Modular representations of the special linear groups with small weight multiplicities

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Dedicated with admiration to A.E. Zalesski on the occasion of his 75*th birthday*

Abstract

We classify irreducible representations of the special linear groups in positive characteristic with small weight multiplicities with respect to the group rank and give estimates for the maximal weight multiplicities. For the natural embeddings of the classical groups, inductive systems of representations with totally bounded weight multiplicities are classified. An analogue of the Steinberg tensor product theorem for arbitrary indecomposable inductive systems for such embeddings is proved.

1 Introduction

In what follows K is an algebraically closed field of characteristic $p > 0$; G_n is a classical algebraic group of rank n over K; $\operatorname{Irr} G_n$ is the set of all rational irreducible representations (or simple modules) of G_n up to equivalence, $\operatorname{Irr}^p G_n \subset \operatorname{Irr} G_n$ is the subset of p-restricted ones; $\operatorname{Irr} M \subset \operatorname{Irr} G_n$ is the set of composition factors of a module M (disregarding the multiplicities), $\omega(M)$ is the highest weight of a simple module M; $L(\omega)$ is the simple G_n-module with highest weight ω; $\omega_1^n, \ldots, \omega_n^n$ are the fundamental weights of G_n; $\omega_0^n =$

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$\omega_{n+1}^n = 0$ by convention. A weight $\sum_{i=1}^n a_i\omega_i^n$ is p-restricted if all $a_i < p$. By the *weight degree* of a module M we mean the maximal dimension of the weight subspaces in M, i.e.

$$\operatorname{wdeg} M = \max_{\mu\in\Lambda(M)} \dim M^\mu$$

where $\Lambda(M)$ is the set of weights of M. In particular, we say that M has a small weight degree if $\operatorname{wdeg} M$ is small with respect to n.

For the classical algebraic groups modular representations of weight degree 1 were classified in [19, 25]. To state the result, first define the following sets of weights of the group $G_n = A_n(K)$, $B_n(K)$, $C_n(K)$, or $D_n(K)$:

$$\begin{aligned}
\Omega_p(A_n(K)) &= \{0, \omega_k^n, (p-1-a)\omega_k^n + a\omega_{k+1}^n \mid 0 \le k \le n,\ 0 \le a \le p-1\},\\
\Omega_p(B_n(K)) &= \{0, \omega_1^n, \omega_n^n\},\\
\Omega_p(C_n(K)) &= \{0, \omega_1^n, \frac{p-1}{2}\omega_n^n, \omega_{n-1}^n + \frac{p-3}{2}\omega_n^n\} \quad (p > 2),\\
\Omega_p(D_n(K)) &= \{0, \omega_1^n, \omega_{n-1}^n, \omega_n^n\},\\
\Omega(G_n) &= \{\sum_{j=0}^k p^j\lambda_j \mid k \ge 0,\ \lambda_j \in \Omega_p(G_n)\}.
\end{aligned}$$

Theorem 1.1 ([19, 6.1], [25, Proposition 2]) *Let G_n be a classical algebraic group of rank $n \ge 4$ and let M be a rational simple G_n-module. Assume $p > 2$ for $G = B_n(K)$ or $C_n(K)$. Then* $\operatorname{wdeg} M = 1$ *if and only if $\omega(M) \in \Omega(G_n)$.*

Obviously, a simple module M is p-restricted with $\operatorname{wdeg} M = 1$ if and only if $\omega(M) \in \Omega_p(G_n)$. The $A_n(K)$-modules $L((p-1-a)\omega_k^n + a\omega_{k+1}^n)$ are truncated symmetric powers of the natural module [26, Proposition 1.2]. Thus, the only p-restricted modules of weight degree 1 for type A are the fundamental modules and truncated symmetric powers of the natural module. Recall that $B_n(K) \cong C_n(K)$ for $p = 2$ (as abstract groups). So we do not consider groups of type B_n in characteristic 2. For groups of type C_n in this case the description of irreducible modules of weight degree 1 is more involved (see details in Section 6).

In this paper we classify irreducible representations of the special linear groups of small weight degree. For other classical groups this was done by the authors earlier. In particular, it was shown that for these groups and odd p no irreducible modules M exist with $1 < \operatorname{wdeg} M < n - 7$.

Theorem 1.2 ([1, Theorem 1.1], [17, Theorem 1],[18, Theorem 1]) *Let $n \ge 8$ and let $G_n = B_n(K)$, $C_n(K)$ or $D_n(K)$. Let M be a rational simple G_n-module with $\omega(M) \notin \Omega(G_n)$. Suppose that $p > 2$ for $G_n = B_n(K)$ or $C_n(K)$. Then* $\operatorname{wdeg} M \ge n - 4 - [n]_4$ *where $[n]_4$ is the residue of n modulo 4. In particular,* $\operatorname{wdeg} M \ge n - 7$.

The main case ($p > 2$ for $G_n = B_n(K)$ or $D_n(K)$ and $p > 7$ for $G_n = C_n(K)$) was settled in [1]; [17] deals with type D for $p = 2$; and [18] gives a new proof for type C for all p. For $G = C_n(K)$ and $p = 2$ a new exceptional series of modules with wdeg $= 2^s$ appears (see details in Section 6).

Now assume that $G_n = A_n(K)$. Let $M \in \operatorname{Irr} G_n$, $\omega(M) = a_1\omega_1^n + \ldots + a_n\omega_n^n$, and M^* be the dual of M. Note that $\omega(M^*) = a_n\omega_1^n + a_{n-1}\omega_2^n + \ldots + a_1\omega_n^n$ and $\operatorname{wdeg} M =$

wdeg M^*. Define the *polynomial degree* of M as the polynomial degree of the corresponding polynomial representation of $GL_{n+1}(K)$, i.e.

$$\operatorname{pdeg} M = \sum_{k=1}^{n} k a_k. \tag{1}$$

Denote by V_n the natural module for G_n. Note that every simple module of polynomial degree d can be obtained as a composition factor of the dth tensor power $V_n^{\otimes d}$. More exactly, we have the following. Set

$$\mathcal{L}_n^d = \cup_{j\le d} \operatorname{Irr} V_n^{\otimes j}, \quad \mathcal{R}_n^d = \cup_{j \le d} \operatorname{Irr}(V_n^*)^{\otimes j}. \tag{2}$$

Then $\mathcal{L}_n^d = \{M \in \operatorname{Irr} G_n \mid \operatorname{pdeg} M \le d\}$ and $\mathcal{R}_n^d = \{M \in \operatorname{Irr} G_n \mid \operatorname{pdeg} M^* \le d\}$ (Proposition 3.2). For $d \le n$, it is not difficult to see that $\operatorname{wdeg} V_n^{\otimes d} = d!$ (Lemma 3.4). This means that modules of small polynomial degree d (with, say, $d! < n$) have small weight degree ($< n$), which gives many more small weight degree modules for type A in addition to those described in Theorem 1.1. This makes situation more difficult than in the case of other classical groups, especially for non p-restricted modules. Our first main result describes p-restricted irreducible representations of the special linear groups of small weight degree.

Theorem 1.3 *Let* $M \in \operatorname{Irr}^p A_n(K)$ *and* $d = \min\{\operatorname{pdeg} M, \operatorname{pdeg} M^*\}$. *Assume* $\omega(M) \notin \Omega_p(A_n(K))$. *Then the following hold.*

(*i*) *If* $n \ge 16$ *and* $d > n$, *then*

$$\operatorname{wdeg} M > \sqrt{n}/p - 1.$$

(*ii*) *If* $d \le n$, *then*

$$d - 2 \le \operatorname{wdeg} M \le d!.$$

Moreover, $M \cong L(a_1\omega_1^n + \ldots + a_d\omega_d^n)$ *or* $L(a_d\omega_{n-d+1}^n + \ldots + a_1\omega_n^n)$ *with* $a_1 + 2a_2 + \cdots + da_d = d$, *and* wdeg M *is determined by the sequence* $(a_1, \ldots, a_d)$ *only and does not depend on* n.

In particular, if $n \ge 16$ *and* $\operatorname{wdeg} M \le \sqrt{n}/p - 1$, *then* M *is as in part* (*ii*) *with* $d \le \sqrt{n}/p + 1$.

The $\sqrt{n}/p - 1$ estimate in part (i) was obtained by applying the Schur functor. It is a quick and rough estimate and can probably be improved if one uses a more thorough analysis, similar to that of [1]. One should expect something close to n, as in Theorem 1.2. Unfortunately, this seems to be very difficult to obtain at the moment as too many modules of small weight degree exist for type A and the methods used in [1] fail to work. But our estimate is good enough to identify the modules with small weight degree and get a full classification of the inductive systems of representations for A_∞ with bounded weight multiplicities (see below).

In what follows for all classical groups Fr is the Frobenius morphism of G_n associated with raising the elements of K to the pth power; $M^{[k]}$ denotes a G_n-module M twisted by the kth power of Fr. Let $M \in \operatorname{Irr} G_n$. Assume that $\omega(M) = \sum_{k=0}^{s} p^k \lambda_k$ with p-restricted dominant weights λ_k of G_n. Put $M_k = L(\lambda_k)$. By the Steinberg tensor product theorem [21],

$$M \cong \otimes_{k=0}^{s} M_k^{[k]}. \tag{3}$$

It is obvious that $\operatorname{wdeg} M \geq \operatorname{wdeg} M_0 \cdot \ldots \cdot \operatorname{wdeg} M_s$ (Lemma 2.14). Therefore, the question of describing non p-restricted G_n-modules of small weight degree is essentially reduced to combining various Frobenius twists of p-restricted modules of small weight degree and making sure that the weight degree does not become too large (see Corollary 3.9, Theorem 3.11, and Proposition 3.12).

Note that the results above can be considered as a modular analogue of the following problem solved by Mathieu [16]: describe all infinite dimensional weight modules with bounded weight multiplicities for a finite dimensional simple Lie algebra over $\mathbb{C}$. Some particular cases, including so-called completely pointed modules (i.e. with one dimensional weight spaces) were previously considered in [5, 6, 8]. It is interesting to note that by specializing p to 0 in the weights in the set $\Omega_p(G_n)$ we get highest weights of completely pointed modules (e.g. $(-1-a)\omega_k^n + a\omega_{k+1}^n$ for type A_n and $\omega_{n-1}^n - \frac{3}{2}\omega_n^n$ and $-\frac{1}{2}\omega_n^n$ for type C_n).

Estimates of weight multiplicities obtained above can be used for recognizing linear groups containing matrices with small eigenvalue multiplicities. Indeed, it occurs that only for some special classes of representations of simple classical algebraic groups, their images can contain matrices all whose eigenvalue multiplicities are small enough with respect to the group rank.

At the end of the paper we classify inductive systems of representations with bounded weight multiplicities for the natural embeddings of the classical groups. In what follows $\mathbb{N}$ is the set of positive integers. For a group G, a subgroup $H \subset G$ and a G-module M denote by $M{\downarrow}H$ the restriction of M to H. Let

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n \subset \ldots \tag{4}$$

be a chain of fixed embeddings of algebraic groups Γ_n over K and let Φ_n, $n \in \mathbb{N}$, be a nonempty finite subset of $\operatorname{Irr}\Gamma_n$, for each n. Recall that the system $\Phi = \{\Phi_n \mid n \in \mathbb{N}\}$ is called an *inductive system* of representations (or modules) for (4) if

$$\bigcup_{\varphi \in \Phi_{n+1}} \operatorname{Irr}(\varphi{\downarrow}\Gamma_n) = \Phi_n$$

for all $n \in \mathbb{N}$. Inductive systems have been introduced by A. Zalesskii in [23]. They can be regarded as an asymptotic version of the branching rules for the embeddings (4). Observe that in positive characteristic one cannot expect to find explicit analogues of the classical branching rules in characteristic 0 which have quite a lot of applications, so their asymptotic versions can be useful. Moreover, inductive systems can be applied to the study of ideals in group algebras of locally finite groups. It is proved in [24] that there exists a bijective correspondence between the inductive systems for a locally finite group and the semiprimitive ideals of the corresponding group algebra. So far we know little about the structure of inductive systems. Minimal and minimal nontrivial inductive systems of modular representations for natural embeddings of algebraic and finite groups of type A_n were classified in [3]. For other classical groups the question on the minimal inductive systems seems substantially more difficult. For natural embeddings of symplectic groups in positive characteristic examples of such systems that have no analogues in the characteristic 0 case were constructed in [25] and [2].

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of G_n labeled as in [7] (it will always be clear from the context what group is considered). It is well known that the root subgroups associated with the roots $\pm\alpha_{n-k+1}, \ldots, \pm\alpha_n$ generate a subgroup isomorphic to G_k. If we identify

G_k with this subgroup, we obtain a sequence of natural embeddings

$$G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots . \tag{5}$$

In this paper we consider only inductive systems for the sequence (5).

Definition 1.4 Let Φ be an inductive system of representations. We say that Φ is a *BWM-system* (bounded weight multiplicities system) if there exists $m \in \mathbb{N}$ such that $\operatorname{wdeg} \varphi \leq m$ for all $\varphi \in \Phi_n$ and all n. For a BWM-system Φ we define $\operatorname{wdeg} \Phi = \max_{\varphi \in \Phi} \operatorname{wdeg} \varphi$.

In Sections 5 and 6 we classify all BWM-systems for all four types of classical groups. To state the main results, we need to introduce some notation. For any dominant weight ω of G_n denote by $\delta(\omega)$ the value of ω on the maximal root of the root system of G_n. For a simple module $M \cong L(\omega)$ put $\delta(M) = \delta(\omega)$. Let $T \subset \mathbb{N}$ be infinite. Assume that $R_t \subset \operatorname{Irr} G_t$ is nonempty for each $t \in T$ and that there exists $k \in \mathbb{N}$ such that $\delta(M) < k$ for all $M \in R_t$ and for all t. Denote by Π_n the set of all G_n-modules Q such that Q is a composition factor of the restriction $Y{\downarrow}G_n$ for some $t > n$, $t \in T$, and $Y \in R_t$. Assume that $R_t \subset \Pi_t$ for all t. By Lemma 4.3, $\Pi = \{\Pi_n \mid n \in \mathbb{N}\}$ is an inductive system for the groups G_n. We will write $\Pi = \langle R_t \mid t \in T \rangle$ and call Π the *inductive system generated by* R_t. If every R_t consists of a single module Y_t, we use a simplified notation $\Pi = \langle Y_t \mid t \in T \rangle$. Let Φ be an inductive system. We say that Φ is a *p-restrictedly generated system* if $\Phi = \langle \Lambda_t \mid t \in T \rangle$ with $\Lambda_t \subset \operatorname{Irr}^p G_t$ for all $t \in T$.

For arbitrary inductive systems Φ and Ψ define the collections $\operatorname{Fr}(\Phi)$ and $\Phi \otimes \Psi$ in a natural way:

$$\begin{aligned} \operatorname{Fr}(\Phi)_n &= \{\varphi^{[1]} \mid \varphi \in \Phi_n\}, \\ (\Phi \otimes \Psi)_n &= \bigcup_{\varphi \in \Phi_n,\ \psi \in \Psi_n} \operatorname{Irr}(\varphi \otimes \psi). \end{aligned}$$

By Lemma 4.2, $\operatorname{Fr}(\Phi)$ and $\Phi \otimes \Psi$ are inductive systems. The union of inductive systems Φ and Ψ and the inclusion relation for such systems are defined in a natural way. An inductive system $\mathcal{T}$ is called *decomposable* if $\mathcal{T}$ is the union of inductive systems Φ and Ψ that do not coincide with $\mathcal{T}$, and *indecomposable* otherwise. For an inductive system Φ put

$$\delta(\Phi_n) = \max\{\delta(\varphi) \mid \varphi \in \Phi_n\}.$$

Then $\delta(\Phi_n)$ does not depend on n (Lemma 4.1), so we can define $\delta(\Phi)$ as $\delta(\Phi_n)$.

In Section 4 we prove the following analogue of the Steinberg product theorem for inductive systems, which is of independent interest.

Theorem 1.5 *Let Φ be an indecomposable inductive system for the sequence (5). Then there exist p-restrictedly generated inductive systems Φ^j, $0 \leq j \leq k$, such that $\Phi = \otimes_{j=0}^{k} \operatorname{Fr}^j(\Phi^j)$.*

Now assume that $G_n = A_n(K)$. Recall the sets $\mathcal{L}_n^d$ and $\mathcal{R}_n^d$ defined in (2). Lemma 5.1 implies that $\mathcal{L}^d = \{\mathcal{L}_n^d \mid n \in \mathbb{N}\}$ and $\mathcal{R}^d = \{\mathcal{R}_n^d \mid n \in \mathbb{N}\}$ are inductive systems. Note that $\mathcal{L}_n^1 = \{L(0), V_n\}$. Set

$$\begin{aligned} \mathcal{F}_n &= \{L(\omega_0^n), L(\omega_1^n), \ldots, L(\omega_n^n)\}, & (6) \\ \mathcal{T}_n &= \{L((p-a-1)\omega_i^n + a\omega_{i+1}^n) \mid 0 \leq a < p, \quad 0 \leq i \leq n\} & (7) \end{aligned}$$

(ω^n_{n+1} is treated as 0). By Lemma 5.1, $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$ and $\mathcal{T} = \{\mathcal{T}_n \mid n \in \mathbb{N}\}$ are inductive systems. Note that the representations of $\mathcal{T}$ are realized exactly in the truncated symmetric powers of the natural module.

Let $d \in \mathbb{N}$. Fix any integers $a_i \geq 0$ for $0 \leq i \leq d$. For $n \geq d$ let $M_{n,L}(a_1, \dots, a_d)$ be a simple G_n-module with highest weight $a_1\omega^n_1 + \ldots + a_d\omega^n_d$ and $M_{n,R}(a_1, \dots, a_d)$ be a simple G_n-module with highest weight $a_d\omega^n_{n-d+1} + \ldots + a_1\omega^n_n$. Set

$$\begin{aligned} C_L(a_1, \dots, a_d) &= \langle M_{n,L}(a_1, \dots, a_d) \mid n \geq d\rangle, \\ C_R(a_1, \dots, a_d) &= \langle M_{n,R}(a_1, \dots, a_d) \mid n \geq d\rangle. \end{aligned}$$

By Lemma 5.2, the systems $C_L(a_1, \dots, a_d)$ and $C_R(a_1, \dots, a_d)$ are well defined.

