Modular representations of classical groups with small weight multiplicities

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Abstract

Lower estimates for the maximal weight multiplicities in irreducible representations of the algebraic groups of types B_n, C_n, and D_n in positive characteristic p are found under some minor restrictions on p. If $G = B_n(K)$, $C_n(K)$ or $D_n(K)$, $n \geq 8$, $p > 2$ for types B_n and D_n and $p > 7$ for type C_n, then either the maximal weight multiplicity for an irreducible representation of G is at least $n - 7$, or all its weight multiplicities are equal to 1.

1 Introduction

In what follows K is an algebraically closed field of characteristic $p > 0$ and G_n is a classical algebraic group of rank n over K. Only rational G_n-modules and representations are considered. Throughout the text M^μ is a weight subspace of weight μ in a G-module M, $\Lambda(M)$ is the set of weights of M, the symbol $\omega(M)$ denotes the highest weight of a simple G_n-module M, and $L(\omega)$ is the simple G_n-module with highest weight ω. Let $\omega_1^n, \ldots, \omega_n^n$ be the fundamental weights of G_n labeled in a standard way. A weight $\sum_{i=1}^n a_i\omega_i$ is p-restricted if $0 \leq a_i < p$ for all i. We denote by $\operatorname{wdeg} M$ the maximal dimension of a weight subspace in M, i.e.

$$\operatorname{wdeg} M = \max_{\mu \in \Lambda(M)} \dim M^\mu.$$

For the classical groups the simple modules M with $\operatorname{wdeg} M = 1$ were classified in [11, 15]. That result was used in the description of maximal subgroups of classical algebraic groups in [11]. In this paper we show that for the groups of types B_n, C_n, and D_n in other

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simple modules the maximal weight multiplicities cannot be too small with respect to the group rank, at least, under some restrictions on the field characteristic. For an algebraic group G denote by $\Omega(G)$ the set of dominant weights ω of G such that $\operatorname{wdeg} L(\omega) = 1$ and by $\Omega_p(G)$ the subset of all p-restricted weights in $\Omega(G)$. By [15, Proposition 2], for $p \neq 2$ and $n \geq 4$ one has

$$\Omega_p(G_n) = \begin{cases} \{0, \omega_1, \omega_n\} & \text{for } G_n = B_n(K), \\ \{0, \omega_1, \omega_{n-1}, \omega_n\} & \text{for } G_n = D_n(K), \\ \{0, \omega_1, \frac{p-1}{2}\omega_n, \omega_{n-1} + \frac{p-3}{2}\omega_n\} & \text{for } G_n = C_n(K). \end{cases}$$

and

$$\Omega(G_n) = \{\sum_{j=0}^{s} p^j \lambda_j \mid s \in \mathbb{Z}_{\geq 0}, \quad \lambda_j \in \Omega_p(G_n)\}$$

Theorem 1.1 *Let $n \geq 8$ and let $G_n = B_n(K)$, $C_n(K)$ or $D_n(K)$. Let M be a rational simple G_n-module with $\omega(M) \notin \Omega(G_n)$. Suppose that $p > 2$ for $G_n = B_n(K)$ or $D_n(K)$ and $p > 7$ for $G_n = C_n(K)$. Then $\operatorname{wdeg} M \geq n - 4 - [n]_4$ where $[n]_4$ is the residue of n modulo 4. In particular, $\operatorname{wdeg} M \geq n - 7$.*

Theorem 1.1 shows that for groups of types B_n, C_n and D_n and not too small p the simple modules M with $\operatorname{wdeg} M = 1$ constitute a class of modules with small weight multiplicities that in a certain sense cannot be extended.

Lemma 2.9 implies that there exists a p-restricted G_n-module M with $\operatorname{wdeg} M = n$. Hence the estimates in Theorem 1.1 are asymptotically exact. It is not clear yet whether the restrictions on p in Theorem 1.1 can be removed. The authors hope to return to this question in the future.

For groups of type A_n the situation is substantially more difficult. For each positive integer a and n large enough with respect to a there exist p-restricted $A_n(K)$-modules M with $\operatorname{wdeg} M > a$ but small enough with respect to n. Indeed, if $\omega(M) = \sum_{i=1}^{d} a_i\omega_i$, then for large n the parameter $\operatorname{wdeg} M$ is bounded by some constant that depends on $a_1, \ldots, a_d$ only and does not depend on n. Naturally, the same holds for representations with highest weights of the form $\sum_{i=1}^{d} a_i\omega_{n-i+1}$. Therefore for groups of type A_n one has to take into account the polynomial degree of a simple module M when trying to estimate $\operatorname{wdeg} M$. For $M = L(a_1\omega_1 + \ldots + a_n\omega_n)$ set $\operatorname{pdeg} M = \sum_{i=1}^{n} ia_i$. For $G = A_n(K)$ and a p-restricted G_n-module M the authors expect to get different estimates for $\operatorname{wdeg} M$ in the case where both $\operatorname{pdeg} M$ and $\operatorname{pdeg} M^* > n + 1$ and in the opposite case. These problems will be considered in a subsequent paper.

Note that Theorem 1.1 can be considered as a modular analog of the famous problem in characteristic zero: describe all infinite dimensional simple weight modules with bounded weight multiplicities for finite dimensional complex semisimple Lie algebras. Fernando [6] proved that such modules exist for types A_n and C_n only. Benkart, Britten and Lemire [1] classified all completely pointed modules (i.e. with one-dimensional weight spaces). A complete solution of the problem was given by Mathieu [10]. Observe that the set of highest weights of simple completely pointed highest weight modules in type C_n is $\{0, \omega_1, -\frac{1}{2}\omega_n, \omega_{n-1} - \frac{3}{2}\omega_n\}$, which is exactly $\Omega_p(C_n)$ if we put $p = 0$.

Estimates of weight multiplicities discussed above and expected results in this direction for groups of type A_n can be used for recognizing linear groups containing matrices with small eigenvalue multiplicities. Indeed, for groups of types B_n, C_n, and D_n it occurs (may be, under some restrictions on the characteristic) that the images of almost all their representations do not contain matrices all whose eigenvalue multiplicities are small enough with respect to the group rank, only the representations lying in certain well-defined classes yield exceptions.

2 Notation and preliminary results

Let G be a simple algebraic group over K. We denote by $\mathrm{Irr}\, G$ the set of all rational irreducible representations (or simple modules) of G up to equivalence and by $\mathrm{Irr}_p\, G \subset \mathrm{Irr}\, G$ the subset of p-restricted ones. Let M be a G-module. Throughout the text, M^* is the dual of the module M; $\mathrm{Irr}\, M \subset \mathrm{Irr}\, G$ is the set of composition factors of M (disregarding the multiplicities); $\Lambda(M)$ is the set of all weights of M; $\Lambda^+(M)$ is the set of all its dominant weights; $\omega(M)$ is the highest weight of M, M^μ is the weight subspace of weight μ in M. We denote by $L(\omega)$ and $V(\omega)$ the simple G-module and the Weyl module with highest weight ω, respectively. The symbol v^+ always denotes a nonzero highest weight vector in a relevant module.

In what follows $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ are the sets of integers and nonnegative integers; the symbols $\Lambda(G)$ and $R(G)$ denote the set of weights and the set of roots of G, respectively, $R^+(G) \subset R(G)$ is the set of positive roots (with respect to a fixed maximal torus $T \subset G$ and a fixed base of $R(G)$); $\langle \lambda, \alpha \rangle$ is the value of a weight $\lambda \in \Lambda(G)$ on a root $\alpha \in R(G)$. We identify $\Lambda(A_1(K))$ with $\mathbb{Z}$ mapping the weight $a\omega_1$ into a. For $\alpha \in R(G)$ the symbols X_α and $\mathfrak{X}_\alpha$ denote the root element of the Lie algebra of G and the root subgroup in G associated with α. If $k \in \mathbb{Z}_{\geq 0}$, then $X_{\alpha,k}$ is the element of the hyperalgebra of G associated with the pair (α, k). For $k < p$ one has $X_{\alpha,k} = (X_\alpha)^k/k!$. The subgroup of G generated by subgroups $\Gamma_1, \ldots, \Gamma_i$ and the subspace of a linear space L spanned by vectors $v_1, \ldots, v_i$ are denoted by $\langle \Gamma_1, \ldots, \Gamma_i \rangle$ and $\langle v_1, \ldots, v_i \rangle$, respectively. For $\beta_1, \ldots, \beta_j \in R^+(G)$ put

$$G(\beta_1, \ldots, \beta_j) = \langle \mathfrak{X}_{\beta_1}, \ldots, \mathfrak{X}_{\beta_j}, \mathfrak{X}_{-\beta_1}, \ldots, \mathfrak{X}_{-\beta_j} \rangle.$$

In all cases where subgroups $H = G(\beta_1, \ldots, \beta_j)$ are considered, the roots $\beta_1, \ldots, \beta_j$ are chosen such that H is semisimple and these roots constitute a base of $R(H)$. In this situation the fundamental weights of H are determined with respect to this base. The intersection $T \cap H$ is a maximal torus in H. If $\omega \in \Lambda(G)$, then $\omega|H$ is the restriction of ω to $T \cap H$. For a G-module M and a weight vector $v \in M$ we denote the weight of v by $\omega(v)$ and set $\omega_H(v) = \omega(v)|H$. If $\alpha \in R(G)$ and $t \in K$, then by [2, Proposition 5.13], for the root element $x_\alpha(t) \in \mathfrak{X}_\alpha$ one has

$$x_\alpha(t)v = \sum_{d=0}^{\infty} t^d X_{\alpha,d} v. \tag{1}$$

We fix a base $\alpha_1, \ldots, \alpha_n$ of $R(G_n)$, the fundamental weights are considered with respect to this base. In what follows ε_j with $1 \leq j \leq n$ are weights of the natural realization of G_n, the labeling of the roots α_i and the weights ε_j is standard and corresponds to [3, Ch. VI, §4] and [4, Ch. VIII, §13]. Set $X_{\pm i} = X_{\pm\alpha_i}$ and define $\mathfrak{X}_{\pm i}$ and $X_{\pm i,k}$ similarly. Put $G_n(i_1, \ldots, i_j) = G_n(\alpha_{i_1}, \ldots, \alpha_{i_j})$ and $b_i(\mu) = \langle \mu, \alpha_i \rangle$ for $\mu \in \Lambda(G_n)$. If $H = G_n(\beta_1, \ldots, \beta_j)$, $\alpha_i \in R(H)$ and $\mu' = \mu|H$, define $b_i(\mu')$ in a similar way.

