

MODULAR BRANCHING RULES FOR 2-COLUMN DIAGRAM REPRESENTATIONS OF GENERAL LINEAR GROUPS

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ABSTRACT. In this paper branching rules for the polynomial irreducible representations of the general linear groups in positive characteristic with highest weights labeled by partitions of the form $(2^a, 1^b, 0^c)$ and their restrictions to the special linear groups are found. The submodule structure of the restrictions of the corresponding irreducible modules for the group $GL_n(F)$ (or $SL_n(F)$) to the naturally embedded subgroup $GL_{n-1}(F)$ (or $SL_{n-1}(F)$) is determined. As a corollary, inductive systems of irreducible representations for $GL_\infty(F)$ and $SL_\infty(F)$ that consist of representations indicated above, are classified. The submodule structure of the relevant Weyl modules is refined.

1. INTRODUCTION

The ground field F is algebraically closed of characteristic $p > 0$. The article is devoted to finding branching rules for the irreducible representations of the general linear groups over F associated with 2-column partitions and the restrictions of these representations to the special linear groups. For the groups $GL_n(F)$ and $SL_n(F)$ the submodule structure of the restrictions of the irreducible modules with these highest weights to the naturally embedded subgroups $GL_{n-1}(F)$ and $SL_{n-1}(F)$ is determined. The submodule structure of relevant Weyl modules for these groups is refined. Results on the submodule structure of the restrictions of $GL_n(F)$ -modules indicated above enable one to determine this structure for the restrictions of the irreducible modules for the symmetric group Σ_n corresponding to 2-row partitions to naturally embedded subgroup Σ_{n-1} (via the Schur functor).

In general, branching rules describe the restrictions of representations of the classical algebraic and symmetric groups to subgroups of smaller ranks. In characteristic 0 for a group of rank n and its fixed irreducible representation φ they yield the composition factors of the restriction of φ to a naturally embedded subgroup of rank $n - 1$ and hence to similar subgroups of smaller ranks, at least algorithmically. These rules provide a basis for induction on rank and have found numerous applications. In positive characteristic one cannot expect to obtain complete general branching rules in an explicit form in a foreseeable future since this problem is closely connected with that of finding the dimensions of arbitrary irreducible representations and the composition factors of the Weyl modules. However, in some important particular cases such rules can be found. So it is worth to investigate these cases and to seek for asymptotic analogs of these rules in the general situation. The notion of an inductive system of representations (see the definition below) introduced by Zalesskii in [20] yields an asymptotic version of the branching rules. It proved to be useful not only within representation theory, but for the study of ideals in group algebras of locally finite groups as well, see, for instance, [21]. For the infinite-dimensional

1991 *Mathematics Subject Classification.* 20G05.

The first author has been supported by the Nuffield Foundation (NAL/00515/G). The second author has been supported by the Institute of Mathematics of the National Academy of Sciences of Belarus in the framework of the program “Mathematical structures”. A part of this research has been done during the second author’s visit to the University of Leicester supported by the Nuffield Foundation. The second author thanks the staff of the Department of Mathematics for their hospitality.

general linear group $GL_\infty(F)$ and special linear group $SL_\infty(F)$ we classify the inductive systems of representations that consist of the representations indicated above.

Let \mathbb{Z} , $\mathbb{Z}_{\geq 0}$, and \mathbb{N} be the sets of integers, nonnegative, and positive integers, respectively. Set $G_n = GL_n(F)$ and $H_n = SL_n(F)$. We use the standard notation Σ_n for the symmetric group of rank n and the notation Γ_n when we do not need to distinguish G_n and H_n . The highest weights of the rational irreducible representations of G_n over F can be identified with n -tuples (a_1, \dots, a_n) where $a_j \in \mathbb{Z}$ and $a_1 \geq a_2 \geq \dots \geq a_n$. As usual, in the notation for such tuples we replace a subsequence a, \dots, a of length l by the expression a^l and so consider tuples $(a_1^{k_1}, \dots, a_s^{k_s})$ with $a_1 > a_2 > \dots > a_s$ and $k_1 + \dots + k_s = n$. Denote by ω_i^n , $0 \leq i \leq n$, the i th fundamental highest weight of H_n putting $\omega_0^n = \omega_n^n = 0$. In what follows $L_{i,j}^n$ and $\Delta_{i,j}^n$ with $0 \leq i \leq j \leq n$ denote the irreducible G_n -module and the Weyl module, respectively, with the highest weight corresponding to the n -tuple $(2^i, 1^{j-i}, 0^{n-j})$. The same symbols are used to denote the restrictions of these modules to H_n . It will always be clear which group is considered. Naturally, the H_n -module $L_{i,j}^n$ is the irreducible module with highest weight $\omega_i + \omega_j$. For stating our results, it is convenient to set $L_{i,j}^n = \Delta_{i,j}^n = 0$ for $i < 0$, for $i > j$, and for $j > n$. The labeling of the fundamental weights of H_n is standard. For an integer $z > 0$ we denote by $\text{lp}(z)$ the maximal i such that $p^i \mid z$. We have $\text{lp}(z) = 0$ if $p \nmid z$. The restriction of a module M to a group Γ is denoted by $M \downarrow \Gamma$. We shall write

$$M \sim N_1 + \dots + N_q$$

if there is a series $0 = M_0 \subset M_1 \subset \dots \subset M_q = M$ of submodules of M and a permutation σ such that $N_{\sigma(i)} \cong M_i/M_{i-1}$ for all $i = 1, \dots, q$. Moreover, if in addition M_i/M_{i-1} coincides with the socle of M/M_{i-1} for $i = 1, \dots, q$, then the sequence

$$N_{\sigma(1)} \prec_s N_{\sigma(2)} \prec_s \dots \prec_s N_{\sigma(q)}$$

is called the *socle series* of M . All notation used to denote particular modules is extended to the representations afforded by these modules.

Theorem 1.1 below describes the branching rules for the Γ_n -modules $L_{i,j}^n$ and the submodule structure of the restrictions of these modules to Γ_{n-1} .

Theorem 1.1. *Let $n \geq 3$ and $0 \leq i \leq j \leq n$. Set $d = \text{lp}(j-i+1)$; $\varepsilon = 0$ if $j-i+1 \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise. Then*

(i)

$$L_{i,j}^n \downarrow \Gamma_{n-1} \sim L_{i,j}^{n-1} + L_{i-1,j-1}^{n-1} + L_{i,j-1}^{n-1} + \left(\sum_{t=0}^{d-1} 2L_{i-p^t,j+p^t-1}^{n-1} \right) + \varepsilon L_{i-p^d,j+p^d-1}^{n-1}$$

(the sum in the brackets is zero whenever $d = 0$);

(ii) $L_{i,j}^n \downarrow \Gamma_{n-1} = L_{i,j}^{n-1} \oplus L_{i-1,j-1}^{n-1} \oplus D$ and the series

$$L_{i-1,j}^{n-1} \prec_s L_{i-p,j+p-1}^{n-1} \prec_s \dots \prec_s L_{i-p^{d-1},j+p^{d-1}-1}^{n-1} \prec_s M \prec_s L_{i-p^d,j+p^d-1}^{n-1} \prec_s \dots \\ \prec_s L_{i-p,j+p-1}^{n-1} \prec_s L_{i-1,j}^{n-1}$$

with $M = L_{i,j-1}^{n-1} \oplus \varepsilon L_{i-p^d,j+p^d-1}^{n-1}$ and $L_{a,b}^{n-1}$ omitted for $a < 0$, $b < a$, or $b > n-1$, is the socle series of D . In particular, $D = M$ if $i = 0$, or $j = n$, or $p \nmid j-i+1$.

Brundan and Kleshchev established a correspondence between the branching coefficients and the coefficients in the direct sum decompositions of tensor products of tilting modules for general linear groups [8, Theorem B(ii)]. Thus, the part (i) of Theorem 1.1 can also be deduced from the result of Erdmann [11] on tensor products of tilting modules for $SL_2(F)$.

Corollary 1.2. *For $n \geq 3$ the restriction $L_{i,j}^n \downarrow \Gamma_{n-1}$ is completely reducible if and only if $i = 0$, or $j = n$, or $p \nmid j-i+1$.*

Corollary 1.3. *For $n \geq 3$ and $0 < i < n$ we have*

$$L_{i,i}^n \downarrow \Gamma_{n-1} = L_{i,i}^{n-1} \oplus L_{i-1,i-1}^{n-1} \oplus L_{i-1,i}^{n-1}$$

if $p > 2$, and $L_{i,i}^n \downarrow \Gamma_{n-1} = L_{i,i}^{n-1} \oplus L_{i-1,i-1}^{n-1}$ for $p = 2$.

For $p > 2$ Corollary 1.3 can be easily deduced from [9, Main Theorem and Theorem 6.2] since in this case the restriction of $L_{i,i}^n$ to each Levi subgroup of Γ_n is completely reducible. For $p = 2$ this corollary follows immediately from the fact that $L_{i,i}^n$ can be obtained from the i th wedge power of the standard module via twisting by the Frobenius morphism associated with taking squares of the elements of F .

Denote by \det_n the 1-dimensional representation of G_n that maps each element into its determinant. It is well known that each rational representation of G_n is equivalent to a tensor product $\varphi \otimes \det_n^l$ where φ is the irreducible representation with highest weight $(a_1^{k_1}, a_2^{k_2}, \dots, 0^{k_s})$, $k_s > 0$, and $l \in \mathbb{Z}$. Since $\det_n^l \downarrow G_{n-1} = \det_{n-1}^l$, Theorem 1.1 in fact yields the branching rules for all irreducible representations of G_n with highest weights of the form $((a+2)^i, (a+1)^{j-i}, a^{n-j})$ with $a \in \mathbb{Z}$.

Observe that for $i = 0$ or for $j = n$ the modules $L_{i,j}^n$ are the fundamental modules or the tensor products of such modules with \det_n , respectively. For these modules the branching rules are characteristic-free and can be easily deduced from their realizations in the wedge powers of the standard module and the tensor products of such wedge powers with \det_n (see [13, Part II, 2.15]). So the class of modules considered in this paper yields examples of representations with highest weights of the simplest form for which the modular branching rules differ from the characteristic 0 case.

The proof of Theorem 1.1 is based on the description of the submodule structure of the Weyl modules $\Delta_{i,j}^n$ (Adamovich [1, 2]). It is shown in [2] that the combinatorics related to the submodule structure of these modules and of the fundamental Weyl modules for the symplectic groups $Sp_{2n}(F)$ is essentially the same. This structure was described in [2, Section 3.2] for both these cases, for the symplectic groups and the fundamental modules the results were announced in [3, 4]. The authors used these results in [5] to find modular branching rules for the fundamental representations of the symplectic groups $Sp_{2n}(F)$ and determine the submodule structure for the restrictions of these representations to the naturally embedded subgroups $Sp_{2n-2}(F)$. The approach and machinery of this paper are inevitably quite similar to those of [5], though main formulas look more complicated as now for each representation of a group of rank n being considered we need two parameters ($L_{i,j}^n$). For the readers' convenience descriptions of some combinatorial objects used in [5] are included. Section 2 contains some new facts on the properties of the modules $\Delta_{i,j}^n$. In particular, an irreducibility criterion for these modules is obtained (Corollary 2.12) and it is proved that their socles are always simple (Corollary 2.13).

The Schur functor allows us to apply Theorem 1.1 for describing the submodule structure of restrictions of the irreducible $F\Sigma_n$ -modules associated with 2-row partitions to naturally embedded subgroups Σ_{n-1} . Recall that the irreducible $F\Sigma_n$ -modules are parametrised by p -regular partitions. Denote by $D^{(n-i,i)}$ the irreducible $F\Sigma_n$ -module corresponding to the 2-row partition $(n-i, i)$ ($i > 0$, $n \geq 2i$, and $n > 2i$ if $p = 2$).

Theorem 1.4. *Let $n \geq 3$, $i > 0$, $n \geq 2i$, and $n > 2i$ if $p = 2$. Set $d = \text{lp}(n - 2i + 1)$; $\varepsilon = 0$ if $n - 2i + 1 \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise. Then the socle series of $D^{(n-i,i)} \downarrow \Sigma_{n-1}$ is*

$$D^{(n-i,i-1)} \prec_s D^{(n-i-1+p,i-p)} \prec_s \dots \prec_s D^{(n-i-1+p^{d-1},i-p^{d-1})} \prec_s M \prec_s$$

$$D^{(n-i-1+p^{d-1},i-p^{d-1})} \prec_s \dots \prec_s D^{(n-i-1+p,i-p)} \prec_s D^{(n-i,i-1)}$$

where $M = D^{(n-i-1,i)} \oplus \varepsilon D^{(n-i-1+p^d,i-p^d)}$ and $D^{(n-1-k,k)}$ is omitted for $k < 0$ and for $n = 2k + 1$ if $p = 2$. In particular, $D^{(n-i,i)} \downarrow \Sigma_{n-1} = M$ if and only if $p \nmid n - 2i + 1$.

The composition factors of such restrictions were first found by Sheth [17]. He also described the inductive systems of 2-row representations for Σ_∞ .

In Section 4 Theorem 1.1 is applied to classify the inductive systems of representations for the groups $GL_\infty(F)$ and $SL_\infty(F)$ that consist of representations $L_{i,j}^n$. Let

$$S_1 \subset S_2 \subset \dots \subset S_n \subset \dots \quad (1)$$

be a sequence of groups, and Ψ_n , $n = 1, 2, \dots$, be a nonempty finite set of (inequivalent) irreducible representations of S_n over a fixed field. The system $\Psi = \{\Psi_n \mid n = 1, 2, \dots\}$ is called an *inductive system* (of representations) for the group $S = \bigcup_{n=1}^\infty S_n$ if each Ψ_n coincides with the union of the sets of composition factors (up to equivalence) of the restrictions $\pi \downarrow S_n$ where π runs over Ψ_{n+1} . For algebraic groups all these representations are assumed to be rational. The union and the intersection of inductive systems Φ and Ψ and the inclusion relation for such systems are defined in a natural way. An inductive system \mathcal{T} is called decomposable if \mathcal{T} is the union of inductive systems Φ and Ψ that do not coincide with \mathcal{T} , and indecomposable otherwise.

