Simple Lie subalgebras of locally finite associative algebras

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Abstract

We prove that any simple Lie subalgebra of a locally finite associative algebra is either finite dimensional or isomorphic to the commutator algebra of the Lie algebra of skew symmetric elements of some involution simple locally finite associative algebra. The ground field is assumed to be algebraically closed of characteristic 0. This result can be viewed as a classification theorem for simple Lie algebras that can be embedded in locally finite associative algebras. We also establish a link between this class of Lie algebras and that of Lie algebras graded by finite root systems.

Key words: Locally finite Lie algebra, simple Lie algebra, involution

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1 Introduction

Throughout the paper we fix an algebraically closed field $\mathbb{F}$ of characteristic 0. We study simple infinite-dimensional Lie subalgebras of locally finite associative algebras. Our main result describes such algebras in terms of algebras of skew-symmetric elements in simple locally finite associative algebras with involution. Recall that an algebra A (associative, Lie, etc) is called locally finite if any finite set $M \subset A$ is contained in a finite-dimensional subalgebra. Locally finite algebras can be alternatively described as direct limits of finite-dimensional algebras.

A well-known theorem of Ado says that any finite-dimensional Lie algebra is isomorphic to a Lie subalgebra of a matrix algebra or, equivalently, to a Lie subalgebra of a finite-dimensional associative algebra. It would be wrong to think that this result remains true if one replaces "finite-dimensional" by "locally finite". In fact, for a Lie algebra the condition of being a subalgebra of a locally finite associative algebra is fairly restrictive. A step in understanding this phenomenon was made by the second author who proved in [5] the following "version of Ado's Theorem" for locally finite Lie algebras.

Theorem 1.1 *Let L be a simple infinite-dimensional Lie algebra. Then L is embeded in a locally finite associative algebra if and only if L is isomorphic to a* diagonal *direct limit of finite-dimensional Lie algebras (see definition below in Section 2).*

This theorem was conjectured by the third author who also introduced the term "diagonal direct limit" ([12]). Theorem 1.1 can be viewed as a local characterization of simple Lie subalgebras of locally finite associative algebras because the diagonal direct limit is defined in terms of embeddings of finite dimensional subalgebras.This definition mimics certain properties of embeddings of finite dimensional associative algebras. Our aim in this paper is to obtain a global characterization, without any reference to finite-dimensional subalgebras.

At this point we turn our reader's attention to Lie subalgebras of associative algebras with involution. Let A be an associative algebra over a field $\mathbb{F}$ which for this paragraph can be arbitrary of characteristic different from 2. Suppose that A has an involution (which will be always denoted by "$*$"), that is, a linear transformation of A such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. Then the set $\mathfrak{u}^*(A) = \{a \in A \mid a^* = -a\}$ of all skew-symmetric elements of A is a Lie subalgebra of A. Let $\mathfrak{su}^*(A) = [\mathfrak{u}^*(A), \mathfrak{u}^*(A)]$ denote the commutator subalgebra of $\mathfrak{u}^*(A)$. It is well-known (see [11]) that if A is involution simple and the dimension of A is greater than 16, then $\mathfrak{su}^*(A)$ is an extension of a simple Lie algebra by an ideal contained in the center of A. This construction yields classical finite-dimensional Lie algebras of types B_n, C_n, and D_n, when A is a matrix algebra over $\mathbb{F}$ with appropriate involution (in this case, already $\mathfrak{u}^*(A)$ is simple). One can adapt the construction to obtain simple Lie algebras of type A_n. Let B be a simple associative over $\mathbb{F}$ with center $Z(B)$. Then $L = [B,B]/([B,B] \cap Z(B))$ is a simple Lie algebra. However, taking $B \oplus B^{op}$ for A and endowing A with the involution $(a,b)^* = (b,a)$ for $a,b \in B$, one has $L \cong \mathfrak{su}^*(A)/(\mathfrak{su}^*(A) \cap Z(A))$. Finite dimensional simple Lie algebras of exceptional types or simple Lie algebras of Cartan type do not appear in this way.

Recall that the ground field is assumed to be algebraically closed and of characteristic 0. One can show that each locally finite Lie algebra constructed as $[A,A]$ or $\mathfrak{su}^*(A)$ for a (involution) simple locally finite associative algebra A is simple and diagonal (see Theorem 2.12). It follows from one of our main results (see Theorem 1.3(2) below) that the converse statement is also true. Combining this with Theorem 1.1, we get the following characterization of simple Lie algebras that are embeddable into locally finite associative algebras.

Theorem 1.2 *Let L be a simple infinite-dimensional Lie algebra. Then L can be embedded into a locally finite associative algebra if and only if L is isomorphic to $\mathfrak{su}^*(A)$ where A is an involution simple locally finite associative algebra.*

It has to be emphasized that A cannot be in general obtained as the associative envelope of L in the original embedding. As in the finite-dimensional case, there are many "non-equivalent" embeddings of L into associative locally finite algebras. Another point to warn the reader is that simple locally finite associative algebras as well as their simple Lie subalgebras are not always expressible as direct limits of simple or even semisimple finite dimensional subalgebras. Some examples of this kind have been provided in Bahturin and Strade [4].

Theorem 1.2 considerably sharpens Theorem 1.1 and can be viewed as a classification theorem for simple Lie algebras that are embeddable into locally finite associative algebras.

We do not see any straightforward way to prove Theorem 1.2. We first use Theorem 1.1 to reformulate the problem in terms of diagonal embeddings of finite dimensional (but not necessarily semisimple) subalgebras L_i. Next we use the results of [8,9] that under certain conditions diagonal Lie subalgebras of a matrix algebra can be obtained as $\mathfrak{su}^*(A_i)$ where A_i is now the envelope of L_i. Finally, roughly speaking, A is obtained as a direct limit of A_i. Thus, the proof is heavily dependent on Theorem 1.1 and the papers [8,9] where the second and the third authors develop a theory of diagonal and plain representations of finite-dimensional Lie algebras (see all the definitions below).

Next we formulate in detail the main results of this paper.

Let A be an associative enveloping algebra of a Lie algebra L (i.e. L is a Lie subalgebra of A and A is generated by L as an associative algebra). We say that A is a $\mathfrak{P}$-*envelope* of L if $[A, A] = L$. We say that A is a $\mathfrak{P}^*$-*envelope* of L if A has an involution such that $\mathfrak{su}^*(A) = L$. Each enveloping algebra A of a Lie algebra L can be considered as a quotient of the augmentation ideal $A(L)$ (i.e. the ideal of codimension 1 of the universal enveloping algebra $U(L)$). Thus there is a 1–1 correspondence $A \to H_A$ between the enveloping algebras A for L and the ideals H_A in $A(L)$ such that $H_A \cap L = 0$ and $A(L)/H_A \cong A$. This gives a partial ordering on the set of enveloping algebras of L: we say that $A \leq B$ if and only if $H_A \supseteq H_B$. To formulate our main result we denote by $\operatorname{Rad} A$ the Jacobson radical of an associative algebra A.

Theorem 1.3 *Let L be an infinite-dimensional simple diagonal locally finite Lie algebra. Then there is a unique (universal) $\mathfrak{P}^*$-envelope $\mathcal{N}$ of L such that the following conditions hold.*

(1) Set $\mathcal{R} = \operatorname{Rad}\mathcal{N}$. Then $\mathcal{R}$ is the annihilator of $\mathcal{N}$.

(2) $\mathcal{M} = \mathcal{N}/\mathcal{R}$ is an involution simple $\mathfrak{P}^*$-envelope of L.

(3) For each $\mathfrak{P}^*$-envelope A of L one has $\mathcal{M} \leq A \leq \mathcal{N}$.

Corollary 1.4 *The mapping $L \mapsto \mathcal{M}$ is a 1-1 correspondence between the set of all (up to isomorphism) infinite-dimensional simple diagonal locally finite Lie algebras and the set of all (up to isomorphism) infinite-dimensional involution simple locally finite associative algebras.* (*The inverse mapping is given by $A \mapsto \mathfrak{su}^*(A)$*).

There is a special class of simple diagonal locally finite Lie algebras, which are called *plain* (see Section 2 for the definition). These algebras play a role similar to that of finite-dimensional simple Lie algebras of type A (see above).

Theorem 1.5 *Let L be an infinite-dimensional simple plain locally finite Lie algebra. Then there are two (universal) $\mathfrak{P}$-envelopes $\mathcal{N}_+$ and $\mathcal{N}_-$ of L such that the following conditions hold.*

(1) Put $\mathcal{R}_\pm = \operatorname{Rad}\mathcal{N}_\pm$. Then $\mathcal{R}_\pm$ is the annihilator of $\mathcal{N}_\pm$.

(2) $\mathcal{M}_\pm = \mathcal{N}_\pm/\mathcal{R}_\pm$ is a simple $\mathfrak{P}$-envelope of L.

(3) For each $\mathfrak{P}$-envelope A of L one has either $\mathcal{M}_+ \leq A \leq \mathcal{N}_+$ or $\mathcal{M}_- \leq A \leq \mathcal{N}_-$.

(4) The mapping $\alpha : \mathcal{N}_+ \to \mathcal{N}_-$ *defined as*

$$\alpha(x_1 \dots x_k) = (-x_k) \dots (-x_1), \quad (k \in \mathbb{N},\ x_1, \dots, x_k \in L)$$

is an antiisomorphism.