Theorem 1.6 *Let $G_n = A_n(K)$. Assume that Φ is a p-restrictedly generated indecomposable BWM-system. Then $\Phi = \mathcal{F}$, $\mathcal{T}$, $C_L(a_1, \dots, a_d)$ or $C_R(a_1, \dots, a_d)$ for some integers $a_1, \dots, a_d < p$.*

Let Φ be an inductive system. Assume that

$$\Phi = \otimes^s_{k=0} \operatorname{Fr}^k(\Phi^k),$$

where Φ^k are p-restrictedly generated systems. We say that Φ is *special* if each Φ^k is equal to one of the systems $C_L(a_1, \dots, a_d)$, $C_R(a_1, \dots, a_d)$, $\mathcal{F}$, or $\mathcal{T}$.

Let Φ be special. Then for every k, either $\Phi^k = \mathcal{F}, \mathcal{T}$ or there exists d such that $\Phi^k \subset \mathcal{L}^d$ or $\mathcal{R}^d$. Therefore, Φ can be represented in the form

$$\Phi = \Psi^0 \otimes \ldots \otimes \Psi^l$$

with

$$\Psi^f = \otimes^{i_f}_{k=i_{f-1}+1} \operatorname{Fr}^k(\Phi^k), \tag{8}$$

where the indices i_f, $0 \leq f \leq l$, satisfy the following: $i_{-1} = -1$ and for each f, either all Φ^k have the form $C_L(a_1, \dots, a_d)$ for $i_{f-1} + 1 \leq k \leq i_f$, or all Φ^k have the form $C_R(a_1, \dots, a_d)$ for $i_{f-1} + 1 \leq k \leq i_f$, or $i_{f-1} + 1 = k = i_f$ and $\Phi^k = \mathcal{F}$ or $\mathcal{T}$. Fix minimal l with this property. Then the systems Ψ^f are uniquely determined.

Theorem 1.7 *Let $G_n = A_n(K)$. Indecomposable BWM-systems are exhausted by special inductive systems with the following property $\delta(\Psi^f) < p^{i_f+1}$ for all Ψ^f with $f < l$ (i_f are such as in* (8)*). An arbitrary BWM-system is a finite union of indecomposable ones.*

Theorems 1.2 and 6.3 allow us to find the BWM-systems for the remaining series of classical groups. Put

$$\mathcal{S}_n = \begin{cases} \{L(\omega^n_n)\} & \text{for } G_n = B_n(K), \\ \{L(\omega^n_{n-1}), L(\omega^n_n)\} & \text{for } G_n = D_n(K), \\ \{L(\frac{p-1}{2}\omega^n_n), L(\omega^n_{n-1} + \frac{p-3}{2}\omega^n_n)\} & \text{for } G_n = C_n(K), p > 2 \end{cases}$$

and $\mathcal{L}_n = \{L(0), L(\omega^n_1)\}$. Lemmas 2.10 and 6.1 imply that $\mathcal{L} = \{\mathcal{L}_n \mid n \in \mathbb{N}\}$ and $\mathcal{S} = \{\mathcal{S}_n \mid n \in \mathbb{N}\}$ are inductive systems. Obviously, the collection $\mathcal{O} = \{\mathcal{O}_n \mid n \in \mathbb{N}\}$ with $\mathcal{O}_n = \{L(0)\}$ is an inductive system for all types.

Theorem 1.8 *Let $G_n = B_n(K)$, $C_n(K)$ or $D_n(K)$, and let $p > 2$ for $G_n \neq D_n(K)$. Set $\mathcal{P} = \{\mathcal{O}, \mathcal{L}, \mathcal{S}\}$. An indecomposable inductive system Φ is a BWM-system if and only if $\Phi = \otimes^s_{j=0} \operatorname{Fr}^j(\Phi^j)$, where $\Phi^j \in \mathcal{P}$. BWM-systems are finite unions of indecomposable ones and consist of modules with one dimensional weight spaces.*

For $G_n = C_n(K)$ and $p = 2$ the answer is more complicated, see Theorem 6.4.

2 Notation and preliminaries

Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. For a simple algebraic group G over K the symbol $\Lambda(G)$ denotes the set of weights of G, $R(G)$ is the set of roots of G; $\langle \lambda, \alpha \rangle$ is the value of a weight $\lambda \in \Lambda(G)$ on a root $\alpha \in R(G)$, and $\operatorname{Irr} G$ is defined as for groups G_n. Throughout the text $\Lambda(M)$ is the set of all weights of a G-module M. For a G-module M denote by v^+ a nonzero highest weight vector of M and by M^μ the weight space in M of a weight μ. The subspace of a linear space L spanned by vectors $v_1, \ldots, v_i$ is denoted by $\langle v_1, \ldots, v_i \rangle$, respectively. For positive roots $\beta_1, \ldots, \beta_j$ denote by $G(\beta_1, \ldots, \beta_j)$ the subgroup of G generated by the root subgroups associated with $\pm\beta_1, \ldots, \pm\beta_j$. In all cases where subgroups of this form are considered, the roots $\beta_1, \ldots, \beta_j$ are chosen such that they constitute a base of the root system of $G(\beta_1, \ldots, \beta_j)$. In this situation the fundamental weights of $G(\beta_1, \ldots, \beta_j)$ are determined with respect to this base. If $H = G(\beta_1, \ldots, \beta_k) \subset G$ and $\omega \in \Lambda(G)$, then $\omega{\downarrow}H$ is the restriction of ω to H. For a G-module M and a weight vector $v \in M$ we denote the weight of v with respect to a subgroup $H \subset G$ by $\omega_H(v)$. Set $\omega(v) = \omega_G(v)$.

In what follows ε_i^n with $1 \leq i \leq n+1$ for $G_n = A_n(K)$ and $1 \leq i \leq n$ otherwise are weights of V_n, their labeling is standard and corresponds to [7, Ch. VIII, §13]. Put $G_n(i_1, \ldots, i_j) = G_n(\alpha_{i_1}, \ldots, \alpha_{i_j})$.

We assume that $n > 1$ in all cases where $n-1$ appears in formulas. For $k < n$ set $G_{n,k} = G_n(n-k+1, \ldots, n)$. As we have mentioned in the Introduction, $G_{n,k} \cong G_k$. Put $\operatorname{Irr}_k M = \operatorname{Irr}(M{\downarrow}G_{n,k})$.

Theorem 2.1 (Jantzen [12], Smith [20]) *Let $H = G_n(i_1, \ldots, i_j) \subset G_n$. Then $KHv^+ \subset L(\omega)$ is an irreducible H-module with highest weight $\omega_H(v^+)$ and a direct summand of the H-module $L(\omega)$.*

Call KHv^+ in the previous theorem the *Smith factor* of $L(\omega)$ (with respect to H).

Lemma 2.2 *Let $M \in \operatorname{Irr} G_n$, and let α be a long root of G_n. Then $\delta(M) = \max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha \rangle$.*

Proof. Denote by $\alpha_{\max}$ the maximal root in $R(G_n)$. As $\alpha_{\max}$ is a dominant weight, $\langle \alpha_i, \alpha_{\max} \rangle \geq 0$. This implies

$$\delta(M) = \langle \omega(M), \alpha_{\max} \rangle = \max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha_{\max} \rangle.$$

Since the Weyl group acts transitively on the set of roots of the same length and $\alpha_{\max}$ is long, $\max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha \rangle = \max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha_{\max} \rangle$ as required. □

Corollary 2.3 *In the assumptions of Lemma 2.2 suppose that α is positive and set $H = G_n(\alpha)$. Then $\delta(M) = \max\{i \mid L(i\omega_1^1) \in \operatorname{Irr}(M{\downarrow}H)\}$.*

Proof. Obviously,

$$\max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha \rangle = \max_{\mu \in \Lambda(M \downarrow H)} \langle \mu, \alpha \rangle = \max\{i \mid L(i\omega_1^1) \in \operatorname{Irr}(M{\downarrow}H)\}.$$

It remains to apply Lemma 2.2. □

Corollary 2.4 *Let $k < n$, $M \in \operatorname{Irr} G_n$, and $N \in \operatorname{Irr}_k M$. Assume that $k > 1$ for $G_n = B_n(K)$. Then $\delta(N) \leq \delta(M)$.*

Proof. Put
$$\Lambda' = \{\lambda{\downarrow}G_k \mid \lambda \in \Lambda(M)\},$$
$\beta = \alpha_{n-1}$ for $G_n = B_n(K)$ and $\beta = \alpha_n$ otherwise. It is clear that $\Lambda(N) \subset \Lambda'$. By Lemma 2.2,
$$\delta(N) = \max_{\lambda \in \Lambda(N)} \langle \lambda, \beta \rangle \leq \max_{\lambda \in \Lambda'} \langle \lambda, \beta \rangle = \delta(M).$$
□

Recall the set of $A_n(K)$-modules $\mathcal{F}_n$ defined in (6).

Lemma 2.5 *Let $G_n = A_n(K)$.*

(i) For $1 \leq i \leq n$ the set $\operatorname{Irr}_{n-1} L(\omega_i^n) = \{L(\omega_{i-1}^{n-1}), L(\omega_i^{n-1})\}$.

(ii) Let $k < i \leq n - k + 1$, $M \in \operatorname{Irr} G_n$, and $\omega(M) = \omega_i^n$. Then $\operatorname{Irr}_k M = \mathcal{F}_k$.

Proof. (i) Denote by $\wedge^i V_n$ the ith wedge power of V_n. One has $L(\omega_i^n) = \wedge^i V_n$ [13, Part II, 2.15]. Let $v_1, \ldots, v_{n+1} \in V_n$ and $\omega(v_i) = \varepsilon_i^n$. Set $\Gamma = G_{n,n-1}$. One can assume that $\varepsilon_1^n{\downarrow}\Gamma = 0$ and Γ fixes $\langle v_2, \ldots, v_{n+1} \rangle$ and v_1. Then the Γ-module $\langle v_2, \ldots, v_{n+1} \rangle$ is isomorphic to V_{n-1}. Set
$$U_1 = \langle v_{k_1} \wedge \ldots \wedge v_{k_i} \mid 1 < k_1 < \ldots < k_i \leq n+1 \rangle$$
and
$$U_2 = \langle v_1 \wedge v_{l_1} \wedge \ldots \wedge v_{l_{i-1}} \mid 1 < l_1 < \ldots < l_{i-1} \leq n+1 \rangle.$$
Then $\wedge^i V_n = U_1 \oplus U_2$. One easily observes that Γ fixes U_1 and U_2, the Γ-module $U_1 \cong L(\omega_i^{n-1})$ and $U_2 \cong L(\omega_{i-1}^{n-1})$.

(ii) Put $H_j = G_n(i-j+1, i-j+2, \ldots, i-j+k)$ for $1 \leq j \leq k$ and $H_0 = G_n(1, \ldots, k)$. The subgroups H_j are conjugate to G_k. Hence $\operatorname{Irr}(M{\downarrow}H_j) = \operatorname{Irr}(M{\downarrow}G_k)$. By Theorem 2.1, $L(\omega_j^k) \in \operatorname{Irr}(M{\downarrow}H_j)$ for $0 \leq j \leq k$. Hence $\mathcal{F}_k \subset \operatorname{Irr}_k M$. It is well known that the maximal root $\alpha_{\max} = \alpha_1 + \ldots + \alpha_n$ for $G_n = A_n(K)$. So $\delta(M) = 1$. By Corollary 2.4, $\delta(N) \leq \delta(M)$ for $N \in \operatorname{Irr}_k M$. Therefore $N \in \mathcal{F}_k$. This completes the proof. □

Lemma 2.6 ([26, Proposition 1.4]) *Let $G_n = A_n(K)$, $H = G_n(1, \ldots, m, m+2, \ldots, n) \subset G_n$, $0 \leq c \leq p-1$, $0 \leq i \leq n$. Then*
$$L(c\omega_i^n + (p-1-c)\omega_{i+1}^n){\downarrow}H =$$
$$= \oplus_{N(i,c)} L(c_1\omega_{i_1}^m + (p-1-c_1)\omega_{i_1+1}^m) \otimes L(c_2\omega_{i_2}^{n-m-1} + (p-1-c_2)\omega_{i_2+1}^{n-m-1})$$
with
$$\begin{aligned} N(i,c) = \{(i_1, c_1), (i_2, c_2) \mid \quad & 0 \leq c_j < p, \quad 0 \leq (p-1)(i_1+1) - c_1 \leq (p-1)(m+1), \\ & 0 \leq (p-1)(i_2+1) - c_2 \leq (p-1)(n-m), \\ & (p-1)(i_1+i_2+2) - c_1 - c_2 = (p-1)i + p - 1 - c\}. \end{aligned}$$

Here $H = H_1 \times H_2$ with $H_1 = G_n(1, \ldots, m) \cong A_m(K)$ and $H_2 = G_n(m+2, \ldots, n) \cong A_{n-m-1}(K)$; and the tensor product is the (external) product of H_1- and H_2-modules.

Recall the set of G_n-modules $\mathcal{T}_n$ defined in (7).

Corollary 2.7 *If $G_n = A_n(K)$, $k+1 \le i < n-k$, and $\omega = c\omega_i^n + (p-1-c)\omega_{i+1}^n$, then* $\operatorname{Irr}_k L(\omega) = \mathfrak{T}_k$.

Proof. In Lemma 2.6 take $m = k$ and observe that $H_1 \cong G_k$. Now the corollary follows immediately from this lemma. □

Corollary 2.8 *Let $G_n = A_n(K)$, $k < n$, and $\omega = a\omega_1^n$ with $0 < a < p$. Then* $\operatorname{Irr}_k L(\omega) = \{L(b\omega_1^k) \mid 0 \le b \le a\}$.

Proof. Argue as in the proof of Corollary 2.7 taking $m = k$, $i = 0$, and $c = p-1-a$. □

Lemma 2.9 ([25, Theorem, part C]) *Let $p > 2$, $n > 1$, and $G_n = C_n(K)$. Set $M_1^n = L(\omega_{n-1}^n + \frac{p-3}{2}\omega_n^n) \in \operatorname{Irr} G_n$ and $M_2^n = L(\frac{p-1}{2}\omega_n^n) \in \operatorname{Irr} G_n$. Then $\operatorname{Irr}_{n-1} M_j^n = \{M_1^{n-1}, M_2^{n-1}\}$ for $j = 1, 2$.*

Lemma 2.10 *Let $n > 2$ for $G_n = B_n(K)$ and $n > 4$ for $G_n = D_n(K)$. Then* $\operatorname{Irr}_{n-1} L(\omega_1^n) = \{L(0), L(\omega_1^{n-1})\}$.

Proof. This is obvious and well known. We put some restrictions on n to avoid complications connected with the isomorphisms between classical groups of small ranks from different series. □

The following lemma is also well known, but we fail to find an explicit reference.

Lemma 2.11 *If $G_n = B_n(K)$ and $n > 2$ or $p = 2$ and $G_n = C_n(K)$, then* $\operatorname{Irr}_{n-1} L(\omega_n^n) = \{L(\omega_{n-1}^{n-1})\}$. *For $G_n = D_n(K)$ with $n > 3$ one has* $\operatorname{Irr}_{n-1} L(\omega_n^n) = \operatorname{Irr}_{n-1} L(\omega_{n-1}^n) = \{L(\omega_{n-1}^{n-1}), L(\omega_{n-2}^{n-1})\}$.

Proof. Let M be one of the modules in question. If $G_n = B_n(K)$ or $D_n(K)$, it is well known that $\omega(M)$ is a microweight and hence $\Lambda(M)$ coincides with the orbit of $\omega(M)$ under the action of the Weyl group. Therefore $\Lambda(M) = \{(\pm\varepsilon_1^n + \ldots + \pm\varepsilon_n^n)/2\}$ with all possible combinations of the "plus" and "minus" signs for $G_n = B_n(K)$. If $G_n = D_n(K)$, then $\Lambda(M)$ consists of all such weights with an odd or even number of the "minus" signs for $M = L(\omega_{n-1}^n)$ or $L(\omega_n^n)$, respectively.

Let $p = 2$ and $G_n = C_n(K)$. It is well known that in this case $\Lambda(M)$ is such as for $B_n(K)$. Indeed, using a special isogeny from $C_n(K)$ to $B_n(K)$, one easily concludes that $\dim M = 2^n$ (as for the relevant $B_n(K)$-module), see [9, Subsection 5.3 and Theorem 5.4]. Hence again $\Lambda(M)$ coincides with the orbit of $\omega(M)$.

The following arguments concern all the groups considered in this lemma. Let $M_+ \subset M$ ($M_- \subset M$) be the sum of all weight subspaces M^λ with $\lambda = \varepsilon_1^n/2 + \mu$ ($\lambda = -\varepsilon_1^n/2 + \mu$, respectively) where μ is a linear combination of the weights $\varepsilon_2^n, \ldots, \varepsilon_n^n$. For $2 \le i \le n$ one can identify the restriction of the weight ε_i^n to G_{n-1} with the weight $\varepsilon_{i-1}^{n-1} \in \Lambda(G_{n-1})$. Taking into account that for $2 \le i \le n$ the roots α_i are linear combinations of the weights ε_i^n with $2 \le i \le n$, one can observe that $G_{n,n-1}$ fixes M_+ and M_-. Analyzing the weight structure of these $G_{n,n-1}$-modules, we conclude that they are irreducible and have desired highest weights. This proves the lemma. □

Corollary 2.12 *Let $p = 2$, $n > 2$, and $G_n = C_n(K)$. Then*

$$\operatorname{Irr}_{n-1} L(\omega_1^n + \omega_n^n) = \{L(\omega_1^{n-1} + \omega_{n-1}^{n-1}), L(\omega_{n-1}^{n-1})\}.$$

Proof. By [22, the corollary of Theorem 41], for $G_k = C_k(K)$ and $k > 1$ the G_k-module $L(\omega_1^k + \omega_k^k) \cong L(\omega_1^k) \otimes L(\omega_k^k)$. It is well known that $L(\omega_1^n){\downarrow}G_{n,n-1}$ is the direct sum of $L(\omega_1^{n-1})$ and two copies of $L(0)$. It has been shown in the proof of Lemma 2.11 that $L(\omega_n^n){\downarrow}G_{n,n-1} \cong L(\omega_{n-1}^{n-1}) \oplus L(\omega_{n-1}^{n-1})$. This yields the corollary. □

Proposition 2.13 *Let $k < n$, $M \in \operatorname{Irr} G_n$, and $N \in \operatorname{Irr}_k M$. Then* $\operatorname{wdeg} N \le \operatorname{wdeg} M$.