Let $M \subset \mathbb{N}$ be infinite. Assume that Ω is an inductive system for S and $R_m \subset \Omega_m$ is nonempty for each $m \in M$. Denote by \mathcal{G}_n the set of all representations ξ of S_n such that ξ is a composition factor of the restriction $\rho \downarrow \Gamma_n$ for some $m > n$, $m \in M$, and $\rho \in R_m$. Assume that $R_m \subset \mathcal{G}_m$ for all m . Then it is easy to check that $\mathcal{G} = \{\mathcal{G}_n \mid n = 1, 2, \dots\}$ is an inductive system for S . We shall write $\mathcal{G} = \langle R_m \mid m \in M \rangle$. If every R_m consists of a single representation ρ_m , we use a simplified notation $\mathcal{G} = \langle \rho_m \mid m \in M \rangle$.

In this article (1) is the sequence of the naturally embedded groups Γ_n (naturally, either all $\Gamma_n = G_n$, or all $\Gamma_n = H_n$), so $\bigcup_{n=1}^\infty \Gamma_n = GL_\infty(F)$ or $SL_\infty(F)$. Set

$$\begin{aligned} \mathcal{F}_n &= \{L_{i,j}^n \mid 0 \leq i \leq j \leq n\}, & \mathcal{F} &= \{\mathcal{F}_n \mid n = 1, 2, \dots\}; \\ \mathcal{L}_n^a &= \{L_{i,j}^n \mid 0 \leq i \leq j \leq n, i \leq a\}, & \mathcal{L}^a &= \{\mathcal{L}_n^a \mid n = 1, 2, \dots\}; \\ \mathcal{R}_n^b &= \{L_{i,j}^n \mid 0 \leq i \leq j \leq n, n - j \leq b\}, & \mathcal{R}^b &= \{\mathcal{R}_n^b \mid n = 1, 2, \dots\}; \\ \mathcal{M}_n^t &= \{L_{i,j}^n \mid 0 \leq i \leq j \leq n, j - i + 1 < p^{t+1}\}, & \mathcal{M}^t &= \{\mathcal{M}_n^t \mid n = 1, 2, \dots\} \end{aligned}$$

(here $a, b, t \in \mathbb{Z}_{\geq 0}$). Put $\mathcal{LR}^{a,b} = \mathcal{L}^a \cap \mathcal{R}^b$.

For a pair of integers a, b with $0 \leq a \leq b$ set $\mathcal{L}^{a,b} = \langle L_{a,b}^n \mid n \geq b \rangle$ and $\mathcal{R}^{a,b} = \langle L_{n-b,n-a}^n \mid n \geq b \rangle$. Theorem 1.1 i) implies that the systems $\mathcal{L}^{a,b}$ and $\mathcal{R}^{a,b}$ are correctly determined.

Theorem 1.5. *Every inductive system $\Phi \subset \mathcal{F}$ is a finite union of indecomposable ones. If $S = GL_\infty(F)$, then \mathcal{F} , \mathcal{L}^a , \mathcal{R}^b , $\mathcal{LR}^{a,b}$, \mathcal{M}^t , $\mathcal{L}^{a,b}$, and $\mathcal{R}^{a,b}$ ($a, b, t \in \mathbb{Z}_{\geq 0}$, $a \leq b$ for $\mathcal{L}^{a,b}$ and $\mathcal{R}^{a,b}$) constitute the complete list of indecomposable inductive systems. For $S = SL_\infty(F)$ one have to exclude $\mathcal{LR}^{0,b}$, $\mathcal{LR}^{a,0}$, one of \mathcal{L}^0 and \mathcal{R}^0 , and one of $\mathcal{L}^{0,0}$ and $\mathcal{R}^{0,0}$ to avoid duplicates.*

In Section 4 we shall show that all systems mentioned in Theorem 1.5 are really inductive systems and check their indecomposability (see Results 4.3, 4.4, and 4.28). Zhilinskii [23] has described inductive systems of representations for sequences of natural embeddings of the classical complex simple Lie algebras. His results can be immediately transferred to classical simple algebraic groups in characteristic 0. Extend the notation \mathcal{F} , \mathcal{L}^a , \mathcal{R}^b , $\mathcal{LR}^{a,b}$, $\mathcal{L}^{a,b}$, and $\mathcal{R}^{a,b}$ to the sequence of the standard embeddings of special linear groups in characteristic 0 in the natural way. Then [23, Theorems 2.3.1 and 2.3.2] yield the following

Theorem 1.6. *Let $\Psi \subset \mathcal{F}$ be an inductive system for the group $SL_\infty(F_0)$ over an algebraically closed field F_0 of characteristic 0. Then Ψ is a finite union of indecomposable such systems that are exhausted by the following ones: \mathcal{F} , \mathcal{L}^a , \mathcal{R}^b , $\mathcal{LR}^{a,b}$, $\mathcal{L}^{a,b}$, and $\mathcal{R}^{a,b}$*

with $a, b \in \mathbb{Z}_{\geq 0}$, $a \leq b$ for $\mathcal{L}^{a,b}$ and $\mathcal{R}^{a,b}$, and the same assumptions for the zero values of a and b as in Theorem 1.5 for $SL_\infty(F)$.

Naturally, the systems \mathcal{M}^t yield a purely modular phenomenon. We also wish to emphasize that in general in the modular case the systems $\mathcal{L}^{a,b}$ and $\mathcal{R}^{a,b}$ differ substantially from their characteristic 0 counterparts. The classical branching rules imply that in the latter case

$$\mathcal{L}_n^{a,b} = \{L_{i,j}^n \mid i \leq a, j \leq b\} \text{ and } \mathcal{R}_n^{a,b} = \{L_{i,j}^n \mid n - i \leq b, n - j \leq a\}. \quad (2)$$

In Section 4 a detailed description of such systems in characteristic p is given (Proposition 4.8 and Corollary 4.11). In particular, we show that usually (2) does not hold (Remark 4.13).

The minimal and the minimal nontrivial inductive systems for the group $SL_\infty(F)$ are found in [6].

2. THE STRUCTURE OF THE WEYL MODULES $\Delta_{i,j}^n$

In this section we present and refine the results of [1, 2] on the structure of these modules for Γ_n . In fact, in [1, 2] the group G_n is considered. However, one has $G_n = H_n Z(G_n)$ and it is well known that the centre $Z(G_n)$ acts on the Weyl modules (on all indecomposable ones) by scalars. Hence the sets of G_n and H_n -submodules in these modules coincide. This enables us to handle G_n and H_n simultaneously, i.e. to cite and establish results for Γ_n . In this section all modules considered are Γ_n -modules.

Throughout the paper for integers i and j with $i \equiv j - 1 \pmod{2}$ we set $\pi_{i,j}^n = L_{(j-i+1)/2, (i+j-1)/2}^n$ and $V_{i,j}^n = \Delta_{(j-i+1)/2, (i+j-1)/2}^n$. Denote by $[a, b]$ the set of all $j \in \mathbb{Z}_{\geq 0}$ with $a \leq j \leq b$. For an integer $a \in \mathbb{Z}_{\geq 0}$ write its p -adic expansion $a = a_0 + a_1 p + \dots + a_s p^s$ with $0 \leq a_i < p$ and set $a_i = 0$ for all such $i \in \mathbb{Z}_{\geq 0}$ that $p^i > a$. We shall write $a = (a_0, a_1, \dots, a_s)$. We say that an integer b contains a to base p and write $a \subset_p b$ if and only if for each i either $a_i = 0$, or $a_i = b_i$. Set $d_a^b = 1$ if $a \subset_p b$, and $d_a^b = 0$ otherwise.

We need some more notation to state Adamovich's results. For $\lambda \in \mathbb{Z}_{\geq 0}$ define maps $s'_\lambda : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ and $s_\lambda : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ setting $s'_\lambda(l) = l + 2k'$ where $l + 1 = a'p^\lambda - k'$, $a' \in \mathbb{Z}_{\geq 0}$, $0 \leq k' < p^\lambda$; $s_\lambda(l) = l + 2k$ where $l = ap^\lambda - k$, $a \in \mathbb{Z}_{\geq 0}$, $0 \leq k < p^\lambda$. We say that the reflection s'_λ or s_λ is l -admissible if $k' \neq 0$ and $p \nmid a'$ or $k \neq 0$ and $p \nmid a$, respectively. We denote by $S(l)$ the set of all $m > l$ that can be written in the form $m = s_{\lambda_u} \dots s_{\lambda_1}(l)$ where $\lambda_u < \dots < \lambda_1$ and for each $i = 0, 1, \dots, u-1$ the reflection $s_{\lambda_{i+1}}$ is $s_{\lambda_i} \dots s_{\lambda_1}(l)$ -admissible. Similarly we define $S'(l)$ (writing s'_{λ_i} instead of s_{λ_i}).

Theorem 2.1. [1, 2] *Let $0 \leq l \leq n$ and $q \in \mathbb{Z}_{\geq 0}$. Then $V_{l+1,q}^n \sim \pi_{l+1,q}^n + \sum_{m \in S'(l)} \pi_{m+1,q}^n$.*

As $s'_\lambda(x-1) = s_\lambda(x) - 1$, the following theorem yields an equivalent statement.

Theorem 2.2. *Let $1 \leq l \leq n+1$ and $q \in \mathbb{Z}_{\geq 0}$. Then $V_{l,q}^n \sim \pi_{l,q}^n + \sum_{m \in S(l)} \pi_{m,q}^n$.*

We would like to have a closed formula for the composition factors of $V_{i,j}^n$. This formula will be written in terms of the coefficients d_a^b introduced at the beginning of the section. We claim that the following theorem holds.

Theorem 2.3. *Let $1 \leq l \leq n+1$ and $q \in \mathbb{Z}_{\geq 0}$. Then $V_{l,q}^n \sim \sum_{k=0}^{\infty} d_k^{l+2k} \pi_{l+2k,q}^n$.*

To prove this, we need some technical facts on the triples k, l, m with $k \subset_p m = l + 2k$ and admissible reflections. Until the end of the section $l \geq 1$. For each $m \in S(l)$ the tuple $(\lambda_1; \dots; \lambda_u)$ is uniquely determined ([2]). If u is odd for some m , set $\lambda_{u+1} = \text{lp}(m)$. Then $s_{\lambda_{u+1}}(m) = m$ and $\lambda_{u+1} < \lambda_u$. Now for every $m \in S(l)$ we have a uniquely determined sequence of reflections $s_{\lambda_1}, \dots, s_{\lambda_{2t}}$. Such sequences will be called l -admissible. For an integer $0 \leq a \leq p-1$ set $\bar{a} = p-1-a$. The following lemma is straightforward.

Lemma 2.4. *Set $q = \text{lp}(l)$. The reflection s_λ is l -admissible if and only if $\lambda > q$ and $l_\lambda \neq p - 1$. In that case $s_\lambda(l) = p^q(\bar{l}_q + 1, \bar{l}_{q+1}, \dots, \bar{l}_{\lambda-1}, l_\lambda + 1, l_{\lambda+1}, \dots)$.*

Two consequent applications of Lemma 2.4 yield

Proposition 2.5. *Let $\text{lp}(l) \leq \mu < \lambda$, $m = s_\mu s_\lambda(l)$, and $k = (m - l)/2$. The pair s_λ, s_μ is l -admissible if and only if $l_\lambda \neq p - 1$ and $l_\mu \neq 0$. In that case*

$$\begin{aligned} m &= (l_0, \dots, l_{\mu-1}, \bar{l}_\mu + 1, \bar{l}_{\mu+1}, \dots, \bar{l}_{\lambda-1}, l_\lambda + 1, l_{\lambda+1}, \dots), \\ k &= (0, \dots, 0, \bar{l}_\mu + 1, \bar{l}_{\mu+1}, \dots, \bar{l}_{\lambda-1}). \end{aligned}$$

In particular, $k \subset_p m = l + 2k$.

We call a tuple $\sigma = (\lambda_1; \dots; \lambda_{2t})$ l -admissible if $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_1 > \dots > \lambda_{2t}$, $l_{\lambda_{2j-1}} \neq p - 1$ and $l_{\lambda_{2j}} \neq 0$ for $j = 1, \dots, t$. For an l -admissible σ set

$$\mathfrak{Q}_l(\sigma) = \bigcup_{j=1}^t [\lambda_{2j}, \lambda_{2j-1} - 1],$$

$\delta_{i\sigma} = 1$ if $i = \lambda_j$ for some j and $\delta_{i\sigma} = 0$ otherwise. Define $l^\sigma \in \mathbb{Z}_{\geq 0}$ putting

$$l_i^\sigma = \begin{cases} \bar{l}_i + \delta_{i\sigma}, & i \in \mathfrak{Q}_l(\sigma) \\ l_i + \delta_{i\sigma}, & i \notin \mathfrak{Q}_l(\sigma). \end{cases}$$

Proposition 2.5 yields the following corollary.

Corollary 2.6. *A sequence $s_{\lambda_1}, \dots, s_{\lambda_{2t}}$ is l -admissible if and only if the tuple $\sigma = (\lambda_1; \dots; \lambda_{2t})$ is l -admissible. In that case $s_{\lambda_{2t}} \dots s_{\lambda_1}(l) = l^\sigma$.*

Proposition 2.7. ([5, Proposition 2.8]) *An integer $m \in S(l)$ if and only if $m - l = 2k > 0$ and $k \subset_p m$.*

Now Theorem 2.3 follows directly from Proposition 2.7.

Next, we rewrite Adamovich's results [2, Section 3.2] on the submodule structure of the Weyl modules in our terms. We fix n and write $V_{l,r}$ and $\pi_{m,r}$ instead of $V_{l,r}^n$ and $\pi_{m,r}^n$. For $m \in S(l)$ or $m = l$ we denote by $P_l(m, r)$ the smallest submodule of $V_{l,r}$ that has a composition factor $\pi_{m,r}$. Since $V_{l,r}$ is multiplicity-free, $P_l(m, r)$ is correctly defined and each submodule of $V_{l,r}$ is a sum of $P_l(m, r)$ for some m . Hence the submodule structure of $V_{l,r}$ is determined by the inclusion relations between the submodules $P_l(m, r)$ (see comments in [16, Section 3]). We shall write $\pi_{m,r} \prec \pi_{q,r}$ if $P_l(m, r) \subset P_l(q, r)$. Let $\sigma = (\lambda_1; \dots; \lambda_{2t})$ be an l -admissible tuple. For $m = l^\sigma$ set

$$\mathfrak{Q}_l(m) = \mathfrak{Q}_l(\sigma) = \bigcup_{j=1}^t [\lambda_{2j}, \lambda_{2j-1} - 1].$$

Put also $\mathfrak{Q}_l(l) = \emptyset$.