Let $M^{[k]}$ denote a G-module M twisted by the kth power of the Frobenius morphism. The following lemma is obvious.

Lemma 2.1 *Let M_1 and M_2 be G-modules. Then* $\mathrm{wdeg}\, M_1^{[k_1]} \otimes M_2^{[k_2]} \geq \mathrm{wdeg}\, M_1 \cdot \mathrm{wdeg}\, M_2$.

Let $M \in \mathrm{Irr}\, G$. Assume that $\omega(M) = \sum_{k=0}^{s} p^k \lambda_k$ with p-restricted λ_k. Put $M_k = L(\lambda_k)$. By the Steinberg tensor product theorem [13],

$$M \cong \otimes_{k=0}^{s} M_k^{[k]}. \tag{2}$$

Therefore by Lemma 2.1, $\mathrm{wdeg}\, M \geq \mathrm{wdeg}\, M_0 \cdot \ldots \cdot \mathrm{wdeg}\, M_s$.

Lemma 2.2 ([2, Lemma 5.14], [11, 1.5], and [14, 2.1]) *(i) For the operators $X_{\alpha,d}$ the following equalities hold:*

$$X_{-\alpha}X_{\alpha,d} = X_{\alpha,d}X_{-\alpha} - H_\alpha X_{\alpha,d-1} + (d-1)X_{\alpha,d-1},$$

$$X_{\alpha,d}X_\beta = X_\beta X_{\alpha,d} + \sum_{t=1}^{d} c_t X_{t\alpha+\beta}X_{\alpha,d-t},\ c_t \in \mathbb{Z} \quad (c_t = 0 \text{ if } t\alpha+\beta \notin R(G_n))$$

(here $H_\alpha = [X_\alpha, X_{-\alpha}]$). In particular, $X_{i,k}X_{-j,d} = X_{-j,d}X_{i,k}$ for $i \neq j$.

(ii) Let V be a G_n-module, $\mu \in \Lambda(G_n)$, $v \in V_\mu \setminus \{0\}$, $\alpha \in R(G_n)$, $X_{\alpha,b}v = 0$ for $b > 0$, and $\langle \mu, \alpha \rangle = c \geq 0$. Then $X_\alpha X_{-\alpha,b}v = (c-b+1)X_{-\alpha,b-1}v$ and $X_{-\alpha,c}v \neq 0$. In particular, if $0 < c < p$, one has $X_{-\alpha,d}v \neq 0$ for $0 < d \leq c$.

Lemma 2.3 *Let $G = A_1(K)$, $0 \leq b < p-1$, and let M be an indecomposable G-module with highest weight $p+b$. Assume that $X_{-\alpha}^{b+1}v^+ \neq 0$ for $\alpha \in R^+(G)$. Then $M \cong V(p+b)$.*

Proof. Set $b_1 = p+b$ and $b_2 = p-b-2$. By the universal property of the Weyl module [8, Part II, Lemma 2.13b)], M is a quotient of $V(b_1)$. It follows from [5] (and can be easily deduced from the weight structure of $V(b_1)$) that $V(b_1)$ has two composition factors: $L(b_1)$ and $L(b_2)$. The Steinberg tensor product theorem [13] (see (2)) forces that $b_2 \notin \Lambda(L(b_1))$. However, $b_2 \in \Lambda(M)$ as $X_{-\alpha}^{b+1}v^+ \neq 0$. This implies that $M \not\cong L(b_1)$ and completes the proof. □

Lemma 2.4 (Jantzen [7], Smith [12]) *Let $H = G_n(i_1, \dots, i_j) \subset G_n$. Then $KHv^+ \subset L(\omega)$ is a simple H-module with highest weight $\omega_H(v^+)$ and a direct summand of the H-module $L(\omega)$.*

Let $H = G_n(\beta_1, \dots, \beta_j) \subset G$ and M be a G-module. Put $U^+(H) = \langle \mathfrak{X}_\alpha \mid \alpha \in R^+(H) \rangle$. Recall that a vector $v \in M$ is called primitive with respect to H if v is a nonzero weight vector and $U^+(H)$ fixes v.

Definition 2.5 ([14]) Let $\omega = \sum_{i=1}^n a_i\omega_i$ be a dominant weight of G_n and let $M = L(\omega)$. Set $y_k = -\langle \alpha_{k-1}, \alpha_k \rangle$, $z_k = -\langle \alpha_{k+1}, \alpha_k \rangle$. Let $1 \leq i, j \leq n$ and let all the roots α_t with t in the interval with the ends i and j form a chain on the Dynkin diagram of G_n. Fix v^+. For an integer d with $0 < d \leq a_j$ define the vector $v(i,j,d)$ as follows. Put $d_j = d$. If $i > j$, put $d_k = a_k + d_{k-1}y_k$ for $i \geq k > j$. If $i < j$, set $d_k = a_k + d_{k+1}z_k$ for $i \leq k < j$. Now define

$$v(i,j,d) = X_{-i,d_i} \dots X_{-k,d_k} \dots X_{-j,d}v^+.$$

Lemma 2.6 ([14, Lemma 2.9]) *The vector $v(i,j,d) \neq 0$ and $X_{l,b}v(i,j,d) = 0$ for $l \neq i$ and $b > 0$. Hence $\mathfrak{X}_l$ fixes $v(i,j,d)$.*

Proposition 2.7 *Let $\Gamma = G_n(i_1, \dots, i_t)$, $M \in \operatorname{Irr} G_n$, and $N \in \operatorname{Irr}(M|\Gamma)$. Then* $\operatorname{wdeg} N \leq \operatorname{wdeg} M$.

Proof. Put $\omega = \omega(M)$. For every $\lambda \in \Lambda(M)$ one has $\lambda = \omega - \sum_{i=1}^n c_i(\lambda)\alpha_i$ with $c_i(\lambda) \in \mathbb{Z}_{\geq 0}$. Set

$$\{j_1, \dots, j_k\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_t\},$$

$$\Delta = \{(f_{j_1}, \dots, f_{j_k}) \mid f_{j_l} \in \mathbb{Z}_{\geq 0}, 1 \leq l \leq k\}.$$

For each k-tuple $s \in \Delta$ put

$$\Lambda(s) = \{\lambda \in \Lambda(M) \mid c_j(\lambda) = f_j \text{ for } j \in \{j_1, \dots, j_k\}\}.$$

It is obvious that almost always $\Lambda(s) = \varnothing$, $\Lambda(s_1) \cap \Lambda(s_2) = \varnothing$ for tuples $s_1 \neq s_2$ and

$$\Lambda(M) = \cup_{s\in\Delta}\Lambda(s).$$

Set

$$U(s) = \oplus_{\lambda\in\Lambda(s)} M^\lambda.$$

Then $U(s)$ are Γ-modules and $M = \bigoplus_{s\in\Delta} U(s)$ as a Γ-module. Since N is simple, it is a composition factor of some $U(s)$. So $\operatorname{wdeg} N \leq \operatorname{wdeg} U(s)$. It remains to observe that for $\mu, \nu \in \Lambda(s)$ and $\nu \neq \mu$ one has $\mu|\Gamma \neq \nu|\Gamma$. Indeed, as $\mu, \nu \in \Lambda(s)$, one gets $c_j(\mu) = c_j(\nu)$ for $j \in \{j_1, \dots, j_k\}$. Hence $c_i(\mu) \neq c_i(\nu)$ for some $i \in \{i_1, \dots, i_t\}$. This yields that $\mu|\Gamma \neq \nu|\Gamma$, $\operatorname{wdeg} U(s) \leq \operatorname{wdeg} M$, and proves the lemma. □