Proof. First assume that $k = n-1$. Put $\omega = \omega(M)$. For every $\lambda \in \Lambda(M)$ one has $\lambda = \omega - \sum_{i=1}^n b_i(\lambda)\alpha_i$ with $b_i(\lambda) \in \mathbb{Z}_{\ge 0}$. For $j \in \mathbb{Z}_{\ge 0}$ put

$$\Lambda_j = \{\lambda \in \Lambda(M) \mid b_1(\lambda) = j\}.$$

It is obvious that $\Lambda_j \cap \Lambda_t = \varnothing$ for $j \ne t$ and

$$\Lambda(M) = \Lambda_0 \cup \ldots \cup \Lambda_l$$

for some l. Set

$$U_j = \oplus_{\lambda \in \Lambda_j} M^\lambda.$$

Then U_j are $G_{n,n-1}$-modules and $M = U_0 \oplus \ldots\ldots \oplus U_l$ as a $G_{n,n-1}$-module. Hence N is realized in a composition factor of some module U_s. So $\operatorname{wdeg} N$ is not bigger then the maximal weight multiplicity of the $G_{n,n-1}$-module U_s. It remains to observe that the restrictions of distinct weights in Λ_s to $G_{n,n-1}$ are distinct. Indeed, assume $\mu, \nu \in \Lambda_s$ and $\nu \ne \mu$. Obviously $b_1(\mu) = b_1(\nu)$. Hence $b_i(\mu) \ne b_i(\nu)$ for some i with $2 \le i \le n$. This yields that $\mu{\downarrow}G_{n,n-1} \ne \nu{\downarrow}G_{n,n-1}$ and proves the lemma for $k = n-1$. To complete the proof, it remains to apply induction on $n-k$. □

The following lemma is obvious.

Lemma 2.14 *Let M_1 and M_2 be G_n-modules. Then*

$$\operatorname{wdeg} M_1^{[k_1]} \otimes M_2^{[k_2]} \ge \operatorname{wdeg} M_1 \cdot \operatorname{wdeg} M_2.$$

3 Modules with small weight multiplicities for groups of type A

In this section $G_n = A_n(K)$. For a module M we assume that $M^{\otimes 0}$ is the trivial module. Recall the pdeg function defined in (1).

Lemma 3.1 *(i) Let $M \in \operatorname{Irr} G_n$ and* $\operatorname{pdeg} M = d$. *Then $M \in \operatorname{Irr} V_n^{\otimes d}$. If $N \in \operatorname{Irr} V_n^{\otimes d}$, then* $\operatorname{pdeg} N \le d$.

(ii) Let $M \in \operatorname{Irr} G_n$ and $\operatorname{pdeg} M^* = d$. *Then $M \in \operatorname{Irr}(V_n^*)^{\otimes d}$. If $N \in \operatorname{Irr}(V_n^*)^{\otimes d}$, then* $\operatorname{pdeg} N^* \le d$.

Proof. (i) By [10, Subsection 5.2], $V_n^{\otimes d}$ has a submodule isomorphic to the Weyl module with highest weight $\omega(M)$. This yields the first claim of (i).

Recall that $\omega_i^n = \varepsilon_1^n + \ldots + \varepsilon_i^n$, $\alpha_i = \varepsilon_i^n - \varepsilon_{i+1}^n$ for $1 \le i \le n$, and $\varepsilon_1^n + \ldots + \varepsilon_{n+1}^n = 0$. This implies that if $\operatorname{pdeg} N = k$ and $\omega(N) = \sum_{i=1}^n b_i \varepsilon_i^n$, then $\sum_{i=1}^n b_i = k$. It is clear that each weight $\mu \in \Lambda(V_n^{\otimes d})$ has the form $d\omega_1^n - \sum_{i=1}^n c_i \alpha_i$ with $c_i \in \mathbb{Z}_{\ge 0}$. This yields that $\operatorname{pdeg} N \le d$ for $N \in \operatorname{Irr} V_n^{\otimes d}$ and completes the proof of (i).

(ii) Take into account that $(V_n^*)^{\otimes d} \cong (V_n^{\otimes d})^*$. □

Recall the sets $\mathcal{L}_n^d$ and $\mathcal{R}_n^d$ defined in (2).

Proposition 3.2 $\mathcal{L}_n^d = \{M \in \operatorname{Irr} G_n \mid \operatorname{pdeg} M \leq d\}$, $\mathcal{R}_n^d = \{M \in \operatorname{Irr} G_n \mid \operatorname{pdeg} M^* \leq d\}$.

Proof. This follows immediately from Lemma 3.1. $\square$

Proposition 3.3 *Let* $M \in \operatorname{Irr}^p G_n$ *and* $\omega(M) \notin \Omega_p(G_n)$. *Assume that* $\operatorname{pdeg} M \leq n$. *Then* $\operatorname{wdeg} M \geq \operatorname{pdeg} M - 2$.

Proof. Put $d = \operatorname{pdeg} M$ and $H = G_n(1, \dots, d-1)$. Then $H \cong SL_d(K)$. Note that $d > 1$ as $\omega(M) \notin \Omega_p(G_n)$. Let $\omega(M) = \sum_{i=1}^n a_i \omega_i^n$. Since $\omega(M)$ is not fundamental, one easily observes that $a_j = 0$ for $j > d-1$. Denote by N the Smith factor of M associated with H (see 2.1). It is clear that $\operatorname{pdeg} M = \operatorname{pdeg} N = d$.

Now we can apply the Schur functor to the H-module N. Let $M(d,d)$ be the category of the polynomial $GL_d(K)$-modules over K which are homogeneous of degree d, Σ_d be the symmetric group of degree d, and let $K\Sigma_d - \operatorname{mod}$ be the category of $K\Sigma_d$-modules. The Schur functor

$$\mathcal{S}_d : M(d,d) \to K\Sigma_d - \operatorname{mod}$$

sends a module $V \in M(d,d)$ to V^0 where V^0 is the $(1, \dots, 1)$-weight subspace in V [10, Chapter 6]. Alternatively, one can regard V as an $SL_d(K)$-module and define $\mathcal{S}_d(V)$ as the 0-weight subspace of V.

Let $\lambda = b_1\varepsilon_1^{d-1} + \ldots + b_d\varepsilon_d^{d-1}$ be the highest weight of N. Note that $b_1 \geq \cdots \geq b_d \geq 0$ and $b_1 + \cdots + b_d = d$. Hence $\lambda = (b_1, \dots, b_d)$ is a partition of d. The functor $\mathcal{S}_d$ is exact and by [10, 6.4],

$$\mathcal{S}_d(N) \cong D^{\lambda'} \otimes \operatorname{sgn}$$

where $D^{\lambda'}$ is the irreducible Σ_d-module corresponding to the partition λ' dual to λ, and sgn is the sign module for Σ_d. Hence by Proposition 2.13, $\operatorname{wdeg} M \geq \operatorname{wdeg} N \geq \dim D^{\lambda'}$. If $\operatorname{wdeg} N < d - 2$, then [11] implies that $D^{\lambda'} \otimes \operatorname{sgn}$ is equal to the trivial module or sgn. So $D^{\lambda'}$ is the trivial module or sgn in this case. If $D^{\lambda'}$ is trivial, then its diagram is the row of d boxes, therefore the diagram for λ is the column of d boxes and N and M are fundamental modules (recall that their highest weights are determined by the same formula). By [14, Section 5, Example], if $d = k(p-1) + r$ with $0 \leq r < p-1$, then the diagram for sgn consists of r rows of length $k+1$ and $p-1-r$ rows of length k. In this case λ has the diagram of k rows of length $p-1$ and 1 row of length r and so $\omega(N) = (p-1-r)\omega_k^{d-1} + r\omega_{k+1}^{d-1}$ which implies that N and M are truncated symmetric powers of the natural modules. In both cases $\omega(M) \in \Omega_p(G_n)$ which yields a contradiction. Hence $\operatorname{wdeg} M \geq \operatorname{wdeg} N \geq d - 2$. $\square$

Lemma 3.4 *Let* $n \geq d$. *Then* $\operatorname{wdeg} V_n^{\otimes d} = d!$.

Proof. Set $T = V_n^{\otimes d}$. Note that each weight λ of T is of the shape $\lambda = b_1\varepsilon_1^n + \cdots + b_d\varepsilon_d^n$ where $(b_1, \dots, b_d)$ runs over all $b_i \geq 0$ with $b_1 + \cdots + b_d = d$ and $\dim T^\lambda = \frac{d!}{b_1!b_2!\ldots b_d!} \leq d!$. On the other hand, for $\lambda = \varepsilon_1^n + \cdots + \varepsilon_d^n$, this dimension is exactly $d!$. Therefore, $\operatorname{wdeg} V_n^{\otimes d} = d!$. $\square$

Recall the G_n-modules $M_{n,L}(a_1, \dots, a_d) = L(a_1\omega_1^n + \ldots + a_d\omega_d^n)$ and $M_{n,R}(a_1, \dots, a_d) = L(a_d\omega_{n-d+1}^n + \ldots + a_1\omega_n^n)$ $(n \geq d)$ defined in the Introduction.

Lemma 3.5 *Let* $n \geq d$ *and* $M_n = M_{n,L}(a_1, \dots, a_d)$ *or* $M_{n,R}(a_1, \dots, a_d)$. *Set* $H_{n,L} = G_{n+1}(1, \dots, n)$ *and* $H_{n,R} = G_{n+1,n}$. *Then* M_n *is isomorphic to the Smith factor of* M_{n+1} *with respect to the subgroup* $H_{n,L}$ *or* $H_{n,R}$ *for* $M_n = M_{n,L}(a_1, \dots, a_d)$ *or* $M_{n,R}(a_1, \dots, a_d)$, *respectively. In particular,* $M_n \in \operatorname{Irr}_n M_{n+1}$.

Proof. This follows directly from Theorem 2.1. □

Proposition 3.6 *Let $n \geq d$ and $M = L(a_1\omega_1^n + \ldots + a_d\omega_d^n) \in \mathcal{L}_n^d$ or $M = L(a_d\omega_{n-d+1}^n + \ldots + a_1\omega_n^n) \in \mathcal{R}_n^d$. Then $\operatorname{wdeg} M \leq d!$. Moreover, $\operatorname{wdeg} M$ is determined by the sequence $(a_1, \ldots, a_d)$ and does not depend on n.*

Proof. Let $M \in \mathcal{L}_n^d$. We have $\operatorname{pdeg} M = j \leq d$ by Lemma 3.1(i). Set $T = V_n^{\otimes j}$. Observe that $M \in \operatorname{Irr} T$ by the same lemma. Therefore $\operatorname{wdeg} M \leq j! \leq d!$ by Lemma 3.4.

Let $\lambda \in \Lambda(M)$ be dominant. As $\lambda \in \Lambda(T)$, we have $\lambda = b_1\varepsilon_1^n + \ldots + b_j\varepsilon_j^n$ with $b_1 \geq \ldots \geq b_j \geq 0$, $b_i \in \mathbb{Z}_{\geq 0}$, and $b_1 + \ldots + b_j = j$. Set $\omega = \omega(M)$. Then $\lambda = j\omega_1^n - \sum_{i=1}^{j-1} c_i\alpha_i = \omega - \sum_{i=1}^{j-1} d_i\alpha_i$ with c_i, $d_i \in \mathbb{Z}_{\geq 0}$. Denote by M_S the Smith factor of M associated with the subgroup $G_n(1, \ldots, j-1) \cong G_{j-1}$. By Theorem 2.1, $\dim M^\lambda = \dim M_S^{\lambda_S}$ for the weight $\lambda_S = \lambda{\downarrow}G_n(1, \ldots, j-1)$. Since each weight in $\Lambda(M)$ lies in the same orbit with a dominant weight under the action of the Weyl group, we conclude that $\operatorname{wdeg} M = \operatorname{wdeg} M_S$ and hence does not depend on n. To handle the case $M \in \mathcal{R}_n^d$, consider M^*. □

Lemma 3.7 *Let $1 \leq j < k \leq n$, and let $\omega = \sum_{s=j}^k a_s\omega_s^n$ be a dominant p-restricted weight of G_n with both a_j and $a_k \neq 0$. Then*

$$\operatorname{wdeg} L(\omega) \geq k - j.$$

Proof. Write $\omega = a_j\omega_j^n + a_{i_1}\omega_{i_1}^n + \ldots + a_{i_t}\omega_{i_t}^n + a_k\omega_k^n$ with $j < i_1 < \ldots < i_t < k$ and $a_{i_1}, \ldots, a_{i_t} \neq 0$ (t can be zero). By [15, Proposition 1.21], $\operatorname{wdeg} L(\omega) \geq f(j, i_1, \ldots, i_t, k)$, where for l-tuples $(u_1, \ldots, u_l)$ with $u_1 < \ldots < u_l$ the integers $f(u_1, \ldots, u_l)$ are determined by the following recurrent relations:

$$\begin{aligned} f(u_1) &= 1; \\ f(u_1, u_2) &= u_2 - u_1; \\ f(u_1, u_2, \ldots, u_l) &= (u_2 - u_1)f(u_2, \ldots, u_l) + f(u_3, \ldots, u_l) \quad \text{for } l > 2. \end{aligned}$$

We claim that $f(j, i_1, \ldots, i_t, k) \geq k - j$. For $t = 0$ this holds by definition. Then apply induction on t. Let $t > 0$. One easily concludes that $f(u_1, \ldots, u_l) \geq 1$ for all positive integers $u_1, \ldots, u_l$. Now the induction hypothesis yields that

$$f(j, i_1, \ldots, i_t, k) = (i_1 - j)f(i_1, \ldots, i_t, k) + f(i_2, \ldots, i_t, k) \geq (i_1 - j)(k - i_1) + 1.$$

(For $t = 1$ we have $f(j, i_1, k) = (i_1 - j)(k - i_1) + 1$.) Note that $ab \geq a + b$ for a and $b \in \mathbb{N}$ and a, $b > 1$. Hence $ab + 1 \geq a + b$ for all a and $b \in \mathbb{N}$. This yields our claim and completes the proof. □

Propositions 3.3 and 3.6 imply that for groups of type A_n there exist classes of simple modules M with $\operatorname{wdeg} M$ arbitrary large, but small with respect to n. Note that for a generic simple p-restricted module $\operatorname{wdeg} M$ grows with the growth of n.

Proposition 3.8 *Let $M \in \operatorname{Irr}^p G_n$, $\omega(M) \notin \Omega_p(G_n)$, and $n \geq 16$. Assume $\operatorname{pdeg} M > n$ and $\operatorname{pdeg} M^* > n$. Then $\operatorname{wdeg} M > \sqrt{n}/p - 1$.*

Proof. Let $\omega = \sum_{t=i}^{j} a_t\omega_t^n$ with $a_i a_j \neq 0$, $1 \leq i \leq j \leq n$. Due to Lemma 3.7 one can assume that $j - i \leq \sqrt{n}/p - 1$ (otherwise $\operatorname{wdeg} L(\omega) \geq j - i > \sqrt{n}/p - 1$ as required). Put $k = j - i + 1$ and $a = \sum_{t=i}^{j} a_t$. Then $k \leq \sqrt{n}/p$ and

$$a \leq k(p-1) < \sqrt{n}. \tag{9}$$

Passing to M^* if necessary, one can assume $i - 1 \leq n - j$. For $1 \leq i \leq s$ denote by H_s the subgroup $G_n(s, \dots, n) \cong A_{n-s+1}(K)$. So $H_1 = G$ and the rank of H_s is equal to $n - s + 1 > n/2$ for all $s \leq i$.

Let L_s be the Smith factor of $L(\omega)$ with respect to H_s. Then $\operatorname{pdeg} L_s = \operatorname{pdeg} L_i + (i - s)a$ for $1 \leq s \leq i$. Note that

$$\operatorname{pdeg} L_i \leq ka \leq k^2(p-1) \leq n(p-1)/p^2 < n/2$$

since $p \geq 2$.

Fix minimal s such that $\operatorname{pdeg} L_s \leq n/2$. Since $\operatorname{pdeg} L_1 = \operatorname{pdeg} L(\omega) > n$, we have $s > 1$. Then $\operatorname{pdeg} L_{s-1} = \operatorname{pdeg} L_s + a > n/2$, so $\operatorname{pdeg} L_s > n/2 - a$. Applying (9), we get $n/2 - a > n/2 - \sqrt{n}$. As the rank of H_s is greater than $n/2$, by Proposition 3.3,

$$\operatorname{wdeg} L(\omega) \geq \operatorname{wdeg} L_s \geq \operatorname{pdeg} L_s - 2 > n/2 - \sqrt{n} - 2 = \sqrt{n}(\sqrt{n}/2 - 1) - 2 \geq \sqrt{n} - 2 > \sqrt{n}/p - 1$$

since $n \geq 16$ and $p \geq 2$. □

Now we are ready to prove our first main result.