Theorem 2.8. [2, Theorem 3.2.1] *Assume that $1 \leq m, q \leq n + 1$, $m, q \in S(l) \cup \{l\}$, and both $\pi_{m,r}$ and $\pi_{q,r} \neq 0$. Then the module $\pi_{m,r} \prec \pi_{q,r}$ (as composition factors of $V_{l,r}$) if and only if $\mathfrak{Q}_l(q) \subset \mathfrak{Q}_l(m)$.*

Remark 2.9. Actually the sets $\mathfrak{P}_l(m)$ which are considered in [2] differ slightly from $\mathfrak{Q}_l(m)$. For $m \in S(l)$ one has $\mathfrak{P}_l(m) = \bigcup_{j=1}^t [\mu_{2j} + 1, \mu_{2j-1}]$ where $\mu_s = \lambda_s$ for $s < 2t$, $\mu_{2t} = \lambda_{2t}$ if $m \neq s_{\lambda_{2t-1}} \dots s_{\lambda_1}(l)$, and $\mu_{2t} = 0$ otherwise. However, Lemma 2.4 enables one to deduce that $\mathfrak{P}_l(m) \subset \mathfrak{P}_l(q)$ if and only if $\mathfrak{Q}_l(m) \subset \mathfrak{Q}_l(q)$. The crucial point is that $\text{lp}(l) = \text{lp}(m)$ for $m \in S(l)$.

Theorem 2.8 is stated in [16, Theorem 3.3] in a slightly different manner.

For l -admissible tuples $\sigma = (\lambda_1; \dots; \lambda_{2t})$ and $\sigma' = (\lambda'_1; \dots; \lambda'_{2s})$ we say that $\sigma \leq \sigma'$ if there exists $f \leq 2t, 2s$ such that $\lambda_i = \lambda'_i$ for $1 \leq i \leq f$ and either $f = 2t$, or $f < 2t, 2s$ and $\lambda_{f+1} < \lambda'_{f+1}$. It is convenient to assume that the empty tuple \emptyset is l -admissible, $\emptyset \leq \sigma$ for all σ , $l^\emptyset = l$, and $\mathfrak{Q}_l(\emptyset) = \emptyset$. The following is obvious.

Lemma 2.10. *Let σ and σ' be l -admissible tuples. Then $l^\sigma \leq l^{\sigma'}$ if and only if $\sigma \leq \sigma'$.*

Until the end of the section we fix l and r such that $\pi_{l,r} \neq 0$ for our fixed n . Set $r' = \min\{r+1, 2n-r+1\}$. We have $r' = r+1$ just when $r \leq n$. Now our goal is to find the maximal integer m such that $\pi_{m,r}$ is a composition factor of $\pi_{l,r}$. Using the definition of $\pi_{m,r}$, one easily observes that $\pi_{m,r} \neq 0$ if and only if $m \leq r'$. Construct an l -admissible tuple $\sigma^{\max} = (\mu_1; \dots; \mu_{2t})$ as follows. Put $\mu_0 = +\infty$. Assume that μ_{2j} is chosen. Set $\mu = \mu_{2j} - 1$. If there is no l -admissible tuple $(\alpha; \beta)$ such that $\mu \geq \alpha > \beta$ and

$$(l_0, \dots, l_\mu)^{(\alpha; \beta)} = (l_0, \dots, \bar{l}_\beta + 1, \bar{l}_{\beta+1}, \dots, \bar{l}_{\alpha-1}, l_\alpha + 1, \dots, l_\mu) \leq (r'_0, \dots, r'_\mu),$$

we stop the process and set $t = j$ ($\sigma^{\max} = \emptyset$ if $t = 0$). Otherwise we choose maximal such pair $(\alpha; \beta)$ (with respect to \leq); set $\mu_{2j+1} = \alpha$ and $\mu_{2j+2} = \beta$; and if

$$(\bar{l}_\beta + 1, \bar{l}_{\beta+1}, \dots, \bar{l}_{\alpha-1}, l_\alpha + 1) < (r'_\beta, \dots, r'_\alpha),$$

we stop the process and determine $(\mu_{2j+3}; \dots; \mu_{2t})$ as the maximal l -admissible tuple with $\mu_{2j+3} < \beta$. Obviously, $l^{\sigma^{\max}}$ is the maximal integer m such that $\pi_{m,r}$ is a composition factor of $V_{l,r}$.

For l -admissible tuples σ and σ' we write $\sigma \prec \sigma'$ if and only if $\mathfrak{Q}_l(\sigma) \supset \mathfrak{Q}_l(\sigma')$. Using Corollary 2.6, Theorem 2.8, and Lemma 2.10, we get the following theorem on the structure of the Weyl modules $V_{l,r}$.

Theorem 2.11. *The map $\sigma \mapsto \pi_{l^\sigma, r}$ is a poset isomorphism between the l -admissible tuples $\sigma \leq \sigma^{\max}$ and the composition factors of $V_{l,r}$ with the partial orders \prec .*

If $l < r'$, we denote by v the maximal integer such that $l_v \neq r'_v$. If $l_v + 1 = r'_v$, we denote by u the maximal integer $< v$ such that $\bar{l}_u \neq r'_u$ setting $u = -1$ if $\bar{l}_i = r'_i$ for all $i < v$. Put $s = \text{lp}(l)$.

Corollary 2.12. *The module $V_{l,r}$ is irreducible (i.e. $\sigma^{\max} = \emptyset$) if and only if one of the following holds.*

- (1) $l = r'$;
- (2) $l < r'$ and $s \geq v$;
- (3) $l < r'$, $s < v$, $l_v + 1 = r'_v$, $\bar{l}_s \geq r'_s$; $l_i = p - 1$ and $r'_i = 0$ for $s < i < v$.

Proof. This follows from Proposition 2.5 and Corollary 2.6. \square

Corollary 2.13. *The socle of $V_{l,r}$ is always simple. For reducible $V_{l,r}$ it has the form $\pi_{l^\gamma, r}$ with $\gamma = (t; s)$ and t as follows.*

- (1) $t = v$ if $s < v$ and either $l_v + 1 < r'_v$, or $\bar{l}_u < r'_u$;
- (2) $t = w$ if $l_v + 1 = r'_v$; $u = -1$ or $\bar{l}_u > r'_u$; $s < w < v$; $l_w \neq p - 1$; and $l_j = p - 1$ for $w < j < v$.

Proof. Applying Results 2.5, 2.6, and 2.11, we conclude that $\pi_{l^\gamma, r}$ is a composition factor of $V_{l,r}$ and for each l -admissible tuple $\tau \leq \sigma^{\max}$ the set $\mathfrak{Q}_l(\tau) \subset [s, t - 1]$, so $\pi_{l^\gamma} \prec \pi_{l^\tau}$. \square

3. BRANCHING RULES AND THE SUBMODULE STRUCTURE OF THE RESTRICTIONS

In this section the main results of the article are proved. We shall need the following simple lemma.

Lemma 3.1. ([5, Lemma 3.1]) *Assume that $d_k^{l+2k} = 1$ (i.e. $k \subset_p l + 2k$).*

- (i) *If $p^s \mid l + 2k$, then $p^s \mid k$ and $p^s \mid l$.*
- (ii) *If $p^s \mid l$, then $p^s \mid k$ and $p^s \mid l + 2k$.*

As in Section 2, we shall omit the superscript n in our notation for modules when it is known what group is considered. Replacing $L_{i,j}$ by $\pi_{j-i+1,i+j}$ and $\Delta_{i,j}$ by $V_{j-i+1,i+j}$, one immediately concludes that Theorem 1.1(i) is equivalent to the following

Theorem 3.2. *Let $n \geq 3$, $1 \leq i \leq r+1 \leq 2n+2-i$, and $i \equiv r+1 \pmod{2}$. Set $d = \text{lp}(i)$. Then*

$$\pi_{i,r}^n \downarrow \Gamma_{n-1} \sim \pi_{i-1,r-1} + \pi_{i,r} + \pi_{i,r-2} + \left(\sum_{t=0}^{d-1} 2\pi_{i-1+2p^t,r-1} \right) + \varepsilon \pi_{i-1+2p^d,r-1} \quad (3)$$

where $\varepsilon = 0$ if $i \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise.

Proof. One can rewrite Formula (3) as follows.

$$\pi_{i,r}^n \downarrow \Gamma_{n-1} \sim \pi_{i-1,r-1} + \pi_{i,r} + \pi_{i,r-2} + \sum_{t=0}^{\infty} b_t^i \pi_{i-1+2p^t,r-1} \quad (4)$$

where

$$b_t^i = \begin{cases} 2, & i \equiv 0 \pmod{p^{t+1}}, \\ 1, & i \equiv ap^t \pmod{p^{t+1}} \text{ and } a \not\equiv 0, -1 \pmod{p}, \\ 0, & i \not\equiv 0 \pmod{p^t} \text{ or } i \equiv -p^t \pmod{p^{t+1}}. \end{cases} \quad (5)$$

Recall that by convention $\pi_{i,t}^n = 0$ for all $i > n+1$ and each t , $\pi_{n+1,t}^n = 0$ for $t \neq n$, and $\pi_{n+1,n}^n = \det_n$ for $\Gamma_n = G_n$. So (4) holds for $i \geq n+1$. Assume now that $1 \leq l < n+1$ and (4) is valid for all $i > l$. We shall prove it for $i = l$. Then the theorem will follow by induction.

It follows from [10, Proposition 3.3.2 and Theorem 4.3.1] that $V_{l,r} \downarrow \Gamma_{n-1}$ has a filtration by Weyl modules for Γ_{n-1} . Then the classical branching rules for characteristic 0 [22] and Theorem 2.3 imply

$$V_{l,r} \downarrow \Gamma_{n-1} \sim V_{-1,r-1} + V_{l,r} + V_{l,r-2} + V_{l+1,r-1} \sim \mathcal{V} + \sum_{t=0}^{\infty} f_t^{l+2t-1} \pi_{l+2t-1,r-1} \quad (6)$$

where $\mathcal{V} = \sum_{k=0}^{\infty} d_k^{l+2k} \pi_{l+2k,r} + \sum_{k=0}^{\infty} d_k^{l+2k} \pi_{l+2k,r-2}$, $f_0^{l-1} = d_0^{l-1}$, and

$$f_t^{l+2t-1} = d_t^{l+2t-1} + d_{t-1}^{l+2t-1} \quad \text{for } t \geq 1.$$

On the other hand, by Theorem 2.3, $V_{l,r} \downarrow \Gamma_{n-1} \sim \sum_{k=0}^{\infty} d_k^{l+2k} (\pi_{l+2k,r} \downarrow \Gamma_{n-1})$. Since $d_0^l = 1$ and the branching rules for $\pi_{i,r}$ with $i > l$ are assumed to satisfy (4), one can determine the branching of $\pi_{l,r}$. Therefore it suffices to check that the right part of (6) is equal to $\sum_{k=0}^{\infty} d_k^{l+2k} U_{l+2k}$ where U_i is the right part of (4) for the restriction of $\pi_{i,r}$ to Γ_{n-1} . The latter sum can be rewritten as follows:

$$\mathcal{V} + \sum_{k=0}^{\infty} d_k^{l+2k} (\pi_{l+2k-1,r-1} + \sum_{s=0}^{\infty} b_s^{l+2k} \pi_{l+2k+2p^s-1,r-1}) = \mathcal{V} + \sum_{t=0}^{\infty} e_t^{l+2t-1} \pi_{l+2t-1,r-1}$$

where

$$e_t^{l+2t-1} = d_t^{l+2t} + \sum_{k,s \geq 0, k+p^s=t} d_k^{l+2k} b_s^{l+2k}. \quad (7)$$

We have to show that $e_t^{l+2t-1} = f_t^{l+2t-1}$ for all $t \geq 0$. Actually this is a formula involving only the coefficients d_k^m for different values of k and m . It is proved in [5] (4 steps at the end of the proof of Theorem 3.2). This completes the proof of the theorem. \square

Now consider the submodule structure of the restriction $\pi_{i,r}^n \downarrow \Gamma_{n-1}$. To determine this structure, we use some arguments connected with the notion of contravariant dual modules. By [13, Part II, Corollary 1.16], for a reductive algebraic group G with a fixed maximal torus T there exists an antiautomorphism τ of G such that $\tau^2 = id_G$, $\tau \downarrow T = id_T$, and $\tau(U_\alpha) = U_{-\alpha}$ for each root subgroup U_α associated with a root α . For $G = G_n$ or H_n we may assume that $\tau(g) = {}^t g$ (the transposed matrix). For a finite dimensional G -module M the module ${}^\tau(M)$ can be defined as follows. As a vector space, one has ${}^\tau(M) = M^*$ (the space of linear functions on M). The action of G on M is defined via $g\varphi = \varphi(\tau(g))$ for $g \in G$ and $\varphi \in M^*$. We call ${}^\tau(M)$ the contravariant dual module for M . For formal characters one has $ch {}^\tau M = ch M$. Hence ${}^\tau M \cong M$ for simple G -modules M . An exact sequence $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ of finite dimensional G -modules yields an exact sequence $0 \rightarrow {}^\tau M_2 \rightarrow {}^\tau M \rightarrow {}^\tau M_1 \rightarrow 0$ ([13, Part II, 2.12]. Denote by τ_n the morphism τ for the group Γ_n . One may assume that $\tau_n \downarrow \Gamma_n = \tau_{n-1}$. Then ${}^\tau(\pi_{i,r} \downarrow \Gamma_{n-1}) \cong \pi_{i,r} \downarrow \Gamma_{n-1}$ since ${}^\tau(\pi_{i,r}) \cong \pi_{i,r}$.