Lemma 2.8 *Let $G = A_n(K)$, $1 \leq j < k \leq n$, and let $\omega = \sum_{s=j}^{k} a_s\omega_s$ be a dominant p-restricted weight of G with both a_j and $a_k \neq 0$. Then*

$$\operatorname{wdeg} L(\omega) \geq k - j.$$

Proof. Write $\omega = a_j\omega_j + a_{i_1}\omega_{i_1} + \ldots + a_{i_t}\omega_{i_t} + a_k\omega_k$ with $j < i_1 < \ldots < i_t < k$ and $a_{i_1}, \dots, a_{i_t} \neq 0$ (t can be zero). By [9, Proposition 1.21], $\operatorname{wdeg} L(\omega) \geq f(j, i_1, \dots, i_t, k)$, where for l-tuples $(u_1, \dots, u_l)$ with $u_1 < \ldots < u_l$ the integers $f(u_1, \dots, u_l)$ are determined by the following recurrent relations:

$$f(u_1) = 1;$$

$$f(u_1, u_2) = u_2 - u_1;$$

$$f(u_1, u_2, \dots, u_l) = (u_2 - u_1)f(u_2, \dots, u_l) + f(u_3, \dots, u_l) \quad \text{for } l > 2.$$

We claim that $f(j, i_1, \dots, i_t, k) \geq k - j$. For $t = 0$ this holds by definition. Then apply induction on t. Let $t > 0$. One easily concludes that $f(u_1, \dots, u_l) \geq 1$ for all positive integers $u_1, \dots, u_l$. Now the induction hypothesis yields that $f(j, i_1, \dots, i_t, k) = (i_1 - j)f(i_1, \dots, i_t, k) + f(i_2, \dots, i_t, k) \geq (i_1 - j)(k - i_1) + 1$. (For $t = 1$ we have $f(j, i_1, k) = (i_1 - j)(k - i_1) + 1$.) Note that $ab \geq a + b$ for a and $b \in \mathbb{Z}_{\geq 0}$ and a, $b > 1$. Hence $ab + 1 \geq a + b$ for a and $b \in \mathbb{Z}_{\geq 0}$ and a, $b > 0$. This yields our claim and completes the proof. □

Lemma 2.9 *Let $p > 2$ and let G_n be one of the following groups: $B_n(K)$ with $n > 2$, $C_n(K)$ or $D_n(K)$ with $n > 3$. Assume that $M \in \operatorname{Irr} G_n$ with $\omega(M) = \omega_2$ for $G_n = B_n(K)$ or $D_n(K)$ and $\omega(M) = 2\omega_1$ for $G_n = C_n(K)$. Then* $\operatorname{wdeg} M = n$.

Proof. Denote by V the standard G_n-module. Recall that $\operatorname{wdeg} V = 1$, $\Lambda(V) = \{\pm\varepsilon_i, 0 \mid 1 \leq i \leq n\}$ for $G_n = B_n(K)$ and $\Lambda(V) = \{\pm\varepsilon_i \mid 1 \leq i \leq n\}$ for $G_n = C_n(K)$ or $D_n(K)$.

It is well known that for $G_n = B_n(K)$ or $D_n(K)$ the wedge square $\wedge^2 V \cong M$ as a G_n-module and that for $G_n = G_n(K)$ the symmetric square $S^2V \cong M$ (see, for instance, [8, Part II, 2.15] and [11, 8.1]). Hence $\Lambda^+(M) = \{\omega_2, \omega_1, 0\}$ for $G_n = B_n(K)$, $\{\omega_2, 0\}$ for $G_n = D_n(K)$, and $\{2\omega_1, \omega_2, 0\}$ for $G_n = G_n(K)$. Naturally, $\dim M^{\omega_2} = 1$ for $G_n = B_n(K)$ or $D_n(K)$ and $\dim M^{2\omega_1} = 1$ for $G_n = C_n(K)$. One easily observes that $\dim M^0 = n$ for all cases, $\dim M^{\omega_1} = 1$ for $G_n = B_n(K)$, and $\dim M^{\omega_2} = 1$ for $G_n = C_n(K)$. This completes the proof. □

3 Reduction to special n

In Sections 3-5 $n \geq 8$, except Lemmas 5.1–5.3, $M \in \operatorname{Irr} G_n$ is a module with highest weight $\omega = \sum_{i=1}^{n} a_i\omega_i$. We assume that $\omega \notin \Omega(G_n)$, $p > 2$, and $p > 7$ for $G_n = C_n(K)$. We will deal with the subgroup H of G_n of the following type: $H = H_1 \times H_2$ where $H_1 = G_n(\alpha_1, \alpha_2, \alpha_3, \beta)$ with $\beta = \varepsilon_3 + \varepsilon_4$ for $D_n(K)$ and $\beta = 2\varepsilon_4$ for $G_n = C_n(K)$ and $H_2 = G_n(5, \dots, n)$. For a weight $\mu \in \Lambda(H)$ we will write $\mu = (\mu_1, \mu_2)$ if $\mu_i = \mu|H_i$. We call μ *special* if at least one of $\mu_i \in \Omega(H_i)$. Otherwise μ is *nonspecial.*

The general scheme of the proof of Theorem 1.1 is as follows. The problem is reduced to the case where $G_n = D_n(K)$ or $C_n(K)$, $n = 4k$ and $M \in \operatorname{Irr}_p G_n$. This case is considered separately for groups of types D_n and C_n in Sections 4 and 5.

Lemma 3.1 *Let $G_n = B_n(K)$ and $S \subset G$ be the subgroup generated by all long root subgroups. Assume that $M \in \operatorname{Irr}_p G_n$. Then $M|S$ has a composition factor F with $\omega(F) \notin \Omega(S)$.*

Proof. Set $\gamma = \alpha_{n-1} + 2\alpha_n$. It is well known that $S \cong D_n(K)$ and that the roots $\alpha_1, \dots, \alpha_{n-1}, \gamma$ constitute a base of $R(S)$. Let $\overline{\omega}_i$, $1 \leq i \leq n$, be the fundamental weights of S with respect to this base. Now one can observe that

$$\omega_i|S = \overline{\omega}_i \quad \text{for } i \leq n-2, \quad \omega_{n-1}|S = \overline{\omega}_{n-1} + \overline{\omega}_n \quad \text{and} \quad \omega_n|S = \overline{\omega}_n. \tag{3}$$

Obviously, $M|S$ has a composition factor with highest weight $\omega|S$. So Formulas (3) imply that it suffices to consider the case where $\omega = \omega_{n-1} + (p-1)\omega_n$. Set $v = X_{-n}v^+$. Then $v \neq 0$ by Lemma 2.2. Naturally, v is fixed by $\mathfrak{X}_i$ for $i < n$ and $\mathfrak{X}_\gamma$. So v is primitive for S. Since $b_{n-1}(\omega(v)) = 2$, the weight $\omega_S(v) \notin \Omega(S)$. This proves the lemma as $M|S$ has a composition factor with highest weight $\omega_S(v)$. □

Lemma 3.2 *Let $G_n = C_n(K)$ or $D_n(K)$ with $n \geq 12$ and H be as above. Assume that $M \in \operatorname{Irr} G_n$ and $M|H$ has a composition factor $N_1 \otimes N_2$ with $N_i \in \operatorname{Irr} H_i$, $\omega(N_1) \notin \Omega(H_1)$ and* $\operatorname{wdeg} N_2 \geq n-8$. *Then* $\operatorname{wdeg} M \geq n-4$.

Proof. Observe that $\mu|H \neq \nu|H$ for distinct weights μ and ν in $\Lambda(G_n)$ since the rank of H is equal to n. Hence $\operatorname{wdeg} M \geq \operatorname{wdeg} N$ for each $N \in \operatorname{Irr}(M|H)$. Let $N = N_1 \otimes N_2$ with N_1 and N_2 such as in the statement of the lemma. Then $\operatorname{wdeg} N = \operatorname{wdeg} N_1 \operatorname{wdeg} N_2 \geq 2(n-8) \geq n-4$ as desired. □

Lemma 3.3 *Let $n \equiv 0 \pmod 4$ and $G_n = C_n(K)$ or $D_n(K)$. If $n > 8$, assume that Theorem 1.1 holds for G_{n-4}. Suppose that $M \in \operatorname{Irr} G_n$ and $M|H$ has a composition factor $N_1 \otimes N_2$ with $N_i \in \operatorname{Irr} H_i$ and $\omega(N_i) \notin \Omega(H_i)$. Then* $\operatorname{wdeg} M \geq n-4$.

Proof. First let $n = 8$. Then $\operatorname{wdeg} N_i \geq 2$ since $\omega(N_i) \notin \Omega(H_i)$. Hence $\operatorname{wdeg} M \geq \operatorname{wdeg} N_1 \otimes N_2 \geq 4 = n-4$. Now let $n > 8$. Then $n \geq 12$. Our assumptions force $\operatorname{wdeg} N_2 \geq n-8$. Apply Lemma 3.2 and conclude that $\operatorname{wdeg} M \geq n-4$. □

Lemma 3.4 *Let $G_n = C_n(K)$ or $D_n(K)$ and let $S = G_n(2, \dots, n)$. Assume that M is p-restricted. Then $\operatorname{Irr}(M|S)$ contains a factor $N \in \operatorname{Irr}_p S$ with $\omega(N) \notin \Omega(S)$.*

Proof. Write $\omega = a_1\omega_1 + \mu$. Set

$$\Pi = \begin{cases} \{0, \omega_2, \frac{p-1}{2}\omega_n, \omega_{n-1} + \frac{p-3}{2}\omega_n\} & \text{for } G_n = C_n(K), \\ \{0, \omega_2, \omega_{n-1}, \omega_n\} & \text{for } G_n = D_n(K). \end{cases}$$

If $\mu \notin \Pi$, the assertion of the lemma follows directly from Theorem 2.4. Suppose that $\mu \in \Pi$. Set $m = v(1,2,1)$ for $\mu = \omega_2$ and $m = X_{-1}^{a_1} v^+$ otherwise. As $\omega \notin \Omega(G_n)$, one concludes that $a_1 > 0$ for $\mu \neq \omega_2$ and $a_1 > 1$ for $\mu = 0$. By Lemmas 2.6 and 2.2, in all cases the vector $m \neq 0$ and is primitive for S. So it generates an indecomposable S-module with highest weight $\omega_S(m) \notin \Omega(S)$. This yields a required factor. □

Lemma 3.4 holds for $G_n = B_n(K)$ as well, but we do not need this.

