Product of Hybrid Logics

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Purpose of This Talk

- By applying modal logic, we can consider: temporal logic, logic of belief, logic of space, etc.
- Multi-diminensional version of modal logic (Gabbay, D., et al. 2003) allows us to talk about various dimensions at the same time.
- Hybrid Logic is an extension of ML that enables us to deal with both ‘global’ and ‘local’ propositions.
- In this talk, we propose two dimensional hybrid logic.
A Big Question

How can we combine two modal logics?

My Answer in This Talk

If you want to obtain a general completeness result, HYBRIDIZE your logics!
Outline

1. What is Hybrid Logic?
2. How Can We Combine Relational Hybrid Logics?
3. How Can We Combine Topological Hybrid Logics?
Modal Formalism on Kripke Semantics

- $p$ is true at $w$:
  - $p$ holds at the world $w$.
  - $p$ holds at the point of time $w$.
  - $p$ holds at the coordinate $w$.

- $\Box p$ is true at $w$:
  - $p$ is true at all possible worlds relative to $w$.
  - $p$ is true at all points of time later than $w$.
  - $p$ is true at all coordinates within 2km from $w$. 
What is Hybrid Logic?  
How Can We Combine Relational Hybrid Logics?  
How Can We Combine Topological Hybrid Logics?  

Fathers of Hybrid Logics

Prior

Gargov-Passy-Tinchev

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Product of Hybrid Logics
Naming Points
Nominals and Satisfaction Operators
Nominals and Satisfaction Operators
Nominals and Satisfaction Operators
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What is Hybrid Logic?
How Can We Combine Relational Hybrid Logics?
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Hybrid Formalism by Examples

- Nominal $i$ is true at $w$ iff $i$ is a name of $w$.
  - time: 15:00, 16/07/2010, 2010, etc.
  - space: SB2.07, Leicester, UK, etc.
- $@i p$ is true at $w$ iff $p$ is true at the world named by $i$.
  - $@_{13:20}(\text{Mary runs})$
  - $@_{\text{South Africa}}(\text{World Cup 2010 is held})$
- We do not assume that all points are named by some nominals.
- Here we deal with each dimension independently.
A Merit of Hybrid Logic

- 17/07/2010 is future.
- Thus: Katsuhiko will drink much.

Within hybrid logic, we can prove the following as a theorem:

\[ \langle \text{Future} \rangle i \land \@_i p \rightarrow \langle \text{Future} \rangle p, \]

where \( i = '17/07/2010' \) and \( p = '\text{Katsuhiko drinks much}' \).

- Hybrid Logic enables us to formalize the inference containing both local & global information!
Hybrid Definability in Kripke Frames

<table>
<thead>
<tr>
<th>Properties</th>
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<th>HL</th>
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<tr>
<td>Reflexivity</td>
<td>$\Box p \rightarrow p$</td>
<td>$@_i\Diamond i$</td>
</tr>
<tr>
<td>Transitivity</td>
<td>$\Box p \rightarrow \Box \Box p$</td>
<td>$\Box i \rightarrow \Box \Box i$</td>
</tr>
<tr>
<td>Irreflexivity</td>
<td>Undefinable</td>
<td>$\neg @_i\Diamond i$</td>
</tr>
<tr>
<td>Antisymmetry</td>
<td>Undefinable</td>
<td>$@_i\Diamond j \land @_j\Diamond i \rightarrow @_ij$</td>
</tr>
</tbody>
</table>

- Note: $@_ij$ expresses ‘$i = j$’ and $@_i\Diamond j$ expresses ‘$iRj$’.
- $\varphi$ is pure if $\varphi$ contains no ordinary proposition variables.
Let $K_H$ be the axiomatization of hybrid logic.

### Pure Completeness wrt Kripke Semantics

For any set $\Lambda$ of pure formulas, $K_H + \Lambda$ (as new axioms) is strongly complete wrt the class of frames defined by $\Lambda$.

- E.g.: $\varphi$ is a theorem of $K_H + \{ \neg @_i \Diamond i, \Box i \rightarrow \Box \Box i \}$ iff $\varphi$ is valid on any SPOs.
A Need for Two Dimensional Hybrid Logics

Suppose that you are in some downstairs room of SB2.07 on 15/06. Let us consider the following scenario:

"There is the following description in your Google calendar: A logic seminar is held on 16/07 at SB2.07. 16/07 is still future. SB2.07 is an upstairs room of this place. So, the logic seminar will be held in the room overhead."