Corollary 1.6 *The mapping* $L \mapsto \mathcal{M}_+(L)$ *is a 1-1 correspondence between the set of all (up to isomorphism) infinite-dimensional simple plain locally finite Lie algebras and the set of all (up to isomorphism and antiisomorphism) infinite-dimensional simple locally finite associative algebras. (The inverse mapping is* $A \mapsto [A, A]$*).*

Remark 1.7 In Example 3.8 we construct a simple plain locally finite Lie algebra L such that $\dim(\operatorname{Rad}\mathcal{N}_\pm) = 1$. Considering the regular representation of $\mathcal{N}_\pm$, we conclude that there exists a non-split extension of a plain L-module V (i.e. the restriction $V{\downarrow}L_i$ is plain for all i) by the trivial one-dimensional module.

Since each simple plain locally finite Lie algebra is also diagonal, it has both $\mathfrak{P}$-envelope and $\mathfrak{P}^*$-envelope. The following theorem describes the relation between these algebras.

Theorem 1.8 *Let* L *be an infinite-dimensional simple diagonal locally finite Lie algebra. Let* $\mathcal{M}$ *be as in Theorem 1.3 and* $\mathcal{M}_\pm$ *as in Theorem 1.5, in the case where* L *is plain. Then* L *is plain if and only if* $\mathcal{M}$ *is not simple. In this case* $\mathcal{M}$ *decomposes into the direct sum of two ideals* $B_+ \oplus B_-$ *where* $B_\pm \cong \mathcal{M}_\pm$ *and* $B_+^* = B_-$*.*

In the final section we link the theory of diagonal locally finite Lie algebras to the theory of root-graded Lie algebras, developing further the results of Bahturin and Benkart [2]. Root-graded Lie algebras have been introduced by Berman and Moody for studying the toroidal algebras and Slodowy's intersection matrix algebras [10] (see also an important monograph [1] by Allison, Benkart, and Gao for more references). We show that each simple root-graded locally finite Lie algebra is diagonal and the converse is also true provided we slightly generalize the notion of root-graded Lie algebras (Theorem 4.3). Actually, we prove that any locally finite diagonal Lie algebra is BC_r-graded in the sense of [1] (which was also noticed in [2] in the case of diagonal locally finite-dimensional simple algebras of types B, C, D). From this result and the results of Allison-Benkart-Gao [1] one can obtain another proof of the fact that any simple diagonal locally finite Lie algebra can be obtained as a Lie subalgebra of skew symmetric elements of a suitable asssociative algebra.

The above results make sense for infinite-dimensional algebras only. Thus, if otherwise is not stated, each locally finite Lie algebra, considered in the paper, is assumed to be infinite-dimensional.

2 Notation and preliminaries

A Lie algebra L is called *perfect* if $L = [L, L]$. If L is a perfect finite-dimensional Lie algebra and V a finite dimensional L-module then $\operatorname{Irr} L$ (resp. $\overline{\operatorname{Irr}}\, L$) stands for the set of all isomorphism classes of irreducible (resp. nontrivial irreducible) finite-dimensional L-modules and by $\operatorname{Irr} V$ (resp. $\overline{\operatorname{Irr}}\, V$) the set of all isomorphism classes of composition factors (resp. nontrivial composition factors) of V. In particular, $\operatorname{Irr} L = \overline{\operatorname{Irr}}\, L \cup \{T_L\}$ and $\operatorname{Irr} V = \overline{\operatorname{Irr}}\, V$ or $\overline{\operatorname{Irr}}\, V \cup \{T_L\}$ where T_L is the trivial 1-dimensional L-module. We denote by $U(L)$ the universal enveloping algebra of L and by $A(L)$ its augmentation ideal, i.e. the ideal of codimension 1 generated by L. In this paper (as well as in [8,9]) we mainly work with $A(L)$ rather than with the whole of $U(L)$, and the notion of $\overline{\operatorname{Irr}}$ is sometimes more suitable for us than that of Irr, used in [5,6]. We need to prove some results from [5], stated in this new setting. To this end, the following lemma will be helpful.

Lemma 2.1 *Let L be a finite-dimensional perfect Lie algebra and V a finite-dimensional L-module. Then $T_L \in \operatorname{Irr} V$ if and only if $\operatorname{Ann}_{U(L)} V \subset A(L)$ (or equivalently, $\operatorname{Ann}_{U(L)} V = \operatorname{Ann}_{A(L)} V$).*

Proof. Assume $T_L \in \operatorname{Irr} V$. As $[L, L] = L$, the algebra L acts trivially on T_L. Therefore $\operatorname{Ann}_{U(L)} V \subset \operatorname{Ann}_{U(L)} T_L = A(L)$, as required.

Assume $T_L \notin \operatorname{Irr} V$. Let E be the image of $U(L)$ in $\operatorname{End} V$, in other words, $E = U(L)/\operatorname{Ann}_{U(L)} V$. Since L is perfect, the dimension of all composition factors of V is greater than 1. In particular, $E/\operatorname{Rad} E$ has no quotients of dimension 1. Since $\dim U(L)/A(L) = 1$, we have $\operatorname{Ann}_{U(L)} V \not\subset A(L)$. □

Denote by $\mathfrak{F}$ (resp. $\bar{\mathfrak{F}}$) the set of all (two-sided) ideals in $U(L)$ (resp. $A(L)$) of finite codimension. Clearly, each ideal of $A(L)$ is also an ideal of $U(L)$, so that $\bar{\mathfrak{F}} \subset \mathfrak{F}$. For any $X \in \mathfrak{F}$ the quotient $U(L)/X$ is a finite dimensional L-module under the left regular action, hence the notation $\operatorname{Irr}(U(L)/X)$ makes sense.

Lemma 2.2 $\bar{\mathfrak{F}} = \{X \in \mathfrak{F} \mid T_L \in \operatorname{Irr}(U(L)/X)\}$.

Proof. Let $X \in \bar{\mathfrak{F}}$. Then obviously $T_L \in \operatorname{Irr}(U(L)/X)$. Assume now that $X \in \mathfrak{F}$ and $T_L \in \operatorname{Irr}(U(L)/X)$. As $X = \operatorname{Ann}_{U(L)} U(L)/X$, by Lemma 2.1 one has $X \subset A(L)$, that is, $X \in \bar{\mathfrak{F}}$. □

Let Φ be a finite subset of $\operatorname{Irr} L$. Set $\bar{\Phi} = \Phi \cap \overline{\operatorname{Irr}}\, L$ and

$$\mathfrak{F}(\Phi) = \{X \in \mathfrak{F} \mid \operatorname{Irr}(U(L)/X) = \Phi\},$$
$$\bar{\mathfrak{F}}(\bar{\Phi}) = \{X \in \bar{\mathfrak{F}} \mid \overline{\operatorname{Irr}}(A(L)/X) = \bar{\Phi}\}.$$

Assume $T_L \in \Phi$. Then it follows from Lemma 2.2 that $\mathfrak{F}(\Phi) = \bar{\mathfrak{F}}(\bar{\Phi})$. The sets $\mathfrak{F}(\Phi)$ have been described in [5, Theorem 3.4]. Our argument immediately yields a similar description for $\bar{\mathfrak{F}}(\bar{\Phi})$.

Theorem 2.3 *Let L be a perfect finite-dimensional Lie algebra, Φ a finite subset of $\overline{\operatorname{Irr}}\, L$, and $\bar{\mathfrak{F}}(\Phi) = \{X \in \bar{\mathfrak{F}} \mid \overline{\operatorname{Irr}}(A(L)/X) = \Phi\}$. Then $\bar{\mathfrak{F}}(\Phi)$ is nonempty and has the smallest element $N(\Phi)$ and the largest element $M(\Phi)$, such that $N(\Phi) \subseteq X \subseteq M(\Phi)$ for all $X \in \bar{\mathfrak{F}}(\Phi)$. The algebra $A(L)/M(\Phi)$ is semisimple, while $M(\Phi)/N(\Phi)$ is nilpotent.*

Recall, in a slightly different form, that a set $\{L_i\}_{i\in I}$ of finite-dimensional subalgebras of a locally finite Lie algebra L is a *local system* of L if $L = \bigcup_{i\in I} L_i$ and for each pair $i, j \in I$ there exists $k \in I$ such that $L_i, L_j \subseteq L_k$. Set $i \leq j$ if $L_i \subseteq L_j$. Then I becomes a *directed set*, i.e. a partially ordered set such that for each pair $i, j \in I$ there exists $k \in I$ satisfying $i, j \leq k$. It is clear that L is the direct limit of the algebras L_i, that is, $L = \varinjlim L_i$. Assume that L is simple. Then by [3, Theorem 3.2], all L_i can be chosen perfect. We shall call such local systems *perfect*. Locally finite Lie algebras admitting perfect local systems are called *locally perfect*. We shall consider only perfect local systems for simple locally finite Lie algebras. So the notation $L = \varinjlim L_i$ always means that $\{L_i\}_{i\in I}$ is a perfect local system of L.

In order to define diagonal locally finite Lie algebras we first need the notion of a *diagonal module*. Let L be a finite-dimensional perfect Lie algebra such that $L/\operatorname{Rad} L = S_1 \oplus \cdots \oplus S_n$ is the sum of simple components S_i. We fix a Cartan subalgebra of each S_i and a base of the root system. Denote by V_i the *standard* S_i-module (i.e. the irreducible S_i-module of highest weight λ_1, which is the first fundamental weight with respect to the standard labeling). Note that our definition depends on the choice of a base of the root system. We can change it in such a way that the dual module V_i^* becomes standard. However, up to duality, V_i doesn't depend on the choice of a Cartan subalgebra and a root system of S_i if rank of S_i is not too small, which will be our typical situation. Indeed, in this case we do not need to consider the components of exceptional types, so each S_i can be always assumed classical, i.e it can be identified with $\mathfrak{sl}(V_i)$, $\mathfrak{o}(V_i)$, or $\mathfrak{sp}(V_i)$, and the standard S_i-module is precisely the unique (up to duality) nontrivial irreducible S_i-module of minimal dimension.