Let $n > 2$ and $1 \leq i \leq n$. (Obviously, there is nothing to prove for the restriction of \det_n). As $\pi_{i,r}^n$ is the top composition factor of $V_{i,r}^n$, it follows from (6) that $\pi_{i,r}^n \downarrow \Gamma_{n-1}$ is a quotient of the Γ_{n-1} -module $V_{i,r}^n \downarrow \Gamma_{n-1} \sim V_{i,r} + V_{i,r-2} + V_{i-1,r-1} + V_{i+1,r-1}$. Applying Smith' theorem [18] both to $V_{i,r}$ and $\pi_{i,r}$, we conclude that $V_{i,r}^n \downarrow \Gamma_{n-1} = V_{i,r} \oplus V_{i,r-2} \oplus V$ where $V \sim V_{i-1,r-1} + V_{i+1,r-1}$, and $\pi_{i,r}^n \downarrow \Gamma_{n-1} = \pi_{i,r} \oplus \pi_{i,r-2} \oplus D$ where D is a quotient of V . Here the summands $V_{i,r-2}^{n-1}$ and $\pi_{i,r-2}^{n-1}$ are the Γ_{n-1} -submodules in $V_{i,r}^n$ and $\pi_{i,r}^n$, respectively, generated by highest weight vectors, and the summands $V_{i,r}^{n-1}$ and $\pi_{i,r}^{n-1}$ are such submodules generated by lowest weight vectors. Now Theorem 1.1(ii) and Corollary 1.2 follow immediately from

Theorem 3.3. *Let $1 \leq i \leq r+1 \leq 2n+2-i$ and $i \equiv r+1 \pmod{2}$. Let $d = \text{lp } i$. Set $\varepsilon = 0$ if $i \equiv -p^d \pmod{p^{d+1}}$ and $\varepsilon = 1$ otherwise; $j_q = i-1+2p^q$. Put $f = \min\{r, 2n-r\}$. Choose minimal $t \in \mathbb{Z}_{\geq 0}$ such that $jt > f$ and set $d' = \min\{d, t\}$. Then*

$$\begin{aligned} \pi_{j_0,r-1} \prec_s \pi_{j_1,r-1} \prec_s \cdots \prec_s \pi_{j_{d'-1},r-1} \prec_s \pi_{i-1,r-1} \oplus \varepsilon \pi_{j_d,r-1} \prec_s \pi_{j_{d'-1},r-1} \prec_s \cdots \\ \prec_s \pi_{j_1,r-1} \prec_s \pi_{j_0,r-1} \end{aligned}$$

is the socle series of D . In particular, $D = \pi_{i-1,r-1} \oplus \varepsilon \pi_{j_d,r-1}$ if $d' = 0$.

Proof. By Theorem 3.2, $D \sim \pi_{i-1,r-1} + 2\pi_{j_0,r-1} + \cdots + 2\pi_{j_{d'-1},r-1} + \varepsilon \pi_{j_d,r-1}$. It follows from Theorem 2.3 that the factors $\pi_{i-1,r-1}, \pi_{j_0,r-1}, \dots, \pi_{j_{d'-1},r-1}$ come from $V_{i-1,r-1}$ and the factors $\pi_{j_0,r-1}, \dots, \pi_{j_{d'-1},r-1}$, and $\varepsilon \pi_{j_d,r-1}$ if nonzero come from $V_{i+1,r-1}$. Note that

$$j_k = i-1+2p^k = i+p^k + (p^k-1) = i + \underbrace{(p-1, \dots, p-1, 1)}_k,$$

so $\mathfrak{Q}^{i-1}(j_k) = [k, d-1]$ for all $0 \leq k \leq d-1$. Therefore by Theorem 2.8,

$$\pi_{j_0,r-1} \prec \pi_{j_1,r-1} \prec \cdots \prec \pi_{j_{d'-1},r-1} \prec \pi_{i-1,r-1} \quad \text{in } V_{i-1,r-1}. \quad (8)$$

Similarly, we get $\mathfrak{Q}^{i+1}(j_k) = [0, k-1]$ for $1 \leq k < d$ (and for $k = d$ if $\varepsilon \neq 0$ and $d > 0$). Hence

$$\varepsilon \pi_{j_d,r-1} \prec \pi_{j_{d'-1},r-1} \prec \cdots \prec \pi_{j_1,r-1} \prec \pi_{j_0,r-1} \quad \text{in } V_{i+1,r-1}. \quad (9)$$

(Here the symbol \prec is extended to the zero module in the natural way.) Since $\pi_{i,r}^n$ is contravariant selfdual, D is such. Let $D_1 \prec_s \cdots \prec_s D_m$ be the socle series of D . Recall that D has a filtration by quotients of $V_{i-1,r-1}$ and $V_{i+1,r-1}$. As the factor $\pi_{i-1,r-1}$ has multiplicity 1 and D is contravariant selfdual, (8) implies that $\pi_{i-1,r-1}$ is a factor of D_q with $d'+1 \leq q \leq m-d'$, so $m \geq 2d'+1$. If $\varepsilon \pi_{j_d,r-1} = 0$, then $m = 2d'+1$ is the composition length of D and the theorem follows from (8) and (9). Assume that $\varepsilon \pi_{j_d,r-1} \neq 0$. As above, by the contravariant selfduality of D and (9), $\pi_{j_d,r-1}$ is a factor

of $D_{q'}$ with $d' + 1 \leq q' \leq m - d'$. Assume that $q' \neq q$. Then $m = 2d' + 2$, so D is uniserial, which contradicts the contravariant selfduality of D . Hence $q' = q$ and the theorem follows from (8) and (9). \square

Remark 3.4. Obviously, if $\varepsilon\pi_{j_d, r-1} = 0$ (i.e. $d' < d$ or $i \equiv -p^d \pmod{p^{d+1}}$), then the module D is uniserial, so has exactly $2d' + 2$ different submodules. Since D is contravariant selfdual, one can easily observe that D has exactly $2d + 4$ different submodules in the case where $\varepsilon\pi_{j_d, r-1} \neq 0$ (i.e. $d = d'$ and $i \not\equiv -p^d \pmod{p^{d+1}}$).

Following [14], we will use the Schur functor to transfer our results on branching rules to symmetric groups. Our aim is to prove Theorem 1.4.

We keep the notation $G_n = GL_n(F)$ and $H_n = SL_n(F)$. Let $M(n, n)$ be the category of the polynomial G_n -modules over F which are homogeneous of degree n . Let $V \in M(n, n)$ and let $\lambda = (a_1, \dots, a_n)$ be the highest weight of V . Note that $a_1 \geq \dots \geq a_n \geq 0$ and $a_1 + \dots + a_n = n$, i.e. λ is a partition of n .

The Schur functor

$$\varphi_n : M(n, n) \rightarrow F\Sigma_n\text{-mod}$$

is defined as $\varphi(V) = V_0$ where V_0 is the $(1, \dots, 1)$ -weight subspace of the module V [12, Chapter 6]. Alternatively, consider V as an H_n -module and denote by V_ω the ω -weight subspace of V . Then $\varphi(V)$ is exactly the 0-weight subspace of the H_n module V .

Assume that V is irreducible, non-trivial (i.e. $a_n = 0$), and p -restricted as an H_n -module. Let $\alpha_1, \dots, \alpha_{n-1}$ be the simple roots of H_n and let μ be the highest weight of V (i.e. $\mu = \sum_{i=1}^{n-1} (a_i - a_{i+1})\omega_i$ where $\omega_1, \dots, \omega_{n-1}$ are the fundamental weights). For $i \geq 0$, the i th level V^i is defined as $V^i = \sum V_\omega$ where ω runs over those weights $\omega = \mu - \sum_{j=1}^{n-1} k_j \alpha_j$ for which $k_{n-1} = i$. Then each V^i is H_{n-1} -invariant and $V = \bigoplus V^i$ as an H_{n-1} -module. By [14, Lemma 1.4], the H_{n-1} -module V^1 is a restriction of a module in $M(n-1, n-1)$ and $V_0 \subset V^1$. Thus, as it was shown in [14] (see also [15, Lemma 4.8]), the restriction of $\varphi_n(V)$ to Σ_{n-1} is isomorphic to $\varphi_{n-1}(V^1)$.

The functor φ_n is exact and by [12, 6.4],

$$\varphi_n(V) \cong D^{\lambda'} \otimes \text{sgn} \quad (10)$$

where $D^{\lambda'}$ is the irreducible Σ_n -module corresponding to the partition λ' , which is dual to λ , and sgn is the sign representation of Σ_n . Note that λ' is p -regular since μ is p -restricted. If μ is not p -restricted, then $\varphi_n(V) = 0$. Let $M(n, n)^c$ be the full subcategory of $M(n, n)$ consisting of all modules with p -restricted socle and head. There is a 'truncated inverse Schur functor' $\varphi_n^c : F\Sigma_n\text{-mod} \rightarrow M(n, n)^c$ such that φ_n and φ_n^c induce an equivalence of the categories $M(n, n)^c \leftrightarrow \Sigma_n\text{-mod}$ [14, Theorem 2.12].

Assume now that λ is a two-column diagram $(2^i, 1^{n-2i}, 0^i)$ with $i \neq 0$, $n \geq 2i$, and $n > 2i$ if $p = 2$. Then λ' is the partition $(n - i, i)$, $V \in M(n, n)$, V is p -restricted, and $V = L_{i, n-i}^n$. It is easy to see that the H_{n-1} -module V^1 is exactly the module D in Theorem 1.1. Note that $D \in M(n-1, n-1)^c$. Moreover, each composition factor of D is p -restricted except in the case $p = 2$ and $n = 2i + 1$. Thus the equivalence of the categories $M(n-1, n-1)^c \leftrightarrow \Sigma_{n-1}\text{-mod}$ yields an isomorphism of the submodule structures of D and $\varphi_{n-1}(D)$. It remains to note that

$$\varphi_{n-1}(D) = \varphi_{n-1}(V^1) \cong (D^{\lambda'} \downarrow_{\Sigma_{n-1}}) \otimes \text{sgn} = (D^{(n-i, i)} \downarrow_{\Sigma_{n-1}}) \otimes \text{sgn}$$

and apply (10) to each composition factor of D . This proves Theorem 1.4 for non-exceptional case. In the exceptional case $p = 2$ and $n = 2i + 1$ (i.e. $V = L_{i, i+1}^n$), D contains three composition factors $L_{i-1, i+1}^{n-1}, L_{i, i}^{n-1}, L_{i-1, i+1}^{n-1}$. The factor $L_{i, i}^{n-1}$ is not 2-restricted, so it is killed by φ_{n-1} . Thus $D^{(i+1, i)} \downarrow_{\Sigma_{2i}}$ has two copies of $D^{(i+1, i-1)}$ as composition factors.

Since the categories $M(n-1, n-1)^c$ and $\Sigma_{n-1}\text{-mod}$ are equivalent and D is indecomposable, $D^{(i+1, i)} \downarrow \Sigma_{2i}$ cannot be a direct sum of these two copies. This completes the proof of Theorem 1.4.

4. INDUCTIVE SYSTEMS

In this section the inductive systems for $GL_\infty(F)$ and $SL_\infty(F)$ that consist of representations $L_{i,j}^n$ are classified. In what follows, except Proposition 4.1, we consider only inductive systems $\Phi \subset \mathcal{F}$ and assume that $l, n \in \mathbb{N}$ and $n > 1$ whenever $n-1$ occurs. For an irreducible representation θ of Γ_m and $1 \leq n \leq m$ denote by $\text{Irr}_n(\theta)$ the set of composition factors (up to equivalence) of the restriction $\theta \downarrow \Gamma_n$. More generally, if Θ is a set of representations of Γ_m , put $\text{Irr}_n(\Theta) = \cup_{\theta \in \Theta} \text{Irr}_n(\theta)$. Recall that for inductive systems Φ and Ψ we say that Ψ is contained in Φ and write $\Psi \subset \Phi$ if $\Psi_n \subset \Phi_n$ for all n . If $\Psi \subset \Phi$, but $\Psi \neq \Phi$, it is clear that $D_n = \Phi_n \setminus \Psi_n \neq \emptyset$ for large enough n and that for each $\delta \in D_n$ there exists $\delta' \in D_{n+1}$ with $\delta \in \text{Irr}_n(\delta')$. Hence the inductive system $\mathcal{D}(\Phi, \Psi) = \langle D_n \mid D_n \neq \emptyset \rangle$ is correctly determined. Obviously, in this case $\Phi = \Psi \cup \mathcal{D}(\Phi, \Psi)$. Throughout this section, if we write down a p -adic expansion $\sum_{i=0}^t x_i p^i$ of an integer x , we assume that $x_t \neq 0$. For $0 \leq j \leq t$ set $x^+(j) = \sum_{i=j}^t x_i p^i$ and $x^-(j) = \sum_{i=0}^{j-1} x_i p^i$; put $x^-(0) = x^+(t+1) = 0$. Set $N_f = (p-1) \sum_{i=0}^f p^i = p^{f+1} - 1$ for $f \in \mathbb{Z}_{\geq 0}$ and $N_{-1} = 0$.

The following proposition reduces the study of relevant inductive systems for $SL_\infty(F)$ to that of such systems for $GL_\infty(F)$. Observe that it concerns a more general situation.

Proposition 4.1. *Let $T_1 \subset T_2 \subset \dots \subset T_n \subset \dots$ and $U_1 \subset U_2 \subset \dots \subset U_n \subset \dots$ be sequences of groups. Set $T = \bigcup_{n=1}^\infty T_n$ and $U = \bigcup_{n=1}^\infty U_n$. Assume that $T_n \subset U_n$ for all n . Let Ω be an inductive system of irreducible representations for U such that the representation $\omega' = \omega \downarrow T_n$ is irreducible for each $\omega \in \Omega_n$. For an inductive system $\Phi \subset \Omega$ set $\Phi'_n = \{\varphi' \mid \varphi \in \Phi_n\}$ and $\Phi' = \{\Phi'_n \mid n = 1, 2, \dots\}$. Then Φ' is an inductive system for T and for every inductive system $\Psi \subset \Omega'$ there exist an inductive system $\Delta \subset \Omega$ such that $\Psi = \Delta'$. If Δ is indecomposable, then Ψ is indecomposable as well.*

Proof. Extend the notation $\text{Irr}_n(\theta)$ to representations of the groups T_m and U_m . For $\omega \in \Omega_n$ use the notation ω' as in the statement of the proposition. For all representations $\rho \in \Phi'_{n+1}$ and $\varphi \in \Phi'_n$ there exist $\xi \in \Phi_{n+1}$ and $\tau \in \Phi_n$ with $\xi' = \rho$ and $\tau' = \varphi$. Since Φ is an inductive system, $\text{Irr}_n(\xi) \subset \Phi_n$ and $\tau \in \text{Irr}_n(\lambda)$ for some $\lambda \in \Phi_{n+1}$. As $\text{Irr}_n(\rho) = \{\sigma' \mid \sigma \in \text{Irr}_n(\xi)\}$ and $\text{Irr}_n(\lambda') = \{\delta' \mid \delta \in \text{Irr}_n(\lambda)\}$, the definition of Φ' implies that $\text{Irr}_n(\rho) \subset \Phi'_n$, $\lambda' \in \Phi'_{n+1}$, and $\varphi \in \text{Irr}_n(\lambda')$. Hence Φ' is an inductive system for T . Obviously, $\Phi' \subset \Omega'$.

