MAXIMAL IDEALS IN MODULAR GROUP ALGEBRAS OF THE FINITARY SYMMETRIC AND ALTERNATING GROUPS

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1. Introduction

A permutation of the set $\mathbb{N} = \{1, 2, 3, \dots\}$ is called *finitary* if it fixes all but finitely many elements. The finitary symmetric group Σ_∞ is the group of all finitary permutations of $\mathbb{N}$. The finitary alternating group A_∞ is defined as the group of all *even* finitary permutations of $\mathbb{N}$. Clearly, Σ_∞ and A_∞ are locally finite groups. They can be represented as the unions

$$\Sigma_\infty = \cup_{n\geq 1}\Sigma_n, \quad A_\infty = \cup_{n\geq 1} A_n$$

where Σ_n and A_n are the groups of all permutations and all even permutations of the set $\{1, 2, \dots, n\}$, respectively.

Let F be an arbitrary field of characteristic $p > 0$.

The main result of this paper is a description of the maximal *two-sided* ideals of the group algebras $F\Sigma_\infty$ and FA_∞ for the case $p > 2$. In particular, we show that there are exactly $p-1$ of them in $F\Sigma_\infty$ and $\frac{p-1}{2}$ in FA_∞.

In the last years there has been a noticeable progress in the theory of group algebras of locally finite groups, see the expository paper [23] and references there. A. E. Zalesskii has shown that the ideals (we always mean two-sided ones) of these algebras are closely related with the so-called *inductive systems*, see Definition 2.1 below. This idea is crucial in our approach here. It allows one to reduce many problems on ideals of $F\Sigma_\infty$ and FA_∞ to the ones on modular representations of finite groups Σ_n and A_n.

To describe our main results in details we need some facts and terminology from the modular representation theory of the symmetric group. The main reference here is [10].

Let $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ be a partition of n (we write $|\lambda| = n$). We define

$$h(\lambda) = m \tag{1}$$

and

$$\chi(\lambda) = l_1 - l_m + m \tag{2}$$

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The irreducible $F\Sigma_n$-modules are parametrized by the p-regular partitions of n. If λ is such a partition we denote by D^λ the corresponding irreducible. For $s = 1, 2, \dots, p-1$ and $n > (p-s)(s-1)$ set

$$\Phi(s)_n = \{D^\mu \mid |\mu| = n, \quad h(\mu) = s, \quad \chi(\mu) \le p\}\,.$$

Let G be a group and M be an FG-module. We write $\mathrm{Ann}_{FG}(M)$ for the annihilator of M in FG. If H is a subgroup of G we denote by $M{\downarrow}H$ the restriction of M to H.

Theorem 1.1. *Let F be a field of characteristic $p > 2$.*

(i) *Set*

$$I(s)_n = \bigcap_{D\in\Phi(s)_n} \mathrm{Ann}_{F\Sigma_n}(D), \quad s = 1, \dots, p-1, \quad n > (p-s)(s-1).$$

Then $I(s)_n = I(s)_{n+1} \cap F\Sigma_n$ for all s, n. Moreover,

$$I(s) = \bigcup_{n>(p-s)(s-1)} I(s)_n, \quad s = 1, \dots, p-1,$$

are exactly all distinct maximal (two-sided) ideals of $F\Sigma_\infty$, and $I(s)\cap F\Sigma_n = I(s)_n$ for $n > (p-s)(s-1)$.

(ii) *Set*

$$J(t)_n = \bigcap_{D\in\Phi(t)_n} \mathrm{Ann}_{FA_n}(D{\downarrow}A_n), \quad t = 1, \dots, \frac{p-1}{2}, \quad n > (p-t)(t-1).$$

Then $J(t)_n = J(t)_{n+1} \cap FA_n$ for all t, n. Moreover,

$$J(t) = \bigcup_{n>(p-t)(t-1)} J(t)_n, \quad t = 1, \dots, \frac{p-1}{2},$$

are exactly all distinct maximal (two-sided) ideals of FA_∞, and $J(t)\cap FA_n = J(t)_n$ for $n > (p-t)(t-1)$.

(iii) *$I(s) \cap FA_\infty = J(t)$ where $t = \min(s, p-s)$, $s = 1, \dots, p-1$.*

Remark. (1) Theorem 1.1 was conjectured by A. E. Zalesskii.

(2) It is known that every group algebra FG contains the maximal ideal

$$\mathrm{Aug}(FG) = \{\sum_{g\in G} \alpha_g g \mid \sum \alpha_g = 0\}$$

called the augmentation ideal. It has codimension 1. One can easily demonstrate another maximal ideal of codimension 1 in $F\Sigma_\infty$:

$$\mathrm{Aug}^\sigma(F\Sigma_\infty) = \{\sum_{g\in\Sigma_\infty} \alpha_g g \mid \sum \mathrm{sign}(g)\alpha_g = 0\}$$

It can be shown that

$$\begin{aligned} I(1) &= \mathrm{Aug}(F\Sigma_\infty), \\ J(1) &= \mathrm{Aug}(FA_\infty), \\ I(p-1) &= \mathrm{Aug}^\sigma(F\Sigma_\infty) \end{aligned}$$

and that ideals $I(s)$, $1 < s < p-1$, and $J(t)$, $t > 1$, are of infinite codimension.

(3) Ideals $I(s)$ of $F\Sigma_\infty$ has been essentially described in [15]. Results from [15] are used in this paper.

(4) One can show that all the restrictions $D{\downarrow}A_n$ in the part (ii) of Theorem 1.1 are actually irreducible.

(5) Tensoring $F\Sigma_n$-modules with the one-dimensional sign representation is one of the crucial methods used in our proofs. This is why the case $p = 2$ happened to be exceptional in this paper. A.E. Zalesskii observed that if $p = 2$ there exists at least one maximal ideal in both $F\Sigma_\infty$ and FA_∞ which is different from the augmentation ideal. But it remains unclear if there are other maximal ideals.

(6) We note that for a ground field of characteristic 0 there is a complete description of the ideal lattice in the group algebra of Σ_∞, see [6, 18]. In particular, $\mathrm{Aug}(F\Sigma_\infty)$ and $\mathrm{Aug}^\sigma(F\Sigma_\infty)$ are the only maximal ideals of $F\Sigma_\infty$. Much less is known about *modular* group rings. Important information is contained in the papers [7, 19, 21, 22].

(7) The ideals we consider play a role in the theory of identities of algebras, see [1, 18, 19, 20].

(8) In this paper we use recent results from modular representation theory of symmetric groups related with branching rules [13] and tensoring with sign (Mullineux Conjecture) [5], [2].

(9) All results of this paper about the groups Σ_∞ and A_∞ can be easily generalized to the finitary symmetric and alternating groups Σ_Ω and A_Ω of an *uncountable* set Ω. One just has to use inductive limits instead of unions. We leave details to the reader.

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Notation

F	a field of characteristic $p > 0$;
Σ_n (resp., A_n)	the symmetric (resp., alternating) group on $\{1, \dots, n\}$;
$\mathcal{P}_n$	the set of all p-regular partitions of n;
$\lvert\lambda\rvert = n$	means "λ is a partition of n";
S^λ	the Specht module over $F\Sigma_n$ corresponding to a partition λ of n [10];
D^λ	the irreducible $F\Sigma_n$-module corresponding to a p-regular partition λ of n [10];

E^λ, $E^\mu_\pm$	denote the irreducible FA_n-modules, see Section 6 below;
sgn	the one-dimensional sign representation of FG_n ($gv = \text{sign}(g)v$ for $g \in \Sigma_n$, $v \in \text{sgn}$;
$M : \mathcal{P}_n \to \mathcal{P}_n$	the Mullineux bijection. Thus $D^\lambda \otimes \text{sgn} \cong D^{M(\lambda)}$, see Section 3 below;
$\chi(\lambda)$, $h(\lambda)$	are defined in (1) and (2) above;
$\text{res}\, A$	the residue of the node A, see [11] and (4) below;
$\text{cont}(\lambda)$	the residue content of the Young diagram λ, see [11] and (5) below;
λ_A	stands for the Young diagram $\lambda\backslash\{A\}$ where λ is a Young diagram and A is a removable node of λ, see Section 3 below.

Let G be a group, H a subgroup of G, A an associative algebra, M an FG-module, S a set of FG-modules, N an FH-module. Then we denote:

Irr G (resp., Irr A)	the set of the isomorphism classes of the irreducible FG-modules (resp., A-modules);
Irr M	the set of isomorphism classes of the composition factors of M;
Irr S	stands for $\cup_{M\in S} \text{Irr } M$;
$\text{Irr}(S{\downarrow}H)$	stands for $\cup_{M\in S} \text{Irr}(M{\downarrow}H)$;
$M{\downarrow}H$ (or $M{\downarrow}^G_H$)	the restriction of M from G to H;
$N{\uparrow}^G$ (or $N{\uparrow}^G_H$)	the module induced from H to G.

2. Inductive systems (general setting)

In this section we shortly discuss and push a bit further the relation between inductive systems and (two-sided) ideals in the group algebra of a locally finite group, revealed by A. E. Zalesskii. Assume for simplicity that a locally finite group G is countable. Then G can be represented as a union

$$G = \cup_{i=1}^\infty G_i$$

where $G_1 \subset G_2 \subset \dots$ are finite subgroups of G (if G is not countable one has to use an inductive limit).

Definition 2.1. [23, 1.1] Let Φ_i be a non-empty subset of Irr G_i, $i = 1, 2, \dots$. We say that the collection $\Phi = \{\Phi_i\}_{i\in\mathbb{N}}$ is an *inductive system* (for G) if for any $i, j \in \mathbb{N}$ with $i < j$ we have

$$\Phi_i = \text{Irr}(\Phi_j{\downarrow}G_i).$$

Let X be a (proper) ideal of the group algebra FG. Set

$$\Phi(X)_i = \text{Irr}((FG/X){\downarrow}G_i).$$

One easily checks that $\Phi(X) = \{\Phi(X)_i\}_{i\in\mathbb{N}}$ is an inductive system. Recall that an ideal I of an algebra A is called *primitive* if it is the annihilator of an irreducible A-module and *semiprimitive* if it is an intersection of primitive ideals. Equivalently, I is semiprimitive if and only if the Jacobson radical $\mathrm{Rad}(A/I)$ is trivial. We shall also use the known fact that the Jacobson radical of a locally finite-dimensional algebra is the largest locally nilpotent ideal. The following result is crucial.