Each V_i can be considered as an L-module. The L-modules $V_1, \ldots, V_n$ are called *standard*. An L-module V is called *diagonal* if each nontrivial composition factor of V is either standard or dual to it. An L-module V is called *plain* if each S_i is of type A (i.e. $S_i \cong \mathfrak{sl}(V_i)$) and each nontrivial composition factor of V is standard. The definition of a plain module slightly differs from that in [8].

Remark 2.4 By changing the base of the root system (or by relabeling the simple roots) we can turn V_i^* into a standard L_i-module. This gives us some freedom in the choice of a standard module, which we are going to use in the future.

Assume we have another perfect Lie algebra L' containing L. Let $V'_1, \ldots, V'_k$ be the standard L'-modules. The embedding $L \subset L'$ is called *diagonal* (resp. *plain*) if the restriction of the direct sum $V'_1 \oplus \cdots \oplus V'_k$ to L is a *diagonal* (resp. *plain*) L-module. We illustrate this definition by the following example. An embedding $\mathfrak{sl}(V) \to \mathfrak{sl}(W)$ is diagonal if and only if one can choose a basis of W such that

$$A \mapsto \mathrm{diag}(\underbrace{A, \ldots, A}_{l}, \underbrace{-A^t, \ldots, -A^t}_{r}, \underbrace{0, \ldots, 0}_{z})$$

for any matrix $A \in \mathfrak{sl}(V)$ where l, r, z do not depend on A, $z + (l+r)\dim V = \dim W$. This embedding is plain if $r = 0$.

By the *rank* of a perfect finite-dimensional Lie algebra L we mean the smallest rank of the simple components of $L/\operatorname{Rad} L$. We need the following.

Lemma 2.5 *Let $L_1 \subseteq L_2 \subseteq L_3$ be three perfect finite-dimensional Lie algebras. Assume that the ranks of L_1 and L_3 are greater than* 10 *and the embedding $L_1 \subseteq L_3$ is diagonal. Then the embedding $L_1 \subseteq L_2$ is diagonal. Moreover, if the restriction of each standard L_2-module to L_1 is nontrivial, then both embeddings $L_1 \subseteq L_2$ and $L_2 \subseteq L_3$ are diagonal.*

Proof. By choosing Levi subalgebras of the L_i, each embedded into another, one can assume that the algebras L_i are semisimple. Moreover, replacing L_2 by the ideal of L_2 generated by L_1, one can assume that the restriction of each standard L_2-module to L_1 is nontrivial. It suffices to show that for each standard L_3-module W the restriction $W{\downarrow}L_2$ is diagonal. Indeed, in that case all standard L_2-modules can obtained as composition factors of such restrictions, so their restrictions to L_1 are diagonal. Let $M \in \overline{\mathrm{Irr}}(W{\downarrow}L_2)$. The module M can be represented in the form $M = M_1 \otimes \cdots \otimes M_k$ where M_i is a nontrivial irreducible module for a simple component S_i of L_2. As the restriction of each standard L_2-module to L_1 is nontrivial and $M{\downarrow}L_1$ is diagonal, $k = 1$. It remains to note that $M = M_1$ can not be non-standard (see [6, Lemma 5.2]). □

Let L be a locally finite Lie algebra and $\mathfrak{L} = \{L_i\}_{i \in I}$ a local system of L. We say that $\mathfrak{L}$ is *diagonal* (resp. *plain*) if it is perfect and for each pair $L_i \subseteq L_j$ the corresponding embedding is diagonal (resp. plain). Note that in [6, Definition 3.7] we use the term "pure diagonal" instead of "diagonal". A locally finite Lie algebra L is called *diagonal* (resp. *plain*) if it has a diagonal (resp. plain) local system.

Let $L = \varinjlim L_i$ be a locally perfect Lie algebra. Associated with the direct limit, there is a local system $\mathfrak{L} = \{L_i\}_{i \in I}$ of perfect subalgebras in L. Let Φ_i be a finite nonempty subset of $\operatorname{Irr} L_i$. The set $\Phi = \{\Phi_i\}_{i \in I}$ is called an *inductive system* (of representations) for L with respect to $\mathfrak{L}$ if

$$\bigcup_{\varphi \in \Phi_j} \operatorname{Irr}(\varphi{\downarrow}L_i) = \Phi_i,$$

for each pair $i < j$. If $\mathfrak{L}$ is fixed then we simply say that Φ is an inductive system for L. An inductive system $\Phi = \{\Phi_i\}_{i \in I}$ is called *diagonal* (resp. *plain*) if for each i, $\bigoplus_{\varphi \in \Phi_i} \varphi$ is a diagonal (resp. plain) L_i-module. We say that Φ is *selfdual*, if $\Phi = \Phi^* := \{\Phi_i^*\}_{i \in I}$ where $\Phi_i^* = \{\varphi^* \mid \varphi \in \Phi_i\}$. Otherwise, Φ is *non-selfdual*. If $\Phi_i = \{T_{L_i}\}$ for all $i \in I$, the system Φ is called *trivial*. Otherwise, Φ is called *nontrivial*. We shall denote by $\Phi \cup \{T_L\}$ the system $\{\Phi_i \cup \{T_{L_i}\}_{i \in I}\}$. More generally, let $\Phi = \{\Phi_i\}_{i \in I}$ and $\Psi = \{\Psi_i\}_{i \in I}$ be inductive systems. Then the union $\Phi \cup \Psi = \{\Phi_i \cup \Psi_i\}_{i \in I}$ is an inductive system as well.

As before, it is sometimes convenient to "forget" about trivial modules and to define a *reduced inductive system* by replacing $\operatorname{Irr} L$ by $\overline{\operatorname{Irr}}\, L$ and $\operatorname{Irr}(\cdot)$ by $\overline{\operatorname{Irr}}(\cdot)$ in the definition above. We denote by $\mathfrak{S}$ (resp. $\bar{\mathfrak{S}}$) the set of inductive systems (resp. reduced inductive systems) of L with respect to $\mathfrak{L}$. We have a mapping $\mathfrak{S} \to \bar{\mathfrak{S}}$ defined by

$$\Phi = \{\Phi_i\}_{i \in I} \mapsto \bar{\Phi} = \{\bar{\Phi}_i\}_{i \in I}. \tag{1}$$

One can easily check that this mapping is surjective (if $\Psi \in \bar{\mathfrak{S}}$, then $\Phi = \Psi \cup \{T_L\} \in \mathfrak{S}$). Denote by $\mathfrak{G}$ (resp. $\bar{\mathfrak{G}}$) the set of (two-sided) ideals of $U(L)$ (resp. $A(L)$) such that the corresponding factor algebra is locally finite. As before $\bar{\mathfrak{G}} \subset \mathfrak{G}$. Let $X \in \mathfrak{G}$. Then the set

$$\Phi(X) = \{\operatorname{Irr}(U(L_i)/X \cap U(L_i))\}_{i \in I}$$

is an inductive system for L ([5, Lemma 3.8]). Let $X \in \bar{\mathfrak{G}}$. Set

$$\bar{\Phi}(X) = \{\overline{\operatorname{Irr}}(A(L_i)/X \cap A(L_i))\}_{i \in I}.$$

Then $\bar{\Phi}(X)$ is a reduced inductive system for L and $\bar{\Phi}(X) = \overline{\Phi(X)}$ in the sense of (1).

Lemma 2.6 $\bar{\mathfrak{G}} = \{X \in \mathfrak{G} \mid T_{L_i} \in \Phi(X)_i,$ *for all* $i\}$.

Proof. Let X be an ideal of $U(L)$. Clearly $X \subset A(L)$ if and only if $X \cap U(L_i) \subset A(L_i)$ for all i. Thus the result follows from Lemma 2.2. □

Let Φ be an inductive system for L and Ψ be a reduced inductive system for L. Set

$$\mathfrak{G}(\Phi) = \{X \in \mathfrak{G} \mid \Phi(X) = \Phi\},$$
$$\bar{\mathfrak{G}}(\Psi) = \{X \in \bar{\mathfrak{G}} \mid \bar{\Phi}(X) = \Psi\}.$$

Assume $T_{L_i} \in \Phi_i$ for all i. Then it follows from Lemma 2.6 that, $\mathfrak{G}(\Phi) = \bar{\mathfrak{G}}(\bar{\Phi})$. Since the sets $\mathfrak{G}(\Phi)$ have been described in [5, Theorem 3.9], we immediately derive a similar description for $\bar{\mathfrak{G}}(\Psi)$, as follows. The same result can also be directly deduced from Theorem 2.3.