Lemma 3.5 *Assume that for all integers l with $8 \leq l = 4k \leq n$ Theorem 1.1 holds for the groups $G_l = C_l(K)$ or $D_l(K)$ and modules $M \in \mathrm{Irr}_p\, G_l$. Then it holds for G_n and M.*

Proof. First assume that $G_n = C_n(K)$ or $D_n(K)$. Let $n = 4k + r$ with $r > 0$ and $M \in \mathrm{Irr}_p\, G_n$. Observe that $4k \geq 8$. Set $S_r = G_n(r+1, \ldots, n)$. Several applications of Lemma 3.4 imply that $\mathrm{Irr}(M|S_r)$ contains a factor N with $\omega_N \notin \Omega(S_r)$. By our assumptions, $\mathrm{wdeg}\, N \geq 4k - 4$. By Proposition 2.7, $\mathrm{wdeg}\, M \geq \mathrm{wdeg}\, N \geq n - r - 4$. For arbitrary M, apply Lemma 2.1.

Finally, let $G_n = B_n(K)$ and $S \subset G_n$ be the subgroup generated by all long root subgroups. Recall that $S \cong D_n(K)$. Since the ranks of S and G coincide, one has $\mathrm{wdeg}\, M \geq \mathrm{wdeg}\, N$ for each $N \in \mathrm{Irr}(M|S)$. It remains to apply the fact proven for $D_n(K)$ in the previous paragraph and Lemmas 3.1 and 2.1. □

Now it remains to show that Theorem 1.1 holds for $G_n = D_n(K)$ or $C_n(K)$ with $n \equiv 0 (\mathrm{mod}\, 4)$ and $M \in \mathrm{Irr}_p\, G_n$.

4 The case of spinor groups

In this section we complete the proof of Theorem 1.1 for $G_n = D_n(K)$ and $B_n(K)$.

Proposition 4.1 *Let $n \geq 8$, $n \equiv 0 (\mathrm{mod}\, 4)$ and $G_n = D_n(K)$. Assume that $M \in \mathrm{Irr}_p\, G_n$ and $\omega(M) \notin \Omega_p(G_n)$. Then* $\mathrm{wdeg}\, M \geq n - 4$.

Proof. Set $\Gamma = G_n(1, \ldots, n-2, n)$. Then $\Gamma \cong A_{n-1}(K)$. The proof is based on the analysis of the restrictions $M|H$ and $M|\Gamma$. Recall that $H = H_1 \times H_2$ where

$$\begin{aligned} H_1 &= G_n(\alpha_1, \alpha_2, \alpha_3, \beta) \cong D_4(K), \\ H_2 &= G_n(5, \ldots, n) \cong D_{n-4}(K), \\ \beta &= \varepsilon_3 + \varepsilon_4 = \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n. \end{aligned}$$

If $n > 8$, apply induction on n, assuming that Theorem 1.1 holds for G_{n-4}. The correctness of such enlarged induction hypothesis where we assume that not only our proposition, but Theorem 1.1 holds for G_{n-4}, follows from Lemma 3.5. Now Lemma 3.3 implies that $\mathrm{wdeg}\, M \geq n - 4$ if $M|H$ has a composition factor N with nonspecial $\omega(N)$.

The following scheme is used to construct a desired factor $N \in \mathrm{Irr}(M|H)$. Set $\omega_H = \omega|H$. If a vector $m \in M$ is primitive for H, it is clear that it generates an indecomposable H-module with highest weight $\omega_H(m)$ and hence $L(\omega_H(m)) \in \mathrm{Irr}(M|H)$. One can observe that m is primitive for H if it is fixed by $\mathfrak{X}_i$ for $i \neq 4$ and by $\mathfrak{X}_\beta$. Naturally, a nonzero highest weight vector generates an indecomposable H-module with highest weight ω_H and therefore $L(\omega_H) \in \mathrm{Irr}(M|H)$. For $j < n$ and $d \leq a_j$ construct the vector $m = v(4, j, d)$ as in Lemma 2.6. Set $\mu = \omega_H(m)$. By Lemma 2.6, $\mathfrak{X}_i$ fixes m for $i \neq 4$. Since $\beta = \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, the group $\mathfrak{X}_\beta$ fixes m as well. Hence m is primitive for H and so $L(\mu) \in \mathrm{Irr}(M|H)$. First we find out when ω_H is nonspecial and settle the question in this case. Then we try to construct a vector m with nonspecial μ. In some situations we cannot construct suitable vectors of this form and use specific arguments to construct other primitive vectors for H with nonspecial weights.

Our assumptions on the labeling of the fundamental weights for the groups $G_n(i_1, \dots, i_k)$ imply that $\omega_n|\Gamma = \omega_{n-1}$. In some situations we construct a composition factor $F \in \mathrm{Irr}(M|\Gamma)$ with $\mathrm{wdeg}\, F \geq n-4$ and apply Proposition 2.7. By Lemma 2.6, for $1 \leq k \leq n-1$ and $l \leq a_k$ the vector $u = v(n-1, k, l) \neq 0$ and is fixed by $\mathfrak{X}_t$ for $t < n-1$. Obviously, $\mathfrak{X}_n$ fixes u. Hence u is primitive for Γ. Set $\nu = \omega_\Gamma(u)$. Then $L(\nu) \in \mathrm{Irr}(M|\Gamma)$. Proposition 2.7 implies that $\mathrm{wdeg}\, M \geq \mathrm{wdeg}\, L(\nu)$. The notation j, d, m, μ, u, and ν is fixed until the end of the proof.

First consider some special cases. Assume that $\omega = a_i\omega_i$ with $a_i > 0$ and $2 \leq i \leq 4$ or $\omega = a_1\omega_1$ with $a_1 > 1$. Set $u = v(n-1, i, a_i)$ in the first case and $u = v(n-1, 1, 1)$ in the second one. Then $\nu = a_i\omega_{i-1} + a_i\omega_{n-1}$ in the first case and $(a_1 - 1)\omega_1 + \omega_{n-1}$ in the second one. Lemma 2.8 yields that $\mathrm{wdeg}\, L(\nu) \geq n - i$ in the first case and $\mathrm{wdeg}\, L(\nu) \geq n-2$ in the second one. Hence in both cases $\mathrm{wdeg}\, M \geq n-4$.

Now let $\omega \neq a_i\omega_i$ with $i \leq 4$. Here in the majority of cases we construct a composition factor $N \in \mathrm{Irr}(M|H)$ with nonspecial $\omega(N)$, but sometimes we consider the restriction $M|\Gamma$ and apply Lemma 2.8 and Proposition 2.7 as above.

Set $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ and $\lambda' = \sum_{i=5}^n a_i\omega_i$. Then $\omega = \lambda + a_4\omega_4 + \lambda'$. Observe that ω_H is nonspecial unless

$$\lambda \in \{0, \omega_1, \omega_3\} \text{ or } \lambda' \in \{0, \omega_5, \omega_{n-1}, \omega_n\}. \tag{4}$$

Hence we can and shall assume that (4) holds. Notice that λ and λ' cannot both be 0 since $\omega \neq a_4\omega_4$. As the graph morphism does not change $\mathrm{wdeg}\, M$, one can assume that $a_n \geq a_{n-1}$, so $\lambda' \neq \omega_{n-1}$.

I) First suppose that $a_4 \geq 2$.

1.) If both $a_3, a_5 < p-2$, set $j = 4$ and $d = 2$. For $a_3 = a_5 = p-2$, put $j = 4$ and $d = 1$. In both cases $1 < b_3(\mu) < p$ and $1 < b_5(\mu) < p$. Hence μ is nonspecial.