Theorem 2.2. [23, 8.1, 1.25]

(i) $\Phi(X)_i = \mathrm{Irr}(FG_i/(X \cap FG_i))$ *where* $FG_i/(X \cap FG_i)$ *is considered as an* FG_i*-module.*

(ii) *The map* $f : X \mapsto \Phi(X)$ *from the set of the proper ideals to the set of inductive systems is surjective.*

(iii) *Let* Φ *be an inductive system. Then the set* $f^{-1}(\Phi)$ *contains the largest and the smallest (by inclusion) ideals* $I(\Phi)$ *and* $K(\Phi)$*, respectively.*

(iv) *The ideals* $I(\Phi)$ *are semiprimitive. The quotient algebras* $I(\Phi)/K(\Phi)$ *are locally nilpotent. In particular, the map* $\Phi \to I(\Phi)$ *is a bijection between the set of the inductive systems and the set of the semiprimitive ideals of* FG*.*

Definition 2.3. Let $\Phi = \{\Phi_i\}_{i\in\mathbb{N}}$ and $\Psi = \{\Psi_i\}_{i\in\mathbb{N}}$ be inductive systems. We write $\Phi \subseteq \Psi$ if $\Phi_i \subseteq \Psi_i$ for all i.

Lemma 2.4. *Let* $\Psi_i \subseteq \mathrm{Irr}\ G_i$*,* $i \in \mathbb{N}$*. Assume*

$$\mathrm{Irr}(\Psi_i{\downarrow}G_{i-1}) \subseteq \Psi_{i-1}.$$

Set

$$\tilde{\Psi}_i = \cap_{j\geq i}\,\mathrm{Irr}(\Psi_j{\downarrow}G_i).$$

Then $\tilde{\Psi} = \{\tilde{\Psi}_i\}_{i\in\mathbb{N}}$ *is the largest (with respect to* $\subseteq$*) inductive system contained in* Ψ*.*

Proof. Note that

$$\Psi_i \supseteq \mathrm{Irr}(\Psi_{i+1}{\downarrow}G_i) \supseteq \mathrm{Irr}(\Psi_{i+2}{\downarrow}G_i) \supseteq \dots$$

Since the set Ψ_i is finite, there exists $n = n(i)$ such that for any $j > n$ we have

$$\tilde{\Psi}_i = \mathrm{Irr}(\Psi_j{\downarrow}G_i).$$

Let $i \in \mathbb{N}$. Choose $j \geq \max(n(i), n(i+1))$. Then

$$\mathrm{Irr}(\tilde{\Psi}_{i+1}{\downarrow}G_i) = \mathrm{Irr}(\mathrm{Irr}(\Psi_j{\downarrow}G_{i+1}){\downarrow}G_i) = \mathrm{Irr}(\Psi_j{\downarrow}G_i) = \tilde{\Psi}_i,$$

so $\tilde{\Psi}$ is an inductive system. Clearly, it is the largest one contained in Ψ. □

Definition 2.5. Let Φ and Ψ be inductive systems. Set $\Theta_i = \Phi_i \cap \Psi_i$. One can easily see that

$$\mathrm{Irr}(\Theta_i{\downarrow}G_{i-1}) \subseteq \Theta_{i-1}.$$

In view of Lemma 2.4, there exists the largest inductive system $\tilde{\Theta}$ such that $\tilde{\Theta} \subseteq \Phi$ and $\tilde{\Theta} \subseteq \Psi$. We shall denote $\tilde{\Theta}$ by $\Phi \wedge \Psi$. Moreover, we denote by $\Phi \vee \Psi$ the inductive system with $(\Phi \vee \Psi)_i = \Phi_i \cup \Psi_i$.

Proposition 2.6.

(i) $\Phi \to I(\Phi)$ *is an order-reversing isomorphism of partially ordered sets between the inductive systems and semiprimitive ideals of* FG.

(ii) $\Phi \to I(\Phi)$ *is a bijection between the set of the minimal (with respect to* $\subseteq$*) inductive systems and the set of the maximal ideals of* FG.

(iii) *Semiprimitive ideals of* FG *satisfy A.C.C. if and only if inductive systems satisfy D.C.C.*

(iv) *For any ideals* I, J *of* FG *we have*

$$\Phi(I \cap J) = \Phi(I) \vee \Phi(J)$$

(v) *For any ideals* I, J *of* FG *we have*

$$\Phi(I + J) = \Phi(I) \wedge \Phi(J)$$

Proof. We first prove (iv). Fix $k \in \mathbb{N}$ and set

$$I_k = I \cap FG_k, \quad J_k = J \cap FG_k.$$

In view of Theorem 2.2 (i), we have to show that

$$\mathrm{Irr}(FG_k/(I_k \cap J_k)) = \mathrm{Irr}(FG_k/I_k) \cup \mathrm{Irr}(FG_k/J_k).$$

But this equality follows from the following exact sequences of G_k-modules

$$0 \to I_k/(I_k \cap J_k) \to FG_k/(I_k \cap J_k) \to FG_k/I_k \to 0,$$
$$0 \to J_k/(I_k \cap J_k) \to FG_k/(I_k \cap J_k) \to FG_k/J_k \to 0,$$

and the embedding $I_k/(I_k \cap J_k) \cong (I_k + J_k)/J_k \subset FG_k/J_k$.

Now we prove (i). By 2.2 (iv), $\Phi \to I(\Phi)$ is a bijection between the inductive systems and the semiprimitive ideals. It follows from the definitions that $I \supseteq J$ implies $\Phi(I) \subseteq \Phi(J)$. This shows that $I(\Phi) \supseteq I(\Psi)$ implies $\Phi \subseteq \Psi$. Conversely, let $\Phi \subseteq \Psi$. By (iv),

$$\Phi(I(\Phi) \cap I(\Psi)) = \Phi(I(\Phi)) \vee \Phi(I(\Psi)) = \Phi \vee \Psi = \Psi,$$

since $\Phi \subseteq \Psi$. It is known that the intersection of semiprimitive ideals is a semiprimitive ideal. So we have two semiprimitive ideals, $I(\Phi) \cap I(\Psi)$ and $I(\Psi)$, corresponding to one inductive system Ψ. In view of 2.2 (iv),

$$I(\Phi) \cap I(\Psi) = I(\Psi).$$

Hence $I(\Phi) \supseteq I(\Psi)$.

(iii) follows immediately from (i).

Let X be a maximal ideal of FG. Then FG/X is simple. Observe that FG/X is not locally nilpotent (for example because it contains the identity). Therefore X is semiprimitive. Now (ii) follows from (i) .

(v) For an ideal X of FG we denote by $\bar{X}$ the smallest semiprimitive ideal of FG containing X (it coincides with the intersection of the semiprimitive ideals containing X). Since $I(\Phi(X))$ is semiprimitive and contains X, we have $\bar{X} \subseteq$

$I(\Phi(X))$. The inclusions $X \subseteq \bar{X} \subseteq I(\Phi(X))$ imply $\Phi(\bar{X}) = \Phi(X)$. By 2.2 (iv), $\bar{X} = I(\Phi(X))$ and $\bar{X}/X$ is locally nilpotent. Set

$$X = I + J, \quad Y = \bar{I} + \bar{J}.$$

Observe that

$$(\bar{I} + X)/X \cong \bar{I}/(\bar{I} \cap X)$$

is a locally nilpotent ideal in Y/X. Analogously $(\bar{J} + X)/X$ is a locally nilpotent ideal in Y/X. Since

$$Y/X = (\bar{I} + X)/X + (\bar{J} + X)/X.$$

we conclude that Y/X is locally nilpotent. Hence $\mathrm{Rad}(FG/X) \supseteq Y/X$.

On the other hand, we have $\mathrm{Rad}(FG/X) \subseteq \bar{X}/X$ since $\mathrm{Rad}(FG/\bar{X}) = 0$. Thus $\bar{X}/X \supseteq Y/X$, whence $\bar{X} \supseteq Y$. Therefore $\bar{X} \supseteq \bar{Y}$. Since $\bar{Y} \supseteq Y \supseteq X$, we conclude that $\bar{X} = \bar{Y}$. So

$$\Phi(\bar{Y}) = \Phi(\bar{X}) = \Phi(X).$$

Since $\bar{Y}$ is the smallest semiprimitive ideal, containing semiprimitive ideals $\bar{I}$ and $\bar{J}$, (i) implies

$$\Phi(\bar{Y}) = \Phi(\bar{I}) \wedge \Phi(\bar{J}).$$

Finally, we get

$$\Phi(X) = \Phi(\bar{Y}) = \Phi(\bar{I}) \wedge \Phi(\bar{J}) = \Phi(I) \wedge \Phi(J).$$

□

Remark. In the proof of Proposition 2.6 (v) we have introduced the operation $X \to \bar{X}$. Note that the set of *all* (not necessarily proper) semiprimitive ideals of FG is a lattice with respect to the operations $I \cap J$ and $\overline{I + J}$. In view of 2.6 (i) this lattice is antiisomorphic to the lattice of all inductive systems with respect to the operations $\vee$ and $\wedge$, if we admit the empty inductive system (which will correspond to the ideal FG).

Definition 2.7. An inductive system $\Phi = \{\Phi_i\}_{i \in \mathbb{N}}$ is called *semisimple* if for any $i \in \mathbb{N}$ and any $D \in \Phi_{i+1}$ the restriction $D{\downarrow}G_i$ is completely reducible.

Lemma 2.8. *Let $\Phi = \{\Phi_i\}_{i \in \mathbb{N}}$ be an inductive system for G. Then Φ is semisimple if and only if*

$$I(\Phi) \cap FG_i = \cap_{D \in \Phi_i} \mathrm{Ann}_{FG_i} D \qquad (3)$$

for all $i \in \mathbb{N}$.