Theorem 2.7 *Let $f : \bar{\mathfrak{G}} \to \bar{\mathfrak{S}}$ be a mapping defined by $f(X) = \bar{\Phi}(X)$. Then for each reduced inductive system Φ the set $\bar{\mathfrak{G}}(\Phi)$ is nonempty and has the smallest element $N(\Phi)$ and the largest element $M(\Phi)$ such that $N(\Phi) \subseteq X \subseteq M(\Phi)$ for each $X \in \bar{\mathfrak{G}}(\Phi)$. The algebra $A(L)/M(\Phi)$ is semisimple, while $M(\Phi)/N(\Phi)$ is locally nilpotent. Moreover, the mapping f produces a 1-1 correspondence between the set of semiprimitive ideals in $\bar{\mathfrak{G}}$ and the set of reduced inductive systems for L (the inverse mapping is given by $\Phi \mapsto M(\Phi)$).*

To proceed further, we need few facts about simple (and involution simple) locally finite associative algebras. They are similar to those for locally finite Lie algebras and are proved in a similar way. Recall that an associative algebra with involution "$*$" is called *involution simple* (or $*$*-simple*) if it has no $*$-invariant ideals. The following trivial observation is often helpful for studying involution simple algebras.

Proposition 2.8 *Let A be an associative algebras with involution. Assume that A is involution simple. Then either A is simple or $A = B \oplus B^*$ where B is a simple ideal of A.*

Proof. Assume that A is not simple and let B be a non-zero proper ideal of A. Then B^* is an ideal of A and $B+B^*$ is a $*$-invariant ideal of A, so $B+B^* = A$. Let $C = B \cap B^*$. Then C is a proper $*$-invariant ideal of A, so $C = 0$. Therefore $A = B \oplus B^*$. □

Let A be an associative algebra. We denote by A^n the linear span of all products $x_1 \cdots x_n$, $x_i \in A$, and call A *perfect* if $A^2 = A$. Assume that A has an involution. Then, if A is finite-dimensional, it is well-known (see for example [9, Lemma 2.4]) that A has a $*$-invariant Levi subalgebra Q. Let A be a locally finite associative algebra with involution and $\mathfrak{A} = \{A_i\}_{i \in I}$ a local system of A. Let $\hat{A}_i$ be the subalgebra of A generated by $A_i + A_i^*$. Then $\hat{\mathfrak{A}} = \{\hat{A}_i\}_{i \in I}$ is a $*$-invariant local system of A (i.e. $\hat{A}_i^* = \hat{A}_i$ for all $i \in I$). Moreover $\hat{\mathfrak{A}}$ is perfect if $\mathfrak{A}$ is perfect.

Now we define a *conical* local system $\mathfrak{A} = (A_i)_{i \in I}$ of ($*$-invariant) perfect subalgebras of an (involution) simple associative algebra A. First of all, we require that I has the smallest element 1 and that

(1) $A_1 \subseteq A_i$ for all $i \in I$;
(2) A_1 is an (involution) simple algebra;
(3) the restriction of any nontrivial A_i-module to A_1 is nontrivial.

Note that (3) implies that

(4) for each (involution) simple component T of $A_i/\operatorname{Rad} A_i$ one has $\dim T \geq \dim A_1$.

Let N be a proper ($*$-invariant) ideal of A_i. Since A_i is perfect, the image of N in $A_i/\operatorname{Rad} A_i$ is a proper ideal. Hence the codimension of N in A_i is not less than the minimal dimension of (involution) simple components of $A_i/\operatorname{Rad} A_i$. Combining this with (4), we get the following property of conical systems:

(5) for each $i \in I$ and each proper ($*$-invariant) ideal N of A_i, one has $\operatorname{codim} N \geq \dim A_1$.

By the *rank* of a conical system we mean the dimension of A_1.

Proposition 2.9 *Let A be an (involution) simple locally finite associative algebra and let $\mathfrak{A} = \{A_i\}_{i\in I}$ be a perfect ($*$-invariant) local system of A. Fix $k \in I$. Let S be a (involution) simple component of a ($*$-invariant) Levi subalgebra of A_k. Denote by A_i^S ($i \geq k$) the two-sided ideal of A_i generated by S and set $A_1^S = S$. Put $I^S = \{i \in I \mid i \geq k\} \cup \{1\}$. Then $\mathfrak{A}^S = \{A_i^S\}_{i\in I^S}$ is a conical local system of A.*

Proof. Clearly, A_i^S is a perfect ($*$-invariant) ideal of A_i and $A_i^S \subset A_j^S$ whenever $i \leq j$. Therefore $A^S = \varinjlim A_i^S$ is a ($*$-invariant) ideal of A. Since A is (involution) simple, $A^S = A$, so $\{A_i^S\}_{i\in I^S}$ is a perfect ($*$-invariant) local system of A. The properties (1), (2), and (3) of the definition of a conical system are satisfied in an obvious manner. □

Now we can apply the argument in [3, Theorem 3.3] and Proposition 2.8 to easily get the following.

Proposition 2.10 *Let A and $\mathfrak{A}$ be as in Proposition 2.9. Then for any $i \in I$ there exists $j \geq i$ and a maximal ($*$-invariant) ideal P_j of A_j such that $P_j \cap A_i = 0$. In particular, a ($*$-invariant) Levi subalgebra of A_j has a (involution) simple component S with $\dim S \geq \dim A_i$.*

Combining this with Proposition 2.9 we derive the following

Corollary 2.11 *Locally finite (involution) simple associative algebras have conical ($*$-invariant) local systems of arbitrary large ranks.*

Finally, we can prove our first important result.

Theorem 2.12 (1) *Let A be a simple locally finite associative algebra. Set $L = [A, A]$. Then L is a simple plain locally finite Lie algebra and A is a $\mathfrak{P}$-envelope of L.*

(2) *Let A be an involution simple locally finite associative algebra. Set $L = \mathfrak{su}^*(A)$. Then L is a simple diagonal locally finite Lie algebra and A is a $\mathfrak{P}^*$-envelope of L.*

Proof. (1) By Corollary 2.11 A has a conical local system $\{A_i\}_{i\in I}$ of rank > 4. Set $L_i = [A_i, A_i]$. By property (5) of conical systems, each A_i has no proper ideals of codimension ≤ 4, i.e. A_i is strongly perfect in the sense of [8]. Hence, by [8, Theorem 6.3(1)], L_i is perfect, A_i is a plain L_i-module with respect to the left regular action, and L_i generates A_i. Therefore L generates A. We also claim that $\{L_i\}_{i\in I}$ is a plain local system of L. Indeed, we need to show that for each pair $L_i \subset L_j$ the corresponding embedding is plain, i.e. the restriction of each standard L_j-module V to L_i is a plain L_i-module. Let $\bar{A}_j = A_j + \mathbb{F}\mathbf{1}$ be the algebra that is obtained from A by external adjointing the identity. Then $\bar{A}_j$ is a faithful plain L_j-module. Thus V is a composition factor of its submodule A_j, so it is enough to show that the restriction of A_j to L_i is a plain L_i-module. The latter is obvious as this restriction factors through A_i.

Denote by $\operatorname{Rad} A$ the maximal locally nilpotent ideal of A and by $\operatorname{Rad} L$ the maximal locally solvable ideal of L. Clearly $\operatorname{Rad} A \cap A_i \subset \operatorname{Rad} A_i$ and $\operatorname{Rad} L \cap L_i \subset \operatorname{Rad} L_i$ for all i. Note that $\operatorname{Rad} L_i \subset \operatorname{Rad} A_i$ for all i. Indeed, the image of L_i in $Q = A_i/\operatorname{Rad} A_i$ is $[Q, Q]$, which is a direct sum of simple Lie algebras of type A. Fix any $x \in \operatorname{Rad} L$. Then there exists k such that $x \in \operatorname{Rad} L_i$ for all $i \geq k$. Hence $x \in \operatorname{Rad} A_i$ for all $i \geq k$, so $x \in \operatorname{Rad} A$. Therefore $\operatorname{Rad} L \subset \operatorname{Rad} A$. As $\operatorname{Rad} A = 0$, this implies that $\operatorname{Rad} L = 0$.

Now, to prove simplicity of L, one could use the method developed in [6], by showing that the Bratteli diagrams of A and L are identical, and applying a simplicity criterion in terms of Bratteli diagrams (see for example [6, Theorem 3.2]). However it is easier to use a much more general result [11, Theorem 4], which claims that for any simple ring R (of characteristic different from 2), each proper ideal of $[R, R]$ is in the center of R. Since $\operatorname{Rad} L = 0$, we get that L is simple.

(2) By Corollary 2.11 A has a conical $*$-invariant local system $\{A_i\}_{i\in I}$ of rank > 36. Set $L_i = \mathfrak{su}^*(A_i)$. Then by [9, Theorem 6.3], L_i is perfect, A_i is a diagonal L_i-module, and L_i generates A_i. Thus L generates A. Moreover, as in (1), we get that $\{L_i\}_{i\in I}$ is a diagonal local system of L.

It remains to check that L is simple. As in (1), $\operatorname{Rad} L_i \subset \operatorname{Rad} A_i$ for all i. (this follows, for example, from [9, Lemmas 2.3 and 2.4]. Thus $\operatorname{Rad} L = 0$, so we could use a simplicity criterion in terms of Bratteli diagrams [6, Theorem 3.2]. The only difference is that we have to modify a bit the notion of Bratteli diagrams for associative algebras with involution in an obvious way: the nodes of the diagram must correspond to the involution simple components of $A_i/\operatorname{Rad} A_i$ rather than to the ordinary simple components. However it is easier to refer to [11, Theorem 10], which claims that each proper ideal of $\mathfrak{su}^*(A)$ must be in the center of A. Since $\operatorname{Rad} L = 0$, we get that L is simple. □

3 Locally finite Lie algebras

To formulate a Lie algebra analog of Theorem 2.9 (see [6]) we have to define the Lie algebra counterpart of the notion of a conical system. If L is a Lie algebra then a local system $\mathfrak{L} = (L_i)_{i\in I}$ of finite-dimensional perfect subalgebras is called *conical* if I contains the smalest element 1 such that

(1) $L_1 \subseteq L_i$ for all $i \in I$;
(2) L_1 is simple;
(3) for each $i \in I$ the restriction of any standard L_i-module to L_1 is nontrivial.