Proof of Theorem 1.6. Part (i) is proved in Proposition 3.8 and part (ii) follows from Lemma 3.1 and Propositions 3.3 and 3.6. □

Corollary 3.9 *Let $M \cong \otimes_{k=0}^{i} M_k^{[k]}$. If at least one of M_k satisfies the assumptions of Proposition* 3.8*, then* $\operatorname{wdeg} M > \sqrt{n}/p - 1$.

Proof. This follows immediately from Lemma 2.14 and Proposition 3.8. □

Now we pass to modules that are not p-restricted.

Lemma 3.10 *Let $M \in \operatorname{Irr} G_n$, $M = N_1 \otimes N_2^{[s]}$, $N_1, N_2 \in \operatorname{Irr} G_n$, and let $\delta(N_1) < p^s$. Then for any weight $\lambda \in \Lambda(M)$ there exists a unique pair (μ, ν) with $\mu \in \Lambda(N_1)$, $\nu \in \Lambda(N_2^{[s]})$, and $\lambda = \mu + \nu$.*

Proof. It is obvious that $\lambda = \mu + \nu$ for some μ and ν. Put $N' = N_2^{[s]}$. Suppose that $\mu + \nu = \mu' + \nu'$ with $\mu' \in \Lambda(N_1)$, $\nu' \in \Lambda(N')$, and $\mu \neq \mu'$. Then $\mu - \mu' = \nu' - \nu$. Acting by the Weyl group, one can assume that $\mu - \mu'$ (and hence $\nu' - \nu$) is dominant. Denote by α_m the maximal root of G_n. Note that $\nu = p^s\xi$ and $\nu' = p^s\xi'$ with ξ and $\xi' \in \Lambda(N_2)$. Therefore

$$p^s\langle \xi' - \xi, \alpha_m \rangle = \langle \nu' - \nu, \alpha_m \rangle = \langle \mu - \mu', \alpha_m \rangle \leq 2\delta(N_1) < 2p^s.$$

This implies that $\langle \xi' - \xi, \alpha_m \rangle = 1$, i.e. $\xi' - \xi$ is a fundamental weight. However, this difference is a radical weight (i.e. a linear combination of roots). This yields a contradiction and proves the lemma. □

Now consider tensor products of certain special modules with relatively small $\operatorname{wdeg} M$.

Theorem 3.11 *Let $d \in \mathbb{N}$ and*

$$M = N_0 \otimes \ldots \otimes N_l \in \operatorname{Irr} G_n.$$

Assume that $\Omega(M) \notin \Omega(G_n)$,

$$N_t = \otimes_{s=i_{t-1}+1}^{i_t} M_s^{[s]} \tag{10}$$

with $i_{-1} = -1$, $i_0 < i_1 < \ldots < i_l$, and for each t, $0 \le t \le l$, one of the following holds: $M_s \in \mathcal{L}_n^d$ for $i_{t-1}+1 \le s \le i_t$, or $M_s \in \mathcal{R}_n^d$ for all these s, or $\omega(N_t) \in \Omega(G_n)$. Let $\delta(N_f) < p^{i_f+1}$ for all N_f with $f < l$ (i_f are such as in (10)). Suppose that $\{u_1, \ldots, u_k\}$ be the set of all indices t for which $\omega(N_t) \notin \Omega(G_n)$. Set $l_j = i_{u_j} - i_{u_j-1} - 1$ for $1 \le j \le k$ and $d_j = d(1+p+\ldots+p^{l_j})$. Assume that $n \ge \max_{1\le j\le k}(d_j)$. Then $\operatorname{wdeg} M \le \prod_{j=1}^k d_j!$*.*

Proof. For $1 \le j \le k$ set $s_j = i_{u_j-1}+1$ and $N'_j = \otimes_{g=0}^{l_j} M_{s_j+g}^{[g]}$. We have $N_{u_j} = (N'_j)^{[s_j]}$. Hence $\operatorname{wdeg} N_{u_j} = \operatorname{wdeg} N'_j$. Apply induction on l. If $l = 0$, it is clear that $k = 1$, $s_1 = 0$, $l_1 = i_0$, and $d_1 = d(1+p+\ldots+p^{i_0})$. Then Proposition 3.2 implies that $M \in \mathcal{L}^{d_1}$ or $\mathcal{R}^{d_1}$. Hence our assertion follows from Proposition 3.6. Assume that $l > 0$ and the assertion holds for $l-1$. Set $M' = N_0 \otimes \ldots \otimes N_{l-1}$. Since $\delta(N_j) < p^{i_j+1}$ for $j < l$, we get $\delta(M') < p^{i_{l-1}+1}$. Then by Lemma 3.10, for each $\lambda \in \Lambda(M)$ there exists a unique pair (μ, ν) with $\mu \in \Lambda(M')$, $\nu \in \Lambda(N_l)$, and $\lambda = \mu + \nu$. Then $\dim M^\lambda = \dim (M')^\mu \dim N_l^\nu$ and hence $\operatorname{wdeg} M = \operatorname{wdeg} M' \operatorname{wdeg} N_l$. By the induction assumptions, $\operatorname{wdeg} M' \le \prod_{j=1}^{k-1} d_j!$ if $u_k = l$ and $\operatorname{wdeg} M' \le \prod_{j=1}^k d_j!$ otherwise. In the first case Proposition 3.2 yields that $N'_l \in \mathcal{L}^{d_k}$ or $\mathcal{R}^{d_k}$. Hence $\operatorname{wdeg} N_l = \operatorname{wdeg} N'_l \le d_k!$ by Proposition 3.6. In the second one $\omega(N_l) \in \Omega(G_n)$ and $\operatorname{wdeg} N_l = 1$. This completes the proof. □

Remark In some cases much stronger estimates can be obtained. In particular, this holds if $n \ge d$, $M = \otimes_{k=0}^f M_k^{[k]}$ with $M_k \in \operatorname{Irr}^p G_n$, and $\delta(M_k) < p$ for all $k < f$. Then, applying Lemma 3.10 and Proposition 3.6, we can deduce that $\operatorname{wdeg} M \le (d!)^N$, where N is the number of indices k for which $\omega(M_k) \notin \Omega_p(G_n)$.

Proposition 3.12 shows that our assumptions on $\delta(N_f)$ play a crucial role in Theorem 3.11.

Proposition 3.12 *Let $i, l \in \mathbb{N}$ with $i < l-1$ and $M, N \in \operatorname{Irr} G_n$. Assume that $\omega(M) = \sum_{t=1}^i a_t\omega_t^n = \sum_{k=0}^j p^k\lambda_k$ with p-restricted λ_k and $\omega(N) = \sum_{t=l}^n b_t\omega_t^n \ne 0$ is p-restricted. Suppose that $\delta(M) \ge p^{j+1}$. Set $Q = M \otimes N^{[j+1]}$. Then* $\operatorname{wdeg} Q \ge l-i-1$*. The same holds if $\omega(M) = \sum_{t=l}^n b_t\omega_t^n$, $\omega(N) = \sum_{t=1}^i a_t\omega_t^n$, and other assumptions of the proposition are valid. In particular, in this situation* $\operatorname{wdeg} Q \ge n-m-i$ *if $M \in \mathcal{L}_n^i$, $N \in \mathcal{R}_n^m$ or vice versa.*

Proof. We will consider the case where $\omega(M) = \sum_{t=1}^i a_t\omega_t^n$ and $\omega(N) = \sum_{t=l}^n b_t\omega_t^n \ne 0$. The proof for the other case is similar.

Taking maximal possible l, we can suppose that $b_l \ne 0$. Put $c = \delta(M)$ and write down the p-adic expansion $c = \sum_{k=0}^u c_kp^k$ with $0 \le c_k < p$.

(a) First assume that $c_{j+1} \ne 0$. Set $\Gamma = G_n(\alpha_1+\ldots+\alpha_i, \alpha_{i+1}, \ldots, \alpha_n)$. Observe that Γ is conjugate to G_{n-i+1}, the group $G_n(i+1, \ldots, n)$ is conjugate to G_{n-i} and $G_n(i+1, \ldots, l)$ is conjugate to G_{l-i}. We have $\langle \omega(M), \alpha_1+\ldots+\alpha_i\rangle = c$. Then one easily concludes that $L(c\omega_1^{n-i+1}) \in \operatorname{Irr}(M{\downarrow}\Gamma) = \operatorname{Irr}_{n-i+1} M$. By the Steinberg tensor product theorem (3), $L(c\omega_1^{n-i+1}) = \otimes_{k=0}^u L(c_k\omega_1^{n-i+1})^{[k]}$. By Corollary 2.8, $L(0) \in \operatorname{Irr}_{n-i} L(c_k\omega_1^{n-i+1})$ for $0 \le k \le u$ and $L(\omega_1^{n-i}) \in \operatorname{Irr}_{n-i} L(c_{j+1}\omega_1^{n-i+1})$. Hence

$$L(p^{j+1}\omega_1^{n-i}) \in \operatorname{Irr}_{n-i} L(c\omega_1^{n-i+1}) \subset \operatorname{Irr}_{n-i} M.$$

So by Theorem 2.1, $L(p^{j+1}\omega_1^{l-i}) \in \mathrm{Irr}_{l-i} M$. Applying Theorem 2.1 to the restriction $N{\downarrow}G_n(i+1,\dots,l)$, we get that $L(b_l\omega_{l-i}^{l-i}) \in \mathrm{Irr}_{l-i} N$. Consequently, $F = L(p^{j+1}(\omega_1^{l-i} + b_l\omega_{l-i}^{l-i})) \in \mathrm{Irr}_{l-i} Q$. Lemma 3.7 and Proposition 2.13 imply that $\operatorname{wdeg} Q \geq l-i-1$.

(b) Now let $c_{j+1} = 0$. Then $\sum_{k=1}^{i} a_k = c \geq p^{j+2}$. Fix minimal s with $\sum_{k=1}^{s} a_k > \sum_{k=0}^{j} c_k p^k$. Put $\Sigma_s = a_1 + \ldots + a_s$ and $c_s = c - \Sigma_s$. Since all $a_k < p^{j+1}$, we get $\Sigma_s < \sum_{k=0}^{j} c_k p^k + p^{j+1}$ and hence $s < i$. Write $\Sigma_s = \sum_{k=0}^{u} d_k p^k$ and $c_s = \sum_{k=0}^{u} g_k p^k$ with $0 \leq d_k < p$ and $0 \leq g_k < p$. One can observe that either $\Sigma_s = \sum_{k=0}^{j} d_k p^k$ or $\Sigma_s = \sum_{k=0}^{j} d_k p^k + p^{j+1}$ with $\sum_{k=0}^{j} d_k p^k < \sum_{k=0}^{j} c_k p^k$. So in both cases $g_{j+1} = p-1$.

Set $H = G_n(s+1,\dots,n)$. Then $H \cong G_{n-s}$. Let M_s be the Smith factor of M with respect to H. Then c_s is the value of $\omega(M_s)$ on the maximal root of H. Now we can proceed as in Part (a) using H, M_s, and the Smith factor of N with respect to H rather than G_n, M, and N. □

4 The Steinberg tensor product theorem for inductive systems

In this section we study arbitrary inductive systems of representations for the sequence (5) and prove an analogue of the Steinberg product theorem for such systems.

Let $\Phi = \{\Phi_n \mid n \in \mathbb{N}\}$ be an inductive system. Put $\delta(\Phi_n) = \{\max \delta(\omega) \mid L(\omega) \in \Phi_n\}$.

Lemma 4.1 *Assume that $n \in \mathbb{N}$ and $n > 2$ for $G_n = B_n(K)$. Then for an inductive system Φ one has $\delta(\Phi_{n+1}) = \delta(\Phi_n)$.*

Proof. Fix any $L(\lambda) \in \Phi_n$ and $L(\mu) \in \Phi_{n+1}$ with $\delta(\Phi_n) = \delta(\lambda)$ and $\delta(\Phi_{n+1}) = \delta(\mu)$. Put $H = G_{n+1}(n)$ for $G_{n+1} = B_{n+1}(K)$ and $H = G_{n+1}(n+1)$ in the other cases. Hence $H \cong A_1(K)$. Recall that G_n is identified with $G_{n+1,n} = G_{n+1}(2,\dots,n+1)$. So we can assume that $H \subset G_n$. Set

$$I_l = \cup_{\varphi \in \Phi_l} \mathrm{Irr}(\varphi{\downarrow}H)$$

for $l = n$ and $n+1$. It is clear that

$$\mathrm{Irr}(\varphi{\downarrow}H) = \cup_{\psi \in \mathrm{Irr}(\varphi{\downarrow}G_n)} \mathrm{Irr}(\psi{\downarrow}H)$$

for $\varphi \in \Phi_{n+1}$. Now it follows from the definition of an inductive system that $I_n = I_{n+1}$. Corollary 2.3 implies that $\delta(\mu) = \max\{i \mid L(i\omega_1^1) \in I_{n+1}\}$ and $\delta(\lambda) = \max\{i \mid L(i\omega_1^1) \in I_n\}$. Hence $\delta(\Phi_{n+1}) = \delta(\Phi_n)$. □

Set $\delta(\Phi) = \delta(\Phi_n)$ for $n > 2$. Lemma 4.1 shows that $\delta(\Phi)$ is well defined.

For the groups of type A the previous lemma was proven in [3, Lemma 2.4]. Note that for any dominant weight $\omega = a_1\omega_1^n + \cdots + a_n\omega_n^n$ of $A_n(K)$ one has $\delta(\omega) = a_1 + a_2 + \cdots + a_n$.

Lemma 4.2 *Let Φ and Ψ be inductive systems of representations. Then $\mathrm{Fr}(\Phi)$ and $\Phi \otimes \Psi$ are inductive systems of representations.*

Proof. The claim on $\mathrm{Fr}(\Phi)$ follows immediately from the definition of an inductive system since for $M \in \mathrm{Irr}\, G_{n+1}$

$$\mathrm{Irr}_n(M^{[1]}) = \{\mu^{[1]} \mid \mu \in \mathrm{Irr}_n M\}.$$

Clearly, the set $(\Phi \otimes \Psi)_n$ is finite. It remains to note that restricting representations to subgroups commutes with taking tensor products. □

Lemma 4.3 *Let $T \subset \mathbb{N}$ be infinite. Assume that $R_t \subset \operatorname{Irr} G_t$ is nonempty for each $t \in T$ and that there exists $k \in \mathbb{N}$ such that $\delta(\varphi) < k$ for all $\varphi \in R_t$ and all t. Denote by Π_n the set of all $\pi \in \operatorname{Irr} G_n$ such that π is a composition factor of the restriction $\mu \downarrow G_n$ for some $t > n$, $t \in T$, and $\mu \in R_t$. Suppose also that $R_t \subset \Pi_t$ for all t. Then $\Pi = \{\Pi_n \mid n \in \mathbb{N}\}$ is an inductive system of representations.*

Proof. Let $\rho \in \Pi_{n+1}$. The construction of Π implies that there exist $t > n+1$ and $\psi \in R_t$ with $\rho \in \operatorname{Irr}_{n+1} \psi$. So if $\varphi \in \operatorname{Irr}_n \rho$, then $\varphi \in \operatorname{Irr}_n \psi$ and hence $\varphi \in \Pi_n$. On the other hand, for each $\mu \in \Pi_n$ there exist $u > n$ and $\nu \in R_u$ with $\mu \in \operatorname{Irr}_n \nu$. If $u > n+1$, the set $\operatorname{Irr}_{n+1} \nu \subset \Pi_{n+1}$ and, obviously, $\mu \in \operatorname{Irr}_n \lambda$ for some $\lambda \in \operatorname{Irr}_{n+1} \nu$. Since $R_{n+1} \subset \Pi_{n+1}$ by the assumptions of the lemma, for $u = n+1$ the representation $\mu \in \operatorname{Irr}_n \lambda$ for some $\lambda \in \Pi_{n+1}$ as well. It remains to show that Π_n is finite. As $\Pi_1 = \bigcup_{\rho \in \Pi_2} \operatorname{Irr}_1 \rho$, we can assume that $n > 1$. It follows from Corollary 2.4 that $\delta(\varphi) \leq k$. It is clear that the number of inequivalent irreducible representations of G_n with this property is finite. □

Corollary 4.4 *Lemma* 4.3 *holds if we replace the condition that $\delta(\varphi) < k$ for all $\varphi \in R_t$ and all t, by the condition that there exists an inductive system Φ with $R_t \subset \Phi_t$ for all t.*

Proof. Corollary 2.4 implies that $\delta(\pi) < \delta(\Phi)$ for all $\pi \in R_t$. So we can apply Lemma 4.3. □

Definition 4.5 Let $\Psi \subset \Phi$ be inductive systems of representations and the embedding be proper. Put $\Xi_n = \Phi_n \setminus \Psi_n$. Denote by $D(\Phi, \Psi)$ the inductive system of representations generated by Ξ_n and call it the *difference* of two inductive systems.

It is shown in [4, Section 4] that $D(\Phi, \Psi)$ is well defined. (We emphasize that though [4] is devoted to general linear and special linear groups, the arguments on the difference of induction systems at the beginning of Section 4 of that paper hold for inductive systems for the sequence (5) for all four series of the classical groups.) Since the embedding is proper, for any $n \in \mathbb{N}$ there exists $n_0 > n$ such that the set $\Xi_{n_0} \neq \varnothing$. Hence $D(\Phi, \Psi)_n \neq \varnothing$ for all n. One obviously has $\Phi = \Psi \cup D(\Phi, \Psi)$.