How can we formalize this inference?

@i@ap \land \langle \text{Future} \rangle i \land \langle \text{Upstairs} \rangle a \rightarrow \langle \text{Future} \rangle \langle \text{Upstairs} \rangle p,

where \( i = '16/07' \), \( a = 'SB2.07' \) & \( p = 'A logic seminar is held' \).
Formalism for Hybrid Product

- Here, we restrict ourselves to two-dimensional talk.
- We need two kinds of Boxes:
  - $\square_1$ (e.g. for time)
  - $\square_2$ (e.g. for space)
- We also need two kinds of nominals:
  - $t$-nominals: $i, j, k, \ldots$
  - $s$-nominals: $a, b, c, \ldots$
- Each kind of nominals has its satisfaction operator:
  - $@_i, @_j, @_k, \ldots$
  - $@_a, @_b, @_c, \ldots$
- But, we have only one kind of proposition letters.
  - Propositions depend on two parameters.
An Example: Product of Kripke Frames
What is Hybrid Logic?
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Product of Kripke Frames

Let $\mathcal{F} = \langle T, R \rangle$ and $\mathcal{G} = \langle X, S \rangle$ be Kripke frames. Then, we define the product of Kripke frames $\mathcal{F} \times \mathcal{G} = \langle T \times X, R_1, R_2 \rangle$ by:

- $\langle t, x \rangle R_1 \langle t', x' \rangle$ iff $tRt'$ and $x = x'$.
- $\langle t, x \rangle R_2 \langle t', x' \rangle$ iff $t = t'$ and $xSx'$.

What is a valuation $V$ on $\mathcal{F} \times \mathcal{G}$?
For any $p$, it suffices to define $V(p) \subseteq T \times X$, i.e. a subset of ‘2D-plane’. How about nominals?
Naming Lines
Semantic Idea behind Two Nominals (1)

To $t$-nominal $i$, we assign a **vertical line** $\{ i^V \} \times T_2$. 
Semantic Idea behind Two Nominals (2)

To s-nominal $a$, we assign a horizontal line $T_1 \times \{a^V\}$. 
Truth Condition for Satisfaction Operators

- $@_i @_a p$ is true at $\langle x, y \rangle$
- iff $@_a p$ is true at $\langle i^V, y \rangle$
- iff $p$ is true at $\langle i^V, a^V \rangle$
- iff $\langle i^V, a^V \rangle \in V(p)$
Semantic Understanding of Our Example (1)

\(@_i@_a p \land \langle \text{Future} \rangle i \land \langle \text{Upstairs} \rangle a \to \langle \text{Future} \rangle \langle \text{Upstairs} \rangle p.\)

@_i@_a p is true at \(\langle t, x \rangle\) iff:

\[ p \]
Semantic Understanding of Our Example (2)

\[ \langle \text{Future} \rangle i \text{ is true at } \langle t, x \rangle \text{ iff:} \]

\[
\begin{align*}
\text{Space:} & \quad \langle X, S \rangle \\
\langle t, x \rangle & \quad i^V \\
\text{Time:} & \quad \langle T, R \rangle
\end{align*}
\]
Semantic Understanding of Our Example (3)

\[ \langle \text{Upstairs} \rangle a \text{ is true at } \langle t, x \rangle \text{ iff:} \]

- Space: \( \langle X, S \rangle \)
- Time: \( \langle T, R \rangle \)

\[ a^V \]

\[ \langle t, x \rangle \]

\[ \text{Time: } \langle T, R \rangle \]
Thus, $\langle \text{Future} \rangle \langle \text{Upstairs} \rangle p$ is true at $\langle t, x \rangle$.  

![Diagram showing the denotation of p in a time-space model with a point at (t, x) marked as the denotation of p.]