Note that (3) implies that

(4) for each $i \in I$ and each simple component T of $L_i/\operatorname{Rad} L_i$, one has $\dim T \geq \dim L_1$.

Let N be a proper ideal of L_i. Since L_i is perfect, the image of N in $L_i/\operatorname{Rad} L_i$ is a proper ideal. Hence the codimension of N in L_i is not less than the minimal dimension of simple components of $L_i/\operatorname{Rad} L_i$. Combining this with (4), we get the following property of conical systems:

(5) for each $i \in I$ and each proper ideal N of L_i, one has $\operatorname{codim} N \geq \dim L_1$.

By the *rank* of the conical system $\mathfrak{L}$ we mean the rank of L_1.

Proposition 3.1 *Let L be a simple locally finite Lie algebra and let $\mathfrak{L} = \{L_i\}_{i\in I}$ be a perfect local system for L. Fix $k \in I$. Let S be a simple component of a Levi subalgebra of L_k. Denote by L_i^S $(i \geq k)$ the ideal of L_i generated by S and set $L_1^S = S$. Put $I^S = \{i \in I \mid i \geq k\} \cup \{1\}$. Then $\mathfrak{L}^S = \{L_i^S\}_{i\in I^S}$ is a (perfect) conical local system of L with the following additional properties.*

(6) *If $\mathfrak{L}$ is diagonal (resp. plain), then $\mathfrak{L}^S$ is diagonal (resp. plain).*
(7) *Let $i \in I$. Fix any $j \in I$ such that $L_i \subset L_j^S$. Assume that the embedding $L_i \subset L_j$ is diagonal (resp. plain). Then the embedding $L_i \subset L_j^S$ is diagonal (resp. plain).*

Proof. Since S is perfect, L_i^S is a perfect ideal of L_i. Indeed, let SL^k denote $[\dots[S,L]\dots L]$ where L occurs k times ($k \geq 0$). Then

$$L_i^S = \sum_{k=0}^{\infty} SL^k = \sum_{k=0}^{\infty} [S,S]L^k = \sum_{k=0}^{\infty} \sum_{i+j=k} [SL^i, SL^j] \subset [L_i^S, L_i^S].$$

Clearly $L_i^S \subset L_j^S$ whenever $i \leq j$. Therefore $L^S = \varinjlim L_i^S$ is an ideal of L. Since L is simple, $L^S = L$, so $\{L_i^S\}_{i \in I^S}$ is a perfect local system of L.

Now the properties (1), (2), and (3) of the definition of the conical system are obvious. The properties (6) and (7) of the Proposition easily follow from the fact that the simple components of $L_i^S / \operatorname{Rad} L_i^S$ are simple components of $L_i / \operatorname{Rad} L_i$. For full details see [6, Section 3]. □

Proposition 3.2 ([3, Theorem 3.3]) *Let L and $\mathfrak{L}$ be as in Proposition 3.1. Then for any $i \in I$ there exists $j \geq i$ and a maximal ideal P_j of L_j such that $P_j \cap L_i = 0$. In particular, a Levi subalgebra of L_j has a simple component S with* $\dim S \geq \dim L_i$.

Corollary 3.3 *Simple locally finite Lie algebras have conical local systems of arbitrary large ranks.*

Let $\mathfrak{L} = \{L_i\}_{i \in I}$ and $\mathfrak{M} = \{M_j\}_{j \in J}$ be perfect local systems of L, and $\Phi = \{\Phi_i\}_{i \in I}$ be an inductive system with respect to $\mathfrak{L}$. For each $j \in J$ fix $k(j) \in I$ such that $M_j \subseteq L_{k(j)}$. Set

$$\Psi_j = \bigcup_{\varphi \in \Phi_{k(j)}} \operatorname{Irr}(\varphi{\downarrow}M_j).$$

Then it is not difficult to see that $\Psi = \{\Psi_j\}_{j \in J}$ is an inductive system with respect to $\mathfrak{M}$. We shall denote this system by $\Phi{\downarrow}\mathfrak{M}$. Clearly,

$$(\Phi{\downarrow}\mathfrak{M}){\downarrow}\mathfrak{L} = \Phi.$$

Proposition 3.4 *Let L be a simple locally finite Lie algebra and let $\mathfrak{L} = \{L_i\}_{i \in I}$ and $\mathfrak{M} = \{M_j\}_{j \in J}$ be two diagonal local systems of L. Let $\Phi = \{\Phi_j\}_{j \in J}$ be a diagonal inductive system with respect to $\mathfrak{M}$. Then $\Phi{\downarrow}\mathfrak{L}$ is a diagonal inductive system with respect to $\mathfrak{L}$.*

Proof. Let $\mathfrak{L}^S = \{L_i^S\}_{i \in I^S}$ be a conical diagonal local system of rank > 10 constructed as in Proposition 3.1. Fix $i \in I^S$ and any $j(i) \in J$ such that $L_i^S \subseteq M_{j(i)}$. Fix any $i' \in I^S$ such that

$$L_i^S \subseteq M_{j(i)} \subseteq L_{i'}^S.$$

Since the embedding $L_i^S \subseteq L_{i'}^S$ is diagonal, by Lemma 2.5, the embedding $L_i^S \subseteq M_{j(i)}$ is diagonal. Therefore the restriction of each standard $M_{j(i)}$-module to L_i^S is diagonal. This implies that $\Psi = \Phi{\downarrow}\mathfrak{L}^S$ is a diagonal inductive system. By Proposition 3.1(7), $\Psi{\downarrow}\mathfrak{L}$ is diagonal. Therefore

$$\Phi{\downarrow}\mathfrak{L} = \Phi{\downarrow}\mathfrak{L}^S{\downarrow}\mathfrak{L} = \Psi{\downarrow}\mathfrak{L}$$

is diagonal, as required. □

Let $\mathfrak{L} = \{L_i\}_{i\in I}$ be a perfect local system of a locally finite Lie algebra L. Let $V_1^i, \ldots, V_{k_i}^i$ be the standard L_i-modules. Set

$$\begin{aligned}
\Pi_i &= \{V_1^i, \ldots, V_{k_i}^i, T_{L_i}\},\\
\Pi_i^* &= \{(V_1^i)^*, \ldots, (V_{k_i}^i)^*, T_{L_i}\},\\
\hat{\Pi}_i &= \Pi_i \cup \Pi_i^* \subset \operatorname{Irr} L_i.
\end{aligned}$$

Lemma 3.5 *Let L be a simple locally finite Lie algebra and $\mathfrak{L} = \{L_i\}_{i\in I}$ be a perfect local system of L. Then $\mathfrak{L}$ is diagonal (resp. plain) if and only if $\hat{\Pi} = \{\hat{\Pi}_i\}_{i\in I}$ (resp. $\Pi = \{\Pi_i\}_{i\in I}$) is an inductive system for L.*

Proof. This follows from the definition of diagonal and plain embeddings. □

We shall need the following easy observation from [5, Section 5].

Lemma 3.6 *Let L be a simple locally finite Lie algebra, $\{L_i\}_{i\in I}$ a perfect local system of L, and $\Phi = \{\Phi_i\}_{i\in I}$ an inductive system. Then for each simple component S of $L_i/\operatorname{Rad} L_i$ there is $\varphi \in \Phi_i$ such that $\varphi{\downarrow}S$ is nontrivial.*

Lemma 3.7 *Let L be a simple locally finite Lie algebra and $\mathfrak{L} = \{L_i\}_{i\in I}$ a perfect local system of L. Suppose that $\Phi = \{\Phi_i\}_{i\in I}$ is a nontrivial diagonal inductive system for $\mathfrak{L}$. Then the following are true.*

(1) *If $\Phi^* = \Phi$, then $\Phi\cup\{T_L\} = \hat{\Pi}$ for all i. In particular, $\mathfrak{L}$ and L are diagonal.*

(2) *If $\Phi^* \neq \Phi$, then L is plain.*

(3) If $\Phi^* \neq \Phi$ and $\mathfrak{L}$ is plain, then $\Phi \cup \{T_L\} = \Pi$ or Π^*.

Proof. (1) It follows from Lemma 3.6 that for each $i \in I$ and each $j = 1, \ldots, k_i$, Φ_i contains V_j^i or $(V_j^i)^*$. Since $\Phi_i^* = \Phi_i$, we have $\Phi_i \cup \{T_{L_i}\} = \hat{\Pi}_i$, as required.

(2) By changing the bases of the root systems (see Remark 2.4) and by using Lemma 3.6, we can assume that Φ_i contains all standard L_i-modules. As $\Phi^* \neq \Phi$, there is $k \in I$ and a simple component S of a Levi subalgebra of L_k such that Φ_k contains the standard S-module W and $W^* \notin \Phi_k$. Let $\mathfrak{L}^S = \{L_i^S\}_{i\in I^S}$ be the conical local system generated by S (see Proposition 3.1). Set $\Psi = \Phi{\downarrow}\mathfrak{L}^S$. Clearly, Ψ is a diagonal inductive system, Ψ_i contains all standard L_i^S-modules,

and $\Psi_1 = \mathrm{Irr}(\Phi_i{\downarrow}S) = \{W\}$. As $\mathfrak{L}^S$ is conical, for each $i \in I^S$ the restriction of any standard M_i-module V to S is nontrivial, so $\mathrm{Irr}(V{\downarrow}S) = \{W\}$. It follows that $\Psi_i \cap \Psi_i^* \subset \{T_{L_i}\}$, so Ψ and $\mathfrak{L}^S$ are plain. Thus L is plain.