Lemma 4.6 *Let Φ be an indecomposable inductive system. Then for each two representations $\varphi \in \Phi_k$ and $\psi \in \Phi_l$ there exist $m > \max\{k, l\}$ and $\xi \in \Phi_m$ such that $\varphi \in \operatorname{Irr}_k \xi$ and $\psi \in \operatorname{Irr}_l \xi$.*

Proof. Set $t = \max\{k, l\}$. For each $n > t$ put $P_n = \{\rho \in \Phi_n \mid \varphi \in \operatorname{Irr}_k \rho\}$. It is clear that $P_n \neq \varnothing$ and for any $\mu \in P_n$ there exists $\nu \in P_{n+1}$ such that $\mu \in \operatorname{Irr}_n \nu$. Hence $\mathcal{P} = \langle P_n \mid n > t \rangle$ is an inductive system by Corollary 4.4. We claim that $\mathcal{P} = \Phi$. Indeed, otherwise $D(\Phi, \mathcal{P}) = \Phi$ as Φ is indecomposable. However, $\varphi \notin \operatorname{Irr}_k \psi$ if $\psi \in \Phi_n \setminus P_n$ by the construction of P_n. This yields a contradiction as $D(\Phi, \mathcal{P})$ is generated by the collection $\Phi_n \setminus P_n$. Hence $\mathcal{P} = \Phi$. So there exists $m > t$ such that $\psi \in \operatorname{Irr}_l \rho$ for $\rho \in P_m$. □

Corollary 4.7 *Let Φ be an indecomposable inductive system and let $\varphi_1 \in \Phi_{n_1}, \ldots, \varphi_l \in \Phi_{n_l}$. Then there exist $m > \max\{n_1, \ldots, n_l\}$ and $\xi \in \Phi_m$ such that $\varphi_j \in \operatorname{Irr}_{n_j} \xi$ for $1 \leq j \leq l$.*

Proof. Use Lemma 4.6 and induction on l. □

Proof of Theorem 1.5. Since $\delta(\varphi) \leq \delta(\Phi)$ for all n and all $\varphi \in \Phi_n$, by the Steinberg tensor product theorem (3) there exists an integer $k = k(\Phi)$ such that $\varphi = \varphi_0 \otimes \varphi_1^{[1]} \otimes \ldots \otimes \varphi_k^{[k]}$ with $\varphi_j \in \operatorname{Irr}^p G_n$, for all n and all $\varphi \in \Phi_n$. Fix minimal such k. Then the representations φ_j, $0 \leq j \leq k$, are uniquely determined (some of them can be trivial). We will use this notation until the end of the proof.

Set

$$\begin{aligned} S_n &= \{\varphi \in \Phi_n \mid \delta(\varphi) = \delta(\Phi)\}, \quad S = \cup_{n=1}^{\infty} S_n, \\ S_n^0 &= \{\varphi \in S_n \mid \delta(\varphi_0) = \max_{\psi \in S} \delta(\psi_0)\}, \quad S^0 = \cup_{n=1}^{\infty} S_n^0, \\ S_n^{0,\ldots,j} &= \{\varphi \in S_n^{0,\ldots,j-1} \mid \delta(\varphi_j) = \max_{\psi \in S^{0,\ldots,j-1}} \delta(\psi_j)\}, \quad S^{0,\ldots,j} = \cup_{n=1}^{\infty} S_n^{0\ldots j} \end{aligned}$$

for $1 \leq j \leq k-1$. Set $T_n = S_n^{0,1,\ldots,k-1}$ and $T_n^j = \{\varphi_j \mid \varphi \in T_n\}$ for $0 \leq j \leq k$. The sets T_n^j will be used to generate tensor factors for Φ.

Since $\delta(\varphi) \leq \delta(\Phi)$ for all $\varphi \in \Phi_l$ and all l, it is clear that T_n is well defined and $T_n \neq \varnothing$ for some n. Choose minimal n with this property and denote it by $n_{\min}$. Now we shall prove the following claim: if $m > n \geq n_{\min}$, $\varphi \in T_n$, $\psi \in \Phi_m$, and $\varphi \in \operatorname{Irr}_n \psi$, then

$$\delta(\varphi_j) = \delta(\psi_j), \varphi_j \in \operatorname{Irr}_n \psi_j \tag{11}$$

for $0 \leq j \leq k$. Hence such $\psi \in T_m$.

Fix $\psi \in \Phi_m$ with $\varphi \in \operatorname{Irr}_n \psi$ (such ψ do exists as Φ is an inductive system). Since restricting to subgroups commutes with the morphism Fr and taking tensor products, one can observe that

$$\operatorname{Irr}_n \psi = \cup_{(\tau^0,\ldots,\tau^k)} \operatorname{Irr}(\otimes_{j=0}^k (\tau^j)^{[j]}),$$

where the union is taken over all tuples $(\tau^0, \ldots, \tau^k)$ with $\tau^j \in \operatorname{Irr}_n \psi_j$. Fix a tuple $(\tau^0, \ldots, \tau^k)$ that yields φ and set $\tau = \otimes_{j=0}^k (\tau^j)^{[j]}$. In fact, we shall show that all $\tau^j \in \operatorname{Irr}^p G_n$ and so $\tau^j = \tau_j$ for $0 \leq j \leq k$, but this requires some explanations. One has $\tau = \tau_0^0 \otimes \rho^{[1]}$, where ρ is a representation of G_n (not necessarily irreducible). The Steinberg tensor product theorem implies that each representation in $\operatorname{Irr} \tau$ has the form $\tau_0^0 \otimes \lambda^{[1]}$ with $\lambda \in \operatorname{Irr} G_n$. Hence $\varphi_0 = \tau_0^0$. Similar arguments yield that if $0 < l \leq k$ and $\tau^0, \ldots, \tau^{l-1} \in \operatorname{Irr}^p G_n$, then $\operatorname{Irr} \tau$ consists of representations of the form $(\otimes_{j=0}^{l-1} (\tau^j)^{[j]}) \otimes (\tau_0^l)^{[l]} \otimes \mu^{[l+1]}$ with $\mu \in \operatorname{Irr} G_n$ and therefore in this case

$$\tau^j = \varphi_j \text{ for } 0 \leq j \leq l-1, \quad \varphi_l = \tau_0^l. \tag{12}$$

Obviously, we have $\delta(\rho) = \sum_{j=0}^k p^j \delta(\rho_j)$ for each $\rho \in \operatorname{Irr} G_l$ and all l. By Corollary 2.4, $\delta(\varphi) \leq \delta(\psi)$ and $\delta(\tau^j) \leq \delta(\psi_j)$ for $0 \leq j \leq k$. This implies that $\delta(\varphi_0) \leq \delta(\psi_0)$ and $\psi \in S_m$ as $\varphi \in S_n$. Now we start proving (11) using the induction on j. At each step we shall also show that $\tau^j \in \operatorname{Irr}^p G_n$. Since $\varphi \in S_n^0$ and $\varphi_0 = \tau_0^0$, we conclude that $\delta(\varphi_0) = \delta(\tau^0) = \delta(\psi_0)$ and $\tau^0 \in \operatorname{Irr}^p G_n$. So $\varphi_0 \in \operatorname{Irr}_n \psi_0$ and (11) holds for $j = 0$. It is clear that $\psi \in S_m^0$. Now let $0 < j < k$ and assume that for $0 \leq l < j$ Formula (11) holds and $\tau^l \in \operatorname{Irr}^p G_n$. The construction of the sets $S^{0,\ldots,t}$ yields that $\psi \in S^{0,\ldots,j-1}$. By (12), $\varphi_j = \tau_0^j$. As $\varphi \in S_n^{0,\ldots,j}$ and $\delta(\varphi_j) \leq \delta(\tau^j) \leq \delta((\psi_j)$, we can deduce that $\delta(\varphi_j) = \delta(\tau^j) = \delta(\psi_j)$ and $\tau^j \in \operatorname{Irr}^p G_n$. So $\varphi_j \in \operatorname{Irr}_n \psi_j$ and (11) holds for j. Finally, suppose that (11) is valid and $\tau^j \in \operatorname{Irr}^p G_n$ for $0 \leq j < k$. The choice of k shows that $\tau^k \in \operatorname{Irr}^p G_n$. Then $\tau^k = \varphi_k$ by (12) and hence $\varphi_k \in \operatorname{Irr}_n \psi_k$. Naturally, $\delta(\varphi_k) = \delta(\psi_k)$ since φ and $\psi \in S$ and $\delta(\varphi_j) = \delta(\psi_j)$ for $0 \le j < k$. This completes the proof of the claim.

Now it is clear that T_n and hence all $T_n^j \ne \varnothing$ for $n \ge n_{\min}$. Let $\mu \in T_n^j$ with $0 \le j \le k$. Then $\mu = \rho_j$ for some $\rho \in T_n$. We have shown above that there exists $\lambda \in T_{n+1}$ with $\rho \in \operatorname{Irr}_n \lambda$ and $\rho_j \in \operatorname{Irr}_n \lambda_j$. Naturally, $\lambda_j \in T_{n+1}^j$. It is clear that $\delta(\mu) \le \delta(\varphi)$. Now Lemma 4.3 yields that the collections $\Theta^j = \langle T_n^j \rangle$ are inductive systems. Put $\Theta = \otimes_{j=0}^k \operatorname{Fr}^j(\Theta^j)$ and prove that $\Phi = \Theta$. As Φ is indecomposable, Lemma 4.6 implies that for every $\varphi \in \Phi_n$ and $\psi \in T_k$ with $k \ge n_{\min}$ there exists $m > \max\{n, k\}$ and $\rho \in \Phi_m$ with $\varphi \in \operatorname{Irr}_n \rho$ and $\psi \in \operatorname{Irr}_k \rho$. It follows from Formula (11) and the phrase just below this formula that $\rho \in T_m$. Hence the construction of Θ yields that $\Phi \subset \Theta$. By the definition of a tensor product of inductive systems, now it suffices to prove the following: if $\rho = \otimes_{j=0}^k \rho_j^{[j]}$ with $\rho_j \in \Theta_n^j$, then $\operatorname{Irr} \rho \subset \Phi_n$. The construction of the systems Θ^j implies that there exist $m > n$ and representations $\theta_j \in T_m^j$ with $\rho_j \in \operatorname{Irr}_n \theta_j$. Set $\theta = \otimes_{j=0}^k \theta_j^{[j]}$. As Φ is an inductive system, now it remains to show that $\theta \in \Phi_m$. By the definition of T_m^j, there exist representations $\psi^j \in T_m$ with $\theta_j = \psi_j^j$. Since Φ is indecomposable, Corollary 4.7 implies that for some $l > m$ there exists $\zeta \in \Phi_l$ with $\psi^j \in \operatorname{Irr}_m \zeta$. By Formula (11), $\psi_j^j \in \operatorname{Irr}_m \zeta_j$ for $0 \le j \le k$. Hence $\theta \in \operatorname{Irr}_m \zeta \subset \Phi_m$ as desired. □

To describe BWM-systems, we also need the following lemma on tensor products of inductive systems that are generated by collections R_n that consist of a single p-restricted representation of G_n.

Lemma 4.8 *Let $j \in \mathbb{N}$ and $M_{nt} \in \operatorname{Irr}^p G_n$ for $0 \le t \le j$ and $n \ge d_t$. Assume that $\delta(M_{nt}) \le c$ for some constant c and $M_{nt} \in \operatorname{Irr}_n M_{n+1,t}$ for $0 \le j \le t$ and $n \ge d_t$. Set $d = \max\{d_t \mid 0 \le t \le j\}$ and $M_n = \otimes_{t=0}^j M_{nt}^{[t]}$ for $n \ge d$. Then $\langle M_{nt} \mid n \ge d_t \rangle$ and $\langle M_n \mid n \ge d \rangle$ are inductive systems and*

$$\langle M_n \mid n \ge d \rangle = \otimes_{t=0}^j \operatorname{Fr}^t \langle M_{nt} \mid n \ge d_t \rangle.$$

Proof. Set $\mathcal{M} = \langle M_n \mid n \ge d \rangle$ and $\mathcal{M}^t = \langle M_{nt} \mid n \ge d_t \rangle$. By the Steinberg tensor product theorem (3), the modules M_n are irreducible. Observe that $\delta(M_n) \le c \sum_{t=0}^j p^t$ and $M_n \in \operatorname{Irr}_n M_{n+1}$. Now Lemma 4.3 implies that $\mathcal{M}$ and $\mathcal{M}^t$ are inductive systems. Put $\mathcal{P} = \otimes_{t=0}^j \operatorname{Fr}^t \langle M_{nt} \mid n \ge d_t \rangle$. As $M_n \in \mathcal{P}_n$ and $\mathcal{P}$ is an inductive system, $\mathcal{M} \subset \mathcal{P}$. Taking into account the definition of a tensor product of inductive systems, it remains to prove that for each collection $(N_0, \dots, N_j)$ with $N_t \in \mathcal{M}_n^t$ the set $S = \operatorname{Irr}(\otimes_{t=0}^j N_t^{[t]}) \subset \mathcal{M}_n$. As $M_{nt} \in \operatorname{Irr}_n M_{n+1,t}$ and the sets $\mathcal{M}_n^t$ are finite, the construction of the systems $\mathcal{M}^t$ implies that for q large enough $\mathcal{M}_n^t \subset \operatorname{Irr}_n M_{qt}$ for all t, $0 \le t \le j$. Hence $S \subset \operatorname{Irr}_n M_q \subset \mathcal{M}_n$. This completes the proof. □

5 Inductive systems with bounded weight multiplicities for special linear groups

In this section we classify the BWM-systems for $G_n = A_n(K)$. We will denote by $\mathbb{N}_j$ the set of integers s with $0 \le s \le j$.

Recall the collections $\mathcal{L}^l$, $\mathcal{R}^l$, $\mathcal{F}$, and $\mathcal{T}$ defined in the Introduction.

Lemma 5.1 *The collections $\mathcal{L}^l$, $\mathcal{R}^l$ ($l \in \mathbb{N}$), $\mathcal{F}$, and $\mathcal{T}$ are inductive systems of representations for the groups $A_n(K)$.*

Proof. By Lemma 2.5(i), $\operatorname{Irr}_{n-1} V_n = \{L(0), V_{n-1}\}$. Hence $\operatorname{Irr} V_n^{\otimes l} \subset \operatorname{Irr}_n V_{n+1}^{\otimes l}$, $\operatorname{Irr}_{n-1} V_n^{\otimes l} \subset \cup_{j\leq l} \operatorname{Irr} V_{n-1}^{\otimes j}$ and $\operatorname{Irr}_{n-1} \varphi \in \mathcal{L}_{n-1}^l$ for any $\varphi \in \mathcal{L}_n^l$. Consequently, $\mathcal{L}^l$ is an inductive system. The proof for $\mathcal{R}^l$ is similar.

For $\mathcal{F}$ and $\mathcal{T}$ the lemma follows from Lemma 2.5 and Lemma 2.6, respectively. This completes the proof. □

Recall the G_n-modules $M_{n,L}(a_1, \ldots, a_d) = L(a_1\omega_1^n + \ldots + a_d\omega_d^n)$ and $M_{n,R}(a_1, \ldots, a_d) = L(a_d\omega_{n-d+1}^n + \ldots + a_1\omega_n^n)$ $(n \geq d)$ defined in the Introduction.

Lemma 5.2 *The systems*

$$C_L(a_1, \ldots, a_d) = \langle M_{n,L}(a_1, \ldots, a_d) \mid n \geq d\rangle$$

and

$$C_R(a_1, \ldots, a_d) = \langle M_{n,R}(a_1, \ldots, a_d) \mid n \geq d\rangle$$

are well defined.

Proof. For $n \geq d$ set $M_n = M_{n,L}(a_1, \ldots, a_d)$ or $M_{n,R}(a_1, \ldots, a_d)$ (the index "L" or "R" is the same for all n). Obviously, $\delta(M_n) = a_1 + \ldots + a_d$. By Lemma 3.5, $M_n \in \operatorname{Irr}_n M_{n+1}$. It remains to apply Lemma 4.3. □

Proposition 5.3 *Assume that $S_1 \cup S_2 \cup S_3 = \mathbb{N}_j$, $S_i \cap S_k = \varnothing$ for $i \neq k$, $S_3 = \varnothing$ if $p = 2$, and $S_2 \cup S_3 \neq \varnothing$. If $S_1 \neq \varnothing$, for each $k \in S_1$ set $M_{n,k} = M_{n,L}(a_{1k}, \ldots, a_{dk})$ or $M_{n,R}(a_{1k}, \ldots, a_{dk})$, where $0 \leq a_{1k}, \ldots, a_{dk} < p$ and the index "L" or "R" and the sequence $a_{1k}, \ldots, a_{dk}$ are the same for all $n \geq d$. Put $\Psi^k = \langle M_{n,k} \mid n \geq d\rangle$ for $k \in S_1$, $\Psi^k = \mathcal{F}$ for $k \in S_2$, $\Psi^k = \mathcal{T}$ for $k \in S_3$, and $\Psi = \otimes_{k=0}^j \operatorname{Fr}^k(\Psi^k)$. Let Φ be an inductive system. Assume that for each l there exist n and a module $\varphi = \otimes_{k=0}^j \varphi_k^{[k]} \in \Phi_n$ with the following properties:*

$$\varphi_k = M_{n,k} \text{ for } k \in S_1; \tag{13}$$
$$\varphi_k \in \mathcal{F}_n \text{ for } k \in S_2; \tag{14}$$
$$\varphi_k \in \mathcal{T}_n \text{ for } k \in S_3; \tag{15}$$
$$\varphi_k \notin \mathcal{L}_n^l \cup \mathcal{R}_n^l \text{ for } k \in S_2 \cup S_3. \tag{16}$$

Then $\Psi \subset \Phi$.

Proof. The construction of Ψ and the definition of a tensor product of inductive systems imply that for each $\psi \in \Psi_t$ there exist $m > \max\{d, t\}$ and a G_m-module $\pi = \otimes_{k=0}^j \pi_k^{[k]}$ with $\pi_k = M_{m,k}$ for $k \in S_1$, $\pi_k \in \mathcal{F}_m$ for $k \in S_2$, and $\pi_k \in \mathcal{T}_m$ for $k \in S_3$, such that $\psi \in \operatorname{Irr}_t \pi$. So it suffices to prove that all such modules $\pi \in \Phi_m$. Put $l = (p-1)(m+1)$ and choose $n > m$ and $\varphi \in \Phi_n$ that satisfies (13)–(16) for this l. Then Lemmas 2.5 and 3.5 and Corollary 2.7 imply that $\pi_k \in \operatorname{Irr}_m \varphi_k$ for all $k \in \mathbb{N}_j$. Hence $\pi \in \operatorname{Irr}_m \varphi \subset \Phi_m$. This completes the proof. □

Note that S_1 can be empty.