2.) In other cases where $a_4 \geq 2$ the following construction is used. For $t = X_{-4}v^+$ or $X_{-4,2}v^+$ and $i = 3, 5$ we analyze the $G_n(i)$-module $M' = KG_n(i)u$. It occurs that $b_i(\omega(t)) = p + y$, $0 \leq y < p-1$. We show that $w = X_{-i}^{y+1}t \neq 0$. Then Lemma 2.3 implies that $M' \cong W(p+y)$. As $b_i(\omega(w)) = p - y - 2$, the vector w is a highest weight vector in the maximal submodule of M' and is fixed by $\mathfrak{X}_i$. For $k \neq 4, i$ the group $\mathfrak{X}_k$ commutes with $X_{-i,l}$ and fixes u. Hence $\mathfrak{X}_k$ fixes w. It is clear that $\mathfrak{X}_\beta$ fixes w as well. Therefore w is primitive with respect to H and $L(\omega_H(w)) \in \mathrm{Irr}(M|H)$. Put $\tau = \omega_H(w)$. Then we compute $b_2(\tau)$ or $b_3(\tau)$ and $b_5(\tau)$ or $b_6(\tau)$ and conclude that τ is nonspecial. Below details for all subcases are presented. In Items a)-d) c is a nonzero constant whose exact value we do not need to know. The calculations are based on Lemmas 2.2 and 2.3.

a.) Suppose that $a_3 = p-2$ and $a_5 = p-1$. Take $i = 5$ and $t = X_{-4}v^+$. Then $w = X_{-5}t$. Observe that

$$X_4 w = X_4 X_{-5} X_{-4} v^+ = c X_{-5} v^+.$$

Hence $w \neq 0$. One has $b_3(\tau) = p-1$ and $b_6(\tau) \neq 0$.

b.) Let $a_3 = p-2$ and $a_5 < p-2$. Take $t = X_{-4,2}v^+$ and $i = 3$. Then $w = X_{-3}t$,

$$X_4^2 w = X_4^2 X_{-3} X_{-4,2} v^+ = c X_{-3} v^+$$

and hence $w \neq 0$. We get $b_2(\tau) \neq 0$ and $1 < b_5(\tau) < p$.

c.) If $a_3 = p-1$, the weight $\omega|H_1 \notin \Omega(H_1)$. Hence we can assume that $\omega|H_2 \in \Omega(H_2)$ and so $a_5 = 0$ or 1 and hence $a_5 < p-1$. Let $a_5 = p-2$ (then $p = 3$, but we do not use this). Take $t = X_{-4}v^+$ and $i = 3$. Then $w = X_{-3}t$,

$$X_4 w = X_4 X_{-3} X_{-4} v^+ = c X_{-3} v^+ \neq 0.$$

We have $b_2(\tau) \neq 0$ and $b_5(\tau) = p-1$. If $a_5 < p-2$, take t and i as in Item (b) and argue as in that item taking into account that now $w = X_{-3}^2 t$. Here $b_2(\tau) \neq 0$ and $1 < b_5(\tau) < p$.

d.) Finally, let $a_3 < p-2$ and $a_5 = p-2$ or $p-1$. Take $t = X_{-4,2}v^+$ and $i = 5$. Then $w = X_{-5}t$ or $X_{-5}^2 t$, respectively. One can check that

$$X_4^2 w = c X_{-5} v^+ \text{ or } c X_{-5}^2 v^+$$

and in both cases $X_4^2 w \neq 0$. Hence $w \neq 0$. We have $1 < b_3(\tau) < p$ and $b_6(\tau) \neq 0$.

II) Now assume that $a_4 = 1$.

1.) Suppose that $\lambda = 0$. If $a_5 \neq 0$, set $j = 5$ and $d = 1$. Then $b_3(\mu) = 2$ and $b_6(\mu) \neq 0$, so μ is nonspecial. Now let $a_5 = 0$. Set $w_1 = X_{-3}X_{-5}X_{-4}v^+$ and $w = X_{-4}w_1$. Applying Lemma 2.2 three times, we get $w_1 \neq 0$. It is clear that $\mathfrak{X}_4$ fixes w_1 as $\omega - \alpha_3 - \alpha_5 \notin \Lambda(M)$. Since $b_4(\omega(w_1)) = 1$, another application of Lemma 2.2 forces $w \neq 0$. Obviously, the groups $\mathfrak{X}_i$ for $i \neq 3, 4, 5$ and $\mathfrak{X}_\beta$ fix w. Since $\omega - \alpha_3 - 2\alpha_4$ and $\omega - 2\alpha_4 - \alpha_5 \notin \Lambda(M)$, the groups $\mathfrak{X}_3$ and $\mathfrak{X}_5$ fix w as well. So w is primitive for H. Put $\delta = \omega_H(w)$. Then δ is nonspecial since $b_2(\delta) \neq 0$ and $b_6(\delta) \neq 0$.

2.) Let $\lambda = \omega_1$ or ω_3. Set $j = 4$ and $d = 1$. Then $b_3(\mu) = a_3 + 1$ and hence $\omega_{H_1}(m) \notin \Omega(H_1)$. For $0 < a_5 < p-1$ the weight μ is nonspecial as $1 < b_5(\mu) < p$. Let $a_5 = 0$. If $\lambda' \neq 0$, μ is also nonspecial since $b_5(\mu) = 1$ and $b_k(\mu) = a_k$ for $k \geq 6$. Suppose that $\lambda' = 0$, i.e. $\omega = \omega_i + \omega_4$ with $i = 1$ or 3. Now we shall construct a factor $F \in \mathrm{Irr}(M|\Gamma)$ with $\mathrm{wdeg}\, F \geq n-4$. Set $u = v(n-1, 4, 1)$. Then $\nu = \omega_1 + \omega_3 + \omega_{n-1}$ for $\lambda = \omega_1$ and $2\omega_3 + \omega_{n-1}$ if $\lambda = \omega_3$. Arguing as at the beginning of the proof, one can conclude that $F \cong L(\nu) \in \mathrm{Irr}(M|\Gamma)$. Lemma 2.8 implies that $\mathrm{wdeg}\, F \geq n-4$.

Assume that $a_5 = p-1$. Put $w = X_{-5}X_{-4}v^+$. Then $w \neq 0$. To prove this, observe that

$$X_4 w = X_4 X_{-5} X_{-4} v^+ = c X_{-5} v^+ \neq 0,$$

where c is a nonzero constant. Applying Lemma 2.3 and arguing as in Item I.2, we conclude that w is primitive with respect to H. Set $\kappa = \omega_H(w)$. Then κ is nonspecial since $\kappa|G_n(1,2,3) = \mu|G_n(1,2,3) = \omega_1 + \omega_3$ or $2\omega_3$ and $b_6(\kappa) \neq 0$.

3.) Suppose that $\lambda' \in \{0, \omega_5, \omega_n\}$. We may assume that $\lambda \notin \{0, \omega_1, \omega_3\}$ since we have already considered another case. Let $\lambda' = 0$. Set $u = v(n-1, 4, 1)$ if $a_3 < p-1$, and $u = v(n-1, 3, 1)$ otherwise. Then $\nu = a_1\omega_1 + a_2\omega_2 + (a_3+1)\omega_3 + \omega_{n-1}$ in the first case and $a_1\omega_1 + (a_2+1)\omega_2 + (p-2)\omega_3 + 2\omega_{n-1}$ in the second one. Lemmas 2.8 and 2.1 imply that in both cases $\mathrm{wdeg}\, L(\mu) \geq n-4$. Using the arguments on the restriction $M|\Gamma$ at the beginning of the proof, we deduce that $\mathrm{wdeg}\, M \geq n-4$.

Now assume that $\lambda' \neq 0$. Take $j = 4$ and $d = 1$. Then $b_5(\mu) = 2$ or $b_5(\mu) = 1$ and $0 < b_n(\mu) < p$. Therefore $\mu|H_2 \notin \Omega(H_2)$. One has $b_3(\mu) = a_3 + 1$. Due to our assumptions, if $a_3 = 0$, then $a_1 \neq 0$ or $a_2 \neq 0$. This implies that μ is nonspecial if $a_3 \neq p-1$.

Let $a_3 = p-1$. Put $w = X_{-3}X_{-4}v^+$ and $\kappa = \omega_H(w)$. Arguing as in case 2 for $a_5 = p-1$, one can show that $w \neq 0$ and is primitive with respect to H. Since $b_2(\kappa) \neq 0$ and $\kappa|H_2 = \mu|H_2$, the weight κ is nonspecial.

III) Finally, let $a_4 = 0$. Then $\lambda \neq 0$ if $\lambda' \in \{0, \omega_n\}$ since $\omega \notin \Omega_p$.

1.) Suppose that $\lambda' = 0$. Choose maximal i for which $a_i \neq 0$. Naturally, $i \leq 3$. For $a_i > 1$ put $u = v(n-1, i, 1)$. Then $\nu = \ldots + (a_i - 1)\omega_i + \omega_{n-1}$. Lemma 2.8 implies that $\mathrm{wdeg}\, L(\nu) \geq n-4$. It remains to apply the arguments from the beginning of the proof concerned with the restriction $M|\Gamma$.

Now let $a_i = 1$. As $\omega \neq \omega_i$, there exists $k < i$ with $a_k \neq 0$. Choose maximal such k. Set $u = v(n-1, k, 1)$. Then $b_{i-1}(\nu) = 1$ if $k = 1$ and $i = 3$, and $b_{i-1}(\nu) = a_k$ otherwise. In both cases $b_{n-1}(\nu) = 1$. Applying Lemma 2.8 and arguing as for $a_i > 1$, we can conclude that $\operatorname{wdeg} M > n - 4$.

2.) Let $\lambda' = \omega_5$. For $\omega = \omega_5$ put $j = 5$ and $d = 1$. Then $b_3(\mu) = b_6(\mu) = \langle \mu, \beta \rangle = 1$. Hence μ is nonspecial.