(3) Assume $\Phi \cup \{T_L\} \neq \Pi^*$. Set $\Gamma = \Phi \cup \Pi$. In view of Lemma 3.6, it suffices to show that $\Gamma = \Pi$. Clearly Γ is a diagonal inductive system containing Π and $\Gamma \neq \Gamma^*$. Arguing as in (1), we see that there is a conical local system $\mathfrak{L}^S$ of L such that $\Gamma{\downarrow}\mathfrak{L}^S$ is plain. Therefore $\Gamma{\downarrow}\mathfrak{L}^S = \Pi{\downarrow}\mathfrak{L}^S$. Hence $\Gamma = \Pi$, as required. □

Proof of Theorem 1.2. This follows from Theorems 1.1, 2.12(2), and 1.3(2) (which we prove below). □

Proof of Theorem 1.3. Let L be a simple diagonal locally finite Lie algebra. By Proposition 3.1 and Corollary 3.3, there is a diagonal conical local system $\mathfrak{L} = \{L_i\}_{i\in I}$ of rank > 10. Then $\hat{\Pi}$ is a diagonal inductive system with respect to $\mathfrak{L}$. Let $\bar{\Pi}$ be the reduced inductive system obtained from $\hat{\Pi}$. Then by Lemma 3.7, $\bar{\Pi}$ is the only diagonal reduced inductive system for L. Let the ideals $N(\bar{\Pi})$ and $M(\bar{\Pi})$ of $A(L)$ be as in Theorem 2.7. Set

$$\mathcal{N} = A(L)/N(\bar{\Pi}),$$
$$\mathcal{M} = A(L)/M(\bar{\Pi}).$$

By Theorem 2.7, $\mathcal{M} = \mathcal{N}/\mathcal{R}$ where $\mathcal{R}$ is the Jacobson radical of $\mathcal{N}$. Since L is simple, $M(\bar{\Pi}) \cap L = 0$, so $\mathcal{N}$ and $\mathcal{M}$ are enveloping algebras of L. Consider the left regular action of $A(L_i)$ on

$$V_i = U(L_i)/(N(\bar{\Pi}) \cap A(L_i)).$$

This makes V_i into a faithful L_i-module with $\overline{\mathrm{Irr}}\, V_i = \bar{\Pi}_i$, so V_i is diagonal. Observe that $N(\bar{\Pi})$ is invariant under the action of the standard involution of $A(L_i)$. Therefore V_i is selfdual (hence $*$-plain in the sense of [9]). Clearly

$$\mathcal{N}^i = A(L_i)/(N(\bar{\Pi}) \cap A(L_i))$$

is an enveloping algebra of L_i in $\mathrm{End}\, V_i$ with involution inherited from the standard involution of $A(L_i)$. Therefore by [9, Theorem 1.3(2)], $\mathcal{N}^i$ is a $\mathfrak{P}^*$-envelope of L_i, i.e. $\mathfrak{su}^*(\mathcal{N}^i) = L_i$. Hence $\mathfrak{su}^*(\mathcal{N}) = L$, i.e. $\mathcal{N}$ is a $\mathfrak{P}^*$-envelope of L. Since $\mathcal{R} \cap L = 0$, we also have that $\mathfrak{su}^*(\mathcal{M}) = L$.

Let us prove that $\mathcal{M}$ is involution simple. Let Q be any proper $*$-invariant ideal of $\mathcal{M}$. As L generates $\mathcal{M}$ and L is simple, one has $Q \cap L = 0$. Thus the reduced inductive system corresponding to the preimage of Q in $A(L)$ is a nontrivial $*$-invariant diagonal subsystem Ψ of $\bar{\Pi}$. Hense by Lemma 3.7(1),

$\Psi = \bar{\Pi}$. Thus by Theorem 2.7, $Q = 0$. Therefore $\mathcal{M}$ is involution simple, and (2) is proved.

Set $\mathcal{R}^i = \mathcal{R} \cap \mathcal{N}^i$. Then $\mathcal{R}^i$ is a $*$-invariant ideal of $\mathcal{N}^i$ with $\mathcal{R}^i \cap L_i = 0$. Therefore $\mathcal{N}^i/\mathcal{R}^i$ is a $\mathfrak{P}^*$-envelope of L. Now [9, Theorem 6.5] implies that $\mathcal{R}^i$ annihilates $\mathcal{N}^i$. Therefore $\mathcal{R}$ annihilates $\mathcal{N}$, so (1) is proved.

Let us prove (3). Represent A as $A(L)/H_A$ where H_A is an ideal of $A(L)$. In view of Theorem 2.7, it suffices to show that the reduced inductive system corresponding to H_A coincides with $\bar{\Pi}$. Let $\{A_i\}_{i\in I}$ be a $*$-invariant local system of A. Since L is simple, by Proposition 3.2, L has a finite dimensional simple subalgebra Q of rank greater than 8. Fix any i such that A_i contains Q and fix any $*$-invariant Levi subalgebra of A_i. As $Q \subset \mathfrak{su}^*(A_i)$, the Levi subalgebra has a $*$-simple component S of sufficiently large dimension (we need > 36). Let $A^S = \varinjlim A_i^S$ be the ideal of A generated by S (see proof of Proposition 2.9). Note that A^S is $*$-invariant and $\{A_i^S\}_{i\in I^S}$ is a conical local system of A^S. Observe that $\mathfrak{su}^*(A^S)$ is an ideal of L. Since L is simple and $\mathfrak{su}^*(A^S) \supset \mathfrak{su}^*(S) \neq 0$, we get that $\mathfrak{su}^*(A^S) \supset L$. As L generates A, we have $A^S = A$. Thus $\{A_i^S\}_{i\in I^S}$ is a conical local system of A of rank greater than 36.

By property (5) of conical systems, each A_i^S has no proper $*$-invariant ideals of codimension ≤ 36. In particular, A_i^S is admissible in the sense of [9]. Thus, by [9, Theorem 6.3], $M_i := \mathfrak{su}^*(A_i^S)$ is a perfect Lie algebra and A_i^S is a selfdual diagonal (or equivalently, $*$-plain) M_i-module (with respect to the regular action). Clearly, $\mathfrak{M} = \{M_i\}_{i\in I^S}$ is a local system of L and $\Phi = \{\Phi_i\}_{i\in I^S}$ with $\Phi_i = \overline{\mathrm{Irr}}(A_i^S)$ is a diagonal selfdual reduced inductive system for L with respect to $\mathfrak{M}$. By Proposition 3.4, $\Phi{\downarrow}\mathfrak{L}$ is a diagonal selfdual inductive system. Therefore by Lemma 3.7(1) $\Phi{\downarrow}\mathfrak{L} = \bar{\Pi}$, as required. □

Proof of Corollary 1.4. Let us denote by f_1 and f_2 the mappings $L \mapsto \mathcal{M}$ and $A \mapsto \mathfrak{su}^*(A)$, respectively. By Theorem 1.3, $f_2 f_1(L) = L$. Let A be an involution simple locally finite associative algebra. Then by Theorem 2.12(2) $L = \mathfrak{su}^*(A)$ is a diagonal simple Lie algebra and A is a $\mathfrak{P}^*$-envelope of L Therefore by Theorem 1.3, $A = \mathcal{M}$, so $f_1 f_2(A) = A$, as required. □

Proof of Theorem 1.5. Let L be a simple plain locally finite Lie algebra. By Corollary 3.3, there is a plain conical local system $\mathfrak{L} = \{L_i\}_{i\in I}$ of rank > 10. Then Π is a plain inductive system with respect to $\mathfrak{L}$. Let Π_+ (resp. Π_-) be the reduced inductive system obtained from Π (resp. Π^*). Then by Lemma 3.7(3), Π_+ and Π_- are the only non-selfdual diagonal reduced inductive systems. Let the ideals $N(\Pi_\pm)$ and $M(\Pi_\pm)$ of $A(L)$ be as in Theorem 2.7. Set

$$\mathcal{N}_\pm = A(L)/N(\Pi_\pm),$$
$$\mathcal{M}_\pm = A(L)/M(\Pi_\pm).$$

By Theorem 2.7, $\mathcal{M}_\pm = \mathcal{N}_\pm/\mathcal{R}_\pm$ where $\mathcal{R}_\pm$ is the Jacobson radical of $\mathcal{N}_\pm$. Since L is simple, $M(\Pi_\pm) \cap L = 0$, so $\mathcal{N}_\pm$ and $\mathcal{M}_\pm$ are enveloping algebras of L. Consider the left regular action of $A(L_i)$ on

$$V_i = U(L_i)/N(\Pi_+) \cap A(L_i).$$

This makes V_i into a faithful L_i-module with $\overline{\operatorname{Irr}}\, V_i = (\Pi_+)_i$, so V_i is plain. Clearly

$$\mathcal{N}_+^i = A(L_i)/(N(\Pi_+) \cap A(L_i))$$

is an enveloping algebra of L_i in $\operatorname{End} V_i$. Therefore by [8, Theorem 1.5], $\mathcal{N}_+^i$ is a $\mathfrak{P}$-envelope of L_i, i.e. $[\mathcal{N}_+^i, \mathcal{N}_+^i] = L_i$. Hence $[\mathcal{N}_+, \mathcal{N}_+] = L$, i.e. $\mathcal{N}_+$ is a $\mathfrak{P}$-envelope of L. Arguing similarly, we conclude that $\mathcal{N}_\pm$ and $\mathcal{M}_\pm$ are $\mathfrak{P}$-envelopes of L.