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Product of Hybrid Logics
Hilbert-style Axiomatization of Hybrid Products

Roughly, we need the two kinds of axioms and rules: $K_{\mathcal{H}}$ for $\square_1$ and $@_i \land K_{\mathcal{H}}$ for $\square_2$ and $@_a$.
Furthermore, we also need the five ‘interaction’ axioms:

- $@_a @_i p \leftrightarrow @_i @_a p$.
- $@_{SB2.07} @_{16/07} p \leftrightarrow @_{16/07} @_{SB2.07} p$.
- $\Diamond_1 @_a p \leftrightarrow @_a \Diamond_1 p$.
- $\langle \text{Future} \rangle @_{SB2.07} p \leftrightarrow @_{SB2.07} \langle \text{Future} \rangle p$.
- $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$.
- $\langle \text{Upstairs} \rangle @_{16/07} p \leftrightarrow @_{16/07} \langle \text{Upstairs} \rangle p$.
- $@_i a \leftrightarrow a$.
- $@_a i \leftrightarrow i$. 
Interaction Axioms for Two kinds of Nominals

We explain $a \rightarrow @_i a$ alone. Assume that $a$ is true at $\langle t, x \rangle$.

Space: $\langle X, S \rangle$

Time: $\langle T, R \rangle$
Interaction Axioms for Two kinds of Nominals (Cont.)

Then, $a$ is true also at $\langle i^V, x \rangle$, i.e. $\@_i a$ is true at $\langle t, x \rangle$. 
Main Result: Pure Completeness of Hybrid Products

Let $[K_H, K_H]$ be our axiomatization of hybrid products. Let’s call $\varphi$ pure when $\varphi$ contains no ordinary proposition letters.

**Pure Completeness wrt Product Frames (S.201x)**

For any set $\Lambda$ of pure formulas, $[K_H, K_H] + \Lambda$ (as new axioms) is strongly complete wrt the class of product Kripke frames defined by $\Lambda$.

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K. Sano

Axiomatizing hybrid products.

Accepted for publication in *Journal of Applied Logic*. 
Heart of Our Proof

Henkin
A Generalization of Hybrid Products

- We may consider that the accessible space-area varies with the time: $yS(t)y'$ rather than $ySy'$.

- $\Diamond_2 \Diamond p \leftrightarrow \Diamond \Diamond_2 p$ is invalid.

- $\Diamond \Diamond_2 \Diamond p \leftrightarrow \Diamond \Diamond_2 p$ is our new interaction axiom.
Nominals and Satisfaction Operators

Structures (relational, topological) on the domain are irrelevant to a hybridization.

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Nominals and Satisfaction Operators

Structures (relational, topological) on the domain are irrelevant to a hybridization.
Our Guiding Example

Suppose: you drink a lot of various alcohol last night in the reception of CICS. As a result, this morning at the room of Hotel X, you realize that you forget the calendar date of today.
Our Guiding Example (Cont.)

You find a memo: CICS 2010 is NOT held on 15th July at Hotel X. You know that here is Hotel X. After going down-stairs, you find out that CICS 2010 is held in a small region around you (say 2.5m). Therefore, you realize that today is not 15th July.

\[ \@ a \@ i \neg p \land a \land \Box_2 p \rightarrow \neg i. \]

where \( i \) = ‘15/7’, \( a \) = ‘Hotel X’, \( p \) = ‘CICS 2010 is held’, \( \Box_2 \varphi \) = ‘At all the coordinates within 2.5m from here, \( \varphi \) holds’. 
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Topological Reading of Our Spatial Modality

Our motivating example is:

\[ @_a @_i \neg p \land a \land \Box_2 p \rightarrow \neg i, \quad \text{where:} \]

\[ \Box_2 \varphi = \text{‘} \varphi \text{ holds at all the coordinates within 2.5m from here’}. \]

If \( S \) is reflexive, this is valid. This is a theorem of the previous axiomatization \([\mathbf{K_H}, \mathbf{K_H}]\) with \(@_a \Diamond_2 a\) (i.e. \( \mathbf{T} \)-axiom).