Corollary 5.4 *Set $n' = \left[\frac{n+1}{2}\right]$ and $F_n = L(\omega_{n'}^n) \in \operatorname{Irr} G_n$. Then $\mathcal{F} = \langle F_n \mid n \in \mathbb{N}\rangle$.*

Proof. Lemma 2.5 implies that $F_n \in \operatorname{Irr}_n F_{n+1}$. Hence $\langle F_n \mid n \in \mathbb{N}\rangle$ is an inductive system by Lemma 4.3. Naturally, for each d there exists n with $F_n \notin \mathcal{L}^d_n \cup \mathcal{R}^d_n$. Now apply Proposition 5.3. □

Corollary 5.5 *Define n' as in Corollary 5.4 and set $T_n = L((p-1)\omega^n_{n'})$. Then $\mathcal{T} = \langle T_n \mid n \in \mathbb{N}\rangle$.*

Proof. Argue as in the proof of Corollary 5.4, applying Lemmas 2.6 and 4.3, and Propositions 3.2 and 5.3. □

Proposition 5.6 *Let $\Phi \subset \mathcal{L}^d$ or $\mathcal{R}^d$. Then Φ is a finite union of systems $C_L(a_1, \dots, a_d)$ or $C_R(a_1, \dots, a_d)$, respectively.*

Proof. We shall prove the claim for $\mathcal{L}^d$. The proof for $\mathcal{R}^d$ is similar. Assume that $\Phi \subset \mathcal{L}^d$. For a d-tuple $s = (a_1, \dots, a_d)$ with $a_j \in \mathbb{Z}_{\geq 0}$ set $M_{n,L}(s) = L(a_1\omega^n_1 + \ldots + a_d\omega^n_d)$ and $C_L(s) = C_L(a_1, \dots, a_d)$. Denote by S_d the set of all such tuples with $a_1+2a_2+\ldots+da_d \leq d$. Obviously, the set S_d is finite. By Proposition 3.2, $\mathcal{L}^d_n = \{M_{n,L}(s) \mid s \in S_d\}$. Let $S(\Phi) = \{s \in S_d \mid C_L(s) \subset \Phi\}$ and $\Psi = \cup_{s\in S(\Phi)} C_L(s)$. We claim that $\Psi = \Phi$. Suppose this is not the case and set $D_n = \Phi_n \setminus \Psi_n$. Then $D_n \neq \varnothing$ for large enough n. Hence there exists $\sigma \in S_d$ for which the set $\{n \mid M_{n,L}(\sigma) \in D_n\}$ is infinite. Lemma 3.5 implies that $M_{n,L}(\sigma) \in \operatorname{Irr}_n M_{k,L}(\sigma)$ for $k > n$. Since Φ is an inductive system, this forces $C_L(\sigma) \subset \Phi$ and yields a contradiction. Hence $\Psi = \Phi$ as desired. □

Corollary 5.7 *If $\Phi \subset \mathcal{L}^d$ or $\mathcal{R}^d$ is an indecomposable inductive system, then $\Phi = C_L(a_1, \dots, a_d)$ or $C_R(a_1, \dots, a_d)$, respectively.*

Lemma 5.8 *Let $\Phi \subset \mathcal{L}^a \cup \mathcal{R}^b$, but $\Phi \not\subset \mathcal{L}^a$ and $\Phi \not\subset \mathcal{R}^b$. Then $\Phi = \Phi^L \cup \Phi^R$ where Φ^L and Φ^R are proper subsystems of Φ, $\Phi^L \subset \mathcal{L}^a$, and $\Phi^R \subset \mathcal{R}^b$.*

Proof. Set $\Pi_n = \Phi_n \cap \mathcal{L}^a_n$, $\Sigma_n = \Phi_n \cap \mathcal{R}^b_n$. Observe that $\Pi_n \cap \Sigma_n = \varnothing$ for $n \geq a+b$. As $\mathcal{L}^a$ and $\mathcal{R}^b$ are inductive systems, this implies the following: if $n \geq a+b$, $\varphi \in \Pi_n$ or Σ_n, $\psi \in \Phi_{n+1}$, and $\varphi \in \operatorname{Irr}_n \psi$, then $\psi \in \Pi_{n+1}$ or Σ_{n+1}, respectively. Since Φ is an inductive system, we conclude that for every $\varphi \in \Pi_n$ or Σ_n there exists $\rho \in \Pi_{n+1}$ or Σ_{n+1}, respectively, with $\varphi \in \operatorname{Irr}_n \rho$. Now Corollary 4.4 yields that the inductive systems $\Phi^L = \langle \Pi_n \mid n \geq a+b\rangle$ and $\Phi^R = \langle \Sigma_n \mid n \geq a+b\rangle$ are well defined. It is clear that $\Phi_n = \Phi^L_n \cup \Phi^R_n$. Hence $\Phi = \Phi^L \cup \Phi^R$. □

Now we start describing BWM-systems for groups of type A_n. Note that $\mathcal{F} = \mathcal{T}$ for $p = 2$, but this does not affect the proofs.

Proposition 5.9 *Let Φ be a p-restrictedly generated BWM-system. Then one of the following holds:*

(1) $\Phi = \mathcal{F}$;
(2) $\Phi = \mathcal{T}$;
(3) $\Phi = \mathcal{F} \cup \mathcal{T}$;
(4) $\Phi \subset \mathcal{L}^d \cup \mathcal{R}^d$;
(5) $\Phi = \Phi' \cup \mathcal{T}$, $\Phi = \Phi' \cup \mathcal{F}$, *or* $\Phi = \Phi' \cup \mathcal{F} \cup \mathcal{T}$ *with* $\Phi' \subset \mathcal{L}^d \cup \mathcal{R}^d$.

In all cases, if $\operatorname{wdeg} \Phi = k$, *then* $\Phi \subset \mathcal{L}^{k+2} \cup \mathcal{R}^{k+2} \cup \mathcal{F} \cup \mathcal{T}$.

Proof. Assume that $\operatorname{wdeg}\Phi = k$. First suppose that $\Phi \not\subset \mathcal{F} \cup \mathcal{T}$. Then $\Phi_n \not\subset \mathcal{F}_n \cup \mathcal{T}_n$ for large enough n. Set $m = (k+1)^2p^2$, fix $n > m$ and a p-restricted $\varphi \in \Phi_n \setminus \{\mathcal{F}_n \cup \mathcal{T}_n\}$. Proposition 3.8 implies that $\operatorname{pdeg}\varphi$ or $\operatorname{pdeg}\varphi^* \leq n$ since otherwise $\operatorname{wdeg}\varphi > \sqrt{n}/p - 1 > k$. Now Proposition 3.3 forces that $\operatorname{pdeg}\varphi$ or $\operatorname{pdeg}\varphi^* \leq k+2$ and hence $\varphi \in \mathcal{L}_n^{k+2}$ or $\mathcal{R}_n^{k+2}$ by Proposition 3.2. This yields the last claim of the proposition.

Now we want to reduce the problem to the situation where both $\mathcal{F} \not\subset \Phi$ and $\mathcal{T} \not\subset \Phi$. Assume that this is not the case. Put $\Psi = \mathcal{F}$ if $\mathcal{F} \subset \Phi$, but $\mathcal{T} \not\subset \Phi$; $\Psi = \mathcal{T}$ if $\mathcal{T} \subset \Phi$, but $\mathcal{F} \not\subset \Phi$; and $\Psi = \mathcal{F} \cup \mathcal{T}$ if $\mathcal{F} \cup \mathcal{T} \subset \Phi$. If $\Psi = \Phi$, the proposition is proved. Assume that $\Psi \neq \Phi$ and put $D = D(\Phi, \Psi)$. We claim that both $\mathcal{F} \not\subset D$ and $\mathcal{T} \not\subset D$.

If $\Psi \neq \mathcal{T} \cup \mathcal{F}$, define an inductive system D' by the equality $\{\Psi, D'\} = \{\mathcal{F}, \mathcal{T}\}$. The arguments in the first paragraph of the proof yield that if $n > m$ and $\varphi \in (\Phi_n \setminus \Psi_n)$, then $\varphi \in \mathcal{L}_n^{k+2} \cup \mathcal{R}_n^{k+2}$ or $\mathcal{L}_n^{k+2} \cup \mathcal{R}_n^{k+2} \cup D'_n$. Since $D = \langle \Phi_n \setminus \Psi_n \mid n > m \rangle$, we observe that $D \subset \mathcal{L}^{k+2} \cup \mathcal{R}^{k+2}$ or $D \subset \mathcal{L}^{k+2} \cup \mathcal{R}^{k+2} \cup D'$ which yields our claim. Replacing Φ by D if necessary, we assume that both $\mathcal{F} \not\subset \Phi$ and $\mathcal{T} \not\subset \Phi$.

Proposition 5.3 implies that for some l the intersections $\Phi_n \cap \mathcal{F}_n$ and $\Phi_n \cap \mathcal{T}_n \subset \mathcal{L}_n^l \cup \mathcal{R}_n^l$ for all n. Put $d = \max(l, k+2)$. Then the last claim of the proposition implies that $\Phi_n \subset \mathcal{L}_n^d \cup \mathcal{R}_n^d$ and hence $\Phi \subset \mathcal{L}^d \cup \mathcal{R}^d$. □

Proof of Theorem 1.6. The theorem follows immediately from Propositions 5.6 and 5.9, Corollaries 5.4, 5.5, and 5.7, and Lemma 5.8. □

Let Φ be an inductive system with $\delta(\Phi) < p^{j+1}$ for some $j \in \mathbb{Z}_{\geq 0}$. Then each $\varphi \in \Phi_n$ can be uniquely represented in the form $\otimes_{k=0}^{j}\varphi_k^{[k]}$ with $\varphi_k \in \operatorname{Irr}^p G_n$. This notation is used in Proposition 5.10.

Proposition 5.10 *Let Φ be a BWM-system with $\delta(\Phi) < p^{j+1}$. Then there exists an integer $N = N(\operatorname{wdeg}\Phi, j)$ with the following properties: if $d \geq N$, U_1, $U_2 \subset \mathbb{N}_j$, $U_1 \cap U_2 = \varnothing$, $U_2 = \varnothing$ for $p = 2$, $\varphi \in \Phi_n$, $\varphi_k \in \mathcal{F}_n$ for all $k \in U_1$, $\varphi_k \in \mathcal{T}_n$ for all $k \in U_2$, $\varphi_k \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$ for each $k \in U_1 \cup U_2$, and $\varphi \in \operatorname{Irr}_n \psi$ with $\psi \in \Phi_q$, $q > n$, then $\psi_k \in \mathcal{F}_q$ for $k \in U_1$, $\psi_k \in \mathcal{T}_q$ for $k \in U_2$, and $\psi_k \notin \mathcal{L}_q^d \cup \mathcal{R}_q^d$ for $k \in U_1 \cup U_2$.*

Proof. Let $\operatorname{wdeg}\Phi = c$. Proposition 3.12 yields that for all a, $b \in \mathbb{N}$ there exists $t = t(a, b)$ such that the following holds: if $n > t$, $M = \otimes_{k=0}^{s} M_k^{[k]}$ with $M_k \in \operatorname{Irr}^p G_n$, all $M_k \in \mathcal{L}_n^a$ or all $M_k \in \mathcal{R}_n^a$, $\delta(M) \geq p^{s+1}$, $F \in \mathcal{F}_n$ or $\mathcal{T}_n$, and $F \notin \mathcal{L}_n^t$ or $\mathcal{R}_n^t$, respectively, then $\operatorname{wdeg}(M \otimes (F^{[s+1]})) > b$. One may assume that $t(a, b) \geq a + 2b$. Now fix

$$t_1 = t(c+2, c) \text{ and } t_k = t(t_{k-1}, c) \text{ for } 1 < k \leq j. \tag{17}$$

Hence

$$t_j > \ldots > t_1 \geq 3c + 2.$$

Set $g = c + 2 + \sum_{k=1}^{j} t_k p^k$ and $N = \max(g, (c+1)^2p^2 + 1)$.

Let $n > N$, $\varphi \in \Phi_n$ and satisfy the assumptions of the proposition with this N and some $d \geq N$. Assume that $\psi \in \Phi_q$ and $\varphi \in \operatorname{Irr}_n \psi$. Arguing as in the first paragraph of the proof of Proposition 5.9, one can conclude that for all k

$$\psi_k \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2} \cup \mathcal{F} \cup \mathcal{T}. \tag{18}$$

We claim that $\varphi_k \in \operatorname{Irr}_n \psi_k$ for $k \in U_1 \cup U_2$. To prove this, we shall show that $\delta(\otimes_{s=0}^{k-1}\psi_s^{[s]}) < p^k$ if $k \in U_1 \cup U_2$ and $k > 0$. For $k > 0$ and $l < k$ put $\pi(l, k) = \otimes_{s=l}^{k-1}\psi_s^{[s]}$,

$\pi(k) = \pi(0,k)$, $\rho(l,k) = \otimes_{s=l}^{k}\psi_s^{[s]}$, and $\rho(k) = \rho(0,k)$. Assume that $\delta(\pi(k)) \geq p^k$ for some $k \in U_1 \cup U_2$. If there exists $i < k$ with $\delta(\pi(i)) < p^i$, choose maximal such i and put $l = i$. Otherwise put $l = 0$. Then $\delta(\pi(l,k)) \geq p^k$. One easily observes that $\delta(\psi_l) \geq p$ since otherwise $\delta(\pi(l+1)) < p^{l+1}$, which contradicts the choice of l. Hence $\omega(\psi_l) \notin \Omega_p(G_q)$. So $\psi_l \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2}$ by (18).

Assume that $\psi_l \in \mathcal{L}_q^{c+2}$. Put $f_u = t_{u-l}$ for $l < u \leq k$. We claim that $\psi_u \in \mathcal{L}_q^{f_u}$ for such u. Using (18), we conclude that $\psi_u \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2}$ if $\omega(\psi_u) \notin \Omega_p(G_q)$. Recall that $t_{u-l} \geq t_1 \geq 3c+2$. First let $u = l+1$. Obviously, $\psi_u \in \mathcal{L}_q^{t_1}$ if $\psi_u \in \mathcal{L}_q^{c+2}$. Observe that $\operatorname{wdeg}\rho(l,u) = \operatorname{wdeg}(\psi_l \otimes (\psi_u^{[1]})) \leq c$. Since $n > N > t_1 \geq 3c+2$ and hence $n \geq 3c+4$, Proposition 3.12 yields that $\psi_u \notin \mathcal{R}_q^{c+2}$ if $\omega(\psi_u) \neq 0$. Let $\psi_u \in \mathcal{F}_q \cup \mathcal{T}_q$. Then Formula (17) and the arguments above that formula yield that $\psi_u \in \mathcal{L}_q^{t_1}$. This completes the proof of the claim for $u = l+1$.

Now assume that $u > l+1$ and apply induction on u. Suppose that $\psi_s \in \mathcal{L}^{f_s}$ for $l < s < u$. Then $\psi_s \in \mathcal{L}^{f_{u-1}}$ for these s as $f_s < f_{u-1}$ if $s < u-1$. The choice of l shows that $\delta(\pi(l,u)) \geq p^u$ since otherwise $\delta(\pi(u)) < p^u$, which yields a contradiction. Write $\rho(l,u) = \rho'^{[l]}$ and observe that $\operatorname{wdeg}\rho' = \operatorname{wdeg}\rho(l,u) \leq c$. Applying Proposition 3.12 and arguing as above, we conclude that $\psi_u \notin \mathcal{R}_q^{c+2}$ and $\psi_u \in \mathcal{L}_q^{f_u}$ if $\psi_u \in \mathcal{F}_q \cup \mathcal{T}_q$. Here it is essential that $n > t_j > t_{u-l} \geq t_{u-1-l} + 2c$ and so $n > t_{u-1-l} + 2c + 2$. Put $g' = c + 2 + \sum_{h=1}^{k-l} t_h p^h$. Then for $u = k$ one has $\rho' \in \mathcal{L}_q^{g'}$. Obviously, $g' \leq g$ (the equality holds only for $l = 0$ and $k = j$).

If $\psi_l \in \mathcal{R}_q^{c+2}$, similar arguments yield that $\rho(l,k) = \rho'^{[l]}$ with $\rho' \in \mathcal{R}_q^{g'}$. Using the Steinberg tensor product theorem, we conclude that $\otimes_{s=0}^{l-1}\varphi_s^{[s]} \in \operatorname{Irr}_n \pi(l)$ if $l > 0$ and in all cases there exists $\mu \in \operatorname{Irr}_n \rho'$ with $\mu = (\otimes_{s=0}^{k-l}\varphi_{s+l}^{[s]}) \otimes (\chi^{[k-l+1]})$, $\chi \in \operatorname{Irr} G_n$. Since $\mathcal{L}^{g'}$ and $\mathcal{R}^{g'}$ are inductive systems, this forces $\varphi_k \in \mathcal{L}_n^{g'}$ or $\mathcal{R}_n^{g'}$ and yields a contradiction as $\mathcal{L}^{g'} \cup \mathcal{R}^{g'} \subset \mathcal{L}^N \cup \mathcal{R}^N$. Hence $\delta(\pi(k)) < p^k$ if $k > 0$ and $k \in U_1 \cup U_2$.