Let us prove that $\mathcal{M}_\pm$ is simple. Let Q be any proper ideal of $\mathcal{M}_\pm$. As L generates $\mathcal{M}_\pm$ and L is simple, one has $Q \cap L = 0$. Thus the reduced inductive system corresponding to the preimage of Q in $A(L)$ is a nontrivial diagonal subsystem Ψ of $\Pi_\pm$. Hense by Lemma 3.7(3), $\Psi = \Pi_\pm$. Thus by Theorem 2.7, $Q = 0$. Therefore $\mathcal{M}_\pm$ is simple, and (2) is proved.

Set $\mathcal{R}_\pm^i = \mathcal{R}_\pm \cap \mathcal{N}_\pm^i$. Then $\mathcal{R}_\pm^i$ is an ideal of $\mathcal{N}_\pm^i$ with $\mathcal{R}_\pm^i \cap L_i = 0$. Therefore $\mathcal{N}_\pm^i/\mathcal{R}_\pm^i$ is a $\mathfrak{P}$-envelope of L. Now [8, Theorem 6.10(6)] implies that $\mathcal{R}_\pm^i$ annihilates $\mathcal{N}_\pm^i$. Therefore $\mathcal{R}_\pm$ annihilates $\mathcal{N}_\pm$, so (1) is proved.

Let us prove (3). Represent A as $A(L)/H_A$ where H_A is an ideal of $A(L)$. In view of Theorem 2.7, it suffices to show that the reduced inductive system corresponding to A coincides with Π_+ or Π_-. Let $\{A_i\}_{i\in I}$ be a local system of A. Since L is simple, by Proposition 3.2, L has a finite dimensional simple subalgebra Q of rank greater than 8. Fix any i such that A_i contains Q and fix any Levi subalgebra of A_i. As $Q \subset [A_i, A_i]$, the Levi subalgebra has a simple component S of sufficiently large dimension (we need > 4). Let $A^S = \varinjlim A_i^S$ be the ideal of A generated by S (see proof of Proposition 2.9). Note that $\{A_i^S\}_{i\in I^S}$ is a conical local system of A^S. Observe that $[A^S, A^S]$ is an ideal of L. Since L is simple and $[A^S, A^S] \supset [S, S] \neq 0$, we get that $[A^S, A^S] \supset L$. As L generates A, we have $A^S = A$. Thus $\{A_i^S\}_{i\in I^S}$ is a conical local system of A of rank greater than 4.

By property (5) of conical systems, each A_i^S has no proper ideals of codimension ≤ 4, i.e. A_i^S is strongly perfect in the sense of [8]. Thus, by [8, Theorem 6.3(1)], $M_i = [A_i^S, A_i^S]$ is a perfect plain Lie algebra and A_i^S is a plain M_i-module (with respect to the regular action). Clearly, $\mathfrak{M} = \{M_i\}_{i\in I^S}$ is a local system of L and $\Phi = \{\Phi_i\}_{i\in I^S}$ with $\Phi_i = \overline{\operatorname{Irr}}(A_i^S)$ is a plain reduced inductive system for L with respect to $\mathfrak{M}$. By Proposition 3.4, $\Phi\downarrow\mathfrak{L}$ is a diagonal inductive system. Since $\Phi = (\Phi\downarrow\mathfrak{L})\downarrow\mathfrak{M}$ and Φ is non-selfdual, $\Phi\downarrow\mathfrak{L}$ is non-selfdual. Therefore $\Phi\downarrow\mathfrak{L} = \Pi_\pm$, as required.

(4) follows from the duality of Π_+ and Π_-. Indeed, consider the opposite algebra $\mathcal{N}_+^{op}$, i.e. the vector space $\mathcal{N}_+$ with new multiplication $\cdot$ defined as $a \cdot b = ba$. Since $a \cdot b - b \cdot a = ba - ab = -[a, b]$, the mapping $x \mapsto -x$ is a Lie homomorphism of L into $\mathcal{N}_+^{op}$, which can be extended to an antiisomorphism of $\mathcal{N}_+$ onto $\mathcal{N}_+^{op}$. Clearly the image of L generates $\mathcal{N}_+^{op}$ as an associative algebra, so $\mathcal{N}_+^{op}$ is a locally finite envelope of L. Observe that the corresponding reduced inductive system is Π_-. Therefore by Theorem 2.7, $\mathcal{N}_+^{op} \cong \mathcal{N}_-$, as required. □

Proof of Corollary 1.6. Let us denote by f_1 and f_2 the mappings $L \mapsto \mathcal{M}_+$ and $A \mapsto [A, A]$, respectively. By Theorem 1.5, $f_2 f_1(L) = L$. Let A be a simple locally finite associative algebra. Then by Theorem 2.12(1), $L = [A, A]$ is a plain simple Lie algebra and A is a $\mathfrak{P}$-envelope of L. Therefore by Theorem 1.5, $A = \mathcal{M}_\pm$, so $f_1 f_2(A) = A$, as required. □

Proof of Theorem 1.8. Let L be a simple plain locally finite Lie algebra and let $\mathcal{M}_\pm$ be as in Theorem 1.5. Set $A = \mathcal{M}_+ \oplus \mathcal{M}_-$. Let $\alpha : \mathcal{M}_+ \to \mathcal{M}_-$ be as in Theorem 1.5(4). Recall that α is an antiisomorphism and $\alpha(x) = -x$ for all $x \in L$. Set $(a, b)^* = (\alpha^{-1}(b), \alpha(a))$. Then $*$ is an involution of A. Note that A is involution simple. The embedding $x \mapsto (x, x)$ $(x \in L)$ turns A into an envelope of L. One can easily check that

$$\mathfrak{u}^*(A) = \{(a, -\alpha(a)) \mid a \in \mathcal{M}_+\}$$

Therefore $L = \mathfrak{su}^*(A)$, so A is a simple $\mathfrak{P}^*$-envelope of L. Theorem 1.3 implies that $A = \mathcal{M}$, as required.

Assume now that $\mathcal{M}$ is not simple. Then by Theorem 2.8, $\mathcal{M} = B \oplus B^*$. One can easily check that $\mathfrak{su}^*(\mathcal{M}) \cong [B, B]$, so B is a simple $\mathfrak{P}$-envelope of L. Therefore by Theorem 1.5, $B \cong \mathcal{M}_\pm$. □

In the example below we construct a simple plain locally finite Lie algebra L such that the radical of $\mathcal{N}_\pm$ is nonzero. Considering the regular representation of $\mathcal{N}_\pm$, we conclude that there exists a non-split extension of a plain L-module V (i.e. the restriction $V\downarrow L_i$ is plain for all i) by the trivial one-dimensional module.

Example 3.8 Let us recall the construction of the algebra L_n $(n \geq 3)$ from [8, Example 6.12]. We denote by L_n the Lie algebra of $(3n + 3) \times (3n + 3)$

matrices of the form

$$x = \begin{pmatrix} 0 & x_2 & x_4 & & & \\ & x_0 & x_1 & & & \\ & & 0 & & & \\ & & & x_0 & x_1 & x_3 \\ & & & & 0 & x_2 \\ & & & & & x_0 \end{pmatrix} \qquad (2)$$

where x_0 runs over all $n \times n$ matrices with zero traces; x_1, x_2, x_3 run over all matrices of sizes $n \times 1$, $1 \times n$, $n \times n$, respectively; $x_4 = -\operatorname{tr} x_3$ (one can see that x_4 is a 1×1 matrix); and all empty spaces are zero matrices. Let $x \in L_n$ and $m = 3n+1$. We denote by $\pi_n(x)$ the element y of L_m such that

$$y_0 = \begin{pmatrix} x_0 & & & \\ & x_0 & x_1 & x_3 \\ & & 0 & x_2 \\ & & & x_0 \end{pmatrix}, \quad y_1 = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad y_3 = \begin{pmatrix} x_3 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

$y_2 = (x_2\, 0\, \dots\, 0)$, and $y_4 = x_4$. Then the mapping $\pi_n : L_n \to L_m$ is an injective homomorphism. We will identify L_n with $\pi_n(L_n)$. Obviously, $\operatorname{Rad} L_m = \{y \in L_m \mid y_0 = 0\}$, so $L_n \cap \operatorname{Rad} L_m = 0$. Let A_n be the enveloping algebra of L_n in our matrix representation (2). It was shown in [8, Example 6.12] that A_n is a universal $\mathfrak{P}$-envelope of L_n and A_n consists of all matrices of the form (2) where $x_0, \dots, x_4$ are arbitrary. Denote by R_n the two-sided annihilator of A_n in A_n. Clearly, R_n is the one-dimensional subspace consisting of matrices with $x_0 = x_1 = x_2 = x_3 = 0$. Note that the associative subalgebra of A_m generated by L_n is isomorphic to A_n. Identifying this subalgebra with A_n, we see that $R_n = R_m$. Let H_n and $\bar{H}_n$ be the kernels of the canonical homomorphisms $A(L_n) \to A_n$ and $A(L_n) \to A_n/R_n$, respectively. It follows from the arguments above that $H_m \cap A(L_n) = H_n$ and $\bar{H}_m \cap A(L_n) = \bar{H}_n$. Let us denote by L the direct limit of the sequence

$$L_n \xrightarrow{\pi_n} L_{3n+1} \xrightarrow{\pi_{3n+1}} L_{9n+4} \xrightarrow{\pi_{9n+4}} \dots$$

Obviously, $A(L)$ has ideals H and $\bar{H}$ such that $H \cap A(L_n) = H_n$ and $\bar{H} \cap A(L_n) = \bar{H}_n$ for each algebra L_n in the sequence. Since $L_n \cap \operatorname{Rad} L_m = 0$, and $L_m / \operatorname{Rad} L_m$ is simple, the algebra L is simple. One can easily check that $A = A(L)/H$ is a universal $\mathfrak{P}$-envelope of L, $R = \bar{H}/H$ is the Jacobson radical (and the annihilator) of A of dimension 1, and $\bar{A} = A/R$ is a simple $\mathfrak{P}$-envelope of L.