Proof. Set

$$X_i = \cap_{D \in \Phi_i} \mathrm{Ann}_{FG_i} D, \quad i = 1, 2, \ldots$$

Let Φ be semisimple. If $i < j$ and $D \in \Phi_j$ we have $D{\downarrow}G_i = D_1 \oplus \cdots \oplus D_s$ with all $D_k \in \Phi_i$. Moreover, $\mathrm{Irr}(\Phi_j{\downarrow}G_i) = \Phi_i$, and the annihilator of a direct sum of modules is the intersection of their annihilators. So we have

$$\begin{aligned} X_j \cap FG_i &= (\cap_{D\in\Phi_j} \mathrm{Ann}_{FG_j} D) \cap FG_i = \cap_{D\in\Phi_j}((\mathrm{Ann}_{FG_j} D) \cap FG_i) \\ &= \cap_{D\in\Phi_j} \mathrm{Ann}_{FG_i}(D{\downarrow}G_i) = X_i. \end{aligned}$$

Therefore $X_i \subseteq X_j$ (we consider both X_i and X_j as subsets of FG via embeddings $FG_l \subset FG$). So $X = \cup_{i=1}^{\infty} X_i$ is an ideal of FG such that $X \cap FG_i = X_i$. So, taking into account Theorem 2.2 (i), we get

$$\Phi(X)_i = \mathrm{Irr}(FG_i/(FG_i \cap X)) = \mathrm{Irr}(FG_i/X_i) = \mathrm{Irr}(FG_i/\,\mathrm{Ann}_{FG_i}(\oplus_{D\in\Phi_i} D)) = \Phi_i,$$

i.e. $\Phi(X) = \Phi$.

Now note that

$$FG/X = \cup_{i=1}^{\infty} FG_i/(FG_i \cap X) = \cup_{i=1}^{\infty} FG_i/X_i.$$

Since every ideal X_i is an intersection of primitive ideals (annihilators of irreducible modules) it is semiprimitive, i.e. FG_i/X_i is semisimple. It follows that $\mathrm{Rad}(FG/X) = 0$, i.e. X is semiprimitive. By Theorem 2.2 (iv), $X = I(\Phi)$, which proves the equality (3).

Conversely, let $I(\Phi) \cap FG_i = X_i$, $i = 1, 2, \dots$. For any $i \in \mathbb{N}$, we consider an arbitrary module $D \in \Phi_{i+1}$. Note that $X_i \subseteq X_{i+1} \subseteq \mathrm{Ann}_{FG_{i+1}}(D)$ by assumption. Hence the restriction $D{\downarrow}G_i$ is an FG_i/X_i-module. However, FG_i/X_i is a semisimple finite-dimensional algebra (because $X_i \supseteq \cap_{D\in \mathrm{Irr}\, G_i} \mathrm{Ann}_{FG_i} D = \mathrm{Rad}\, FG_i$). Therefore $D{\downarrow}G_i$ is semisimple. $\square$

3. Modular representations of symmetric groups

We start from some notation concerning Young diagrams.

Let us fix an arbitrary partition $\lambda = (l_1 \geq \cdots \geq l_m > 0)$ of n. We do not distinguish between λ and its Young diagram which is defined as a subset

$$\{(i,j) \mid 1 \leq i \leq m, \quad 1 \leq j \leq l_i\}$$

of $\mathbb{N} \times \mathbb{N}$. Elements (i,j) of $\mathbb{N} \times \mathbb{N}$ are called nodes. If $A = (i,j)$ is a node we define

$$\mathrm{res}\, A = (j - i) \pmod{p}. \tag{4}$$

If a diagram λ contains exactly c_i nodes of residue i, $i = 0, 1, \dots, p-1$, we define

$$\mathrm{cont}(\lambda) = (c_0, c_1, \dots, c_{p-1}). \tag{5}$$

If $l_i > l_{i+1}$ then the node (i, l_i) is called a *removable* node for λ (l_{m+1} is interpreted as 0). If $A = (i, l_i)$ is a removable node for λ we denote by λ_A the partition $(l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_m)$ of $n-1$ whose Young diagram is $\lambda \backslash \{A\}$. We call (i,j) an *indent* node for λ if $l_i < l_{i-1}$, $j = l_i + 1$ (l_0 is interpreted as $+\infty$).

We list some known results on representations of Σ_n for future reference. All modules over Σ_n are $F\Sigma_n$-modules.

The Specht module S^λ labelled by partitions λ of n are defined in [10, Section 4]. The irreducible modules D^λ labelled by the p-regular partitions λ of n are defined as the top composition factors of the corresponding S^λ, see [10, Section 11]. So

Lemma 3.1. *Let $\lambda \in \mathcal{P}_n$. Then $D^\lambda \in \mathrm{Irr}(S^\lambda)$.*

The following well known result follows from the "Nakayama Conjecture" [11, 6.1] and Lemma 3.1.

Lemma 3.2. *If $D^\mu \in \mathrm{Irr}(S^\lambda)$ then*

$$\mathrm{cont}(\mu) = \mathrm{cont}(\lambda).$$

We need the following Branching Theorem for the Specht modules.

Theorem 3.3. [10, 9.3] *The restriction $S^\lambda{\downarrow}\Sigma_{n-1}$ has a filtration whose factors are*

$$\{S^{\lambda_A} \mid A \text{ is a removable node for } \lambda\}.$$

We shall use a special case of the branching rule for irreducible modules obtained in [13]. To state the result we need a concept of a normal node.

Definition 3.4. Let $A = (i, l_i)$ be a removable node for λ. We call A *normal* (for λ) if and only if for every $j = 1, 2, \dots, i-1$ the following condition holds: the number of the removable nodes B in the rows $j, j+1, \dots, i-1$ such that $\mathrm{res}\, B = \mathrm{res}\, A$ is greater than or equal to the number of the indent nodes C in the rows $j, j+1, \dots, i-1$ such that $\mathrm{res}\, C = \mathrm{res}\, A$.

Theorem 3.5. [13, 16] *Let $\lambda \in \mathcal{P}_n$ and A be a removable node of λ such that $\lambda_A \in \mathcal{P}_{n-1}$. Then $D^{\lambda_A} \in \mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n-1})$ if and only if A is normal.*

If $d \in \mathbb{N}$ we define the *p-exponent* of d as the largest $i \in \mathbb{Z}_+$ such that p^i divides d.

Theorem 3.6. [9, 12] *Let $\lambda \in \mathcal{P}_n$. Then S^λ is irreducible if and only if the p-exponents of the hook length are constant along the columns of λ.*

We denote by sgn the one-dimensional sign representation of Σ_n. The Mullineux bijection $M : \mathcal{P}_n \to \mathcal{P}_n$ is defined via

$$D^\lambda \otimes \mathrm{sgn} \cong D^{M(\lambda)}.$$

The main result of [5] and [2] provides an algorithm for calculating $M(\lambda)$ which we now describe.

The *rim* of a Young diagram λ is its south-east border — in other words, a node (i, j) of λ belongs to the rim if and only if $(i+1, j+1) \notin \lambda$.

Let us number the nodes of the rim moving from the "top-right" to the "left-bottom" (see the example below). Define the first p-segment of the rim as the set consisting of the nodes with numbers $\leq p$. If the last node B of the first p-segment is in the last row of λ then λ has only one p-segment. If not, let r be the row containing B. The first node of the second p-segment is the node which has the smallest number, say d, among the nodes of the rim in row $r+1$. The second

p-segment is now defined as the set consisting of the nodes whose numbers i satisfy $d \le i \le d+p-1$. Repeating this procedure sufficiently many times we reach the bottom row of the diagram. It is clear that all p-segments except possibly the last one contain p nodes. The *p-edge* is defined as the union of the p-segments.

Now define diagrams $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \dots$ as follows.

$$\lambda^{(1)} = \lambda, \quad \lambda^{(i)} = \lambda^{(i-1)} \setminus \{p - \text{edge of } \lambda^{(i-1)}\}$$

for $i > 1$. We choose z to be maximal with respect to $\lambda^{(z)} \neq \emptyset$. The *Mullineux symbol* of λ is an array

$$G(\lambda) = \begin{pmatrix} a_1 & a_2 & \dots & a_z \\ r_1 & r_2 & \dots & r_z \end{pmatrix}$$

where a_i is the number of nodes of the p-edge of $\lambda^{(i)}$, and $r_i = h(\lambda^{(i)})$ is the number of rows in $\lambda^{(i)}$, $i = 1, 2, \dots, z$.

The following theorem was first proved in [5]. An easier proof was recently found by Bessenrodt and Olsson [2].

Theorem 3.7. *Let $\lambda \in \mathcal{P}_n$, and*

$$G(\lambda) = \begin{pmatrix} a_1 & a_2 & \dots & a_z \\ r_1 & r_2 & \dots & r_z \end{pmatrix}.$$

Put $\varepsilon_i = 0$ if $p | a_i$, and $\varepsilon_i = 1$ otherwise. Then

$$G(M(\lambda)) = \begin{pmatrix} a_1 & a_2 & \dots & a_z \\ s_1 & s_2 & \dots & s_z \end{pmatrix}$$

where $s_i = a_i + \varepsilon_i - r_i$, $i = 1, 2, \dots, z$.

Thus the Mullineux symbol of $M(\lambda)$ can be easily found from that of λ. On the other hand, a p-regular partition can be reconstructed from its Mullineux symbol—we just add p-edges starting from the last one.

Example. Let $\lambda = (6, 4^2, 2, 1)$. Then the rim of λ contains the nodes represented by numbers in the following picture.

$$\begin{array}{llllll} \circ & \circ & \circ & 3 & 2 & 1 \\ \circ & \circ & \circ & 4 & & \\ \circ & 7 & 6 & 5 & & \\ 9 & 8 & & & & \\ 10 & & & & & \end{array}$$

Let $p = 5$. The nodes of the p-edge (which consists of two p-segments) are coloured in black in the following picture.

$$\begin{array}{llllll} \circ & \circ & \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & & \\ \circ & \circ & \circ & \bullet & & \\ \bullet & \bullet & & & & \\ \bullet & & & & & \end{array}$$

We have $\lambda^{(2)} = (3^3)$, $\lambda^{(3)} = (2^2)$, $\lambda^{(4)} = (1)$, $z = 4$, and

$$G(\lambda) = \begin{pmatrix} 8 & 5 & 3 & 1 \\ 5 & 3 & 2 & 1 \end{pmatrix}, \quad G(M(\lambda)) = \begin{pmatrix} 8 & 5 & 3 & 1 \\ 4 & 2 & 2 & 1 \end{pmatrix}.$$

Finally, $M(\lambda) = (7^2, 2, 1)$.

4. Some lemmas on branching and tensoring with sign

Let $m, f \in \mathbb{N}$. Represent m in the form

$$m = (p-1)d + r, \quad d \in \mathbb{Z}, \quad 0 < r \le p-1.$$

We need to consider certain classes of partitions. Set

$$\begin{aligned} \beta(m,f) &= ((f+d)^{p-1}, (f+d-1)^{p-1}, \ldots, (f+1)^{p-1}, f^r); \\ \alpha(m,f) &= ((f+d)^r, (f+d-1)^{p-1}, (f+d-2)^{p-1}, \ldots, f^{p-1}); \\ \alpha(m) &= \alpha(m,1). \end{aligned}$$

Obviously $h(\beta(m,f)) = h(\alpha(m,f)) = m$.