Next, consider an arbitrary inductive system $\Psi \subset \Omega'$. Set $D_n = \{\nu \in \Omega_n \mid \nu' \in \Psi_n\}$. Then $\text{Irr}_n(D_{n+1}) \subset D_n$ for all n . Put $\Delta_n = \cap_{k>n} \text{Irr}_n(D_k)$. In fact, since Δ_n is finite, there exists $m = m(n)$ such that $\Delta_n = \text{Irr}_n(D_k)$ for all $k > m$. It follows that $\Delta = \{\Delta_n \mid n = 1, 2, \dots\}$ is an inductive system and $\Delta' = \Psi$.

Finally, assume that Ψ is decomposable. Then $\Psi = \Psi^1 \cup \Psi^2$ where Ψ^1 and Ψ^2 are proper inductive subsystems of Ψ . As above, set $D_n^i = \{\nu \in \Omega_n \mid \nu' \in \Psi_n^i\}$ and $\Delta_n^i = \cap_{k>n} \text{Irr}_n(D_k^i)$, $i = 1, 2$, so $\Delta^i = \{\Delta_n^i \mid n = 1, 2, \dots\}$ is an inductive system with $(\Delta^i)' = \Psi^i$. Clearly, $D_n = D_n^1 \cup D_n^2$ for all n and $D_n^i \neq D_n$ whenever $\Psi_n^i \neq \Psi_n$. This implies that Δ^1 and Δ^2 are proper subsystems of Δ and $\Delta = \Delta^1 \cup \Delta^2$ is decomposable. This completes the proof. \square

It is clear that the groups G_n and H_n and the collection \mathcal{F} satisfy the assumptions of Proposition 4.1. So it suffices to describe the inductive systems contained in \mathcal{F} for $GL_\infty(F)$ and then to find out when two distinct such systems give the same under restricting to $SL_\infty(F)$. To simplify notation, we use the same symbols mentioned in Theorem 1.5 both

for $GL_\infty(F)$ and $SL_\infty(F)$. This will cause no confusion. Until the end of the proof of Theorem 1.5 we mean inductive systems for $GL_\infty(F)$ contained in \mathcal{F} whenever speaking on inductive systems.

In this section we consider $\pi_{l,k}^n$ (and $L_{i,j}^n$) both as modules and as representations. Set $\mathbb{N}(l) = \{k \in \mathbb{Z} \mid k \geq l-1, k \equiv l-1 \pmod{2}\}$. Let $\Phi \subset \mathcal{F}$ be an inductive system. If $k \in \mathbb{N}(l)$, put $\mathbb{N}_{l,k}(\Phi) = \{n \mid \pi_{l,k}^n \neq 0, \pi_{l,k}^n \in \Phi_n\}$. Set $\mathbb{N}_l(\Phi) = \{r \in \mathbb{N}(l) \mid |\mathbb{N}_{l,r}(\Phi)| = \infty\}$. Finally, put $\mathbb{N}(\Phi) = \{l \mid |\mathbb{N}_l(\Phi)| = \infty\}$. It is clear that the sets $\mathbb{N}_{l,k}(\Phi)$, $\mathbb{N}_l(\Phi)$, and $\mathbb{N}(\Phi)$ are correctly defined. A priori, they can be empty, finite or infinite. By Theorem 3.2, if $k \in \mathbb{N}_l(\Phi)$, then

$$\pi_{l,k}^n \in \Phi_n \quad \text{for all } n \geq (l+k-1)/2. \quad (11)$$

Proposition 4.2. *Let $\Phi \subset \mathcal{F}$ be an inductive system of representations. Then $\Phi = \mathcal{F}$ if and only if $\mathbb{N}(\Phi)$ is infinite.*

Proof. Obviously, $\mathbb{N}(\mathcal{F})$ is infinite. Assume that $\mathbb{N}(\Phi)$ is infinite. In fact, we need to show that $\pi_{l,k}^n \in \Phi_n$ if $k \in \mathbb{N}(l)$ and $\pi_{l,k}^n \neq 0$. Fix such l, k , and n . Since $|\mathbb{N}(\Phi)| = \infty$, there exists $a \in \mathbb{N}(\Phi)$ with $a > l$. As $|\mathbb{N}_a(\Phi)| = \infty$, there exists $b \in \mathbb{N}_a(\Phi)$ such that $b-k > a-l$. Observe that $b-k = a-l+2t$ with $t \in \mathbb{Z}_{\geq 0}$ since $b \equiv a+1 \pmod{2}$ and $k \equiv l+1 \pmod{2}$. As $|\mathbb{N}_{a,b}(\Phi) = \infty$, by (11), $\pi_{a,b}^m \in \Phi_m$ if $m-n > t+a-l$ and $2m \geq a+b-1$. Set $n_1 = m-t$ and $n_2 = n_1 - (a-l)$. We have $n_2 > n$ and $b-2t = k+a-l$. Several applications of Theorem 3.2 yield that $\pi_{a,k+a-l}^{n_1} \in \Phi_{n_1}$, $\pi_{l,k}^{n_2} \in \Phi_{n_2}$, and $\pi_{l,k}^n \in \Phi_n$ as required. Observe that all these modules are nonzero since $\pi_{l,k}^n \neq 0$. This completes the proof. \square

Now we shall show that the collections of representations indicated in Theorem 1.5 are really inductive systems.

Lemma 4.3. *\mathcal{M}^t is an inductive system for each $t \in \mathbb{Z}_{\geq 0}$.*

Proof. Recall that $\mathcal{M}_n^t = \{\pi_{l,k}^n \mid \pi_{l,k}^n \neq 0, l < p^{t+1}\}$. As $\pi_{l,k}^n \in \text{Irr}_n(\pi_{l,k}^{n+1})$ by Theorem 3.2, it suffices to show that for such pairs l, k the set $\text{Irr}_{n-1}(\pi_{l,k}^n)$ consists of representations $\pi_{a,b}^{n-1}$ with $a < p^{t+1}$. Write the p -adic expansion $l = \sum_{i=0}^t l_i p^i$ and assume that $l_p(l) = d$. First suppose that $l_d < p-1$. If $d=0$, we have $l+1 < p^{t+1}$. Now let $d > 0$. Then $l-1 = l^+(d+1) + (l_d-1)p^d + N_{d-1}$ and hence $l-1+2p^d < p^{t+1}$. If $l_d = p-1$ and $d > 0$, we get $l-1+2p^{d-1} = l^+(d) + p^{d-1} + N_{d-2} < p^{t+1}$. Now Theorem 3.2 yields the required claim on the composition factors of the restriction considered. \square

Lemma 4.4. *\mathcal{L}^a , \mathcal{R}^b , and $\mathcal{LR}^{a,b}$ are inductive systems.*

Proof. This follows immediately from Theorem 1.1 (i). \square

Now assume that $\mathbb{N}(\Phi)$ is finite. This means that $\mathbb{N}_l(\Phi)$ is finite for large enough l . We introduce some terminology to analyze this situation. For an inductive system Ψ write $\pi_{l,k} \in \Psi$ (or " Ψ contains $\pi_{l,k}$ ") if $\pi_{l,k}^n \in \Psi_n$ for some n . We need to consider different types of restricted systems in the sense of Definition 4.5 given below. In Definition 4.5 in all cases it is assumed that there exists a constant B with the relevant property.

Definition 4.5. An inductive system Ψ is called l -restricted if $l < B$ whenever $\pi_{l,k} \in \Psi$ for some k ; Ψ is called (k, k') -restricted if $k < B$ or $2n-k < B$ whenever $\pi_{l,k}^n \in \Psi_n$; we say that Ψ is k -restricted if $k < B$ for all $\pi_{l,k} \in \Psi$ and Ψ is k' -restricted if $2n-k < B$ whenever $\pi_{l,k}^n \in \Psi_n$; Ψ is called d -restricted if $k-l < B$ for all $\pi_{l,k} \in \Psi$ and d' -restricted if $2n-k-l < B$ whenever $\pi_{l,k}^n \in \Psi_n$.

It is clear that (k, k') -restricted systems are l -restricted. We shall see later that d -restricted and d' -restricted systems can be not l -restricted.

Now one can conclude that the following possibilities can be distinguished:

- i) Φ is l -restricted;
- ii) Φ is not l -restricted, but for large enough l there exist integers $m(l)$ such that $k < m(l)$ if $\pi_{l,k} \in \Phi$;
- iii) For all a and $b \in \mathbb{Z}_{\geq 0}$ there exist $\pi_{l,k} \in \Phi$ with $l > a$ and $k - l > b$, but for large enough l there exist integers $k(l)$ with $\mathbb{N}_{l,k}$ finite for $k > k(l)$.

Next, we shall describe l -restricted inductive systems. The first step is the description of the systems $\mathcal{L}^{a,b}$ and $\mathcal{R}^{a,b}$. Fix a and b as in the definition of these systems. In what follows $\psi_{l,k}^n = \pi_{l,2n-k}^n$. Set $l = b - a + 1$ and $k = a + b$. Put $\Pi^{l,k} = \langle \pi_{l,k}^n \mid n \geq b \rangle$ and $\Psi^{l,k} = \langle \psi_{l,k}^n \mid n \geq b \rangle$. Theorem 1.1 i) implies that $\Pi^{l,k}$ and $\Psi^{l,k}$ are correctly determined. Obviously, $\mathcal{L}^{a,b} = \Pi^{l,k}$ and $\mathcal{R}^{a,b} = \Psi^{l,k}$.

Write the p -adic expansion $l = \sum_{i=0}^s l_i p^i$. If $l = p^{s+1} - 1$, take $M = \{1, 2, \dots, l\}$. Now assume that $l < p^{s+1} - 1$. Set $c = k + 1 - l$. We have $l_i < p - 1$ for some of the coefficients l_i . If $c < 2p^i$ for all i with $l_i < p - 1$, again put $M = \{1, 2, \dots, l\}$. If there exists $r \leq s$ with $l_r < p - 1$ and $c \geq 2p^r$, fix maximal such r and the maximal integer u with $l_r < u \leq p - 1$ such that $c \geq 2(u - l_r)p^r$. Set $l' = l^+(r + 1) + up^r + N_{r-1}$. Then $l' < p^{s+1}$. Now put $M = \{1, 2, \dots, l'\}$. For $x \in M$ define $k(x)$ as follows. If $x \leq l$, set $k(x) = k - (l - x)$. Now let $x > l$. In this case r and l' are defined and $x \leq l'$. Fix the p -adic expansion $x = \sum_{i=0}^s x_i p^i$ and maximal v such that $x_v > l_v$. Obviously, $v \leq r$ and $x_r \leq u$ if $v = r$. Put

$$k(x) = k - l^-(v) - (x_v - l_v - 1)p^v - (p^v - x^-(v)) = k - (x_v - l_v)p^v + (x^-(v) - l^-(v)). \quad (12)$$

It is clear that $k(x) < k$ for $x \neq l$. We need the following technical lemma.

Lemma 4.6. *Let $e = \sum_{i=0}^s e_i p^i$ and $f = \sum_{i=0}^s f_i p^i$ be the p -adic decompositions of integers e and $f > e$ and $g \in \mathbb{N}(e)$. Fix maximal j with $f_j > e_j$. Assume that $g - e + 1 \geq 2(f_j - e_j)p^j$. Set $d = e^-(j) + (f_j - e_j)p^j - f^-(j)$, $h = g - d$, and $m = n + d$. Suppose that $2n + 1 \geq f + h$. Then $h \in \mathbb{N}(f)$, $2m + 1 \geq e + g$, and*

$$\pi_{f,h}^n \in \text{Irr}_n(\pi_{e,g}^m). \quad (13)$$

Proof. The second claim is obvious since $e < f$. Observe that $h - f = g - e - 2(f_j - e_j)p^j \geq -1$ and $h - f$ is odd. This yields the first claim.

To prove (13), apply induction by $f_j - e_j$. First assume that $j = 0$. Then Theorem 3.2 and inductive arguments imply that $\pi_{e+a, g-a}^{m-a} \in \text{Irr}_{m-a}(\pi_{e,g}^m)$ for $1 \leq a \leq d$. For $a = d$ one gets (13). Now let $j > 0$. Suppose that $f_j - e_j = 1$. Set $m_1 = m - e^-(j)$, $e_1 = e - e^-(j)$, $g_1 = g - e^-(j)$, $m_2 = m_1 - 1$, $e_2 = e_1 - 1 + 2p^j$, $g_2 = g_1 - 1$, $m_3 = m_2 - N_{j-1} + f^-(j)$, $e_3 = e_2 - N_{j-1} + f^-(j)$, and $g_3 = g_2 - N_{j-1} + f^-(j)$. One easily concludes that $m_3 = n$, $e_3 = f$, $g_3 = h$, and $\text{lp}(e_1) \geq j$. Theorem 3.2 forces that $\pi_{e_1, g_1}^{m_1} \in \text{Irr}_{m_1}(\pi_{e, g}^m)$ and $\pi_{e_i, g_i}^{m_i} \in \text{Irr}_{m_i}(\pi_{e_{i-1}, g_{i-1}}^{m_{i-1}})$ for $i = 2, 3$. Hence (13) holds.

