3.3), so $FG$ is not almost simple. The converse is also true. We have the following general result.

**Theorem 3.13.** Let $\{(G_i, N_i) | i \in \mathbb{N}\}$ be a Kegel sequence for $G$ with $G_i/N_i \cong \text{Alt}(\Omega_i)$ for all $i$. Assume that $G$ is non-diagonal and one of the following holds:

1. $\text{char } F = 0$ and each $N_i$ is abelian;
2. $\text{char } F = p > 0$, each $N_i$ is abelian and is not a $p$-group;
3. $\text{char } F = p > 2$ and each $N_i$ is a $p$-group.

Then $FG$ is almost simple.

**Proof.** The cases (a) and (b) have been proved in [16, 1.6]. Consider the case (c). By [16, 1.7 and 3.1], for each $i$ and $k$ there exists $n > i$ such that $G_i$ has at least $k$ regular orbits on $\Omega_n$. Now, using Corollary 3.11 as in the proof of Theorem 3.12, we conclude that $G$ has no proper inductive systems different from the unitary one, so $FG$ is almost simple. 

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