Now assume that $\lambda \neq 0$. Fix maximal $i < 4$ with $a_i > 0$. If $i = 1$ or 2, define the vector m as for $\omega = \omega_5$. Then $b_3(\mu) = b_6(\mu) = 1$ and $0 < b_i(\mu) < p$. Hence μ is nonspecial. For $i = 3$ set $j = 3$ and $d = 1$. Then μ is nonspecial since $b_2(\mu) \neq 0$ and $b_5(\mu) = 2$.

3.) Suppose that $\lambda' = \omega_n$. Fix minimal i with $a_i \neq 0$. It is clear that $i < 4$. If $i = 3$, argue as in the previous paragraph for $i = 3$ taking into account that now $b_5(\mu) = b_n(\mu) = 1$. Let $i < 3$. Put $w = X_{-n}v^+$ and $\Gamma_1 = G_n(1, \ldots, n-1)$. Then $w \neq 0$ by Lemma 2.2 and is primitive for Γ_1. Set $\gamma = \omega_{\Gamma_1}(w)$. We have $\gamma = a_i\omega_i + \ldots + \omega_{n-2}$. Naturally, $L(\gamma) \in \operatorname{Irr}(M|\Gamma_1)$. Applying Proposition 2.7 and Lemma 2.8, we get $\operatorname{wdeg} M \geq n - 2 - i \geq n - 4$.

4.) Let $\lambda = 0$ and $\lambda' \notin \{0, \omega_5, \omega_n\}$. Fix minimal i with $a_i \neq 0$. If $i = n$, then $a_n \neq 1$ and hence the weight ω_H is nonspecial.

Now let $i < n - 1$. Recall that $i > 4$. First assume that $a_i > 1$. If $i \leq n - 2$, set $j = i$ and $d = 2$. Then $b_3(\mu) = 2$ and $\mu|H_1 \notin \Omega(H_1)$. The weight $\mu|H_2 \notin \Omega(H_2)$ as well. For $i \leq n - 3$ this follows from the fact that $b_{i+1}(\mu) \neq 0$. For $i = n - 2$ we have

$$2 \leq b_{n-1}(\mu), b_n(\mu) \leq p + 1$$

and hence $\mu|H_2 \notin \Omega(H_2)$.

Next, let $a_i = 1$. If $\omega = \omega_i$, our assumptions yield that $i < n - 1$. Then the weight ω_H is nonspecial since $\langle \omega, \beta \rangle = 2$. Next, suppose that $\omega \neq \omega_i$.

Let $\omega = \omega_i + a_n\omega_n$ with $a_n \neq 0$. Set $w = X_{-4,2} \ldots X_{-i,2}X_{-(i+1)} \ldots X_{-(n-2)}X_{-n}v^+$ for $i < n - 2$ and $w = X_{-4,2} \ldots X_{-i,2}X_{-n}v^+$ for $i = n - 2$. Applying Lemma 2.6 to the Γ-module $K\Gamma v^+$, one can check that $w \neq 0$ and is fixed by $\mathfrak{X}_l$ for $l \neq 4, n-1$. Naturally, $\mathfrak{X}_\beta$ and $\mathfrak{X}_{n-1}$ fix w. Hence w is primitive for H. Put $\delta = \omega(w)$. Observe that $b_3(\delta) = 2$, $0 < b_{i+1}(\delta) < p$ and that $b_n(\delta) = a_n$ if $i = n - 2$. This yields that $\delta|H$ is nonspecial.

Now assume that $a_k \neq 0$ for some k with $i < k < n$. Fix minimal such k. Set $j = k$ and $d = 1$. Then $b_3(\mu) = 2$, $b_{i+1}(\mu) > 0$, and for $i = n - 2$ the coefficient $b_{n-1}(\mu) = a_{n-1}$ and $b_n(\mu) = a_n + 2$. Hence μ is nonspecial unless $\omega = \omega_{n-2} + \omega_{n-1} + (p-2)\omega_n$. In this exceptional case define the vector w and the weight δ as for $\omega = \omega_{n-2} + a_n\omega_n$. Arguing as above, conclude that w is primitive for H. We have $b_3(\delta) = 2$, $b_{n-1}(\delta) = 3$, and $b_n(\delta) = p - 2$. Hence $\delta|H$ is nonspecial if $p > 3$. For $p = 3$ put $t = X_{-(n-1)}w$. Using Lemma 2.2, we get

$$X_n X_{n-2}^2 \ldots X_4^2 t = cX_{-(n-1)}v^+ \neq 0,$$

where c is a nonzero constant. Hence $t \neq 0$. Applying Lemma 2.3 and arguing as in Item I.2, we deduce that t is primitive for H. Here it is essential that the $G_n(n-1)$-module $KG_n(n-1)w \cong V(3)$. Set $\tau = \omega(t)$. Then $\tau|H$ is nonspecial since $b_3(\tau) = 2$ and $b_{n-2}(\tau) > 0$. This completes the proof for $i < n - 1$.

Let $i = n - 1$. Recall that by our assumptions $a_n \neq 0$. Set

$$t = X_{-4,2} \ldots X_{-(n-2),2}X_{-(n-1)}X_{-n}v^+.$$

Since $p > 2$, several applications of Lemma 2.2 yield that $t \neq 0$. We want to show that $\mathfrak{X}_k$ fixes t for $k \neq 4$. Naturally this holds for $k < 4$. So suppose that $k > 4$. By Formula (1), it suffices to prove that $X_k t = 0$ for such k. First let $4 < k \leq n - 2$. Put

$$t_1 = X_{-(k-1),2} \ldots X_{-(n-2),2}X_{-(n-1)}X_{-n}v^+$$

and $\tau = \omega(X_k t_1)$. We have $b_{k-1}(\tau) = -3$. Hence the weight $\tau' = \tau + 3\alpha_{k-1}$ lies in the same orbit with τ under the action of the Weyl group of G_n. One easily observes that $\tau' \notin \Lambda(M)$. Therefore $\tau \notin \Lambda(M)$ and $X_k t_1 = 0$. So $X_k t = X_{-4,2} \dots X_{-(k-2),2} X_k t_1 = 0$ (for $k = 5$ we have $t_1 = t$). If $k = n-1$ or n, set

$$t_1 = X_{-(n-2),2} X_{-(n-1)} X_{-n} v^+.$$

One easily observes that $\omega(t_1) + \alpha_k \notin \Lambda(M)$. Hence $X_k t = X_{-4,2} \dots X_{-(n-3),2} X_k t_1 = 0$. Therefore t is fixed by $\mathfrak{X}_k$ with $k \neq 4$. Since $\beta = \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, the group $\mathfrak{X}_\beta$ fixes t. This yields that t is primitive for H. Set $b_l = b_l(\omega(t))$ for $1 \leq l \leq n$. Then $b_3 = 2$, $b_{n-1} = a_{n-1}$, and $b_n = a_n$. Hence $\omega_H(t)$ is nonspecial. The case where $\lambda = 0$ is settled.

5.) Finally, suppose that $\lambda = \omega_1$ or ω_3 and $\lambda' \notin \{0, \omega_5, \omega_n\}$. Choose maximal i with $a_i > 0$. Observe that $i \geq 5$.

Recall that $i \neq n-1$ since $a_n \geq a_{n-1}$. If $i < n-1$, put $u = v(n-1, i, 1)$. Then $b_n(\nu) = 1$. Hence Proposition 2.7 and Lemma 2.8 yield that $\operatorname{wdeg} M \geq n-4$. If $i = n$, the same results imply that $\operatorname{wdeg} M \geq \operatorname{wdeg} L(\omega|\Gamma) \geq n-4$. The proposition is proved. $\square$

5 The case of symplectic groups

In this section $G_n = C_n(K)$, the proof of Theorem 1.1 for these groups is completed. Recall that $p > 7$. We emphasize that in Lemmas 5.1–5.3 below we do not assume that $n \geq 8$.

Lemma 5.1 *Let $n \geq 2$, and either $a_{n-1} + a_n = \frac{p-1}{2}$ with $a_{n-1} = 2$ or 3, or $a_{n-1} = 1$ and $a_n = \frac{p-5}{2}$. Then*

$$\dim M^{\omega - \alpha_{n-1} - \alpha_n} = 2.$$

Proof. Since $p > 7$, we have $a_n \neq 0$. Set $\Delta = G_n(n-1, n)$, $\omega' = \omega|\Delta$, $M_0 = L(\omega')$, $\mu = \omega - \alpha_{n-1} - \alpha_n$, and $\mu' = \mu|\Delta$. By Theorem 2.4, $\dim M^\mu = \dim M_0^{\mu'}$. It is proved in [11, 6.11], that $\dim M_0^{\mu'} = 2$. $\square$

Lemma 5.2 *Let $n \geq 3$, $\omega = a_{n-1}\omega_{n-1} + a_n\omega_n$ with $a_{n-1} + a_n = \frac{p-1}{2}$ and $a_{n-1} = 2$ or 3. Set $S = G_n(2, \dots, n)$. Then $M|S$ has a composition factor with highest weight $(a_{n-1}+1)\omega_{n-2} + (a_n - 1)\omega_{n-1}$.*

Proof. Denote $M^{\omega-\alpha_{n-1}-\alpha_n}$ by U. By Lemma 5.1, $\dim U = 2$. Since $\dim X_n U = 1$, there exists a nonzero vector $u \in U$ such that $X_n u = 0$. Set

$$w = X_{-1} \dots X_{-(n-2)} u.$$

It is clear that $G_n(1, \dots, n-2) \cong A_{n-2}(K)$ and for this subgroup the vector u generates an indecomposable and hence a simple module V with highest weight ω_{n-2}. By Lemma 2.2, $w \neq 0$. Analyzing the weight structure of V, one can see that w is primitive with respect to $G_n(2, \dots, n-2)$. Moreover,

$$X_n w = X_{-1} \dots X_{-(n-2)} X_n u = 0$$

and $X_{n-1} w = 0$ as $\omega - \alpha_1 - \ldots - \alpha_{n-2} - \alpha_n \notin \Lambda(M)$. Now Formula (1) implies that w is primitive with respect to S. Hence $M|S$ has a composition factor with highest weight $\omega_S(w)$. One can see that $\omega_S(w) = (a_{n-1}+1)\omega_{n-2} + (a_n - 1)\omega_{n-1}$. $\square$

Lemma 5.3 *Let $\omega = a_{n-1}\omega_{n-1} + a_n\omega_n$ with $a_{n-1} + a_n = \frac{p-1}{2}$ and $a_{n-1} \geq 2$. Assume that $n = 4k$ with $a_{n-1} \geq 4$ or $n = 4k+2$. Then* $\operatorname{wdeg} M \geq 2^{k+1}$.