Next, assume that $f_j - e_j = b > 1$ and that (13) holds in all cases where $f_j - e_j < b$. Put $f_1 = e^+(j + 1) + (f_j - 1)p^j$, $d_1 = e^-(j) + (f_j - e_j - 1)p^j$, $h_1 = g - d_1$, $m_1 = m - d_1$, $f_2 = f_1 - 1 + 2p^j$, $h_2 = h_1 - 1$, $m_2 = m_1 - 1$, $f_3 = f_2 - N_{j-1} + f^-(j)$, $h_3 = h_2 - N_{j-1} + f^-(j)$, and $m_3 = m_2 - N_{j-1} + f^-(j)$. Observe that $f_3 = f$, $h_3 = h$, $m_3 = n$, and $\text{lp}(f_1) = j$. By the inductive hypothesis, $\pi_{f_1, g_1}^{m_1} \in \text{Irr}_{m_1}(\pi_{e, g}^m)$. Then apply Theorem 3.2 and conclude that $\pi_{f_i, g_i}^{m_i} \in \text{Irr}_{m_i}(\pi_{f_{i-1}, g_{i-1}}^{m_{i-1}})$ for $i = 2$ and 3 . This yields (13). Notice also that all modules that appear in this proof are nonzero since $g - e + 1 \geq 2(f_j - e_j)p^j$ and $2n + 1 \geq f + h$. \square

Corollary 4.7. *We have $k(x) \in \mathbb{N}(x)$ for all $x \in M$.*

Proof. This is obvious for $x \leq l$ and follows directly from Lemma 4.6 for $x > l$. \square

Proposition 4.8. *We have*

$$\Pi_n^{l,k} = \{\pi_{x,y}^n \mid x \in M, y \geq x - 1, y = k(x) - 2t, t \in \mathbb{Z}_{\geq 0}, x + y \leq 2n + 1\}. \quad (14)$$

Proof. We use the notation s, r, u, v, M introduced before Lemma 4.6. For $x \in M$ denote by $S(x)$ the set of all pairs (y, n) such that the triple (x, y, n) satisfies (14). Set $I_n = \{\pi_{x,y}^n \mid x \in M, (y, n) \in S(x)\}$ and $\mathcal{I} = \{I_n, n = 1, 2, \dots\}$. We claim that \mathcal{I} is an inductive system and $\mathcal{I} \subset \Pi^{l,k}$. Since $\pi_{l,k}^n \in I_n$, this would imply the assertion of the proposition.

Fix $\pi = \pi_{x,y}^n \in I_n$. It is clear that $\pi' = \pi_{x,y}^{n+1} \in I_{n+1}$. Since $\pi \in \text{Irr}_n(\pi')$ by Theorem 3.2, it remains to check that

$$\text{Irr}_{n-1}(\pi) \subset I_{n-1} \quad (15)$$

and that

$$\pi \in \text{Irr}_n(\pi_{l,k}^m) \quad (16)$$

for some $m > n$. Observe that $(y - 2j, n) \in S(x)$ if $(y, n) \in S(x)$ and $y \geq x + 2j - 1$, $(y - 2, n - 1) \in S(x)$ if $(y, n) \in S(x)$ and $y \geq x + 1$, and $(y - 1, n - 1) \in S(x - 1)$ if $(y, n) \in S(x)$. For $1 < x \leq M$ we claim that $k(x - 1) \geq k(x) - 1$. Indeed, this is obvious if $x \leq l$. Now let $x > l$. If $\text{lp}(x) \neq v$ or $v = 0$, Formula (12) forces $k(x - 1) = k(x) - 1$. Assume that $\text{lp}(x) = v > 0$. One has $x - 1 = l^+(v + 1) + (x_v - 1)p^v + N_{v-1}$. Suppose that $x_v > l_v + 1$. Then $k(x - 1) = k - l^-(v) - (x_v - l_v - 2)p^v - 1 > k(x)$. Finally, assume that $x_v = l_v + 1$. Then $x - 1 = l^+(v) + N_{v-1}$. If $x - 1 = l$, we have $k(x - 1) = k > k(x)$. Otherwise there exists $w < v$ with $l_w < p - 1$. Fix maximal such w . Then $k(x - 1) = k - l^-(w) - (p - 2 - l_w)p^w - 1$ and $k(x) = k - l^-(v) - p^v$. Since $l^-(w) \leq l^-(v)$ and $(p - 2 - l_w)p^w + 1 < p^v$, we conclude that $k(x - 1) > k(x)$. This completes the proof of the claim. Now the construction of \mathcal{I} implies that $\pi' = 0$ or $\pi' \in I_{n-1}$ for $\pi' \in \{\pi_{x,y}^{n-1}, \pi_{x,y-2}^{n-1}, \pi_{x-1,y-1}^{n-1}\}$.

First prove (15) and (16) for $y = k(x)$. Set $l_p(x) = d$. Let $x' = x - 1 + 2p^t$ with $t \in \mathbb{Z}_{\geq 0}$, $t \leq d$, and $t < d$ for $x_d = p - 1$. Then $x' = x^+(d + 1) + x_d p^d + p^t + N_{t-1}$. Put $\rho = \pi_{x',y-1}^{n-1}$. Now our goal is to show that $\rho \in I_{n-1}$ if $\rho \neq 0$. Actually it suffices to check that $x' \in M$ and $k(x') \geq y - 1$ if $x' \leq y$ since, obviously, $\rho = 0$ for $x' > y$. Assume that $x > l$. Then $d \leq v$. The definition of M forces that one can get $x' \notin M$ only for $t = d = v = r$ and $x_r = u$. If $d < v$ or $t < d$, Formula (12) yields that $k(x') > y$. Next, for $t = d = v$ we have $y = k - l^-(v) - (x_v - l_v)p^v$ by (12). Suppose that also $v = r$ and $x_r = u$. Then $y - x' = c - 2(u + 1 - l_r)p^r < 0$ since $x^+(v + 1) = l^+(v + 1)$. In all other situations where $t = d = v$ one gets $y = k(x') + 1$. Now let $x \leq l$. Then $y - x = k - l$ and $y = k - l^-(d)$. If $x' \leq l$, we have $k(x') > y$. Assume that $x' > l$. Then $x_a = l_a$ for $a > d$. First let $t < d$. Then $x_d = l_d$, $l_b = 0$ for $t < b < d$, and either $l_t = 0$, or $l_t = 1$ and $l_j < p - 1$ for some $j < t$. Let $l_t = 0$. Then $c < 2p^t$ if $x' \notin M$. In this case $y < x'$. Otherwise we get $y - 1 = k(x')$ by (12). Next, assume that $l_t = 1$ and $l_j < p - 1$ for some $j < t$. Fix maximal such j . If $x' \notin M$, one has $c < 2(p - 1 - l_j)p^j < 2p^t$ and hence $y < x'$. If $x' \in M$, Formula (12) forces that $k(x') = k - l^-(j) - (p - 2 - l_j)p^j - 1 < y$ as desired. Now suppose that $t = d$. Then either $x_d = l_d$, or $x_d = l_d - 1$ and $l_j < p - 1$ for some $j < d$. Let $x_d = l_d$. If $x' \notin M$, we conclude that $c < 2p^d$ and $y < x'$. Otherwise $y - 1 = k(x')$ by (12). Finally, assume that $x_d = l_d - 1$ and $l_j < p - 1$ for some $j < d$. Fix maximal such j and argue as for $t < d$ and $l_t = 1$ distinguishing the cases where $x' \notin M$ and $x' \in M$. In the latter case $k(x') > y$ since $y = k - l^-(d) - p^d$. Now all the possibilities have been considered and Theorem 3.2 implies that (15) holds for $y = k(x)$.

For $x \geq l$ Formula (16) follows directly from Lemma 4.6 and Theorem 3.2. If $x \leq l$, set $m = n + l - x$. Our assumptions on x, y , and n yield that $\pi_{l,k}^m \neq 0$. Applying Theorem 3.2, one can conclude that $\pi_{l-j,k-j}^{m-j} \in \text{Irr}_{m-j}(\pi_{l,k}^m)$ for $1 \leq j \leq l - x$. Hence (16) holds for $y = k(x)$.

Finally, assume that $y < k(x)$. Set $b = k(x) - y$ and apply induction on b using as the induction base the results proven just above for $b = 0$. Assume that $b > 0$ and (15) and (16) hold if $k(x) - y < b$. Observe that $(y + 2, n + 1) \in S(x)$. Set $\pi^+ = \pi_{x,y+2}^{n+1}$. Now the induction hypothesis implies that $\text{Irr}_n(\pi^+) \subset I_n$ and $\pi^+ \in \text{Irr}_{n+1}(\pi_{l,k}^m)$ for some m .

It follows from Theorem 3.2 that $\text{Irr}_{n-1}(\pi) = \{\pi_{a,b}^{n-1} \mid \pi_{a,b+2}^n \in \text{Irr}_n(\pi^+), b \geq a-1\}$ and $\pi \in \text{Irr}_n(\pi^+)$. Hence π satisfies (16) and $\pi \in I_n$ as required. This completes the proof of the proposition. \square

Lemma 4.9. *Let $k \leq 2n$. Then $\text{Irr}_{n-1}(\psi_{l,k}^n) = \{\psi_{a,b}^{n-1} \mid \pi_{a,b}^{n-1} \in \text{Irr}_{n-1}(\pi_{l,k}^n)\}$.*

Proof. Observe that $\pi_{l,2n-k}^{n-1} = \psi_{l,k-2}^{n-1}$, $\pi_{x,2n-k-1}^{n-1} = \psi_{x,k-1}^{n-1}$, and $\pi_{l,2n-k-2}^{n-1} = \psi_{l,k}^{n-1}$. Now the lemma follows immediately from Theorem 3.2. \square

Corollary 4.10. *Let Π_n , $n = 1, 2, \dots$, be nonempty sets of irreducible representations of G_n and $\Pi = \{\Pi_n \mid n = 1, 2, \dots\}$. Set $\Psi_n = \{\psi_{l,k}^n \mid \pi_{l,k}^n \in \Pi_n\}$ and $\Psi = \{\Psi_n \mid n = 1, 2, \dots\}$. Then Ψ is an inductive system if and only if Π is such.*

Corollary 4.11. *Let the integers l , k , and M be as in the description of the system $\Pi^{l,k}$. Then $\Psi_n^{l,k} = \{\psi_{x,y}^n \mid x \in M, y \geq x-1, y = k(x) - 2t, t \in \mathbb{Z}_{\geq 0}, x+y \leq 2n+1\}$ where $k(x)$ is determined by (12).*

The assertions of these two corollaries follow immediately from Lemma 4.9.

Remark 4.12. In the assumptions of Corollary 4.10, if Π and Ψ are inductive systems, then Π is k' - or d' -restricted if and only if Ψ is k - or d -restricted, respectively.

Remark 4.13. Let $a, b \in \mathbb{Z}_{\geq 0}$, $b \geq a$, $l = b - a + 1$, and $k = a + b$. Assume that $k - l + 1 \geq 2p^t$ and $l_t < p - 1$ for some $t > 0$. Then for large enough n the set $\mathcal{L}_n^{a,b} = \Pi_n^{l,k}$ contains representations $L_{x,y}^n$ with $y > b$, i.e. (2) does not hold.

Lemma 4.14. *Let Π be a k -restricted inductive system. Then Π is a finite union of systems $\Pi^{l,k}$.*

Proof. For a k -restricted inductive system Δ denote by $k(\Delta)$ the maximal integer m such that $\pi_{l,m} \in \Delta$ for some l . Set $k = k(\Delta)$. Proceed by induction on k . Obviously, $\Pi = \Pi^{1,0}$ for $k = 0$ since $l \leq k + 1$ if $k \in \mathbb{N}(l)$. Assume that $k > 0$ and that the assertion of the lemma holds for k -restricted inductive systems Δ with $k(\Delta) < k$. As Π is an inductive system, Theorem 3.2 implies that $k \in \mathbb{N}_l(\Pi)$ if $\pi_{l,k}^n \in \Phi_n$. Denote by S the set of all such l and put $\Pi' = \bigcup_{l \in S} \Pi^{l,k}$. Then (11) yields that $\Pi' \subset \Pi$. If $\Pi' = \Pi$, we are done. Otherwise set $\mathcal{D} = \mathcal{D}(\Pi, \Pi')$. It is clear that \mathcal{D} is a k -restricted inductive system with $k(\mathcal{D}) < k$. This completes the proof since the assertion of the theorem holds for \mathcal{D} by the induction hypothesis. \square

Lemma 4.15. *Let Ψ be a k' -restricted inductive system. Then Ψ is a finite union of systems $\Psi^{l,k}$.*

Proof. Set $\Pi_n = \{\pi_{l,k}^n \mid \psi_{l,k}^n \in \Psi_n\}$ and $\Pi = \{\Pi_n \mid n = 1, 2, \dots\}$. By Corollary 4.10, Π is a k -restricted inductive system. To complete the proof, apply Lemma 4.14. \square

Lemma 4.16. *Let Φ be a (k, k') -restricted inductive system that is not k -restricted or k' -restricted. Then $\Phi = \Pi \cup \Psi$ where Π is k -restricted and Ψ is k' -restricted.*

Proof. As Φ is (k, k') -restricted, but is not k' -restricted, there exist constants m and M such that for all $\pi_{l,k}^n \in \Phi_n$ one has $k \leq m$ or $2n - k \leq M$ and the set $\{n \mid \pi_{l,m}^n \in \Phi_n \text{ for some } l\}$ is infinite. Since $l \leq m + 1$ if $\pi_{l,m}^n \neq 0$, we conclude that $m \in \mathbb{N}_l(\Phi)$ for some l . Set $P_n = \{\pi_{x,k}^n \mid \pi_{x,k}^n \in \Phi_n, k \leq m, k \in \mathbb{N}_x(\Phi)\}$. Theorem 3.2 implies that $P_n \neq \emptyset$ for all n and that for every $\pi \in P_n$ there exists $\rho \in P_{n+1}$ with $\pi \in \text{Irr}_n(\rho)$. Put $\Pi = \langle P_n \mid n = 1, 2, \dots \rangle$. The arguments above yield that Π is correctly determined. Observe that if $\pi_{y,k}^n \in \Phi_n \setminus P_n$ and $k \leq m$, then $n \leq N$ for some constant N as $k \notin \mathbb{N}_y(\Phi)$ and $1 \leq y \leq m + 1$. So since Φ is not k -restricted, we can deduce from Theorem 3.2 that Π is k -restricted, $\Pi \neq \Phi$, and $\mathcal{D}(\Phi, \Pi)$ is k' -restricted. This completes the proof. \square

The following proposition completes the description of l -restricted inductive systems.