For $k = 0$ it follows from the Steinberg tensor product theorem that there exists $\mu \in \operatorname{Irr}_n \psi_k$ with $\mu = \varphi_k \otimes (\mu'^{[1]})$, where $\mu' \in \operatorname{Irr} G_n$. Since $\delta(\pi(k)) < p^k$ if $k > 0$ and $k \in U_1 \cup U_2$, one can conclude that the same holds for all such k. Obviously, $\mu = \varphi_k$ if $\omega(\psi_k) \in \Omega_p(G_n)$. Assume this is not the case. Then $\psi_k \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2}$ by (18). But then $\varphi_k \in \mathcal{L}_n^{c+2} \cup \mathcal{R}_n^{c+2} \subset \mathcal{L}_n^N \cup \mathcal{R}_n^N$ which yields a contradiction. Hence $\psi_k \in \mathcal{F}^q \cup \mathcal{T}^q$ and $\varphi_k \in \operatorname{Irr}_n \psi_k$. Naturally, $\psi_k \notin \mathcal{L}_q^d \cup \mathcal{R}_q^d$ as otherwise $\varphi_k \in \mathcal{L}_n^d \cup \mathcal{R}_n^d$ since $\mathcal{L}^d$ and $\mathcal{R}^d$ are inductive systems. Now Lemmas 2.5 and 2.6 imply that $\psi_k \in \mathcal{F}^q$ if $\varphi_k \in \mathcal{F}^q$ and $\psi_k \in \mathcal{T}^q$ if $\varphi_k \in \mathcal{T}^q$. This completes the proof. □

Proof of Theorem 1.7. **(1) Indecomposable systems.** Recall that an inductive system $\Phi = \otimes_{k=0}^{j}\operatorname{Fr}^k(\Phi^k)$ is special if each $\Phi^k = C_L(a_1,\dots,a_s)$, $C_R(a_1,\dots,a_s)$, $\mathcal{F}$, or $\mathcal{T}$. Let Φ be special. We can write $\Phi = \otimes_{f=0}^{l}\Psi^f$, where Ψ^f are determined as before the statement of this theorem in the Introduction. Define the parameters i_f with $0 \leq f \leq l$ as in (8). Let $\delta(\Psi^f) < p^{i_f+1}$ for all $f < l$. If all systems $\Phi^k \in \{\mathcal{F}, \mathcal{T}\}$, it is clear that $\operatorname{wdeg}\varphi = 1$ for every $\varphi \in \Phi_n$. Otherwise one can conclude that for some d and $N \in \mathbb{N}$ the system Φ is generated by a collection $\{R_n \mid n \geq N\}$ that consists of representations satisfying the assumptions of Theorem 3.11 for this d. Now Theorem 3.11 and Proposition 2.13 imply that Φ is a BWM-system if $\delta(\Psi^f) < p^{i_f+1}$ for all $f < l$.

Next, suppose that $\delta(\Psi^f) \geq p^{i_f+1}$ for some $f < l$. The definition of the systems Ψ^f implies that one of the following holds:

(a) $\Phi^k = C_L(a_{1,k},\dots,a_{d_k,k})$ for $i_{f-1}+1 \leq k \leq i_f$ and $\Phi^{i_f+1} = C_R(b_1,\dots,b_t)$, $\mathcal{F}$, or $\mathcal{T}$;

(b) $\Phi^k = C_R(a_{1,k}, \ldots, a_{d_k,k})$ for $i_{f-1}+1 \le k \le i_f$ and $\Phi^{i_f+1} = C_L(b_1, \ldots, b_t)$, $\mathcal{F}$, or $\mathcal{T}$.

Here $0 \le a_{i,j} < p$, $0 \le b_m < p$, and Φ^{i_f+1} is nontrivial. Consider Case (a). Set $i = i_{f-1}+1$, $h = i_f - i$, and $q = \max\{d_k \mid i \le k \le i_f\}$. Let $n > q+t$ if $\Phi^{i_f+1} = C_R(b_1, \ldots, b_t)$ and $n > q+1$ otherwise. Put $M_n^u = M_{n,L}(a_{1,u+i}, \ldots, a_{d_{u+i},u+i})$ for $0 \le u \le h$ and $M_n = \otimes_{u=0}^h (M_n^u)^{[u]}$. Set $T_n = M_{n,R}(b_1, \ldots, b_t)$ if $\Phi^{i_f+1} = C_R(b_1, \ldots, b_t)$ and $T_n = L(\omega_n^n)$ otherwise. Let $Q_n = M_n \otimes T_n^{[h+1]}$. Obviously, $L(\omega_n^n) \in \mathcal{F}_n$ and $\mathcal{T}_n$. Hence in all cases

$$Q_n^{[i]} \in \left(\otimes_{k=i}^{i_f+1} \operatorname{Fr}^k(\Phi^k)\right)_n.$$

So if $f > 0$, the set Φ_n contains a module of the form $L_n \otimes Q_n^{[i]} \otimes S_n^{[i_f+2]}$ with $L_n, S_n \in \operatorname{Irr} G_n$ and $\omega(L_n) = c_1\omega_1^n + \ldots + c_n\omega_n^n$ with $c_y < p^i$ (the module S_n is trivial if $i_f + 1 = j$). If $f = 0$, then Φ_n contains a module of the form $Q_n^{[i]} \otimes S_n^{[i_f+2]}$.

Now we estimate $\operatorname{wdeg} Q_n$. It is clear that

$$\delta(M_n) = \sum_{u=0}^{h} p^u (a_{1,u+i} + \ldots + a_{d_{u+i},u+i}).$$

It follows from the construction of the system Ψ^f that $\delta(\Psi^f) = p^i \delta(M_n)$. Hence $\delta(M_n) \ge p^{h+1}$. Obviously, $\omega(M_n) = \sum_{r=1}^q g_r \omega_r^n$ and $\omega(Q_n) = \sum_{r=n-t+1}^n m_r \omega_r^n$ if $\Phi^{i_f+1} = C_L(b_1, \ldots, b_t)$. Hence Proposition 3.12 yields that $\operatorname{wdeg} Q_n \ge n - t - q$ if $\Phi^{i_f+1} = C_L(b_1, \ldots, b_t)$ and $\operatorname{wdeg} Q_n \ge n - q - 1$ otherwise. So $\operatorname{wdeg} Q_n$ is not bounded. Now Lemma 2.14 implies that Φ is not a BWM-system. In Case (b) the arguments are similar.

Lemmas 3.5 and 4.8 and Corollaries 5.4 and 5.5 yield that each special inductive system Φ has the form $\Phi = \langle \varphi_n \mid n \ge A \rangle$ where $\varphi_n \in \operatorname{Irr} G_n$, $A \in \mathbb{N}$. Hence special systems are indecomposable.

Now we will show that every indecomposable BWM-system is a special system with $\delta(\Psi^f) < p^{i_f+1}$ for $f < l$. Let Φ be an indecomposable inductive system and $\operatorname{wdeg} \Phi = c$. By Theorem 1.5, $\Phi = \otimes_{k=0}^j \operatorname{Fr}^k(\Phi^k)$, where Φ^k are p-restrictedly generated inductive systems. It follows from Lemma 2.14 that $\operatorname{wdeg} \Phi^k \le c$ for $0 \le k \le j$. One easily concludes that Φ^k are indecomposable. By Theorem 1.6, each $\Phi^k = C_L(a_1, \ldots, a_d)$, $C_R(a_1, \ldots, a_d)$, $\mathcal{F}$, or $\mathcal{T}$, i.e. Φ is special. This completes the proof of the theorem for indecomposable systems.

(2) Arbitrary systems. Let $\mathcal{B}$ be an arbitrary BWM-system. We describe a procedure that allows one either to show that $\mathcal{B} \subset \mathcal{L}^d \cup \mathcal{R}^d$ for some d, or to construct explicitly a subsystem $\mathcal{S} \subset \mathcal{B}$ such that $\mathcal{S}$ is a finite union of indecomposable inductive systems and $D(\mathcal{B}, \mathcal{S}) \subset \mathcal{L}^d \cup \mathcal{R}^d$. Then Proposition 5.6 and Lemma 5.8 imply that $\mathcal{B}$ is a finite union of indecomposable BWM-systems.

Fix minimal j with $\delta(\mathcal{B}) < p^{j+1}$. Then for all n and each $\varphi \in \mathcal{B}_n$ we have $\varphi = \otimes_{k=0}^j \varphi_k^{[k]}$ with $\varphi_k \in \operatorname{Irr}^p G_n$. Until the end of this proof for a module $\psi \in \mathcal{B}_n$ we denote by ψ_k, $0 \le k \le j$, the modules in $\operatorname{Irr}^p G_n$ that occur in such decomposition. Set

$$\Delta_{n,k} = \{M \in \operatorname{Irr}^p G_n \mid M = \varphi_k \text{ for some } \varphi \in \Phi_n\}, \quad 0 \le k \le j.$$

Assume that $\operatorname{wdeg} \mathcal{B} = c$. By Lemma 2.14, $\operatorname{wdeg} M \le c$ for all $M \in \Delta_{n,k}$. Arguing as in the proof of Proposition 5.9, one concludes that

$$\Delta_{n,k} \subset \mathcal{F}_n \cup \mathcal{T}_n \cup \mathcal{L}_n^{c+2} \cup \mathcal{R}_n^{c+2} \tag{19}$$

for $n > (c+1)^2p^2$ and $0 \le k \le j$. First assume that

$$\text{for every } d \text{ there exist } n \text{ and } k \text{ with } \Delta_{n,k} \cap (\mathcal{F}_n \cup \mathcal{T}_n) \not\subset \mathcal{L}_n^d \cup \mathcal{R}_n^d. \tag{20}$$

If $p \neq 2$, denote by $\mathcal{C}$ the collection of pairs (V_1, V_2), $V_i \subset \mathbb{N}_j$ with the following properties:

(i) $V_1 \cap V_2 = \varnothing$, $V_1 \cup V_2 \neq \varnothing$;

(ii) for each d there exist n and $\varphi \in \mathcal{B}_n$ such that $\varphi_k \in \mathcal{F}_n$ for $k \in V_1$, $\varphi_k \in \mathcal{T}_n$ for $k \in V_2$, and $\varphi_k \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$ for $k \in V_1 \cup V_2$;

(iii) there is no pair (V_1', V_2') such that V_1' and V_2' satisfy (i) and (ii), $V_1 \subset V_1'$, $V_2 \subset V_2'$, and $V_1' \cup V_2' \neq V_1 \cup V_2$.

As $\mathcal{L}_n^d \cup \mathcal{R}_n^d \subset \mathcal{L}_n^m \cup \mathcal{R}_n^m$ if $d < m$, Formula (20) yields that for certain fixed k the following holds: for each d there exists n with $\Delta_{n,k} \cap \mathcal{F}_n \not\subset \mathcal{L}_n^d \cup \mathcal{R}_n^d$ or for each d there exists n with $\Delta_{n,k} \cap \mathcal{T}_n \not\subset \mathcal{L}_n^d \cup \mathcal{R}_n^d$. So $\mathcal{C}$ is nonempty.

If $(V_1, V_2) \in \mathcal{C}$ and $V_1 \cup V_2 = \mathbb{N}_j$, set $\Psi(V_1, V_2) = (\otimes_{k \in V_1} \mathrm{Fr}^k(\mathcal{F})) \otimes (\otimes_{k \in V_2} \mathrm{Fr}^k(\mathcal{T}))$. Assume that $(V_1, V_2) \in \mathcal{C}$ and $V_1 \cup V_2 \neq \mathbb{N}_j$. Set $V_0 = \mathbb{N}_j \setminus (V_1 \cup V_2)$. Fix $t \in V_0$.

The construction of $\mathcal{C}$ implies that there exist $u = u(t)$ with the following properties: if $\varphi \in \Phi_n$, $\varphi_k \in \mathcal{F}_n$ for $k \in V_1$, $\varphi_k \in \mathcal{T}_n$ for $k \in V_2$, $\varphi_k \notin \mathcal{L}_n^u \cup \mathcal{R}_n^u$ for $k \in V_1 \cup V_2$, and $\varphi_t \in \mathcal{F}_n \cup \mathcal{T}_n$, then $\varphi_t \in \mathcal{L}_n^u \cup \mathcal{R}_n^u$ (otherwise (iii) would not hold for (V_1, V_2)). These arguments and Formulas (19) and (20) yield that there exists d such that $\varphi_k \in \mathcal{L}_n^d \cup \mathcal{R}_n^d$ if $\varphi \in \Phi_n$, $n > (c+1)^2p^2$, $k \in V_0$, $\varphi_a \in \mathcal{F}_n \setminus (\mathcal{L}_n^d \cup \mathcal{R}_n^d)$ for all $a \in V_1$, and $\varphi_b \in \mathcal{T}_n \setminus (\mathcal{L}_n^d \cup \mathcal{R}_n^d)$ for all $b \in V_2$. Naturally, we can enlarge d and guarantee that $n > (c+1)^2p^2$ if $\varphi_s \notin (\mathcal{L}_n^d \cup \mathcal{R}_n^d)$ for some s. Denote by $S = S(V_1, V_2)$ the set of all inductive systems $\Pi = \otimes_{k \in V_0} \mathrm{Fr}^k(\Pi^k)$ with the following properties: $\Pi^k = C_L(a_{1k}, \ldots, a_{dk})$ or $C_R(a_{1k}, \ldots, a_{dk})$, $0 \le a_{ik} < p$, $\Pi^k \subset \mathcal{L}^d$ or $\mathcal{R}^d$, and for each m there exist n and $\varphi \in \Phi_n$ with $\varphi_k = M_{n,L}(a_{1k}, \ldots, a_{dk})$ or $M_{n,R}(a_{1k}, \ldots, a_{dk})$ if $k \in V_0$ and $\Pi^k = C_L(a_{1k}, \ldots, a_{dk})$ or $C_R(a_{1k}, \ldots, a_{dk})$, respectively, $\varphi_k \notin \mathcal{L}_n^m \cup \mathcal{R}_n^m$ for $k \in V_1 \cup V_2$, $\varphi_k \in \mathcal{F}_n$ for $k \in V_1$, and $\varphi_k \in \mathcal{T}_n$ for $k \in V_2$. Since the number of inductive systems $C_L(a_{1k}, \ldots, a_{dk}) \subset \mathcal{L}^d$ and $C_R(a_{1k}, \ldots, a_{dk}) \subset \mathcal{R}^d$ is finite and (V_1, V_2) satisfies the assumptions (i)-(iii), one can observe that S is nonempty and finite. For $\Pi \in S$ set

$$\Psi(\Pi) = \Pi \otimes (\otimes_{k \in V_1} \mathrm{Fr}^k(\mathcal{F})) \otimes (\otimes_{k \in V_2} \mathrm{Fr}^k(\mathcal{T})).$$

Put $\Psi(V_1, V_2) = \cup_{\Pi \in S} \Psi(\Pi)$ and $\Psi = \cup_{(V_1, V_2) \in \mathcal{C}} \Psi(V_1, V_2)$. Proposition 5.3 implies that $\Psi(V_1, V_2) \subset \mathcal{B}$ if $V_1 \cup V_2 = \mathbb{N}_j$ and $\Psi(\Pi) \subset \mathcal{B}$ for all $\Pi \in S(V_1, V_2)$ if $V_1 \cup V_2 \neq \mathbb{N}_j$. Hence $\Psi \subset \mathcal{B}$.

For $p = 2$ let $\mathcal{C}$ be the collection of all nonempty sets V such that for each d there exist n and $\varphi \in \mathcal{B}_n$ with $\varphi_k \in \mathcal{F}_n$ for $k \in V$ and V is a maximal subset in $\mathbb{N}_j$ with this property. Using Formula (20) as for $p > 2$, we conclude that $\mathcal{C}$ is nonempty. If $\mathcal{C}$ consists of the set $\mathbb{N}_j$, put $\Psi = \otimes_{k=0}^{j} \mathrm{Fr}^k(\mathcal{F})$. Assume this is not the case. For each $V \in \mathcal{C}$ construct the set $S(V)$ and the system $\Psi(V)$ in the same way as we have constructed the sets $S(V_1, V_2)$ and the systems $\Psi(V_1, V_2)$ for $p \neq 2$. Put $\Psi = \cup_{V \in \mathcal{C}} \Psi(V)$. Using Proposition 5.3 as before, one concludes that $\Psi \subset \mathcal{B}$ for $p = 2$ as well. It is clear that in all cases Ψ is a finite union of indecomposable BWM-systems. So we are done if $\Psi = \mathcal{B}$.

Assume that $\Psi \neq \mathcal{B}$ and set $\mathcal{B}^1 = D(\mathcal{B}, \Psi)$. Obviously, $\operatorname{wdeg} \mathcal{B}^1 \le c$. Denote by $\Delta_{n,k}^1$ the analogues of the sets $\Delta_{n,k}$ for the system $\mathcal{B}^1$. It is clear that (19) holds for $\Delta_{n,k}^1$.

Assume that (20) holds for $\Delta_{n,k}^1$. Then one can define the collection $\mathcal{C}^1$ for the system $\mathcal{B}^1$ in the same way as we have defined $\mathcal{C}$ for $\mathcal{B}$. Put $q(\mathcal{C}) = \max\{|V_1 \cup V_2| \mid (V_1, V_2) \in \mathcal{C}\}$

for $p > 2$, $q(\mathcal{C}) = \max\{|V| \mid V \in \mathcal{C}\}$ for $p = 2$, and define $q(\mathcal{C}^1)$ similarly. We claim that $q(\mathcal{C}^1) < q(\mathcal{C})$. Indeed, let $p > 2$ and $(U_1, U_2) \in \mathcal{C}^1$. We will show that there exists a pair $(V_1, V_2) \in \mathcal{C}$ with $U_i \subset V_i$ and $|V_1 \cup V_2| > |U_1 \cup U_2|$. First we will prove that $(U_1, U_2) \notin \mathcal{C}$. Suppose that $(U_1, U_2) \in \mathcal{C}$ for some pair $(U_1, U_2) \in \mathcal{C}^1$. Let $U_1 \cup U_2 \neq \mathbb{N}_j$. The construction of the subsystem $\Psi(U_1, U_2) \subset \Psi$ above yields that for some $m = m(U_1, U_2)$ if $\varphi \in \Phi_n$, $\varphi_k \in \mathcal{F}_n$ for all $k \in U_1$, $\varphi_k \in \mathcal{T}_n$ for every $k \in U_2$, and $\varphi_k \notin \mathcal{L}_n^m \cup \mathcal{R}_n^m$ for each $k \in U_1 \cup U_2$, then $\varphi \in \Psi(U_1, U_2)_n$.