Proof. First let $k = 0$. Then $\omega \notin \Omega(G_n)$. Hence $\operatorname{wdeg} M \geq 2$. Now suppose that $k > 0$. Apply induction on n assuming that the lemma holds for the groups $C_r(K)$ with even $r < n$.

Set $S_1 = G_n(2, \ldots, n)$, $S = G_n(3, \ldots, n)$, $\Delta = G_n(\alpha_1, 2\varepsilon_2)$, and $\Delta = \Delta_1 \times S$. We have $\Delta_1 \cong C_2(K)$, $S_1 \cong C_{n-1}(K)$, and $S \cong C_{n-2}(K)$. Since the rank of Δ is equal to n, arguing as for H in Section 3, one can conclude that $\operatorname{wdeg} M \geq \operatorname{wdeg} F$ for each factor $F \in \operatorname{Irr}(M|\Delta)$.

Let $n = 4k$ and $a_{n-1} \geq 4$. Put $v = v(2, n-1, 2)$. By Lemma 2.6, $v \neq 0$ and is fixed by $\mathfrak{X}_i$ for $i \neq 2$. Furthermore, since $2\varepsilon_2 = 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$, this vector is fixed by $\mathfrak{X}_{2\varepsilon_2}$. Hence v is primitive for Δ. Set $\lambda = \omega(v)$. One can deduce that $b_1(\lambda) = 2$, $b_{n-1}(\lambda) = a_{n-1} - 2 \geq 2$, $b_n(\lambda) = a_n + 2$, and $\langle \lambda, 2\varepsilon_2 \rangle = \frac{p-5}{2}$; if $n > 4$, we have $b_i(\lambda) = 0$ for $3 \leq i \leq n-2$. Hence $\lambda|\Delta_1 \notin \Omega(\Delta_1)$ and therefore $\operatorname{wdeg} L(\lambda|\Delta_1) \geq 2$. The induction hypothesis yields that $\operatorname{wdeg} L(\lambda|S) \geq 2^k$. So $\operatorname{wdeg} M \geq \operatorname{wdeg} L(\lambda|\Delta) \geq 2^{k+1}$.

Now let $n = 4k+2$. We claim that $M|S$ has a composition factor M_S with highest weight $a\omega_{n-3} + b\omega_{n-2}$ where $a \geq 4$ and $a + b = (p-1)/2$. Then the induction hypothesis implies that $\operatorname{wdeg} M_S \geq 2^{k+1}$. By Proposition 2.7, in this case $\operatorname{wdeg} M \geq 2^{k+1}$. If $a_{n-1} > 3$, apply Theorem 2.4 to get a required factor.

Let $a_{n-1} = 3$. Then Lemma 5.2 yields that the module $M' = L(4\omega_{n-2} + \frac{p-9}{2}\omega_{n-1}) \in \operatorname{Irr}(M|S_1)$. Applying Theorem 2.4 to M', we obtain a required factor in $\operatorname{Irr}(M|S)$.

Finally, let $a_{n-1} = 2$. Naturally, $\operatorname{Irr}(L|S) \subset \operatorname{Irr}(M|S)$ for each $L \in \operatorname{Irr}(M|S_1)$. Two applications of Lemma 5.2 imply that $L = L(3\omega_{n-2} + \frac{p-7}{2}\omega_{n-1}) \in \operatorname{Irr}(M|S_1)$ and $L(4\omega_{n-3} + \frac{p-9}{2}\omega_{n-2}) \in \operatorname{Irr}(M|S)$. This completes the proof. □

Proposition 5.4 *Let $n \geq 8$, $p > 7$, $n \equiv 0 (\operatorname{mod} 4)$ and $G_n = C_n(K)$. Assume that $M \in \operatorname{Irr}_p G_n$ and $\omega(M) \notin \Omega_p(G_n)$. Then* $\operatorname{wdeg} M \geq n - 4$.

Proof. The proof is similar to that of Proposition 4.1. Put $\Gamma = G_n(1, \ldots, n-1)$. Recall that $H = H_1 \times H_2$ where

$$\begin{aligned} H_1 &= G_n(\alpha_1, \alpha_2, \alpha_3, \beta) \cong C_4(K), \\ H_2 &= G_n(5, \ldots, n) \cong C_{n-4}(K), \\ \beta &= 2\varepsilon_4 = 2\alpha_4 + \ldots + 2\alpha_{n-1} + \alpha_n. \end{aligned}$$

We use the same notation m, j, d, and μ as in the proof of Proposition 4.1, but now we assume that $j < n$, so $m = v(4, j, d)$ with $d \leq a_j$ and the vector m is primitive for H. Put $u = v(n, k, l) \neq 0$ for $1 \leq k \leq n$, $l \leq a_k$. In all cases we shall construct a composition factor $F \in \operatorname{Irr}(M|H)$ or $\operatorname{Irr}(M|\Gamma)$ such that $\operatorname{wdeg} F \geq n - 4$. As in Proposition 4.1, first we consider the cases where $\omega = a_i\omega_i$ with $1 \leq i \leq 4$. Here the arguments are similar to those of Proposition 4.1 with $u = v(n, i, a_i)$ for $2 \leq i \leq 4$ and $u = v(n, 1, 1)$ for $i = 1$.

Now assume that $\omega \neq a_i\omega_i$ with $1 \leq i \leq 4$. Define λ and λ' as in Proposition 4.1. Then $\omega = \lambda + a_4\omega_4 + \lambda'$ and ω_H is nonspecial unless

$$\lambda \in \{0, \omega_1, \omega_3\} \text{ or } \lambda' \in \{0, \omega_5, \omega_{n-1} + \frac{p-3}{2}\omega_n, \frac{p-1}{2}\omega_n\}.$$

If either $a_4 \neq 0$, or $\lambda' = 0$, or $\lambda' = \omega_5$, argue as in the proof of Proposition 4.1, in each case replacing the vector $u = v(n-1, k, l)$ by $u = v(n, k, l)$. Observe that for $\omega = \omega_5$ and $m = v(4, 5, 1)$ the weight $\mu = \omega_H(m)$ is nonspecial since $b_3(\mu) = b_6(\mu) = 1$, $\langle \mu, \beta \rangle = 0$, and $p > 3$.

Now assume that $a_4 = 0$ and $\lambda' \neq 0$ or ω_5.

1.) Suppose that $\lambda' = \omega_{n-1} + \frac{p-3}{2}\omega_n$ or $\frac{p-1}{2}\omega_n$. Then $\lambda \neq 0$ since $\omega \notin \Omega_p$. Fix minimal i with $a_i \neq 0$. Obviously, $i \leq 3$. Put $w = X_{-n}v^+$. Then the vector $w \neq 0$ by Lemma 2.2 and is primitive for Γ. Naturally, $M|\Gamma$ has a composition factor with highest weight $\omega_\Gamma(w)$. We have $\omega_\Gamma(w) = a_i\omega_i + \ldots + c\omega_{n-1}$ with $c = 2$ or 3. Applying Proposition 2.7 and Lemma 2.8, we get $\operatorname{wdeg} M \geq n-1-i \geq n-4$.

2.) Let $\lambda = 0$ and $\lambda' \notin \{0, \omega_5, \frac{p-1}{2}\omega_n, \omega_{n-1} + \frac{p-3}{2}\omega_n\}$. Fix minimal i with $a_i \neq 0$. Then $i > 4$. If $i = n$, then $a_n \neq \frac{p-1}{2}$ and hence the weight ω_H is nonspecial.

Now suppose that $i < n$. First assume that $a_i > 1$. Let $i < n-1$. Set $j = i$ and $d = 2$. Then $b_3(\mu) = 2$, $b_i(\mu) = a_i - 2$, and $b_{i+1}(\mu) = a_{i+1} + 2$. Now one concludes that μ is nonspecial. This is quite clear for $i \leq n-3$ or $i = n-2$ and $a_i > 2$. Let $i = n-2$ and $a_{n-2} = 2$. Observe that $b_{n-1}(\mu) = a_{n-1} + 2$, consider the cases $a_{n-1} < p-2$ and $a_{n-1} \geq p-2$ separately, and take into account that $p\omega_{n-1} \notin \Omega(G_n)$ for $p > 3$. Hence μ is nonspecial in this situation as well.