If $\mu = (\mu_1, \ldots, \mu_t)$ is a p-regular partition and $f > \mu_1$, put

$$\begin{aligned} \beta(m,f;\mu) &= ((f+d)^{p-1}, (f+d-1)^{p-1}, \ldots, (f+1)^{p-1}, f^r, \mu_1, \ldots, \mu_t); \\ \alpha(m,f;\mu) &= ((f+d)^r, (f+d-1)^{p-1}, (f+d-2)^{p-1}, \ldots, f^{p-1}, \mu_1, \ldots, \mu_t) \end{aligned}$$

(glue μ to the bottom of $\beta(m,f)$ and $\alpha(m,f)$, respectively).

Lemma 4.1. *The Specht module $S^{\alpha(m)}$ is irreducible.*

Proof. This follows from Theorem 3.6. □

Let us fix a p-regular partition λ with k distinct parts, i.e.

$$\lambda = (l_1^{a_1}, l_2^{a_2}, \ldots, l_k^{a_k}), \quad l_1 > l_2 > \cdots > l_k > 0, \quad 0 < a_i < p.$$

Then λ has k removable and $k+1$ indent nodes. Let $A_1, \ldots, A_k$ (resp., $B_1, \ldots, B_{k+1}$) be all removable (resp., indent) nodes of λ numbered from top to bottom, i.e. $A_i = (a_1 + \cdots + a_i, l_i)$, $B_j = (a_1 + \cdots + a_{j-1} + 1, l_j + 1)$, $i = 1, \ldots, k$, $j = 1, \ldots, k+1$ (l_{k+1} is interpreted as 0).

Lemma 4.2. *Assume* $\operatorname{res} A_1 = \operatorname{res} A_2 = \cdots = \operatorname{res} A_r$ *for some* $r \in \{1, \ldots, k\}$. *Then* $A_1, A_2, \ldots, A_r$ *are normal nodes.*

Proof. Since λ is p-regular, we have $\operatorname{res} A_i \ne \operatorname{res} B_i$ for all $i = 1, \ldots, k$. Now use 3.4. □

Lemma 4.3. *Let $r \ge 1$ be the largest integer such that*

$$l_1 = l_2 + 1 = l_3 + 2 = \cdots = l_r + (r-1)$$

and

$$a_2 = a_3 = \cdots = a_r = (p-1).$$

Then r is minimal such that λ_{A_r} is p-regular. Moreover, A_r is normal for λ.

Proof. The first part of the lemma follows from the definitions. To prove that A_r is normal it suffices to note that

$$\operatorname{res} A_1 = \operatorname{res} A_2 = \cdots = \operatorname{res} A_r$$

and use Lemma 4.2. □

Corollary 4.4. *Let $\lambda = (l_1^{a_1}, l_2^{a_2}, \dots, l_k^{a_k}) \in \mathcal{P}_n$, with $l_k = 1$. Assume that the bottom removable node A_k is the only removable node A_i such that λ_{A_i} is p-regular. Then $\lambda = \alpha(a_1 + a_2 + \cdots + a_k)$.*

Proof. It follows from Lemma 4.3 that

$$l_i = l_k + (k - i), \quad i = 1, 2, \dots, k,$$

and

$$a_2 = a_3 = \cdots = a_k = p - 1,$$

as desired. □

Definition 4.5. Let λ be a p-regular partition. We denote by $A(\lambda)$ the top removable node of λ such that $\lambda_{A(\lambda)}$ is p-regular. Define

$$\varphi(\lambda) = \lambda_{A(\lambda)}.$$

Lemma 4.6. *Let $\lambda \in \mathcal{P}_n$. Then*

$$D^{\varphi(\lambda)} \in \operatorname{Irr}(D^\lambda{\downarrow}\Sigma_{n-1}).$$

Proof. By Lemma 4.3, $A(\lambda)$ is normal for λ. So it suffices to apply Theorem 3.5. □

Remark. Lemma 4.6 is known [8]. We reprove it here for completeness only. Note however that the result is insufficient to prove Proposition 4.14. There we do have to use Theorem 3.5.

Lemma 4.7. *Let μ be a p-regular partition. Assume that for some m we have $\alpha(m) \subseteq \mu$. Set $n = |\alpha(m)|$. Then*

$$D^{\alpha(m)} \in \operatorname{Irr}(D^\mu{\downarrow}\Sigma_n)$$

Proof. In view of Lemma 4.6, it suffices to show that $\alpha(m) = \varphi^i(\mu)$ for some i. Since $\varphi^{|\mu|}(\mu) = \emptyset$ and $(m, 1) \in \alpha(m) \subseteq \mu$, there exists i such that $A(\varphi^i(\mu)) = (m, 1)$. So $(m, 1)$ is the top removable node of $\varphi^i(\mu)$ leaving a p-regular diagram after its removal. Moreover, $(m, 1)$ belongs to the first column, so it is the bottom removable node of $\varphi^i(\mu)$. By Corollary 4.4 we now get $\varphi^i(\mu) = \alpha(m)$. □

Proposition 4.8. *Let $n \in \mathbb{N}$ and λ be a p-regular partition with $h(\lambda) \geq n(p-1)$. Then*

$$\operatorname{Irr}(D^\lambda{\downarrow}\Sigma_n) = \operatorname{Irr} \Sigma_n.$$

Proof. Since λ is p-regular and $h(\lambda) \geq n(p-1)$ we have $\alpha(n(p-1)) \subseteq \lambda$. Let $j = |\alpha(n(p-1))|$. Then Lemma 4.7 implies

$$D^{\alpha(n(p-1))} \in \mathrm{Irr}(D^\lambda{\downarrow}\Sigma_j).$$

Let $\mu \in \mathcal{P}_n$. Then $\mu \subseteq \alpha(n(p-1))$. In view of Lemma 4.1,

$$D^{\alpha(n(p-1))} = S^{\alpha(n(p-1))}.$$

Now Theorem 3.3 implies that $D^{\alpha(n(p-1))}{\downarrow}\Sigma_n$ has a filtration such that S^μ occurs as its factor. Since μ is p-regular, $D^\mu \in \mathrm{Irr}(S^\mu)$ by Lemma 3.1, and the result follows. □

We need to somewhat develop Lemma 4.7. Our goal now is Proposition 4.14.

Lemma 4.9. *Let $\mu = (\mu_1, \dots, \mu_s)$ be a p-regular partition (μ may be $\emptyset$) and f be an integer with $f > \mu_1 + 1$. Set $\alpha = \alpha(m, f; \mu)$, $\beta = \beta(m, f-1; \mu)$, $n = |\beta|$. Then*

$$D^\beta \in \mathrm{Irr}(D^\alpha{\downarrow}\Sigma_n).$$

Proof. Let $m = (p-1)d + r$, $d \in \mathbb{Z}$, $0 < r \leq p-1$. Define the partitions $\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(r)}$ and the sets $M_0, M_1, \dots, M_{r-1}$ of nodes inductively as follows. Set $\alpha^{(0)} = \alpha$ and $\alpha^{(i+1)} = \alpha^{(i)} \setminus M_i$ where M_i is the set of the top $(d+1)$ removable nodes of $\alpha^{(i)}$, $i = 0, 1, \dots, r-1$. Note that all nodes of the set M_i are of the same residue. So Lemma 4.2 implies that the nodes of M_i are normal nodes for the partition $\alpha^{(i)}$ ($i = 0, 1, \dots, r-1$). Therefore

$$D^{\alpha^{(i+1)}} \in \mathrm{Irr}\left(D^{\alpha^{(i)}} \downarrow \Sigma_{|\alpha^{(i+1)}|}\right)$$

for all $i = 0, 1, \dots, r-1$. It remains to observe that $\alpha^{(r)} = \beta$. □

Lemma 4.10. *Let $\mu = (\mu_1, \dots, \mu_s)$ be a p-regular partition (μ may be $\emptyset$), $f > \mu_1 + 1$. Set $\beta = \beta(m, f; \mu)$, $\gamma = \beta(m, f-1; \mu)$, $n = |\gamma|$. Then*

$$D^\gamma \in \mathrm{Irr}(D^\beta{\downarrow}\Sigma_n).$$

Proof. The proof is similar to that of Lemma 4.9. □

Lemma 4.11. *Let $\mu = (\mu_1, \dots, \mu_s)$ be a p-regular partition, $f > \mu_1 + 1$. Set $\nu = \varphi(\mu)$ (see 4.5), $\beta = \beta(m, f; \mu)$, $\gamma = \beta(m, f-1; \nu)$, $n = |\gamma|$. Then*

$$D^\gamma \in \mathrm{Irr}(D^\beta{\downarrow}\Sigma_n).$$

Proof. Let $m = d(p-1) + r$, $d \in \mathbb{Z}$, $0 < r \leq p-1$. It follows from the definition of β that in the first m rows there are exactly $d+1$ indent nodes for β. Moreover, these indent nodes all have the same residue, say ω. Let A be the removable node for β defined from the equality

$$\beta = \beta(m, f; \nu) \sqcup \{A\}.$$

Assume $\mathrm{res}\, A \neq \omega$. Then A is normal for β. Therefore, setting $\beta' = \beta(m, f; \nu)$, $n' = |\beta'|$, we have

$$D^{\beta'} \in \mathrm{Irr}(D^\beta{\downarrow}\Sigma_{n'}),$$

in view of Theorem 3.5. Moreover, Lemma 4.10 implies $D^\gamma \in \mathrm{Irr}(D^{\beta'}{\downarrow}\Sigma_n)$, and we are done.

Now let $\mathrm{res}\, A = \omega$. Put $\beta'' = \beta(m, f-1, \mu)$, $n'' = |\beta''|$. (Actually $n'' = n+1$.) By Lemma 4.10, $D^{\beta''} \in \mathrm{Irr}(D^{\beta}{\downarrow}\Sigma_{n''})$. Moreover, the removable nodes for β'' in the first m rows all have residue $\omega - 1$. So A is normal for β''. Since $\gamma = \beta''_A$, we get from Theorem 3.5

$$D^\gamma \in \mathrm{Irr}(D^{\beta''}{\downarrow}\Sigma_n),$$

and the result follows. □

Lemma 4.12. *Let μ be a p-regular partition and f be an integer with $f > |\mu| + 1$. Set $\beta = \beta(m, f; \mu)$, $e = f - |\mu|$, $\gamma = \beta(m, e)$, $n = |\gamma|$. Then*

$$D^\gamma \in \mathrm{Irr}(D^{\beta}{\downarrow}\Sigma_n).$$

Proof. Apply Lemma 4.11 sufficiently many times. □

The following result is a generalization of Lemma 4.7.