Let $N = N(c, j)$ be such as in Proposition 5.10. Let $d \geq N$ if $U_1 \cup U_2 = \mathbb{N}_j$ and $d \geq \max\{N, m(U_1, U_2)\}$ otherwise. Since $(U_1, U_2) \in \mathcal{C}^1$, some $\mathcal{B}_n^1$ contains a representation φ such that $\varphi_k \in \mathcal{F}_n$ for $k \in U_1$, $\varphi_k \in \mathcal{T}_n$ for $k \in U_2$, and $\varphi_k \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$ for each $k \in U_1 \cup U_2$. The construction of $\mathcal{B}^1$ implies that for some $t > n$ there exists a representation $\rho \in \mathcal{B}_t \setminus \Psi_t$ with $\varphi \in \mathrm{Irr}_n\, \rho$. By Proposition 5.10, $\rho_k \in \mathcal{F}_t$ for $k \in U_1$, $\rho_k \in \mathcal{T}_t$ for $k \in U_2$, and $\rho_k \notin \mathcal{L}_t^d \cup \mathcal{R}_t^d$ for $k \in U_1 \cup U_2$. This yields a contradiction. Indeed, if $U_1 \cup U_2 \neq \mathbb{N}_j$, all such representations $\rho \in \Psi(U_1, U_2)_t$ by the arguments above. If $U_1 \cup U_2 = \mathbb{N}_j$, the construction of $\Psi(U_1, U_2)$ implies that for $\rho \notin \Psi(U_1, U_2)_t$ some $\rho_k \notin \mathcal{F}_t$ with $k \in U_1$ or some $\rho_s \notin \mathcal{T}_t$ for $s \in U_2$. Observe that in all cases $\Psi(U_1, U_2) \subset \Psi$. Hence $(U_1, U_2) \notin \mathcal{C}$.

The construction of $\mathcal{C}$ and $\mathcal{C}^1$ implies that the pair (U_1, U_2) satisfies the assumptions (i) and (ii) that we used to define $\mathcal{C}$, but does not satisfy (iii). Hence there exists a pair (U_1', U_2') mentioned in (iii).

Take for (V_1, V_2) such pair with the maximal $|U_1' \cup U_2'|$. For $p = 2$ similar arguments yield that each $U \subset \mathcal{C}^1$ is the proper subset of some $M \subset \mathcal{C}$. Hence in all cases $q(\mathcal{C}^1) < q(\mathcal{C})$.

Now construct an inductive system $\Psi^1 \subset \mathcal{B}^1$ in the same way as Ψ was constructed for $\mathcal{B}$. If $\Psi^1 \neq \mathcal{B}^1$, set $\mathcal{B}^2 = D(\mathcal{B}^1, \Psi^1)$. Continue the process until this is possible, constructing for a system $\mathcal{B}^i$ the collection $\mathcal{C}^i$ and the subsystem Ψ^i in the same way as $\mathcal{C}^1$ and Ψ^1 were constructed. By the arguments above, if $\mathcal{C}^i$ is determined, then $q(\mathcal{C}^i) < q(\mathcal{C}^{i-1}) < \ldots < q(\mathcal{C})$. Hence for some i either $\Psi^i = \mathcal{B}^i$ or (20) does not hold for $\mathcal{B}^{i+1}$. Here our procedure is finished. In the first case $\mathcal{B} = \Psi \cup (\cup_{1 \leq k \leq i} \Psi^k)$ and hence is a finite union of indecomposable BWM-systems. Now assume that (20) does not hold for $\mathcal{B}$ or $\mathcal{B}^{i+1}$. Set $\Sigma = \mathcal{B}$ or $\mathcal{B}^{i+1}$, respectively. As Σ is an inductive system, Formula (19) yields that $\Sigma \subset \mathcal{L}^d \cup \mathcal{R}^d$ for some d. Therefore our goal is reached. The theorem is proved. □

6 Inductive systems with bounded weight multiplicities for symplectic and spinor groups

In this section $G_n = B_n(K)$, $C_n(K)$, or $D_n(K)$. Recall the collections $\mathcal{S}$ and $\mathcal{L}$ defined in the Introduction. By Lemma 2.10, $\mathcal{L}$ is an inductive system in all cases.

Lemma 6.1 *Let $p > 2$ for $G_n \neq D_n(K)$. The collection $\mathcal{S}$ is an inductive system.*

Proof. This follows from Lemma 2.11 for $G_n = B_n(K)$ or $D_n(K)$ and Lemma 2.9 for $G_n = C_n(K)$. □

Now we state our results on the BWM-systems in the special case where $p = 2$ and $G_n = C_n(K)$. These assumptions on p and G_n are valid until the proof of Theorems 1.8 and 6.4.

Set $\mathcal{S}_n' = \{L(\omega_n^n)\}$, $\mathcal{S}' = \{\mathcal{S}_n'\}_{n \in \mathbb{N}}$,

$$\mathcal{Q}_n = \{L(\omega_1^n + \omega_n^n), L(\omega_n^n)\}$$

for $n > 1$, $\mathcal{Q}_1 = \mathrm{Irr}_1\, \mathcal{Q}_2$, and $\mathcal{Q} = \{\mathcal{Q}_n\}_{n \in \mathbb{N}}$.

Lemma 6.2 *Let $p = 2$ and $G_n = C_n(K)$. Then $\mathcal{S}'$ and $\mathcal{Q}$ are inductive systems.*

Proof. The result follows from Lemma 2.11 and Corollary 2.12. □

We need some notation to describe irreducible representations of G_n with small weight multiplicities. Put

$$\Omega_2(G_n) = \{0, \omega_1^n, \omega_n^n\} \text{ and } \Omega_2'(G_n) = \Omega_2 \cup \{\omega_1^n + \omega_n^n\}.$$

For any dominant weight ω of G_n we can write its "2-adic expansion"

$$\omega = \lambda_0 + 2\lambda_1 + \ldots + 2^k\lambda_k,$$

where weights λ_i are 2-restricted for $0 \leq i \leq k$. This expansion is uniquely determined if we assume that $k = 0$ for $\omega = 0$ and $\lambda_k \neq 0$ otherwise. Set

$$S(\omega) = (\lambda_0, \ldots, \lambda_k).$$

Put

$$\Omega(G_n) = \left\{ \sum_{j=0}^{k} 2^j\lambda_j \mid k \geq 0,\ \lambda_j \in \Omega_2(G_n),\ (\lambda_j, \lambda_{j+1}) \neq (\omega_n^n, \omega_1^n) \text{ for } j < k \right\}$$

and

$$\Omega'(G_n) = \left\{ \sum_{j=0}^{k} 2^j\lambda_j \mid k \geq 0,\ \lambda_j \in \Omega_2'(G_n) \right\}.$$

By [25, Proposition 2], $\mathrm{wdeg}(L(\omega)) = 1$ if and only if $\omega \in \Omega(G_n)$. Thus, in this case a connection between the sets $\Omega(G_n)$ and $\Omega_p(G_n)$ is more complicated than for other classical groups or odd p.

Theorem 6.3 ([18, Theorem 2]) *Let $p = 2$, $G_n = C_n(K)$, $n \geq 8$, and let $M \in \mathrm{Irr}\, G_n$ with $\omega(M) \notin \Omega(G_n)$. Then the following hold:*

(i) if $\omega \in \Omega'(G_n)$, the weight $\omega_1^n + \omega_n^n$ occurs in the sequence $S(\omega)$ exactly l times, and for $0 \leq j < k$

$$(\lambda_j, \lambda_{j+1}) \notin \{(\omega_n^n, \omega_1^n), (\omega_1^n + \omega_n^n, \omega_1^n), (\omega_n^n, \omega_1^n + \omega_n^n), (\omega_1^n + \omega_n^n, \omega_1^n + \omega_n^n)\},$$

then $\mathrm{wdeg}\, M = 2^l$*;*

(ii) otherwise $\mathrm{wdeg}\, M \geq n - 4 - [n]_4$*, where $[n]_4$ is the residue of n modulo 4; in particular,* $\mathrm{wdeg}\, M \geq n - 7$.

Theorem 6.4 *Let $p = 2$ and $G_n = C_n(K)$. Set $\mathcal{P} = \{\mathcal{O}, \mathcal{L}, \mathcal{Q}, \mathcal{S}'\}$. An indecomposable inductive system Φ is a BWM-system if and only if $\Phi = \otimes_{j=0}^{s} \mathrm{Fr}^j(\Phi^j)$ with $\Phi^j \in \mathcal{P}$ and $(\Phi^j, \Phi^{j+1}) \notin \{(\mathcal{S}', \mathcal{L}), (\mathcal{Q}, \mathcal{L}), (\mathcal{S}', \mathcal{Q}), (\mathcal{Q}, \mathcal{Q})\}$. BWM-systems are finite unions of indecomposable ones.*

Though the description of BWM-systems is more complicated for $p = 2$ and $G_n = C_n(K)$, the proofs of Theorems 1.8 and 6.4 are based on similar arguments. So we prove them simultaneously.

Proof of Theorems 1.8 *and* 6.4. In this proof we say that we are in a special case if $p = 2$ and $G_n = C_n(K)$ and in the general case otherwise. Assume that $n > 3$. Set $\tau_n = L(0) \in \operatorname{Irr} G_n$ and $\lambda_n = L(\omega_1^n)$ for all three types. Put

$$\mu_n = \begin{cases} L(\frac{p-1}{2}\omega_n^n) & \text{for } G_n = C_n(K), p > 2, \\ L(\omega_n^n) & \text{otherwise.} \end{cases}$$

In the special case also set $\xi_n = L(\omega_1^n + \omega_n^n)$.

Let Φ be a BWM-system. Lemma 4.1 implies that there exists $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and each $\varphi \in \Phi_n$ the representation $\varphi = \otimes_{k=0}^{l} \varphi_k^{[k]}$ with $\varphi_k \in \operatorname{Irr}^p G_n$, $0 \leq k \leq l$. Fix such l. Theorems 1.2 and 6.3 imply that there exists a constant N such that for $n > N$ and $\varphi \in \Phi_n$ the weight $\omega(\varphi) \in \Omega(G_n)$ in the general case and $\omega(\varphi) \in \Omega'(G_n)$ in the special case.

Now we construct a collection of inductive systems for the groups G_n that actually yield all indecomposable BWM-systems. In the general case for a triple of subsets $A, B, C \subset \mathbb{N}_l$ such that $A \cup B \cup C = \mathbb{N}_l$ and $A \cap B = A \cap C = B \cap C = \varnothing$ put $\pi_n(A, B, C) = \otimes_{k=0}^{l} \varphi_k^{[k]}$ with $\varphi_k = \tau_n$ for $k \in A$, $\varphi_k = \lambda_n$ for $k \in B$, and $\varphi_k = \mu_n$ for $k \in C$. In the special one for a quadruple of subsets $A, B, C, D \subset \mathbb{N}_l$ such that $A \cup B \cup C \cup D = \mathbb{N}_l$ and $U \cap V = \varnothing$ for $U, V \in \{A, B, C, D\}$ with $U \neq V$ put $\rho_n(A, B, C, D) = \otimes_{k=0}^{l} \varphi_k^{[k]}$ with $\varphi_k = \tau_n$ for $k \in A$, $\varphi_k = \lambda_n$ for $k \in B$, $\varphi_k = \mu_n$ for $k \in C$, and $\varphi_k = \xi_n$ for $k \in D$.

We need some notation to expose arguments common for the both cases. Let $\mathcal{A} = (A, B, C)$, $\psi_n(\mathcal{A}) = \pi_n(A, B, C)$, $\mathcal{P} = \{\mathcal{O}, \mathcal{L}, \mathcal{S}\}$ in the general case and $\mathcal{A} = (A, B, C, D)$, $\psi_n(\mathcal{A}) = \rho_n(A, B, C, D)$, $\mathcal{P} = \{\mathcal{O}, \mathcal{L}, \mathcal{Q}, \mathcal{S}'\}$ in the special one where a triple (A, B, C) or a quadruple (A, B, C, D) satisfies the relevant assumptions above. In what follows we shall call such tuples $\mathcal{A}$ admissible tuples. Using Lemmas 2.9, 2.10, and 2.11 and Corollary 2.12, one easily observes that $\psi_n(\mathcal{A}) \in \operatorname{Irr}_n \psi_{n+1}(\mathcal{A})$. It is clear that

$$\delta(\psi_n(\mathcal{A})) \leq \begin{cases} \frac{p^{l+1}-1}{2} & \text{for } G_n = C_n(K) \text{ and } p > 2, \\ 2^{l+2} - 2 & \text{for } G_n = C_n(K) \text{ and } p = 2, \\ 1 + p + \ldots + p^l & \text{otherwise.} \end{cases}$$

Hence Lemma 4.3 implies that the inductive system $\Psi(\mathcal{A}) = \langle \psi_n(\mathcal{A}) \mid n > 3 \rangle$ is well defined. Lemmas 2.9, 2.10, 2.11, and 4.8 and Corollary 2.12 yield that

$$\Psi(\mathcal{A}) = \otimes_{k=0}^{l} \operatorname{Fr}^k(\Psi^k), \quad \Psi^k \in \mathcal{P}, \quad 0 \leq k \leq l,$$

and that each inductive system

$$\Theta = \otimes_{k=0}^{j} \operatorname{Fr}^k(\Theta^k), \quad \Theta^k \in \mathcal{P}, \quad 0 \leq k \leq j,$$

coincides with $\Psi(\mathcal{A})$ for some admissible tuple $\mathcal{A}$. Hence all these systems Θ are indecomposable.

In the general case for all admissible tuples $\mathcal{A}$ one has $\operatorname{wdeg} \psi_n(\mathcal{A}) = 1$ by Theorem 1.1. In the special case for fixed $\mathcal{A} = (A, B, C, D)$ and $0 \leq k < l$ we shall write $X(k) = (U, V)$ with $U, V \in \{A, B, C, D\}$ if $k \in U$ and $k+1 \in V$. Theorem 6.3 and [25, Proposition 2] force that $\operatorname{wdeg} \psi_n(\mathcal{A}) \geq n - 7$ if for some $k < l$ the pair $X(k) \in \{(C, B), (D, B), (C, D), (D, D)\}$ and $\operatorname{wdeg} \psi_n(\mathcal{A}) \leq 2^{l+1}$ otherwise. Now Proposition 2.13 yields that in the general case all systems Θ introduced above are BWM-systems and in the special one such system is a BWM-system if and only if $(\Theta^k, \Theta^{k+1}) \notin \{(\mathcal{S}', \mathcal{L}), (\mathcal{Q}, \mathcal{L}), (\mathcal{S}', \mathcal{Q}), (\mathcal{Q}, \mathcal{Q})\}$ for all $k < j$.

Now assume that $n > N$. We claim that for every $\varphi \in \Phi_n$ there exists an admissible tuple $\mathcal{A}$ such that

$$\psi_{n+1}(\mathcal{A}) \in \Phi_{n+1} \quad \text{and} \quad \varphi \in \operatorname{Irr}_n(\psi_{n+1}(\mathcal{A})). \tag{21}$$

Indeed, since Φ is an inductive system, the representation $\varphi \in \operatorname{Irr}_n \chi$ for some $\chi \in \Phi_{n+2}$. One has $\chi = \otimes_{k=0}^{l} \chi_k^{[k]}$ with $\chi_k \in \Omega_p(G_{n+2})$ in the general case and $\chi_k \in \Omega_2'(G_{n+2})$ in the special one, $0 \leq k \leq l$.

Lemmas 2.9, 2.10, and 2.11 and Corollary 2.12 imply the following: $\operatorname{Irr}_n \chi_k \subset \operatorname{Irr}^p G_n$ and hence $\phi_k \in \operatorname{Irr}_n \chi_k$; $\chi_k \in \mathcal{L}_{n+2}$ if $\varphi_k \in \mathcal{L}_n$, $\chi_k = \lambda_{n+2}$ for $\varphi_k = \lambda_n$, $\chi_k \in \mathcal{S}_{n+2}$ if $\varphi_k \in \mathcal{S}_n$; in the special case $\chi_k \in \mathcal{Q}_{n+2}$ if $\varphi_k \in \mathcal{Q}_n$ and $\chi_k = \xi_{n+2}$ if $\varphi_k = \xi_n$. Then another application of those lemmas permits us to find an admissible tuple $\mathcal{A}$ such that $\psi_{n+1}(\mathcal{A}) \in \operatorname{Irr}_{n+1} \chi$ and $\varphi \in \operatorname{Irr}_n(\psi_{n+1}(\mathcal{A}))$. Naturally, $\psi_{n+1}(\mathcal{A}) \in \Phi_{n+1}$ as Φ is an inductive system. This proves the claim.

Since the set of admissible tuples is finite, Formula (21) yields that for every $\phi \in \Phi_n$ there exist an infinite set $S \subset \mathbb{N}$ and an admissible tuple $\mathcal{A}$ such that S consists of some integers greater than N, $\psi_m(\mathcal{A}) \in \Phi_m$ for $m \in S$, and $\phi \in \operatorname{Irr}_n \psi_m(\mathcal{A})$. Define by I the collection of all tuples $\mathcal{A}$ that have this property for some ϕ and n, and set $\Sigma = \bigcup_{\mathcal{A} \in I} \Psi(\mathcal{A})$. Observe that $\Sigma = \Phi$. Naturally, $\Sigma \subset \Phi$ since Φ is an inductive system and $\Psi(\mathcal{A}) = \langle \psi_m(\mathcal{A}) \mid m \in S \rangle$ for every admissible $\mathcal{A}$ and infinite set $S \in \mathbb{N}$. On the other hand, the construction of Σ yields that $\Phi_n \subset \Sigma_n$ for $n > N$ as Σ is an inductive system. This completes the proof. □

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