Next, assume that $i = n-1$. If $a_{n-1} > 3$, construct m as before. Then $b_3(\mu) = 2$ and $2 < b_{n-1}(\mu) < p$. Therefore μ is nonspecial. Let $a_{n-1} = 2$ or 3. Observe that $\langle \omega, \beta \rangle = a_{n-1} + a_n$. As $a_{n-1} + a_n < 2p$ and $p > 3$, the weight $\omega|H_1 \in \Omega(H_1)$ if and only if $a_{n-1} + a_n = \frac{p-1}{2}$. It is clear that $\omega|H_2 \notin \Omega(H_2)$. Hence ω_H is nonspecial if $a_{n-1} + a_n \neq \frac{p-1}{2}$.

Now suppose that $a_{n-1} = 2$ or 3 and $a_{n-1} + a_n = \frac{p-1}{2}$. Put $S = G_n(3, \ldots, n)$ and $M_S = KSv^+$. By Theorem 2.4, the S-module $M_S \cong L(\omega|S)$. We have $n = 4k$ with $k \geq 2$ and $n - 2 = 4(k-1) + 2$. By Lemma 5.3 and Proposition 2.7, $\operatorname{wdeg} M \geq \operatorname{wdeg} M_S \geq 2^k \geq 4k - 4$.

Finally, let $a_i = 1$. If $\omega = \omega_i$, we can assume that $i > 5$ as the opposite case has been considered already. Since $p > 3$ and $\langle \omega, \beta \rangle = 1$, the weight ω_H is nonspecial.

Now suppose that $\omega = \omega_i + a_n\omega_n$ with $a_n \neq 0$. Then $\langle \omega, \beta \rangle = a_n + 1 \leq p$. Since $p > 3$, the weight $\omega|H$ is nonspecial if $a_n + 1 \neq \frac{p-1}{2}$.

Now let $a_n + 1 = \frac{p-1}{2}$. Then $i \neq n-1$ since $\omega \notin \Omega(G_n)$. Therefore $\omega = \omega_i + \frac{p-3}{2}\omega_n$ with $4 < i < n-1$. Applying Lemma 2.2 several times, we deduce that the vector

$$w = X_{-4}^2 \ldots X_{-i}^2 X_{-(i+1)} \ldots X_{-n}v^+ \neq 0.$$

Formula (1) implies that $\mathfrak{X}_\beta$ fixes w. We claim that for $k \neq 4$ the subgroup $\mathfrak{X}_k$ fixes w as well. This is obvious for $k < 4$. For $4 < k \leq i+1$ set

$$w_k = X_{-(k-1)}^2 \ldots X_{-i}^2 X_{-(i+1)} \ldots X_{-n}v^+,$$

$w'_k = X_k w_k$, and $\gamma = \omega(w'_k)$. Obviously, $w_5 = w$. For $k > 5$ we have

$$X_k w = X_{-4}^2 \ldots X_{-(k-2)}^2 w'_k.$$

Observe that $b_{k-1}(\gamma) = -3$. Hence the weight $\gamma' = \gamma + 3\alpha_{k-1}$ lies in the same orbit with γ with respect to the action of the Weyl group of G_n. Obviously, $\gamma' \notin \Lambda(M)$. Hence $\gamma \notin \Lambda(M)$ and $X_k w = 0$. Therefore $\mathfrak{X}_k$ fixes w by Formula (1). For $i+1 < k \leq n$ the arguments are similar. Here we take $w_k = X_{-(k-1)} \ldots X_{-n}v^+$ and observe that $b_{k-1}(\gamma) = -2$ and hence γ lies in the same orbit with $\gamma + 2\alpha_{k-1}$ with respect to the action of the Weyl group of G_n and does not belong to $\Lambda(M)$. Hence $\mathfrak{X}_k$ fixes w for $k \neq 4$. Therefore w is primitive for H. Set $\tau = \omega_H(w)$. We have $b_3(\tau) = 2$, $b_{i+1}(\tau) = 1$ for $i < n-2$, and $b_{n-1}(\tau) = 2$ for $i = n-2$. So in all cases τ is nonspecial.

Now assume that $a_l \neq 0$ for some l with $i < l < n$. Fix minimal such l. For $i < n-2$ or $a_{n-1} > 1$, set $j = l$ and $d = 1$. Then $b_3(\mu) = 2$ and $b_{i+1}(\mu) \neq 0$. Furthermore, $b_{n-1}(\mu) = a_{n-1}$ for $i = n-2$. Hence μ is nonspecial.

Next, let $\omega = \omega_{n-2} + \omega_{n-1} + a_n\omega_n$. If $a_n \neq \frac{p-5}{2}$, take m as above and conclude that μ is nonspecial since $b_3(\mu) = 2$, $b_{n-1}(\mu) = 1$, and $b_n(\mu) = a_n + 1$. Here it is essential that $p > 3$.

Now suppose that $\omega = \omega_{n-2} + \omega_{n-1} + \frac{p-5}{2}\omega_n$. Set $\tau = \omega - \alpha_{n-1} - \alpha_n$. By Lemma 5.1, $\dim M^\tau = 2$. Hence there exists a nonzero vector $v \in M^\tau$ such that $X_n v = 0$. Set $w = X_{-4}^2 \dots X_{-(n-2)}^2 v$. Applying Lemma 2.2 and arguing as for $\omega = \omega_i + \frac{p-3}{2}\omega_n$, one can deduce that $w \neq 0$ and is primitive for H. Set $\delta = \omega_H(w)$. Then $b_3(\delta) = 2$ and $b_{n-1}(\delta) = 3$. This completes the analysis of the case where $\lambda = 0$.

3.) Finally, suppose that $\lambda = \omega_1$ or ω_3 and $\lambda' \notin \{0, \omega_5, \frac{p-1}{2}\omega_n, \omega_{n-1} + \frac{p-3}{2}\omega_n\}$. If $a_{n-1} \neq 0$, put $v = v^+$. Otherwise choose maximal i with $a_i > 0$ and set $v = v(n, i, 1)$. Put $\rho = \omega_\Gamma(v)$. In all cases $0 < b_{n-1}(\rho) < p$. Lemma 2.6 implies that $v \neq 0$ and is primitive for Γ. Therefore $L(\rho) \in \mathrm{Irr}(M|\Gamma)$. Then Proposition 2.7 and Lemma 2.8 yield that $\mathrm{wdeg}\, M \geq n - 4$. The proposition is proved. □

This completes the proof of Theorem 1.1.

References

[1] Benkart, G., Britten, D., Lemire, F. Modules with bounded weight multiplicities for simple Lie algebras, *Math. Z.* **225** (1997), no. 2, 333–353.

[2] Borel, A. Properties and linear representations of Chevalley groups. In *Seminar on algebraic groups and related finite groups (eds. A. Borel et al.)*; Lecture Notes in Mathematics **131**, Springer, Berlin, 1970, 1–55.

[3] Bourbaki, N. *Groupes et algèbres de Lie, Chaps. IV–VI*; Hermann: Paris, 1968.

[4] Bourbaki, N. *Groupes et algèbres de Lie, Chaps. VII–VIII*; Hermann: Paris, 1975.

[5] Carter R.W.; Cline, E. The submodule structure of Weyl modules for groups of type A_1. In *Proceedings of the conference on finite groups (ed. W.R. Scott and F. Gross)*; Park City: Utah, 1975; Academic Press: New York/London, 1976, 303–311.

[6] Fernando, S.L. Lie algebra modules with finite-dimensional weight spaces. I, *Trans. Amer. Math. Soc.* **322** (1990), no. 2, 757–781.

[7] Jantzen, J.C. Darstellungen halbeinfacher algebraicher Gruppen und zugeordnetekontravariante Formen, *Bonner math. Schr.* **67** (1973).

[8] Jantzen, J.C. *Representations of Algebraic Groups, Second Edition*; American Mathematical Society: Providence, 2003.

[9] Kleshchev, A.S. On decomposition numbers and branching coefficients for symmetric and specil linear groups, *Proc. Lond. Math. Soc.* **75** (1997), 497–558.

[10] Mathieu, O. Classification of irreducible weight modules, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 2, 537–592.

[11] Seitz, G.M. The maximal subgroups of classical algebraic groups. *Memoirs of the AMS* **365** (1987), 1–286.

[12] Smith, S. Irreducible modules and parabolic subgroups. *J. Algebra* **75**, (1982), 286–289.

[13] Steinberg, R. Representations of algebraic groups. *Nagoya Math. J.* **22** (1963), 33–56.

[14] Suprunenko, I.D. On Jordan blocks of elements of order p in irreducible representations of classical groups with p-large highest weights. *J. Algebra* **191** (1997), 589–627.

[15] Zalesskii, A.E.; Suprunenko, I.D. Representations of dimensions $(p^n \pm 1)$ of a symplectic group of degree $2n$ over a finite field (in Russian). *Vestsi AN BSSR, Ser. Fiz.-Mat. Navuk*, no. 6 (1987), 9–15.