Lemma 4.13. *Let*

$$\lambda = (l_1, l_2, \dots, l_m, l_{m+1}, \dots, l_s)$$

be a p-regular partition, with $l_m > l_{m+1}$. Set

$$\begin{aligned} \mu &= (l_{m+1}, \dots, l_s) \\ \alpha &= \alpha(m, l_m; \mu), \quad n = |\alpha| \end{aligned}$$

($\mu = \emptyset$ if $m = s$). Then

$$D^\alpha \in \mathrm{Irr}(D^{\lambda}{\downarrow}\Sigma_n).$$

Proof. We choose the maximal $i \geq 0$ such that $\varphi^i(\lambda)$ has the form

$$\varphi^i(\lambda) = (k_1, \dots, k_{m-1}, l_m, \dots, l_s),$$

see Definition 4.5. This means that the node $A(\varphi^i(\lambda))$ lies in the m-th row or below. Since $l_m \neq l_{m+1}$, it follows from Lemma 4.3 that $(k_1, \dots, k_{m-1}, l_m) = \alpha(m, l_m)$. Therefore $\varphi^i(\lambda) = \alpha$, and the result follows from Lemma 4.6. □

Proposition 4.14. *Let $\lambda = (l_1, l_2, \dots, l_s)$ be a p-regular partition, $m \in \{1, 2, \dots, s\}$, and*

$$l_m > 2 + \sum_{i=m+1}^{s} l_i.$$

Set $f = l_m - 1 - \sum_{i=m+1}^{s} l_i$, $\beta = \beta(m, f)$, $n = |\beta|$. Then

$$D^\beta \in \mathrm{Irr}(D^{\lambda}{\downarrow}\Sigma_n).$$

Proof. Set

$$\begin{aligned}\mu &= (l_{m+1},\dots,l_s),\\ \alpha &= \alpha(m,l_m;\mu),\\ n_1 &= |\alpha|.\end{aligned}$$

By Lemma 4.13,

$$D^\alpha \in \mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n_1}).$$

Let

$$\beta_1 = \beta(m,l_m-1;\mu), \quad n_2 = |\beta_1|.$$

Then

$$D^{\beta_1} \in \mathrm{Irr}(D^\alpha{\downarrow}\Sigma_{n_2})$$

by Lemma 4.9. Now apply Lemma 4.12 with $\beta = \beta_1$ and $\gamma = \beta$. □

Lemma 4.15. *Let $\lambda \in \mathcal{P}_n$ and*

$$\mathrm{cont}(\lambda) = (c_0, c_1, \dots, c_{p-1}).$$

Assume that $D^\mu \in \mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n-1})$ and

$$\mathrm{cont}(\mu) = (d_0, d_1, \dots, d_{p-1}).$$

Then there exists $j \in \{0,1,\dots,p-1\}$ such that $d_j = c_j - 1$ and $d_i = c_i$ for $i \neq j$.

Proof. By Theorem 3.3, the restriction $S^\lambda{\downarrow}\Sigma_{n-1}$ has a filtration with factors S^{λ_A} where A is a removable node for λ. Note that

$$\mathrm{cont}(\lambda_A) = (c_0, \dots, c_j - 1, \dots, c_{p-1})$$

if $\operatorname{res} A = j$. By 3.1, $D^\lambda \in \mathrm{Irr}(S^\lambda)$ so

$$\mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n-1}) \subseteq \mathrm{Irr}(S^\lambda{\downarrow}\Sigma_{n-1}) = \cup_A \mathrm{Irr}(S^{\lambda_A}).$$

Now it remains to use Lemma 3.2. □

Remark. Actually, much more is known about $\mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n-1})$. For example, the contents of the factors $\mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n-1})$ are described in [14]. See also [16] for more information. One can use the argument from the proof of 4.15 to show that $D^\mu \in \mathrm{Irr}(D^\lambda{\downarrow}\Sigma_{n-1})$ implies μ dominates some λ_A.

The rest of this section is devoted to certain results concerning the Mullineux map described at the end of Section 3.

Lemma 4.16. *Let $e, m \in \mathbb{N}$, $\beta = \beta(m, pe)$. Represent m in the form*

$$m = (p-1)d + r, \quad d \in \mathbb{Z}, \quad 0 < r \leq p-1.$$

Let $M(\beta) = \gamma = (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s > 0)$. Then $s = p + d - r$ and $\gamma_s \geq re$.

Proof. One can easily see that the Mullineux symbol of β has the form

$$G(\beta) = \left(\underbrace{\begin{matrix} (d+1)p & (d+1)p & \dots & (d+1)p \\ d(p-1)+r & d(p-1)+r & & d(p-1)+r \end{matrix}}_{re \text{ times}} \quad \text{the rest} \right).$$

So, by Theorem 3.7, the Mullineux symbol $G(\gamma)$ has the form

$$G(\gamma) = \left(\underbrace{\begin{matrix} (d+1)p & (d+1)p & \dots & (d+1)p \\ p+d-r & p+d-r & & p+d-r \end{matrix}}_{re \text{ times}} \quad \text{the rest} \right).$$

It follows from the Mullineux algorithm that $h(\gamma) = p+d-r$. Moreover, the p-edge of a partition has nodes in every row. In particular, the p-edge of a partition with s rows has at least one node in the sth row. It follows from the form of $G(\gamma)$ found above that the partitions $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(re)}$ (defined at the end of Section 3) all have s rows. Hence $\gamma_s \geq re$. $\square$

Corollary 4.17. *Let $m, e \in \mathbb{N}$, $\beta = \beta(m, p(p-1)e)$, $\gamma = M(\beta)$. Represent m in the form*

$$m = (p-1)d + r, \quad d \in \mathbb{Z}, \quad 0 < r \leq p-1.$$

Define $m_1 = p+d-r$. Assume that $f \in \mathbb{Z}$ satisfies $0 < f < re$ and set $\delta = \beta(m_1, f)$, $n = |\delta|$. Then

$$D^\delta \in \mathrm{Irr}(D^\gamma{\downarrow}\Sigma_n).$$

Proof. Follows immediately from Lemma 4.16, Proposition 4.14 (with $m = s = p+d-r$), and Lemma 4.10 (with $\mu = \emptyset$, $m = p+d-r$). $\square$

Lemma 4.18. *Let $e \in \mathbb{N}$, $\beta = \beta(p, p(p-1)e)$, $\gamma = M(\beta)$. Then*

$$\gamma = ((p-1) + pe(p-1)^2, (pe)^{p-1}).$$

Proof. One can easily see that the Mullineux symbol of β is

$$G(\beta) = \left(\underbrace{\begin{matrix} 2p & \dots & 2p \\ p & & p \end{matrix}}_{(p-1)e \text{ times}} \quad \underbrace{\begin{matrix} p & \dots & p \\ p-1 & & p-1 \end{matrix}}_{(p-1)(p-2)e \text{ times}} \quad \begin{matrix} p-1 \\ p-1 \end{matrix} \right).$$

So, by Theorem 3.7, the Mullineux symbol of γ is

$$G(\gamma) = \left(\underbrace{\begin{matrix} 2p & \dots & 2p \\ p & & p \end{matrix}}_{(p-1)e \text{ times}} \quad \underbrace{\begin{matrix} p & \dots & p \\ 1 & & 1 \end{matrix}}_{(p-1)(p-2)e \text{ times}} \quad \begin{matrix} p-1 \\ 1 \end{matrix} \right),$$

and the lemma follows. $\square$

Corollary 4.19. *Let* $p > 2$, $n = (p-2)+pe(p-1)(p-2)$, $e \in \mathbb{N}$, $\beta = \beta(p, p(p-1)e)$, $\gamma = M(\beta)$. *Then*

$$D^{(n)} \in \mathrm{Irr}(D^\gamma\downarrow\Sigma_n).$$

Proof. By Lemma 4.18, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ where $\gamma_1 = (p-1) + pe(p-1)^2$, $\gamma_2 = \gamma_3 = \cdots = \gamma_p = pe$. Observe that $\gamma_1 - (\gamma_2 + \cdots + \gamma_p) = (p-1) + pe(p-1)(p-2) > 2$. Moreover, $(n) = \beta(1, \gamma_1 - (\gamma_2 + \cdots + \gamma_p) - 1)$. Now, by Proposition 4.14 (with $m = 1$, $\lambda = \gamma$) we have the desired result. □

5. Inductive systems for the finitary symmetric group

Now we apply the general notions of Section 2 to the case $G = \Sigma_\infty = \cup_{i=1}^\infty \Sigma_i$.

Definition 5.1. Let $\Phi = \{\Phi_i\}_{i \in \mathbb{N}}$ be an inductive system for Σ_∞. We define its height $h(\Phi)$ as

$$h(\Phi) = \sup\{h(\lambda) \mid D^\lambda \in \Phi_i \quad \text{for some } i \in \mathbb{N}\}$$

(see (1)).

Proposition 5.2. *Let* $\Phi = \{\Phi_i\}_{i \in \mathbb{N}}$ *be an inductive system for* Σ_∞. *Assume that* $h(\Phi) = +\infty$. *Then* $\Phi_i = \mathrm{Irr}\ \Sigma_i$ *for all* $i \in \mathbb{N}$.

Proof. By assumption, for any $i \in \mathbb{N}$ there exists $j > i$ and $D^\lambda \in \Phi_j$ with $h(\lambda) \geq i(p-1)$. Then

$$\Phi_i \supseteq \mathrm{Irr}(D^\lambda\downarrow\Sigma_i) = \mathrm{Irr}\ \Sigma_i,$$

by Proposition 4.8. □

In [15] the second author has described all semisimple inductive systems for Σ_∞ (see Definition 2.7). Surprisingly, it will turn out that the minimal inductive systems (i.e. those which correspond to the maximal ideals of $F\Sigma_\infty$) are exactly indecomposable semisimple inductive systems. We formulate some results from [15].