Proposition 4.17. *Let Φ be an l -restricted inductive system that is not (k, k') -restricted. Then $\Phi = \mathcal{M}^t$ or $\mathcal{M}^t \cup \mathcal{D}$ where \mathcal{D} is a (k, k') -restricted inductive system.*

Proof. We claim that there exists l with infinite $\mathbb{N}_l(\Phi)$. Indeed, as Φ is l -restricted, otherwise for some constant $C \in \mathbb{N}$ we get $|\mathbb{N}_{l,k}(\Phi)| < \infty$ if $k > C$. Therefore, for $k > C$ one can define $f(k) = \max\{2n - k \mid \pi_{l,k}^n \in \Phi_n\}$. Let $L = \max\{l \mid \pi_{l,k} \in \Phi\}$. We can assume that $C \geq L - 2$. By Theorem 3.2, $\pi_{l,k}^n \in \Phi_n$ if $\pi_{l,k+2}^{n+1} \in \Phi_{n+1}$. Hence $f(k+2) \leq f(k)$ for $k > C$. Since $f(k) \geq 0$, considering odd and even k , we can find C_1 and d such that $2n - l - k \leq d$ if $k > C_1$ and $\pi_{l,k}^n \in \Phi_n$. But then Φ is (k, k') -restricted which yields a contradiction. Hence $|\mathbb{N}_l(\Phi)| = \infty$ for some l . Fix maximal such l . We shall show that $l = p^{t+1} - 1$ for some $t \in \mathbb{Z}_{\geq 0}$. Suppose this is false. Fix the p -adic expansion $l = \sum_{i=0}^t l_i p^i$, minimal j with $l_j < p - 1$, and $k \in \mathbb{N}_l(\Phi)$ with $k \geq l - 1 + 2p^j$. Assume that $\pi_{l,k}^n \in \Phi_n$. Set $l_1 = l - N_{j-1}$, $k_1 = k - N_{j-1}$, $n_1 = n - N_{j-1}$, $l_2 = l + p^j$, $k_2 = k - p^j$, and $n_2 = n - p^j$. Observe that $p^j | l_1$, $l_2 = l_1 - 1 + 2p^j$, and $\pi_{l_2, k_2}^{n_2} \neq 0$. By Theorem 3.2, $\pi_{l_1, k_1}^{n_1} \in \Phi_{n_1}$ and $\pi_{l_2, k_2}^{n_2} \in \Phi_{n_2}$. This implies that $k_2 \in \mathbb{N}_{l_2}(\Phi)$. Since there are infinitely many such k , we conclude that $|\mathbb{N}_{l_2}(\Phi)| = \infty$ and come to a contradiction as $l_2 > l$. Hence $l = p^{t+1} - 1$.

Now we shall prove that $\mathcal{M}^t \subset \Phi$. Fix a, b , and n with $a \in \mathbb{N}$, $a \leq l$, $b \in \mathbb{N}(a)$, and $2n \geq a + b - 1$. We have to check that $\pi_{a,b}^n \in \Phi_n$. Since $|\mathbb{N}_l(\Phi)| = \infty$, there exist u and r such that $u - b = l - a + 2x$, $x \in \mathbb{N}$, $r - n > u - b + x$, and $\pi_{l,u}^r \in \Phi_r$. Several applications of Theorem 3.2 yield that $\pi_{l, u-2x}^{r-x} \in \Phi_{r-x}$, $\pi_{a,b}^{r-x-l+a} \in \Phi_{r-x-l+a}$, and $\pi_{a,b}^n \in \Phi_n$. Hence $\mathcal{M}^t \subset \Phi$. If $\Phi = \mathcal{M}^t$, we are done.

Next, assume that $\Phi \neq \mathcal{M}^t$. Set $\Theta_n = \Phi_n \setminus \mathcal{M}_n^t$ and $\mathcal{D} = \mathcal{D}(\Phi, \mathcal{M}^t)$. It remains to prove that \mathcal{D} is (k, k') -restricted. Set $T = \{c \mid \pi_{c,d}^n \in \Theta_n\}$ for some d and n . By our assumptions, $\mathbb{N}_c(\Phi)$ is finite for $c \in T$. Since Φ is l -restricted, T is finite. Choose the maximal integer m in $\bigcup_{c \in T} \mathbb{N}_c(\Phi)$. Then $\mathbb{N}_{c,d}(\Phi)$ is finite for $c \in T$ and $d > m$. Hence for each such pair (c, d) one can define $f(c, d) = \max\{2n - d \mid \pi_{c,d}^n \in \Phi_n\}$. Arguing as at the beginning of the proof, we deduce that there exists a constant L with $2n - d < L$ for all triples (c, d, n) with $c \in T$, $d > m$, and $\pi_{c,d}^n \in \Phi_n$. Now Theorem 3.2 yields that \mathcal{D} is (k, k') -restricted as desired. This completes the proof. \square

Now we describe inductive systems that satisfy ii).

Lemma 4.18. *Let ii) hold. Then Φ is d -restricted or $\Phi = \Psi \cup \mathcal{D}$ where Ψ is d -restricted and \mathcal{D} is l -restricted.*

Proof. We claim that there exist constants L and D such that $k - l < D$ if $\pi_{l,k} \in \Phi$ and $l \geq L$. According to our assumptions, Φ contains $\pi_{l,k}$ with arbitrarily large l and for $l \geq L'$ (a constant) there exist integers $m(l)$ with $k < m(l)$ if $\pi_{l,k} \in \Phi$. Choose minimal $m(l)$ for all such l . Since by Theorem 3.2, $\pi_{l,k}^n \in \Phi^n$ if $\pi_{l+1, k+1}^{n+1} \in \Phi^{n+1}$, we get $m(l+1) \leq m(l)$ for $l \geq L'$. Hence for some constant $L \geq L'$ we have $m(l+1) = m(l)$ if $l \geq L$. This yields the claim on L and D . One can assume that $L = p^{t+1}$ for some t (increasing L if necessary). Set $P_n = \{\pi_{l,k}^n \mid l > L, \pi_{l,k}^n \in \Phi_n\}$ and $\Psi = \langle P_n \mid P_n \neq \emptyset \rangle$. As Φ is not l -restricted, it is clear that $P_n \neq \emptyset$ for large enough n . As Φ is an inductive system, it follows from Lemma 4.3 that each $\pi \in P_n$ belongs to $\text{Irr}_n(\psi)$ for some $\psi \in P_{n+1}$. Hence the inductive system Ψ is correctly determined. Theorem 3.2 forces that $b - a \leq k - l$ if $\pi_{a,b}^{n-1} \in \text{Irr}_n(\pi_{l,k}^n)$. This implies that Ψ is d -restricted. If $\Psi = \Phi$, we are done. Otherwise set $\mathcal{D} = \mathcal{D}(\Phi, \Psi)$. Lemma 4.3 implies that $\mathcal{D} \subset \mathcal{M}^t$. Hence \mathcal{D} is l -restricted. This completes the proof. \square

Lemma 4.19. *Let $a, b \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{LR}^{a,b} = \langle L_{a,n-b}^n \mid n \geq \max\{a, b\} \rangle$.*

Proof. Since $L_{a,n-b}^n \in \mathcal{LR}_n^{a,b}$, it is clear that the inductive system generated by these representations is a subsystem of $\mathcal{LR}^{a,b}$. Now it suffices to prove that for each triple x, y, n with $x \leq a$, $y \leq b$, and $n \geq \max\{x, y\}$ the representation $L_{x,n-y}^n \in \text{Irr}_n(L_{a,m-b}^m)$ for some $m \geq n$. Set $c = a - x$, $d = b - y$, and $m = n + c + d$. Several applications of Theorem 1.1 i) yield that $L_{x,m-c-b}^{m-c} \in \text{Irr}_{m-c}(L_{a,m-b}^m)$ and $L_{x,n-y}^n \in \text{Irr}_n(L_{x,m-c-b}^{m-c})$ since $m - c - b = n - y$. Hence $L_{x,n-y}^n \in \text{Irr}_n(L_{a,m-b}^m)$, and we are done. \square

Lemma 4.20. *Let Φ be both d -restricted and d' -restricted. Then $\Phi = \bigcup_{i=1}^s \mathcal{LR}^{a_i, b_i}$.*

Proof. For a d -restricted inductive system Ψ set $D(\Psi) = \max\{k - l \mid \pi_{l,k} \in \Psi\}$ and put $D = D(\Phi)$. Let $M = \max\{2n - k - l \mid \pi_{l,k}^n \in \Phi_n, k - l = D\}$. Such integer M exists as Φ is d' -restricted. By Theorem 3.2, $y - x \leq k - l$ and $2n - 2 - x - y \leq 2n - k - l$ if $\pi_{x,y}^{n-1} \in \text{Irr}_{n-1}(\pi_{l,k}^n)$. Hence for large enough n we have $k - l = D$ and $2n - k - l = M$ for some $\pi_{l,k}^n \in \Phi_n$. Denote such $\pi_{l,k}^n$ by ρ^n . Observe that D and M are odd. Set $a = (D+1)/2$ and $b = (M+1)/2$. Then $\rho^n = L_{a,n-b}^n$ and $\rho^n \in \Phi_n$ whenever $n \geq \max\{a, b\}$. Furthermore, if $\pi_{x,y}^n \neq 0$, $y - x = D$, and $2n - x - y \leq M$, we have $\pi_{x,y}^n = L_{a,n-b'}^n$ with $b' \leq b$ and hence $\pi_{x,y}^n \in \mathcal{LR}_n^{ab}$. Set $\Phi' = \langle \rho^n \mid n \geq \max\{a, b\} \rangle$. By Lemma 4.19, $\Phi' = \mathcal{LR}^{ab}$. If $\Phi' = \Phi$, we are done. Otherwise set $\Delta = \mathcal{D}(\Phi, \Phi')$ and apply induction by D . In any case it is clear that $\pi_{x,y}^n \in \Phi'_n$ if $\pi_{x,y}^n \in \Phi_n$ and $y - x = D$. Naturally, Δ is d - and d' -restricted. The arguments above imply that $D(\Delta) < D$. Obviously, $D \geq -1$. If $D = -1$, we have $k - l = D$ for all $\pi_{l,k} \in \Phi$ and hence $\Phi = \Phi'$. Next, assume that $D > -1$ and that our assertion holds for inductive systems Ψ such that Ψ is both d - and d' -restricted and $D(\Psi) < D$. Then it holds for Δ , and we are done. \square

To handle d -restricted, but not d' -restricted inductive systems, we need to consider systems containing \mathcal{L}^a .

Lemma 4.21. *An inductive system $\Phi \supset \mathcal{L}^a$ if and only if $|\{l \mid l + 2a - 1 \in \mathbb{N}_l(\Phi)\}| = \infty$.*

Proof. Set $d = 2a - 1$. Then $\pi_{l,l+d}^n \in \mathcal{L}_n^a$ if $\pi_{l,l+d}^n \neq 0$. This proves "only if". Next, assume that $|\{l \mid l + d \in \mathbb{N}_l(\Phi)\}| = \infty$. Let $\pi_{x,y}^m \in \mathcal{L}_m^a$. Observe that $y - x = d - 2t$ with $t \in \mathbb{Z}_{\geq 0}$. Fix $l \geq x$ such that $l + d \in \mathbb{N}_l(\Phi)$. Then choose n such that $n - m \geq l - x + t$ and $\pi_{l,l+d}^n \in \Phi_n$. Now Theorem 3.2 forces that $\pi_{x,x+d}^{n-l+x} \in \Phi_{n-l+x}$, $\pi_{x,y}^{n-l+x-t} \in \Phi_{n-l+x-t}$, and $\pi_{x,y}^m \in \Phi_m$ as desired. This proves the lemma. \square

Lemma 4.22. *Let Φ be d -restricted, but not l -restricted and not d' -restricted. Then either $\mathcal{L}^a \subset \Phi$ for some a , or $\Phi = \Psi \cup \mathcal{D}$ where Ψ is both d -restricted and d' -restricted and \mathcal{D} is l -restricted.*

Proof. Set $S = \{l \mid \mathbb{N}_l(\Phi) \neq \emptyset\}$. First assume that S is infinite. We claim that in this case $\mathcal{L}^a \subset \Phi$. Since Φ is d -restricted, one can fix the maximal integer b such that the set $\{l \mid l + b \in \mathbb{N}_l(\Phi)\}$ is infinite. Recall that b is odd. Set $a = (b + 1)/2$. By Lemma 4.21, $\mathcal{L}^a \subset \Phi$.

Now let S be finite. Then for some constant L' the sets $\mathbb{N}_{l,k}(\Phi)$ are finite if $l > L'$. As Φ is d -restricted, for $l > L'$ one can define $m(l) = \max\{2n - k - l \mid \pi_{l,k}^n \in \Phi_n\}$. By Theorem 3.2, $\pi_{l,k}^n \in \Phi_n$ if $\pi_{l+1,k+1}^{n+1} \in \Phi_{n+1}$. This implies that $m(l+1) \leq m(l)$. Since $m(l) \geq 0$, one can conclude that $m(l) = m(l+1)$ if $l \geq L$ where $L > L'$ is a constant. One can assume that $L = p^{t+1}$. Let $A_n = \{\pi_{l,k}^n \mid l > L, \pi_{l,k}^n \in \Phi_n\}$. Using Lemma 4.3 and arguing as in the proof of Lemma 4.18, one can conclude that the inductive system $\Psi = \langle A_n \mid A_n \neq \emptyset \rangle$ is correctly determined. Applying Theorem 3.2 and Lemma 4.3 and taking into account that Φ is not d' -restricted, we deduce that Ψ is both d - and d' -restricted, $\Phi \neq \Psi$ and $\mathcal{D}(\Phi, \Psi)$ is l -restricted. This yields the lemma. \square

The following proposition completes the description of inductive systems satisfying ii).

Proposition 4.23. *Let Φ be d -restricted, but not l -restricted and d' -restricted. Assume that $\mathcal{L}^a \subset \Phi$. Then one of the following holds:*

- 1) $\Phi = \mathcal{L}^b$ with $b \geq a$;
- 2) $\Phi = \mathcal{L}^b \cup \Psi$ where $b \geq a$ and Ψ is both d -restricted and d' -restricted';
- 3) $\Phi = \Phi^1 \cup \Pi$ where 1) or 2) holds for Φ^1 and Π is l -restricted.