4 Generalized root-graded locally finite Lie algebras

In this section we link diagonal locally finite Lie algebras to root-graded Lie algebras. We show that each simple root-graded locally finite Lie algebra is diagonal and the converse is also true, provided we generalize slightly the notion of root-graded Lie algebras.

Definition 4.1 Let Δ be a root system of type X_n $(X = A, \dots, G)$ and let $P(\Delta)$ be the group of integral weights of Δ. Let Γ be a subset of $P(\Delta)$ containing Δ and 0. A Lie algebra L is called Γ-*graded* if

(Γ1) L contains as a Lie subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ whose root system is Δ relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;

(Γ2) $L = \bigoplus_{\mu\in\Gamma} L_\mu$ where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$;

(Γ3) $L_0 = \sum_{-\mu,\mu\in\Gamma\setminus\{0\}}[L_{-\mu}, L_\mu]$.

The subalgebra $\mathfrak{g}$ is called the *grading subalgebra* of L.

The assumption (Γ3) is included for nondegeneracy (e.g. consider $L = \mathfrak{g} \oplus M$ where $\mathfrak{g}$ and M are ideals of L). However the following trivial observation can be useful.

Lemma 4.2 *Let $\mathfrak{g}$ and Γ be as in Definition 4.1. Assume that a Lie algebra L satisfies* (Γ1) *and* (Γ2)*. Then the space*

$$S = \Big(\sum_{\mu\in\Gamma\setminus\{0\}} L_\mu\Big) \bigoplus \sum_{-\mu,\mu\in\Gamma\setminus\{0\}} [L_{-\mu}, L_\mu]$$

is a nonzero Γ-graded ideal of L. In particular, if L is simple then L is Γ-graded.

Proof. Using $[L_\nu, L_\mu] \subset L_{\nu+\mu}$, one can easily check that $[x, S] \subset S$ for all $x \in L_\nu$, so S is an ideal of L. Clearly S is Γ-graded. □

Let $\mathfrak{g}$ and Δ be as in Definition 4.1. Asssume that Δ is classical of type A_n, B_n, C_n, or D_n. We denote by $\bar{\Delta}$ all weights of the $\mathfrak{g}$-module

$$\bar{V} = (V \otimes V) \oplus (V \otimes V^*) \oplus (V^* \otimes V^*) \oplus V \oplus V^* \oplus T_\mathfrak{g}$$

where V is the standard $\mathfrak{g}$-module (of weight λ_1) and $T_\mathfrak{g}$ is the trivial 1-dimensional $\mathfrak{g}$-module. If $\mathfrak{g}$ is of type A_n, denote by $\bar{\Delta}_A$ all weights of the $\mathfrak{g}$-module

$$\bar{V}_A = (V \otimes V^*) \oplus V \oplus V^* \oplus T_\mathfrak{g}.$$

Obviously $\bar{\Delta}$ and $\bar{\Delta}_A$ contain Δ and 0.

Theorem 4.3 *Let L be an infinite-dimensional simple locally finite Lie algebra.*

(1) *Assume that L is Γ-graded where Γ is finite. Then L is diagonal.*

(2) *Assume that L is diagonal. Then L is $\bar{\Delta}$-graded for each root system Δ of classical type (A_n, B_n, C_n, or D_n). Moreover, if L is plain then L is $\bar{\Delta}_A$-graded.*

Proof. (1) Let $\mathfrak{g}$ be the grading subalgebra of L and let Δ be its root system. Fix any root $\alpha \in \Delta$ and pick any non-zero $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Then

$$S = \langle x_\alpha, x_{-\alpha}, [x_\alpha, x_{-\alpha}] \rangle_{\mathbb{F}}$$

is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. Let Γ_1 be the restriction of Γ to the Cartan subalgebra $\mathbb{F}[x_\alpha, x_{-\alpha}]$ of S. Then Γ_1 is finite and L is Γ_1-graded with the grading subalgebra S of type A_1. Indeed the properties (Γ1) and (Γ2) obviously hold and (Γ3) follows from Lemma 4.2.

Let $\mathfrak{L} = \{L_i\}_{i \in I}$ be a perfect local system for L containing S. Let $\mathfrak{L}^S = \{L_i^S\}_{i \in I^S}$ be the conical local system generated by S, so $L_1^S = S$. Put

$$\Phi_i = \operatorname{Irr}(L{\downarrow}L_i^S) := \bigcup_{j \geq i} \operatorname{Irr}(L_j^S{\downarrow}L_i^S)$$

where L and L_j^S are considered as L_i^S-modules with respect to the adjoint action. Then clearly

$$\cup_{\varphi \in \Phi_j} \operatorname{Irr}(\varphi{\downarrow}L_i^S) = \Phi_i$$

for each pair $i < j$. Thus $\Phi = \{\Phi_i\}_{i \in I^S}$ is an inductive system for L if we show that each Φ_i is a finite set. Note that $\Phi_1 = \Gamma_1$, so Φ_1 is finite. Assume that Φ_i is infinite for some i. Since $S \cap \operatorname{Rad} L_i = 0$ and $\operatorname{Rad} L_i$ annihilates all irreducible L_i-modules, without loss of generality we can assume that L_i is semisimple. Let $L_i = Q_1 \oplus \cdots \oplus Q_k$ be the decomposition of L_i into a sum of simple components Q_j. Then each $\varphi \in \Phi_i$ can be written as $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_k$ where φ_j is an irreducible Q_j-module. Clearly

$$\varphi{\downarrow}S = (\varphi_1{\downarrow}S_1) \otimes \cdots \otimes (\varphi_k{\downarrow}S_k)$$

where S_j is the projection of S into Q_j. By Proposition 3.1(3), each projection S_j is nontrivial, so $S_j \cong S \cong \mathfrak{sl}_2$. Since Φ_i is infinite, there exists j such that the set

$$\Phi_i^j = \{\varphi_j \mid \varphi \in \Phi_i\} \subset \operatorname{Irr} Q_j$$

is infinite. Hence the set $\operatorname{Irr}(\Phi_i^j{\downarrow}S_j)$ is also infinite (this follows, for example, from [5, Lemma 6.5]). This implies that

$$\Gamma_1 = \Phi_1 = \operatorname{Irr}(\Phi_i{\downarrow}S) = \operatorname{Irr}(\operatorname{Irr}(\Phi_i^1{\downarrow}S_1) \otimes \cdots \otimes \operatorname{Irr}(\Phi_i^k{\downarrow}S_k))$$

is infinite, which contradicts the assumption. Thus each Φ_i is finite, so $\Phi = \{\Phi_i\}_{i\in I^S}$ is a nontrivial inductive system for L. Thus by [6, Corollary 3.9], L is diagonal.

(2) Let Δ be a root system of classical type. By Proposition 3.1 and Corollary 3.3, L has a conical diagonal local system $\mathfrak{L} = \{L_i\}_{i\in I}$ of sufficiently large rank, so that L_1 contains a diagonally embedded simple Lie algebra $\mathfrak{g}$ with root system Δ.

Fix any $i \in I$. By Proposition 3.2, there exists $j > i$ such that the $\mathfrak{g}$-module L_i is isomorphic to a submodule of the $\mathfrak{g}$-module L_j/P_j where P_j is a maximal ideal of L_j. Now L_j/P_j is isomorphic to a submodule of the $\mathfrak{g}$-module $W\otimes W^*$ where W is a standard L_j-module. Since the embedding $\mathfrak{g} \to L_j$ is diagonal,

$$\mathrm{Irr}(W{\downarrow}\mathfrak{g}) \subset \{V, V^*, T_{\mathfrak{g}}\}.$$

Therefore the weights of the $\mathfrak{g}$-module $W \otimes W^*$ belong to $\bar{\Delta}$, so the weights of the $\mathfrak{g}$-module L_i belong to $\bar{\Delta}$. Thus L is $\bar{\Delta}$-graded.

If L is plain, then we can assume that $\mathfrak{L} = \{L_i\}_{i\in I}$ is a plain local system for L. Thus

$$\mathrm{Irr}(W{\downarrow}\mathfrak{g}) \subset \{V, T_{\mathfrak{g}}\},$$

so the weights of the $\mathfrak{g}$-module $W \otimes W^*$ belong to $\bar{\Delta}_A$. Therefore L is $\bar{\Delta}_A$-graded.

□

Notice that in the case where $\Delta = B_n$, C_n, or D_n the set $\bar{\Delta}$ reduces to the set of all weights of the $\mathfrak{g}$-module

$$\bar{V} = (V \otimes V) \oplus V \oplus T_{\mathfrak{g}}.$$

This means that $\bar{\Delta}$-graded algebras are actually BC_r-graded where r is the rank of $\mathfrak{g}$. Therefore, the following is true.

Corollary 4.4 *Each simple diagonal locally finite Lie algebra is BC_r-graded for all $r \geq 1$.*

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