Recall the constant $\chi(\lambda)$ defined in (2). For $s = 1, 2, \ldots, p-1$ put

$$\begin{aligned}\Theta_i^s &= \{D^\mu \mid |\mu| = i, \quad h(\mu) = s, \quad \chi(\mu) \leq p\},\\ \Omega_i^s &= \{D^\mu \mid \mu = (\mu_1, \mu_2, \ldots), \quad |\mu| = i, \quad h(\mu) < s, \quad \mu_1 \leq p - s\}.\end{aligned}$$

Note that $\Theta_i^s \cap \Omega_i^s = \emptyset$ and $\Omega_i^s = \emptyset$ for $i > (p-s)(s-1)$. Set

$$\Phi(s)_i = \Theta_i^s \sqcup \Omega_i^s. \tag{6}$$

Theorem 5.3. [15]

(i) $\Phi(s) = \{\Phi(s)_i\}_{i \in \mathbb{N}}$, $s = 1, 2, \ldots, p-1$, *are all distinct semisimple minimal inductive systems for* Σ_∞.

(ii) *Let* $i \in \mathbb{N}$. *If* $j \gg i$ *then for any* $D^\lambda \in \Phi(s)_j$ *we have*

$$\Phi(s)_i = \mathrm{Irr}(D^\lambda\downarrow\Sigma_i).$$

Proof. (ii) is proved in [15, 2.6, 2.7], (i) is an immediate consequence of (ii) and [15, 2.8]. □

An important specific feature of the representation theory of symmetric groups is the presence of the sign module (denoted by sgn). Tensoring with sgn is often an effective trick.

Definition 5.4. Let $\Phi = \{\Phi_i\}_{i\in\mathbb{N}}$ be an inductive system for Σ_∞. Define the inductive system $\Phi^\sigma = \{\Phi_i^\sigma\}_{i\in\mathbb{N}}$ by

$$\Phi_i^\sigma = \{D^\lambda \otimes \mathrm{sgn} \mid D^\lambda \in \Phi_i\},$$

where sgn is the sign representation of Σ_i.

Φ^σ is well defined in view of the following lemma.

Lemma 5.5. *Let $\Phi = \{\Phi_i\}_{i\in\mathbb{N}}$ be an inductive system. Then*

(i) $\Phi^\sigma = \{\Phi_i^\sigma\}_{i\in\mathbb{N}}$ *is an inductive system;*
(ii) Φ *is minimal if and only if* Φ^σ *is so;*
(iii) Φ *is semisimple if and only if* Φ^σ *is so.*

Proof. (i) and (iii) follow from the isomorphism

$$(D \otimes \mathrm{sgn}){\downarrow}\Sigma_{n-1} \cong (D{\downarrow}\Sigma_{n-1}) \otimes \mathrm{sgn} \tag{7}$$

where D is an $F\Sigma_n$-module and sgn in the left (resp., right) hand side means the sign representation of Σ_n (resp., Σ_{n-1}). The second part follows from the equality $(\Phi^\sigma)^\sigma = \Phi$. □

Lemma 5.6. *Let $s \in \{1, 2, \dots, p-1\}$. Then $\Phi(s)^\sigma = \Phi(p-s)$.*

Proof. By Lemma 5.5 (ii), (iii), Φ is minimal and semisimple if and only if Φ^σ is so. So Theorem 5.3 (i) implies $\Phi(s)^\sigma = \Phi(t)$ for some $t \in \{1, 2, \dots, p-1\}$. Note that $D^{((pe)^s)} \in \Phi(s)_{pes}$ for any $e \in \mathbb{N}$. However $h(M((pe)^s)) = p-s$ by Lemma 4.16, which implies the demanded result. □

Theorem 5.7. *Let $p > 2$, and Φ be an inductive system for Σ_∞. Then Φ is minimal if and only if $\Phi = \Phi(s)$ for some $s \in \{1, 2, \dots, p-1\}$.*

Proof. The "if-part" follows from Theorem 5.3 (i).

Let Φ be minimal. If $h(\Phi) = +\infty$ then, by Proposition 5.2,

$$\Phi_i = \mathrm{Irr}\ \Sigma_i, \quad i \in \mathbb{N}.$$

However this system is not minimal (for example because it contains the trivial inductive system $\Phi(1)$). So $h(\Phi) = h < \infty$. It follows that for any $N \in \mathbb{N}$ there exists $i \in \mathbb{N}$ and $D^\lambda \in \Phi_i$ with $\lambda = (l_1 \geq l_2 \geq \dots \geq l_h \geq 0)$ such that $l_1 > N$. This shows that the set

$$V(\Phi) = \{v \in \mathbb{N} \mid \quad \text{for any } N \in \mathbb{N} \text{ there exists } i \in \mathbb{N} \text{ and } D^\lambda \in \Phi_i \text{ with } \lambda = (l_1 \geq l_2 \geq \dots \geq l_h \geq 0) \text{ such that } l_v > N\}$$

is not empty. Since also $V(\Phi) \subseteq \{1, 2, \dots, h\}$ we can define

$$m = m(\Phi) = \max V(\Phi).$$

It follows from the definition of m that there exists $c \in \mathbb{N}$ such that

$$\sum_{j=m+1}^{h} l_j \leq c$$

for any $i \in \mathbb{N}$ and any $D^{(l_1,\dots,l_h)} \in \Phi_i$. Let $N_0 \geq c+2$. By definition of m, there exists $i \in \mathbb{N}$ and $D^{(l_1,\dots,l_h)} \in \Phi_i$ such that $l_m > N_0$. Set

$$\begin{aligned} f &= l_m - \sum_{j=m+1}^{h} l_j - 1, \\ \beta &= \beta(m, f), \quad n = |\beta|. \end{aligned}$$

By Proposition 4.14,

$$D^\beta \in \operatorname{Irr}(D^{(l_1,\dots,l_h)}\downarrow\Sigma_n) \subseteq \Phi_n.$$

Note that $f > N_0 - c - 1$. Set

$$\begin{aligned} f' &= N_0 - c - 1, \\ \beta' &= \beta(m, f'), \quad n' = |\beta'|. \end{aligned}$$

By Lemma 4.10,

$$D^{\beta'} \in \operatorname{Irr}(D^\beta\downarrow\Sigma_{n'}) \subseteq \Phi_{n'}.$$

Choosing $N_0 = c+2, c+3, \dots$ (then $f' = 1, 2, \dots$) we conclude that

$$D^{\beta(m,e)} \in \Phi_{|\beta(m,e)|} \quad \text{for any } e \in \mathbb{N}. \tag{8}$$

We call a number $t \in \mathbb{N}$ *admissible* for Φ if for any $N \in \mathbb{N}$ there is $f > N$ such that $D^{\beta(t,f)}$ belongs to $\Phi_{|\beta(t,f)|}$. By Lemma 4.10, we have

$$\begin{gathered} t \in \mathbb{N} \text{ is admissible for } \Phi \text{ if and only if} \\ D^{\beta(t,f)} \in \Phi_{|\beta(t,f)|} \text{ for all } f \in \mathbb{N}. \end{gathered} \tag{9}$$

By (8), $m(\Phi)$ is admissible for Φ. This allows us to define a constant $s(\Phi)$ as follows

$$s(\Phi) = \min\{t \in \mathbb{N} \mid t \text{ is admissible for } \Phi\}.$$

Let us consider the inductive system Φ^σ (see Definition 5.4). By Lemma 5.5 (ii), Φ^σ is minimal. So $s(\Phi^\sigma)$ is also well defined. We prove the following intermediate fact.

(*) Assume that $s(\Phi) \leq s(\Phi^\sigma)$. Then $\Phi = \Phi(t)$ for some $t \in \{1, 2, \dots, p-1\}$.

Note that (*) immediately implies our theorem. Indeed, if $s(\Phi) \leq s(\Phi^\sigma)$ then (*) is just the theorem's claim. Let $s(\Phi) > s(\Phi^\sigma)$. Since $(\Phi^\sigma)^\sigma = \Phi$, (*) implies $\Phi^\sigma = \Phi(t)$ for some $t \in \{1, 2, \dots, p-1\}$. Then $\Phi = \Phi(p-t)$ by Lemma 5.6.

Proof of (*). Set $s = s(\Phi)$. We consider three cases.
(1) $s \leq p-1$. Then $\beta(s,f) = (f^s)$, and so $\chi(\beta(s,f)) = s < p$. Hence $D^{\beta(s,f)} \in \Phi(s)_{|\beta(s,f)|}$, see (6). By (9), the modules $D^{\beta(s,f)}$, $f \in \mathbb{N}$, belong to the corresponding

$\Phi_{|\beta(s,f)|}$. It now follows from Theorem 5.3 (ii) that $\Phi(s) \subseteq \Phi$. Since Φ is minimal, we have $\Phi = \Phi(s)$.
(2) $s = p$. Set $\beta(e) = \beta(p, p(p-1)e)$. By (9),

$$D^{\beta(e)} \in \Phi_{|\beta(e)|}$$

for any $e \in \mathbb{N}$. Set $n(e) = (p-2) + ep(p-1)(p-2)$. Then, by Corollary 4.19,

$$D^{(n(e))} \in \Phi^{\sigma}_{n(e)}$$

Note that $(n) = \beta(n, 1)$. So $n(e) \to +\infty$ as $e \to +\infty$ implies that 1 is admissible for Φ^{σ}. This contradicts the assumption $s(\Phi^{\sigma}) \geq s(\Phi)$.
(3) $s > p$. Represent s in the form

$$s = (p-1)d + r, \quad d \in \mathbb{Z}, \quad 0 < r \leq p-1.$$

By (9),

$$D^{\beta(s,p(p-1)e)} \in \Phi_{|\beta(s,p(p-1)e)|} \quad \text{for any } e \in \mathbb{N}.$$

By Corollary 4.17,

$$D^{\beta(p+d-r,f)} \in \Phi^{\sigma}_{|\beta(p+d-r,f)|}$$

for any $f < re$. This shows that $p + d - r$ is admissible for Φ^{σ}, whence $s(\Phi^{\sigma}) \leq p + d - r$. However, this contradicts the assumption $s(\Phi^{\sigma}) \geq s(\Phi)$ because

$$p + d - r < (p-1)d + r = s$$

when $s > p$. The contradiction obtained completes the proof of (*) and the theorem. $\square$

Corollary 5.8. *Let $p > 2$. Set*

$$I(s)_n = \bigcap_{D \in \Phi(s)_n} \mathrm{Ann}_{F\Sigma_n}(D), \quad s = 1, \dots, p-1, \quad n \in \mathbb{N}.$$

Then $I(s)_1 \subset I(s)_2 \subset \dots$. Moreover,

$$I(s) = \cup_{n=1}^{\infty} I(s)_n, \quad s = 1, \dots, p-1,$$

are exactly all distinct maximal ideals of $F\Sigma_\infty$, and $I(s) \cap F\Sigma_n = I(s)_n$.