But, a weaker reading of \( \Box_2 \) is sufficient for our aim, i.e.:

\[ \Box_2 \varphi = \text{‘} \text{there exists } \varepsilon > 0 \text{ such that } \varphi \text{ holds at all the coordinates within } \varepsilon \text{ m from here’}. \]
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Semantics Based on Products of Topologies

- Assume that both of space and time are topological.
- Put ‘time’ := \( \langle T, \tau \rangle \) and ‘space’ := \( \langle X, \sigma \rangle \), where \( \tau \) (or \( \sigma \)) is a neighborhood map on \( T \) (or \( X \), resp.).
- Define a valuation \( V \) as before.
- Then, we can define topological semantics for \( \square_1 \) and \( \square_2 \) as follows (Van Benthem, et al. 2008):

\[
\langle t, x \rangle \models \square_1 \varphi \text{ iff } \exists N \in \tau(t). \forall t' \in N. \langle t', x \rangle \models \varphi \\
\langle t, x \rangle \models \square_2 \varphi \text{ iff } \exists P \in \sigma(x). \forall x' \in P. \langle t, x' \rangle \models \varphi
\]
Semantic Understanding of Our Guiding Example

Let us check the validity of $\diamond_a \diamond_i \neg p \land a \land \Box_2 p \rightarrow \neg i$.
Assume that the antecedent is true at $\langle x, y \rangle$:
Semantic Understanding of Our Guiding Example

\[ \langle x, y \rangle \models a \iff \]

\[
V(p) \\
\vdash \quad i \\
\langle \vdash i, a \rangle \\
\langle \vdash x, y \rangle \]

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Semantic Understanding of Our Guiding Example

\[ \langle x, y \rangle \Vdash \Box_2 p \text{ iff:} \]

\[ a^V \langle x, y \rangle \vDash \Box^V (p) \]

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Semantic Understanding of Our Guiding Example

⟨x, y⟩ ⊩ □₂p iff:

\[ a^V \langle x, y \rangle \in a^V \langle x, y \rangle \in V(p) \]
Semantic Understanding of Our Guiding Example

\( \langle x, y \rangle \models @i@_a \neg p \iff: \)

\[ V(p) \]

\[ \langle i^V, a^V \rangle \]

\[ i^V \]

\[ a^V \]
We conclude that $x \neq i^V$, i.e., $\langle x, y \rangle \not\models \neg i$: 

![Diagram of the semantic understanding of the guiding example]
Our five interaction axioms are all valid on any product of topologies. But there is one problem in $K_H$.

Here we need to recall what is $BG$. 
Intuitive Meaning of $\mathbf{BG}$ on Kripke Frame

- $\mathbf{BG}$: If $@_i \Diamond \varphi$ is consistent, then $@_i \Diamond j \land @_j \varphi$ is consistent for some fresh $j$.

$$V$$

$$V'$$

extend

$\Diamond \varphi$

$i$

$j$: fresh

$\varphi$

$i$
‘BG = Kripke Frames’ in Topological Setting (Cont.)

- Topologically, Kripke ($\mathbf{S4}$-) frame is equivalent to a space, all of whose states have the smallest neighborhood. Such space is called Alexandrov.
- Let’s say: A space admits $\mathbf{BG}$ if every valuation satisfying $\Diamond_i \varphi$ at some point can be extended to a valuation satisfying $\Diamond_i j \land \Diamond_j \varphi$ ($j$: fresh) at some point.

Ten Cate & Litak (2007)

A space is Alexandrov iff it admits $\mathbf{BG}$.

So, we should drop two kinds of $\mathbf{BG}$ from our axiomatization.
Pure completeness wrt Products of Topologies

Let $\mathbf{S}^{-}_4 = (\mathbf{K}_H - \mathbf{BG}) + \{ \square p \to p, \square p \to \square \square p \}$. Define $[\mathbf{S}^{-}_4, \mathbf{S}^{-}_4]$ as a combination of two $\mathbf{S}^{-}_4$ with our five interaction axioms.

Pure Completeness for Product of Topologies

For any set $\Lambda$ of pure formulas, $[\mathbf{S}^{-}_4, \mathbf{S}^{-}_4] + \Lambda$ is strongly complete wrt the class of product of topologies defined by $\Lambda$.

If we put $\Lambda = \emptyset$, we obtain:

Cor.

$[\mathbf{S}^{-}_4, \mathbf{S}^{-}_4]$ is strongly complete wrt the class of all products of topologies.
Topological Definability in Hybrid Logic

Here ‘definability’ means *definability by a single formula*.