Proof. Fix maximal b with $\mathcal{L}^b \subset \Phi$. If $\mathcal{L}^b = \Phi$, 1) holds. Otherwise set $\mathcal{D} = \mathcal{D}(\Phi, \mathcal{L}^b)$. Naturally, \mathcal{D} is d -restricted. If \mathcal{D} is l -restricted or d' -restricted, 3) or 2) holds, respectively. So assume that \mathcal{D} is not l -restricted and not d' -restricted. Set $D_n = \Phi_n \setminus \mathcal{L}_n^b$. Observe that $k - l > 2b - 1$ if $\pi_{l,k}^n \in D_n$. Next, we claim that there exists a constant L such that for $l > L$ and each $k \in \mathbb{N}(l)$ the set $\{n \mid \pi_{l,k}^n \in D_n\}$ is finite. Suppose this is false. Observe that the sets D_n contain $\pi_{l,k}^n$ with arbitrarily large l since \mathcal{D} is not l -restricted and that by Theorem 3.2, $k \in \mathbb{N}_l(\Phi)$ if $k \in \mathbb{N}(l)$ and $k + 2 \in \mathbb{N}_l(\Phi)$. This implies that $l + 2b + 1 \in \mathbb{N}_l(\Phi)$ for arbitrarily large l and yields a contradiction by Lemma 4.21 as $\mathcal{L}^{b+1} \not\subset \Phi$. Hence a desired constant L exists. Arguing as in the last paragraph of the proof of Lemma 4.22, we can show that $\mathcal{D} = \Psi \cup \Pi$ where Ψ is both d - and d' -restricted and Π is l -restricted. Hence 3) holds. This completes the proof. \square

Now assume that iii) holds.

Lemma 4.24. *If iii) holds, then $\pi_{l,k} \in \Phi$ for all l and k with $k \in \mathbb{N}(l)$.*

Proof. Argue as in the proof of Proposition 4.2, but do not consider n . \square

Proposition 4.25. *If iii) holds, there exist L, M , and $T \in \mathbb{N}$ such that $\max\{2n - l - k \mid n \in \mathbb{N}_{l,k}(\Phi)\} = T$ for $l > L$ and $k - l > M$.*

Proof. Assume that iii) holds. First we claim that in this case there exist L' and M' such that $|\mathbb{N}_{l,k}(\Phi)| < \infty$ for $l > L'$ and $k - l > M'$. Indeed, according to our assumptions, for all $l > L''$ there exist integers $m(l)$ with $|\mathbb{N}_{l,k}(\Phi)| < \infty$ for $k > m(l)$. Choose minimal possible values of $m(l)$ with $m(l) \geq l - 2$. Now we have to show that there exist L' and M' such that $m(l) - l = M'$ for $l > L'$. Our assumptions imply that $m(l) + 1 \in \mathbb{N}(l)$ and $m(l) + 2 \in \mathbb{N}(l + 1)$. As by Theorem 3.2, $\pi_{l,r}^n \in \Phi_n$ if $\pi_{l,r+2}^{n+1} \in \Phi_{n+1}$ and $r \in \mathbb{N}(l)$, we conclude that $|\mathbb{N}_{l,k}(\Phi)| = \infty$ is infinite if $k \in \mathbb{N}(l)$ and $k < m(l)$. Another application of Theorem 3.2 forces that $\pi_{l,m(l)+1}^n \in \Phi_n$ if $\pi_{l+1,m(l)+2}^{n+1} \in \Phi_{n+1}$. Hence $|\mathbb{N}_{l+1,m(l)+2}(\Phi)| < \infty$. So $m(l + 1) \leq m(l) + 1$ and $m(l + 1) - (l + 1) \leq m(l) - l$ for $l > L''$. As $m(l) - l \geq -2$, this implies the existence of required L' and M' .

Now for each pair (l, k) with $l > L'$, $k - l > M'$, and $k \in \mathbb{N}(l)$ define $t(l, k) = \max\{2n - l - k \mid n \in \mathbb{N}_{l,k}(\Phi)\}$. By Theorem 3.2, $\pi_{l,k}^n \in \Phi_n$ if $k \geq l - 1$ and $\pi_{l,k+2}^{n+1} \in \Phi_{n+1}$ or $\pi_{l+1,k+1}^{n+1} \in \Phi_{n+1}$. Hence $t(l, k) \leq t(l, k + 2)$ and $t(l, k) \leq t(l + 1, k + 1)$. As $t(l, k) \geq -1$, this yields the assertion of the proposition. \square

Fix the notation L, M , and T for the arguments below concerning Case iii). Observe that T is odd.

Proposition 4.26. *Set $b = (T + 1)/2$. Then $\mathcal{R}^b \subset \Phi$.*

Proof. Observe that $\mathcal{R}_n^b = \{\pi_{l,k}^n \mid \pi_{l,k}^n \neq 0, 2n - l - k \leq T\}$. Fix l, k , and n with $\pi_{l,k}^n \in \mathcal{R}_n^b$. We have $2n - l - k = T - 2b$ with $b \in \mathbb{Z}_{\geq 0}$. Proceed by induction on b . First let $b = 0$. If $l > L$ and $k - l > M$, one gets $\pi_{l,k}^n \in \Phi_n$ by the definition of T and Lemma 4.24. Otherwise fix u and v such that $u > \max\{l, L\}$, $v - u > \max\{M, k - l\}$, and $u + v \equiv T \pmod{2}$. Then $u + v + T = 2s > 2n$. By our assumptions, $\pi_{u,v}^s \in \Phi_s$. We have $v - u = k - l + 2a$ with $a \in \mathbb{N}$. Observe that $2s - 2n = 2a + 2(u - l)$ and therefore $s - n = a + (u - l)$. Several

applications of Theorem 3.2 yield that $\pi_{u,v-2a}^{s-a} \in \Phi_{s-a}$ and $\pi_{l,k}^n \in \Phi_n$ as required. Next, let $b > 0$. Assume that our assertion holds for $b-1$. This forces that $\pi_{l,k}^{n+1} \in \Phi_{n+1}$. Then Theorem 3.2 implies that $\pi_{l,k}^n \in \Phi_n$ and completes the proof. \square

The following proposition completes the analysis of Case iii).

Proposition 4.27. *Let $\Phi \neq \mathcal{R}^b$. Then $\Phi = \mathcal{R}^b \cup \mathcal{D}$ where \mathcal{D} is an inductive system satisfying i) or ii).*

Proof. Set $\mathcal{D} = \mathcal{D}(\Phi, \mathcal{R}^b)$ and $D_n = \Phi_n \setminus \mathcal{R}_n^b$. The choice of b yields that $l \leq L$ or $k-l \leq M$ if $\pi_{l,k}^n \in D_n$. Denote by L_p the minimal integer of the form $p^s - 1$ that is $\geq L$. It follows from Lemma 4.3 and Theorem 3.2 that $l \leq L_p$ or $k-l \leq M$ for each $\pi_{l,k} \in \mathcal{D}$. Therefore either \mathcal{D} is l -restricted and i) holds, or ii) holds. \square

Set $\mathbf{S} = \{\mathcal{L}^a, \mathcal{R}^b, \mathcal{LR}^{a,b}, \mathcal{M}^t, \mathcal{L}^{a,b}, \mathcal{R}^{a,b} \mid a, b, t \in \mathbb{Z}_{\geq 0}\}$. Results 4.2, 4.14–4.18, 4.20, 4.22, 4.23, and 4.27 yield that each proper inductive subsystem $\Phi \subset \mathcal{F}$ is a finite union of some members of \mathbf{S} . Now to complete the proof of Theorem 1.5 for $GL_\infty(F)$, it remains to prove the following proposition.

Proposition 4.28. *\mathcal{F} and all inductive systems $\Phi \in \mathbf{S}$ are indecomposable.*

Proof. Notice the following obvious inclusions:

$$\mathcal{L}^{a_1} \subset \mathcal{L}^a, \mathcal{R}^{b_1} \subset \mathcal{R}^b, \mathcal{L}^{a_1, b_1} \subset \mathcal{L}^{a, b}, \mathcal{M}^{t_1} \subset \mathcal{M}^t \quad (17)$$

if $a_1 < a$, $b_1 < b$, and $t_1 < t$. First we claim that a finite union of members of \mathbf{S} is not equal to \mathcal{F} . The arguments just before this proposition show that this would imply the indecomposability of \mathcal{F} . Due to (17), it suffices to show that

$$\Phi = \mathcal{M}^t \bigcup \mathcal{L}^a \bigcup \mathcal{R}^b \bigcup \bigcup_{i=1}^s \mathcal{LR}^{a_i, b_i} \bigcup \bigcup_{j=1}^u \mathcal{L}^{c_j, e_j} \bigcup \bigcup_{m=1}^v \mathcal{R}^{f_m, g_m} \neq \mathcal{F}. \quad (18)$$

Denote by t_r the minimal integer such that $p^{t_r+1} - 1 \geq e_r - c_r + 1$ for $1 \leq r \leq u$ and $\geq g_{r-u} - f_{r-u} + 1$ for $u < r \leq u+v$. Set $l_0 = p^{t+1} - 1$, $l_r = p^{t_r+1} - 1$ for $1 \leq r \leq u+v$, $d_0 = 2a - 1$, $d_i = 2a_i - 1$ for $1 \leq i \leq s$, and $d' = 2b - 1$. Next, put $l^+ = \max\{l_r \mid 0 \leq r \leq u+v\}$ and $d = \max\{d_r \mid 0 \leq r \leq s\}$. Fix a triple l, k, n such that $l > l^+$, $k \in \mathbb{N}(l)$, $k-l > d$, and $2n-l-k > d'$. It is clear that $\pi_{l,k}^n \notin \Phi$ since $\pi_{l,k}^n$ does not lie in any member of \mathbf{S} that appears in (18). Hence $\Phi \neq \mathcal{F}$ as required.

Next, put $\Psi = \mathcal{L}^{a,b}$, $\mathcal{R}^{a,b}$, or $\mathcal{LR}^{a,b}$. Using the definition of the systems $\mathcal{L}^{a,b}$ and $\mathcal{R}^{a,b}$, Lemma 4.19, and Theorem 1.1i), we conclude that there exist an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ and representations $\rho_n \in \Psi_n$ for $n \in \mathbb{N}_1$ such that $\Psi = \langle \rho_n \mid n \in \mathbb{N}_2 \rangle$ for every infinite subset $\mathbb{N}_2 \subset \mathbb{N}$. Hence Ψ is indecomposable since a proper subsystem of Ψ cannot contain infinitely many representations ρ_n .

Now Corollary 4.10 yields that it remains to prove the indecomposability of \mathcal{M}^t and \mathcal{L}^a . Let Υ be one of these systems. If Υ is decomposable, it is a union of two proper subsystems that are l -restricted for $\Upsilon = \mathcal{M}^t$ and d -restricted for $\Upsilon = \mathcal{L}^a$. Now Results 4.14–4.17, 4.20, 4.22, and 4.23 and Formula (17) imply that it suffices to prove the following:

$$\mathcal{M}^0 \not\subset \bigcup_{i=1}^r \Pi^{l_i, k_i} \bigcup \bigcup_{j=1}^s \Psi^{h_j, m_j}, \quad (19)$$

for $t > 0$

$$\mathcal{M}^t \not\subset \mathcal{M}^u \bigcup \bigcup_{i=1}^r \Pi^{l_i, k_i} \bigcup \bigcup_{j=1}^s \Psi^{h_j, m_j} \quad (20)$$

with $u < t$,

$$\mathcal{L}^0 \not\subset \bigcup_{i=1}^r \mathcal{LR}^{a_i, b_i} \cup \Delta, \quad (21)$$

and for $a > 0$

$$\mathcal{L}^a \not\subset \mathcal{L}^v \bigcup_{i=1}^r \mathcal{LR}^{a_i, b_i} \cup \Delta \quad (22)$$

if $v < a$ and Δ is an l -restricted inductive system.

To prove (19) and (20), set $l = p^{t+1} - 1$, $k = \max\{k_i \mid 1 \leq i \leq r\}$, and $k' = \max\{m_j \mid 1 \leq j \leq s\}$. Then fix $y > k$, $y \in \mathbb{N}(l)$, and n with $2n - y > k'$ and $2n + 1 \geq l + y$ and observe that $\pi_{l,y}^n \in \mathcal{M}_n^t$ for $t \geq 0$. It is clear that $\pi_{l,y}^n$ does not lie in the union in the right side of (19) or (20). This implies (19) and (20).

Finally, let $\lambda = \max\{l \mid \pi_{l,z} \in \Delta\}$ and $b = \max\{b_i \mid 1 \leq i \leq r\}$. Set $c = a + \lambda$ and $n = c + b + 1$. Now observe that $L_{a,c}^n \in \mathcal{L}^a$, but does not lie in the union of inductive systems in the right part of (21) or (22). This shows that (21) and (22) hold and completes the proof. \square

Now Theorem 1.5 is proved for $GL_\infty(F)$. Consider the case of $SL_\infty(F)$. Proposition 4.1 and Results 4.2, 4.14–4.18, 4.20, 4.22, 4.23, 4.27, and 4.28 imply that it remains to find out which members of \mathbf{S} yield the same inductive sequences for $SL_\infty(F)$. Until the end of this proof all modules and inductive sequences considered are H_n -modules and inductive sequences for $SL_\infty(F)$, respectively. Let $1 \leq a \leq b \leq n$, $1 \leq c \leq d \leq n$, and $(a, b) \neq (c, d)$. Then $L_{a,b}^n \cong L_{c,d}^n$ if and only if $\{(a, b), (c, d)\} = \{(0, i), (i, n)\}$ for some i . Proposition 4.8 and Corollary 4.11 force that $\mathcal{L}_n^{0,b} = \{L_{0,j}^n \mid 0 \leq j \leq \min\{b, n\}\}$ and $\mathcal{R}_n^{0,b} = \{L_{0,j}^n \mid 0 \leq n - j \leq \min\{b, n\}\}$. Hence we get $\mathcal{L}^0 = \mathcal{R}^0$, $\mathcal{LR}^{a,0} = \mathcal{L}^{0,a}$, $\mathcal{LR}^{0,b} = \mathcal{R}^{0,b}$, and $\mathcal{L}^{0,0} = \mathcal{R}^{0,0}$. One can conclude that there are no other such equalities for the members of \mathbf{S} , except obvious corollaries of these ones. This completes the proof of Theorem 1.5 for $SL_\infty(F)$.

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