Proof. In view of Proposition 2.6 (ii) and Theorem 5.7, the maximal ideals of $F\Sigma_\infty$ are exactly $I(\Phi(s))$, $s = 1, 2, \dots, p-1$. However, the inductive systems $\Phi(s)$ are semisimple. So the result follows from 2.8. $\square$

6. Inductive systems for the finitary alternating group

Throughout the section we assume that $p > 2$.

We first briefly discuss how to label irreducible FA_n-modules. It follows easily from the Clifford theory [3, 49.2] (see [4]) that the restriction $D^\lambda{\downarrow}^{\Sigma_n}_{A_n}$ is either irreducible or splits into a direct sum of two irreducibles. In the former case we write

$$D^\lambda{\downarrow}^{\Sigma_n}_{A_n} = E^\lambda,$$

and in the latter case we write

$$D^\lambda{\downarrow}^{\Sigma_n}_{A_n} = E^\lambda_+ \oplus E^\lambda_-.$$

It is of importance that $D^\lambda{\downarrow}^{\Sigma_n}_{A_n}$ is irreducible provided $\lambda \neq M(\lambda)$. (In fact, if F is a splitting field for A_n then $D^\lambda{\downarrow}^{\Sigma_n}_{A_n}$ is irreducible if and only if $\lambda \neq M(\lambda)$. However we want to work over an arbitrary field). Every irreducible FA_n-module is of the form E^λ or $E^\mu_\pm$. Furthermore, $E^\lambda_+ \not\cong E^\lambda_-$, $E^\lambda_\pm \not\cong E^\mu_\pm$, $E^\lambda \not\cong E^\mu_\pm$, and $E^\lambda \cong E^\mu$ if and only if $\lambda = M(\mu)$ (for all permissible $\lambda \neq \mu$).

The following result was first proved in [17].

Lemma 6.1. [17] *Let $\lambda \in \mathcal{P}_n$, and*

$$\mathrm{cont}(\lambda) = (c_0, c_1, \dots, c_{p-1}).$$

Then

$$\mathrm{cont}(M(\lambda)) = (d_0, d_1, \dots, d_{p-1}).$$

where $d_0 = c_0$ and $d_i = c_{p-i}$ for $i = 1, 2, \dots, p-1$.

Lemma 6.2. *Let $p > 2$. If λ is a Young diagram with cont$(\lambda) = (c_0, c_1, \dots, c_{p-1})$ then $c_0 \leq c_{p-1} + 1$.*

Proof. Let $A = (i, j)$ be an arbitrary node. Set

$$f(A) = \begin{cases} (i, j-1), & \text{if } j > 1; \\ (i+1, j), & \text{if } j = 1. \end{cases}$$

If $\operatorname{res} A = 0$ then $\operatorname{res} f(A) = p - 1$. Also, if $A \in \lambda$ but $f(A) \notin \lambda$ then $A = (m, 1)$ where $m = h(\lambda)$. Moreover, $A \neq B$ and $\operatorname{res} A = \operatorname{res} B$ imply $f(A) \neq f(B)$ since $p > 2$. Thus, to all nodes of λ of residue 0, except possibly one, we associate different nodes of λ of residue $p - 1$. □

Lemma 6.3. *Let $\lambda \in \mathcal{P}_n$, $\nu \in \mathcal{P}_{n+1}$ be such that*

$$D^\lambda{\downarrow}^{\Sigma_n}_{A_n} = E^\lambda_+ \oplus E^\lambda_- \quad \textit{and} \quad D^\nu{\downarrow}^{\Sigma_{n+1}}_{A_{n+1}} = E^\nu_+ \oplus E^\nu_-.$$

Assume that $E^\lambda_+ \in \mathrm{Irr}(E^\nu_+ {\downarrow}^{A_{n+1}}_{A_n})$ or $E^\lambda_+ \in \mathrm{Irr}(E^\nu_- {\downarrow}^{A_{n+1}}_{A_n})$. Then $D^\lambda \in \mathrm{Irr}(D^\nu{\downarrow}^{\Sigma_{n+1}}_{\Sigma_n})$.

Proof. If $D^\lambda \notin \mathrm{Irr}(D^\nu{\downarrow}^{\Sigma_{n+1}}_{\Sigma_n})$ then

$$E^\lambda_+ \notin \mathrm{Irr}((D^\nu{\downarrow}^{\Sigma_{n+1}}_{\Sigma_n}){\downarrow}^{\Sigma_n}_{A_n}) = \mathrm{Irr}((D^\nu{\downarrow}^{\Sigma_{n+1}}_{A_{n+1}}){\downarrow}^{A_{n+1}}_{A_n}) = \mathrm{Irr}(E^\nu_+{\downarrow}^{A_{n+1}}_{A_n}) \cup \mathrm{Irr}(E^\nu_-{\downarrow}^{A_{n+1}}_{A_n}),$$

giving a contradiction. □

Lemma 6.4. *Let $p > 2$, and $\lambda \in \mathcal{P}_n$ be such that $D^\lambda{\downarrow}^{\Sigma_n}_{A_n} = E^\lambda_+ \oplus E^\lambda_-$. Assume $\Phi = \{\Phi_i\}_{i\in\mathbb{N}}$ is an inductive system for A_∞. Then $E^\lambda_+ \in \Phi_n$ if and only if $E^\lambda_- \in \Phi_n$.*

Proof. Assume $E^\lambda_+ \in \Phi_n$ and show that $E^\lambda_- \in \Phi_n$ (the implication $E^\lambda_- \in \Phi_n \Rightarrow E^\lambda_+ \in \Phi_n$ is obtained similarly). Note that $M(\lambda) = \lambda$. We shall prove the following intermediate fact.

(**) There is $k > n$ and $E^\mu \in \Phi_k$ such that $E^\lambda_+ \in \mathrm{Irr}(E^\mu{\downarrow}^{A_k}_{A_n})$.

This will imply the lemma. Indeed, $E^\mu = D^\mu{\downarrow}^{\Sigma_k}_{A_k}$. Therefore

$$E^\mu{\downarrow}^{A_k}_{A_n} \cong (D^\mu{\downarrow}^{\Sigma_k}_{\Sigma_n}){\downarrow}^{\Sigma_n}_{A_n} \tag{10}$$

So $D^\lambda \in \mathrm{Irr}(D^\mu{\downarrow}^{\Sigma_k}_{\Sigma_n})$ because otherwise we would have $E^\lambda_+ \notin \mathrm{Irr}(E^\mu{\downarrow}^{A_k}_{A_n})$. It now follows from (10) that $E^\lambda_- \in \mathrm{Irr}(E^\mu{\downarrow}^{A_k}_{A_n})$.

We now prove (**). Let $\mathrm{cont}(\lambda) = (c_0, c_1, \dots, c_{p-1})$. Since Φ is an inductive system there exists an irreducible module $E \in \Phi_{n+1}$ such that $E^\lambda_+ \in \mathrm{Irr}(E{\downarrow}^{A_{n+1}}_{A_n})$. If $E = E^\nu$ we can take $\mu = \nu$, $k = n+1$. Otherwise $E = E^\nu_+$ or E^ν_- for some ν with $M(\nu) = \nu$. In view of Lemma 6.3, we have

$$D^\lambda \in \mathrm{Irr}(D^\nu{\downarrow}^{\Sigma_{n+1}}_{\Sigma_n}).$$

So by Lemma 4.15,

$$\mathrm{cont}(\nu) = (c_0, \dots, c_j + 1, \dots, c_{p-1})$$

for some $j \in \{0, 1, \dots, p-1\}$. Since $M(\lambda) = \lambda$ and $M(\nu) = \nu$, Lemma 6.1 forces $j = 0$. Thus,

$$\mathrm{cont}(\nu) = (c_0 + 1, c_1, \dots, c_{p-1}).$$

Repeating this argument i times we either arrive to a module $E^\mu \in \Phi_{n+i}$ such that $E^\lambda_+ \in \mathrm{Irr}(E^\mu{\downarrow}^{A_{n+i}}_{A_n})$ or we get a partition ν with

$$\mathrm{cont}(\nu) = (c_0 + i, c_1, \dots, c_{p-1}).$$

However, Lemma 6.2 shows that i can not be too large. $\square$

Lemma 6.5. *Let $\Phi = \{\Phi_i\}_{i\in\mathbb{N}}$ be an inductive system for Σ_∞. Set*

$$\bar\Phi_i = \mathrm{Irr}(\Phi_i{\downarrow}^{\Sigma_i}_{A_i}).$$

Then $\bar\Phi = \{\bar\Phi_i\}_{i\in\mathbb{N}}$ is an inductive system for A_∞.

Proof. Note that

$$\begin{aligned}\mathrm{Irr}(\bar\Phi_i{\downarrow}^{A_i}_{A_{i-1}}) &= \mathrm{Irr}(\mathrm{Irr}(\Phi_i{\downarrow}^{\Sigma_i}_{A_i}){\downarrow}^{A_i}_{A_{i-1}}) = \mathrm{Irr}(\Phi_i{\downarrow}^{\Sigma_i}_{A_{i-1}})\\ &= \mathrm{Irr}(\mathrm{Irr}(\Phi_i{\downarrow}^{\Sigma_i}_{\Sigma_{i-1}}){\downarrow}^{\Sigma_{i-1}}_{A_{i-1}}) = \mathrm{Irr}(\Phi_{i-1}{\downarrow}^{\Sigma_{i-1}}_{A_{i-1}}) = \bar\Phi_{i-1},\end{aligned}$$

as demanded. $\square$

Lemma 6.5 allows us to give the following definition.

Definition 6.6. Let Φ be an inductive system for Σ_∞. Define the inductive system $\Phi{\downarrow}A_\infty$ for A_∞ as $\Phi{\downarrow}A_\infty = \bar\Phi$ (see Lemma 6.5).