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<tr>
<td>$T_0$</td>
<td>Undef.</td>
<td>$\neg @ i j \rightarrow \neg @ i \diamond j \lor \neg @ j \diamond i$</td>
</tr>
<tr>
<td>$T_1$</td>
<td>Undef.</td>
<td>$\diamond i \rightarrow i$</td>
</tr>
<tr>
<td>$T_2$</td>
<td>Undef.</td>
<td>Undef. by Sustretov (2005)</td>
</tr>
<tr>
<td>density-in-itself</td>
<td>Undef.</td>
<td>$\neg \square i$</td>
</tr>
<tr>
<td>compactness</td>
<td>Undef.</td>
<td>Undef.</td>
</tr>
<tr>
<td>discreteness</td>
<td>$\diamond p \rightarrow p$</td>
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- Undef. in ML is due to the result of McKinsey-Tarski.
- $T_1$ says that $\{x\}$ (e.g. $[x, x]$) is closed.
- Density-in-itself says $\{x\} \notin \tau(x)$. 
How to Capture the Dependence of Space on Time

- Spatial topology might depend on Time.
- Then, our spatial topology and semantics should be defined as:

  \[ (\sigma_t : X \rightarrow \mathcal{P}(X))_{t \in T}. \]

\[ \langle t, x \rangle \models \Box_2 \varphi \text{ iff } \exists P \in \sigma_t(x). \forall x' \in P. \langle t, x' \rangle \models \varphi. \]

- Syntactically, this change corresponds to the following change in our interaction axioms:
  - $\diamond_2 \@_i p \leftrightarrow @_i \diamond_2 p$: Invalid
  - $@_i \diamond_2 @_i p \leftrightarrow @_i \diamond_2 p$: Valid

- We can still establish a pure completeness result.
Overview of Hybrid Product Methods

- There is no need to assume that time and space have the same type of structure.

<table>
<thead>
<tr>
<th>Time \ Space</th>
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<th>Top. $\langle X, \sigma \rangle$</th>
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1. $\@_a \@_i p \leftrightarrow \@_i \@_a p$
2. $\Diamond_1 @_a p \leftrightarrow @_a \Diamond_1 p$: Time str. is independent of Space str.
3. $\Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$: Space str. is independent of Time str.
4. $@_i a \leftrightarrow a$
5. $@_a i \leftrightarrow i$
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1. $@a \Diamond ai p \leftrightarrow @i @a p$
2. $\Diamond_1 @a p \leftrightarrow @a \Diamond_1 p$: Time str. is independent of Space str.
3. $@i \Diamond_2 @i p \leftrightarrow @i \Diamond_2 p$: Space str. depends on Time str.
4. $@i a \leftrightarrow a$
5. $@a i \leftrightarrow i$
Overview of Hybrid Product Methods

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1. $@_a@ip \leftrightarrow @_i@_a p$
2. $@_a \Diamond_1 @_a p \leftrightarrow @_a \Diamond_1 p$: Time str. depends on Space str.
3. $@_i \Diamond_2 @_i p \leftrightarrow @_i \Diamond_2 p$: Space str. depends on Time str.
4. $@_ia \leftrightarrow a$
5. $@_ai \leftrightarrow i$
Summary

- I have shown you how to combine two hybrid logics. A key idea is: **Naming Lines**.
- My method of hybrid product is **modular**. We can cover various ways of ‘combining’ logics.
- My method is also **robust** for completeness results. ANY way of ‘combining’ hybrid logics always enjoys a general completeness result (**pure completeness**).
Further Directions

- **Decidability** is still open, even for hybrid product of Kripke frames.

- Can we find the complete axiomatization of the 2D hybrid logic of $\mathbb{Q} \otimes \mathbb{Q}$ or $\mathbb{R} \otimes \mathbb{R}$?
  - Even if we add $\neg \Box i$ and $\Diamond i \rightarrow i$ to each $\mathbf{S}4_H$, the resulting hybrid product logic is **incomplete** wrt $\mathbb{R} \otimes \mathbb{R}$.

- We can generalize the notion of product to **monotone nbhd frames**.
  - Can we drop the assumption: **monotonicity**?
  - Can we give a **coalgebraic generalization** to this study?
Take-home Message

Naming Lines provides a modular and robust way of combining two hybrid logics

Thank You