Definition 6.7. Let $S \subseteq \mathrm{Irr}(\Sigma_n)$. Define the set $S^\sigma \subseteq \mathrm{Irr}(\Sigma_n)$ as

$$S^\sigma = \{D \otimes \mathrm{sgn} \mid D \in S\}.$$

Lemma 6.8. *Let $S, T \subseteq \mathrm{Irr}(\Sigma_n)$, $S^\sigma = S$, $T^\sigma = T$. If $\mathrm{Irr}(S\downarrow^{\Sigma_n}_{A_n}) = \mathrm{Irr}(T\downarrow^{\Sigma_n}_{A_n})$ then $S = T$.*

Proof. Assume $D^\lambda \in S$ but $D^\lambda \notin T$. Then $D^{M(\lambda)} \notin T$. Hence $E \in \mathrm{Irr}(S\downarrow^{\Sigma_n}_{A_n})$ but $E \notin \mathrm{Irr}(T\downarrow^{\Sigma_n}_{A_n})$ where $E = E^\lambda$ or $E = E^\lambda_+$. The contradiction obtained shows that $S \subseteq T$. Similarly, $T \subseteq S$. □

Lemma 6.9. *Let $S \subseteq \mathrm{Irr}(\Sigma_n)$. If $S^\sigma = S$ then $(\mathrm{Irr}(S\downarrow\Sigma_{n-1}))^\sigma = \mathrm{Irr}(S\downarrow\Sigma_{n-1})$.*

Proof. Follows from (7). □

Theorem 6.10. *Let $p > 2$. The map $\Phi \to \Phi\downarrow A_\infty$ defines an isomorphism between the poset of the inductive systems Φ for Σ_∞ such that $\Phi^\sigma = \Phi$ and the poset of the inductive systems for A_∞.*

Proof. The map is injective by Lemma 6.8. We construct a right inverse map. Let $\Psi = \{\Psi_i\}_{i\in\mathbb{N}}$ be an inductive system for A_∞. Set

$$\hat{\Psi}_i = \{D^\lambda \in \mathrm{Irr}\ \Sigma_i \mid \mathrm{Irr}(D^\lambda\downarrow^{\Sigma_i}_{A_i}) \subseteq \Psi_i\}, \quad i \in \mathbb{N}.$$

We first show that

$$\Psi_i = \mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{A_i}), \quad i \in \mathbb{N} \tag{11}$$

Indeed, $\Psi_i \supseteq \mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{A_i})$ by definition of $\hat{\Psi}_i$. On the other hand, let $E \in \Psi_i$. There are two cases:

(1) $E = E^\lambda$;
(2) $E = E^\lambda_+$ or $E = E^\lambda_-$.

In the case (1), $E = D^\lambda\downarrow^{\Sigma_i}_{A_i}$ hence $D^\lambda \in \hat{\Psi}_i$ and $E \in \mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{A_i})$. In the case (2), by Lemma 6.4, both E^λ_+ and E^λ_- belong to Ψ. So $D^\lambda\downarrow^{\Sigma_i}_{A_i} = E^\lambda_+ \oplus E^\lambda_-$ implies that $D^\lambda \in \hat{\Psi}_i$ hence $E \in \mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{A_i})$.

It follows from the definition of $\hat{\Psi}_j$ that

$$(\hat{\Psi}_j)^\sigma = \hat{\Psi}_j, \quad j \in \mathbb{N}. \tag{12}$$

Taking $j = i - 1$ gives

$$(\hat{\Psi}_{i-1})^\sigma = \hat{\Psi}_{i-1}. \tag{13}$$

Taking $j = i$ and using Lemma 6.9, we get

$$(\mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{\Sigma_{i-1}}))^\sigma = \mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{\Sigma_{i-1}}). \tag{14}$$

Now we show that

$$\mathrm{Irr}(\mathrm{Irr}(\hat{\Psi}_i\downarrow^{\Sigma_i}_{\Sigma_{i-1}})\downarrow^{\Sigma_{i-1}}_{A_{i-1}}) = \mathrm{Irr}(\hat{\Psi}_{i-1}\downarrow^{\Sigma_{i-1}}_{A_{i-1}}). \tag{15}$$

Indeed, using (11) twice, we obtain

$$\begin{aligned}\mathrm{Irr}(\mathrm{Irr}(\hat{\Psi}_i{\downarrow}^{\Sigma_i}_{\Sigma_{i-1}}){\downarrow}^{\Sigma_{i-1}}_{A_{i-1}}) &= \mathrm{Irr}(\hat{\Psi}_i{\downarrow}^{\Sigma_i}_{A_{i-1}}) = \mathrm{Irr}(\mathrm{Irr}(\hat{\Psi}_i{\downarrow}^{\Sigma_i}_{A_i}){\downarrow}^{A_i}_{A_{i-1}}) \\ &= \mathrm{Irr}(\Psi_i{\downarrow}^{A_i}_{A_{i-1}}) = \Psi_{i-1} = \mathrm{Irr}(\hat{\Psi}_{i-1}{\downarrow}^{\Sigma_{i-1}}_{A_{i-1}}).\end{aligned}$$

In view of Lemma 6.8, equations (13), (14) and (15) imply

$$\mathrm{Irr}(\hat{\Psi}_i{\downarrow}^{\Sigma_i}_{\Sigma_{i-1}}) = \hat{\Psi}_{i-1},$$

i.e. $\hat{\Psi}$ is an inductive system. Moreover, (11) shows that $\hat{\Psi}{\downarrow}A_\infty = \Psi$. It remains to note that $\Psi \subseteq \Phi$ if and only if $\hat{\Psi} \subseteq \hat{\Phi}$. □

Theorem 6.11. *Let $p > 2$. The inductive systems*

$$\Phi(s){\downarrow}A_\infty, \quad s = 1, 2, \dots, \frac{p-1}{2},$$

are exactly all distinct minimal inductive systems for A_∞.

Proof. Let $\mathcal{S}$ be the set of all inductive systems Φ for Σ_∞ which satisfy the following two conditions:

(1) $\Phi^\sigma = \Phi$.

(2) If Ψ is an inductive system for Σ_∞, $\Psi \subseteq \Phi$, and $\Psi^\sigma = \Psi$ then $\Psi = \Phi$.

By Theorem 6.10,

$$\{\Phi{\downarrow}A_\infty \mid \Phi \in \mathcal{S}\}$$

is the set of all distinct minimal inductive systems for A_∞.

We claim that $\Phi \in \mathcal{S}$ if and only if $\Phi = \Phi(s) \cup \Phi(s)^\sigma$ for some $s \in \{1, 2, \dots, p-1\}$ (see (6)). Indeed, let $\Phi \in \mathcal{S}$. By Theorem 5.7, $\{\Phi(s) \mid 1 \le s \le p-1\}$ are exactly all minimal inductive systems for Σ_∞. So Φ contains some $\Phi(s)$. Since $\Phi^\sigma = \Phi$, we also have $\Phi \supseteq \Phi(s)^\sigma$. Thus $\Phi(s) \cup \Phi(s)^\sigma \subseteq \Phi$. Since $(\Phi(s) \cup \Phi(s)^\sigma)^\sigma = \Phi(s) \cup \Phi(s)^\sigma$ and $\Phi \in \mathcal{S}$, we conclude that $\Phi = \Phi(s) \cup \Phi(s)^\sigma$.

On the other hand, if $\Phi(s) \cup \Phi(s)^\sigma \notin \mathcal{S}$ then there exists an inductive system Φ with $\Phi^\sigma = \Phi$ such that $\Phi \underset{\neq}{\subset} \Phi(s) \cup \Phi(s)^\sigma$. As above, we find $t \in \{1, 2, \dots, p-1\}$ such that $\Phi(t) \cup \Phi(t)^\sigma \subseteq \Phi$. Therefore,

$$\Phi(t) \cup \Phi(t)^\sigma \underset{\neq}{\subset} \Phi(s) \cup \Phi(s)^\sigma$$

for some $s, t \in \{1, 2, \dots, p-1\}$, which is false in view of Lemma 5.6 and the definition of $\Phi(u)$. Since $(\Phi(s) \cup \Phi(s)^\sigma){\downarrow}A_\infty = \Phi(s){\downarrow}A_\infty$ it remains to use Lemma 5.6 another time. □

Lemma 6.12. *The inductive systems $\Phi(s){\downarrow}A_\infty$, $s = 1, 2, \dots, \frac{p-1}{2}$ are semisimple.*

Proof. This follows immediately from the fact that $\Phi(s)$ are semisimple. □

Corollary 6.13. *Let $p > 2$. Set*

$$J(t)_n = \bigcap_{D\in\Phi(t)_n} \mathrm{Ann}_{FA_n}(D{\downarrow}A_n), \quad t = 1,\dots,\frac{p-1}{2}, \quad n \in \mathbb{N}.$$

Then

(i) $J(t)_1 \subset J(t)_2 \subset \dots$. *Moreover,*

$$J(t) = \cup_{n=1}^{\infty} J(t)_n, \quad t = 1,\dots,\frac{p-1}{2},$$

are exactly all distinct maximal ideals of FA_∞, and $J(t)\cap FA_n = J(t)_n$.

(ii) $I(s)\cap FA_\infty = J(t)$ *where $t = \min(s, p-s)$, $s = 1,\dots,p-1$.*

Proof. The proof of (i) is similar to that of Corollary 5.8.

(ii) Let $t = \min(s, p-s)$. In view of Corollary 5.8, Lemma 5.6 and (i), we have

$$I(s)\cap FA_i = (I(s)\cap F\Sigma_i)\cap FA_i = (\cap_{D\in\Phi(s)_i} \mathrm{Ann}_{F\Sigma_i}(D))\cap FA_i$$
$$= \cap_{D\in\Phi(s)_i} \mathrm{Ann}_{FA_i}(D{\downarrow}A_i) = \cap_{D\in\Phi(t)_i} \mathrm{Ann}_{FA_i}(D{\downarrow}A_i) = J(t)_i = J(t)\cap FA_i$$

for any $i\in\mathbb{N}$. This implies (ii). □

Remark. In view of Corollaries 5.8 and 6.13 (i),

$$I(s) = \cup_{j>(p-s)(s-1)} I(s)_j$$

and

$$J(t) = \cup_{i>(p-t)(t-1)} J(t)_i.$$

This allows one to describe the ideals $I(s)$ and $J(t)$ in a more compact form, because $\Omega_i^s = \emptyset$ for $i > (p-s)(s-1)$, and so

$$\Psi(s)_i = \Theta_i^s,$$

see (6). This is done in Theorem 1.1.

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